

ON BOREL SETS BELONGING TO EVERY INVARIANT CCC
 σ -IDEAL ON $2^{\mathbb{N}}$

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ABSTRACT. Let I_{ccc} be the σ -ideal of subsets of the Cantor group $2^{\mathbb{N}}$ generated by Borel sets which belong to every translation invariant σ -ideal on $2^{\mathbb{N}}$ satisfying the countable chain condition (ccc). We prove that I_{ccc} strongly violates ccc. This generalizes a theorem of Balcerzak-Roslanowski-Shelah stating the same for the σ -ideal on $2^{\mathbb{N}}$ generated by Borel sets $B \subseteq 2^{\mathbb{N}}$ which have perfectly many pairwise disjoint translates. We show that the last condition does not follow from $B \in I_{ccc}$ even if B is assumed to be compact. Various other conditions which for a Borel B imply that $B \in I_{ccc}$ are also studied. As a consequence we prove in particular that:

- If A_n are Borel sets, $n \in \mathbb{N}$, and $2^{\mathbb{N}} = \bigcup_n A_n$, then there is n_0 such that every perfect set $P \subseteq 2^{\mathbb{N}}$ has a perfect subset Q a translate of which is contained in A_{n_0} .
- CH is equivalent to the statement that $2^{\mathbb{N}}$ can be partitioned into \aleph_1 many disjoint translates of a closed set.

1. INTRODUCTION

A σ -ideal on an uncountable Polish space X is a family $I \subseteq \mathcal{P}(X)$ which is closed under taking subsets and countable unions. Throughout the paper we assume that I is *proper*, i.e., $X \notin I$, contains all singletons and every set from I is covered by a Borel set from I . We say that a σ -ideal I on X is *ccc* if there is no uncountable family of disjoint Borel subsets of X outside I .

Our starting point is a question stated in [2] by Balcerzak, Roslanowski and Shelah. Looking at σ -ideals that are *not* ccc they asked what the reasons for the failure of ccc there could be. They considered properties (M) and (D), introduced and investigated earlier by Balcerzak in [1].

We say that a σ -ideal I in a Polish space X has *property (M)* if there is a Borel surjective function $f : X \rightarrow 2^{\mathbb{N}}$ such that all fibers of f are not in I .

If $(G, +)$ is a Polish abelian group, following [22] we denote by $\mathcal{F}_0(G)$ the family of all Borel sets $B \subseteq G$ such that there exists a perfect set $P \subseteq G$ with $(B + x) \cap (B + y) = \emptyset$ for $x, y \in P$ and $x \neq y$. We say that an invariant σ -ideal I on G has *property (D)* if there is a Borel set $B \in \mathcal{F}_0(G) \setminus I$.

Balcerzak (see [1]) observed that (D) implies (M) for every invariant σ -ideal I on G and posed the problem whether the converse holds true. Bukovsky reformulated the Balcerzak's question about (M) \Rightarrow (D) by considering the σ -ideal I_0 on $2^{\mathbb{N}}$

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(where $2^{\mathbb{N}}$ is considered with the coordinatewise addition modulo 2 and referred to as the *Cantor group*) generated by $\mathcal{F}_0(2^{\mathbb{N}})$ and asking if I_0 has property (M). The positive answer to the latter is the content of [2, Theorem 2.1].

A σ -ideal I on a Polish abelian group G is *translation invariant* (shortly: *invariant*), if

$$\forall x \in G \forall A \subseteq G (A \in I \Rightarrow x + A \in I).$$

The idea behind (D) is to single out a particular property (namely, \mathcal{F}_0) of a Borel subset of, say, $2^{\mathbb{N}}$ that prevents it from being a member of *any* invariant ccc σ -ideal and then to ask whether a failure of ccc of an invariant σ -ideal I (even in the strong form of (M)) is always witnessed by an I -positive Borel set with the property under consideration. Balcerzak, Rosłanowski and Shelah gave the negative answer in the case of property \mathcal{F}_0 . Generalizing this result we prove that the idea indicated above cannot be accomplished by *any* property. Namely, in Section 2 we show (see Theorem 2.1) that the σ -ideal I_{ccc} generated by the Borel subsets of $2^{\mathbb{N}}$ which belong to every invariant ccc σ -ideal on $2^{\mathbb{N}}$ has property (M). This is done by adapting the main idea of a simplified proof of the Balcerzak-Rosłanowski-Shelah theorem, due to Solecki [22]. The point is that it is actually easier to prove that I_{ccc} has (M) than it is to prove that so has I_0 . In effect, our argument provides a still simpler proof of the latter as well.

In Section 3 we consider the question which Borel sets belong to I_{ccc} . In particular, we give an example (see Theorem 3.8) of a compact set $C \in I_{ccc} \setminus \mathcal{F}_0(2^{\mathbb{N}})$ (in fact no two different translates of C are disjoint). An example of a compact subset of \mathbb{R} which is not in $\mathcal{F}_0(\mathbb{R})$ but belongs to every invariant ccc σ -ideal on \mathbb{R} is the standard middle 1/3 Cantor set (see Remark 3.10).

