Exam in Mathematical Logic
February 3, 2016

Problem 1. Let φ be the propositional formula $p \wedge \neg q$ and let η be a propositional formula with no connectives other than $\Leftrightarrow, \neg$. Prove that $\varphi \Leftrightarrow \eta$ is not a tautology.

Problem 2. Let ξ be the sentence:

$$\forall x\, \forall y\, [f(x) = f(y) \Rightarrow x = y] \Rightarrow [\forall x\, \exists y\, R(f(x), y) \Rightarrow \forall x\, \exists y\, R(x, y)].$$

(a) Find a sentence ψ in prenex normal form logically equivalent to ξ.
(b) Prove that ψ is true in all finite structures.
(c) Prove that $\neg\psi$ is satisfiable.

Problem 3. Decide whether the following sets are definable without parameters in the structure $(\mathbb{Q}, \leqslant^{\mathbb{Q}}, +^{\mathbb{Q}})$:
(a) $\mathbb{Q} \cap (1, +\infty)$,
(b) $\mathbb{Q} \cap (0, +\infty)$.

Problem 4. A *graph* $G = (V, E)$ is a non-empty set V with a symmetric relation $E \subseteq V^2$. Graph G is *connected* if for any $x, y \in V$ there is some $k \in \mathbb{N}$ and a sequence $x_0 = x, ..., x_k = y$ such that $(x_{i-1}, x_i) \in E$ for $i = 1, \ldots, k$.

Let $G_n = (\{0, \ldots, n\}, E_n)$, where $(x, y) \in E_n$ iff $y \equiv x + 1 \pmod{n}$, be a graph consisting of a cycle of $n+1$ elements. Let G be $\prod_{n \in \mathbb{N}} G_n/\mathcal{U}$, where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$. Show that G is not connected.

Problem 5. Prove that there exist structures $\mathbb{A} = (A, +^{\mathbb{A}}, \cdot^{\mathbb{A}}), \mathbb{B} = (B, +^{\mathbb{B}}, \cdot^{\mathbb{B}})$ such that: $(\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}) \equiv \mathbb{A} \preccurlyeq \mathbb{B}$, both A and B have cardinality $\mathfrak{c}$, and $A \neq B$.

Problem 6. Let A be a set and let S be a binary relation on A. Prove that if Player II has a winning strategy in the EF game $\mathrm{G}_3((A, S), (\mathbb{R}, \leqslant))$, then $(A, S) \equiv (\mathbb{R}, \leqslant)$.