Finally, in Section 4 we indicate some links between the subject of this paper and that of covering $2^{\mathbb{N}}$ (or, more generally, an uncountable Polish abelian group) by less than \mathfrak{c} many translates of its closed nowhere dense subset. The latter has been recently dealt with by many authors (see [6], [7], [8], [9], [19], [15]). We prove (see Theorem 4.3) that CH is equivalent to the statement that $2^{\mathbb{N}}$ can be partitioned into \aleph_1 many disjoint translates of a closed set.

2. PROPERTY (M)

Let I_{ccc} be the σ -ideal on $2^{\mathbb{N}}$ generated by Borel sets which belong to every invariant ccc σ -ideal on $2^{\mathbb{N}}$.

Theorem 2.1. I_{ccc} has property (M).

Proof. Following Solecki [22], for each $n \in \mathbb{N}$ we fix a partition of $2^{\mathbb{N}}$ into an F_σ -set A_n^0 of measure 1 and a dense G_δ -set A_n^1 . Next we define $f : (2^{\mathbb{N}})^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by $f((x_n))(i) = 0$ if $x_i \in A_i^0$ and $f((x_n))(i) = 1$ if $x_i \in A_i^1$. Using the canonical topological group isomorphism we view f as a function with domain $2^{\mathbb{N}}$. Our goal is to show that the fibers of f are not only outside I_0 , as shown in [22], but actually even outside I_{ccc} .

To that end, for each $y \in 2^{\mathbb{N}}$ we shall find an invariant ccc σ -ideal I_y on $2^{\mathbb{N}}$ such that $f^{-1}(y) \notin I_y$.

So fix $y \in 2^{\mathbb{N}}$. Let $Y_0 = \{n \in \mathbb{N} : y(n) = 0\}$ and $Y_1 = \{n \in \mathbb{N} : y(n) = 1\}$.

First assume that $Y_0 \neq \emptyset$ and $Y_1 \neq \emptyset$. Using the canonical topological group isomorphism identify $2^{\mathbb{N}}$ with the product group $G = G_0 \times G_1$ where $G_0 = (2^{\mathbb{N}})^{Y_0}$ and $G_1 = (2^{\mathbb{N}})^{Y_1}$. Then $f^{-1}(y) = \prod_{n \in Y_0} A_n^0 \times \prod_{n \in Y_1} A_n^1$ and it suffices to find a

G -invariant ccc σ -ideal I_y on G such that $f^{-1}(y) \notin I_y$. Let $\mathcal{N}(G_0)$ be the σ -ideal of null sets in G_0 (with respect to the product of ordinary measures on 2^{\aleph_1}) and let $\mathcal{M}(G_1)$ be the σ -ideal of meager subsets of G_1 . Note that $\prod_{n \in Y_0} A_n^0 \notin \mathcal{N}(G_0)$ (in fact it is of measure 1) and $\prod_{n \in Y_1} A_n^1 \notin \mathcal{M}(G_1)$ (in fact it is comeager). It follows that $f^{-1}(y) \notin \mathcal{N}(G_0) \otimes \mathcal{M}(G_1)$, where $\mathcal{N}(G_0) \otimes \mathcal{M}(G_1)$, the Fubini product of $\mathcal{N}(G_0)$ and $\mathcal{M}(G_1)$, is the σ -ideal generated by the Borel sets $B \subseteq G_0 \times G_1$ with $\{x \in G_1 : B_x \notin \mathcal{M}(G_1)\} \in \mathcal{N}(G_0)$. By a theorem of Gavalec [12] (see also [11] for a more general result), the σ -ideal $\mathcal{N}(G_0) \otimes \mathcal{M}(G_1)$ is ccc. As it is also easy to see that it is G -invariant, it suffices to let $I_y = \mathcal{N}(G_0) \otimes \mathcal{M}(G_1)$.

If $Y_0 = \emptyset$ or $Y_1 = \emptyset$, then it suffices to let $I_y = \mathcal{M}(G_1)$ in the first case and $I_y = \mathcal{N}(G_0)$ in the second case. □

The following corollary was originally proved by Balcerzak, Roslanowski and Shelah [2, Theorem 2.1] and a simplified proof was found by Solecki [22]. Our approach, though closely related to [22], provides a still simpler proof of it.

Corollary 2.2. I_0 has property (M) .

Proof. As, clearly, $I_0 \subseteq I_{ccc}$, the corollary is an immediate consequence of Theorem 2.1. □

Remark 2.3. Exactly as in [22], Theorem 2.1 can be generalized to the case of the countable product of compact, metric, uncountable abelian groups.

3. WHICH BOREL SETS ARE IN I_{ccc} ?

For a cardinal $0 < \kappa \leq \aleph_1$ we say that a subset A of an abelian, uncountable group G is κ -small, if there is an uncountable set $P \subseteq G$, called a *witness* (of κ -smallness of A), such that $|(g + A) \cap P| < \kappa$ for every $g \in G$ (cf. [6] and [8]). If, moreover, G is Polish and there exists a perfect witness, then we say that A is *perfectly* κ -small. Clearly, \emptyset is the only 1-small subset of G and if $0 < \lambda < \kappa \leq \aleph_1$ and A is λ -small, then it is also κ -small with the same witness. We say that A is small (perfectly small respectively) if it is \aleph_1 -small (perfectly \aleph_1 -small, respectively).

Perfectly small subsets of \mathbb{R} have been studied by Darji and Keleti [6] who proved (see [6, Theorem 2.5]) that every compact subset of \mathbb{R} with packing dimension less than 1 is perfectly small. An example of such a set is the standard middle 1/3 Cantor set.

Following an idea from [6] let us consider the action of G on the product group G^κ by coordinate-wise translations, i.e., the function $F_\kappa : G \times G^\kappa \rightarrow G^\kappa$ defined by:

$$F_\kappa(g, \langle g_\alpha : \alpha < \kappa \rangle) = \langle g + g_\alpha : \alpha < \kappa \rangle.$$

For a set $P \subseteq G$ let

$$IS_\kappa(P) = \{\langle g_\alpha : \alpha < \kappa \rangle \in P^\kappa : \forall \alpha, \beta (\alpha \neq \beta \Rightarrow g_\alpha \neq g_\beta)\}$$

be the set of all injective sequences of length κ of elements of P .

The following simple lemma gives useful characterizations of smallness (cf. [6, Lemma 2.3]).

Lemma 3.1. For a non-zero cardinal $\kappa \leq \aleph_1$ and subsets A and P , $|P| \geq \aleph_1$, of an uncountable abelian group G the following are equivalent:

- (1) P is a witness that A is κ -small,
- (2) $\bigcap_{\alpha < \kappa} (-p_\alpha + A) = \emptyset$ for every κ many different elements $p_\alpha \in P$,
- (3) $F_\kappa[G \times A^\kappa] \cap IS_\kappa(P) = \emptyset$.

Proof. Note that P is *not* a witness that A is κ -small iff for a certain $g \in G$ there are $p_\alpha \in P$, $\alpha < \kappa$, with $\alpha \neq \beta \Rightarrow p_\alpha \neq p_\beta$ such that

$$(A) \{p_\alpha : \alpha < \kappa\} \subseteq g + A.$$

But (A) is equivalent to:

$$(B) -g \in \bigcap_{\alpha < \kappa} (-p_\alpha + A).$$

and also to:

$$(C) \langle p_\alpha : \alpha < \kappa \rangle \in F_\kappa[\{g\} \times A^\kappa].$$

□

In particular, by (2), A is 2-small with a witness of cardinality λ iff there are λ many pairwise disjoint translations of A . Likewise, a Borel subset A of a Polish abelian group G is perfectly 2-small iff $A \in \mathcal{F}_0(G)$.

Also, A is \aleph_0 -small (\aleph_1 -small respectively) iff there is an uncountable set $P \subseteq G$ such that the family $\{g + A : g \in P\}$ is point-finite (point-countable respectively). In the terminology of Kubiś [14] and Mátrai [16], condition (3) says that P is C -homogeneous where C is the complement of $F_\kappa[G \times A^\kappa]$ in G^κ .

Given a group G and a non-empty set $A \subseteq G$, the supremum of cardinalities λ such that there are λ many pairwise disjoint translations of A is sometimes called (see [3]) the *packing index* of A and denoted by $\text{ind}_P(A)$. Therefore, A is 2-small iff $\text{ind}_P(A)$ is uncountable. The packing indices of analytic subsets of Polish group have been studied by Banach, Lyaskovska and Repovš [3].

The following result sheds some light on interesting (and to some extent still unclear) connections between the various notions of smallness introduced above. The particular case of $n = 2$ is implicit in [3, Theorem 1] (cf. Remark 3.3).

Proposition 3.2. Let A be a subset of a Polish, abelian group G .

- (1) Assume that G is σ -compact. If A is F_σ and n -small for a certain n , $0 < n < \aleph_0$, then A is perfectly n -small.
- (2) If A is analytic and \aleph_0 -small with a non-meager witness, then A is perfectly \aleph_0 -small.
- (3) Let $\aleph_1 < \lambda \leq \mathfrak{c}$. It is consistent with ZFC that if A is analytic and \aleph_0 -small with a witness of cardinality λ , then A is perfectly \aleph_0 -small.

Proof. In order to deal with κ -small sets, where $0 < \kappa \leq \aleph_0$, let $C = G^\kappa \setminus F_\kappa[G \times A^\kappa]$. By Lemma 3.1(3), in each case assuming the existence of a C -homogeneous set with certain properties we want to prove the existence of a perfect one.

In case (1) ($\kappa = n$), G and A being σ -compact and F_n being continuous, C is a G_δ subset of G^n and the existence of an uncountable C -homogeneous set implies the existence of a perfect one, by a theorem of Kubiś [14, Corollary 2.3].

In case (2) and (3) ($\kappa = \aleph_0$), A being analytic, C is a co-analytic subset of G^{\aleph_0} , and the existence of a perfect C -homogeneous set is guaranteed by theorems 1.2 and 4.10 of Mátrai [16] respectively. □

As mentioned above, the particular case of 2-small and perfectly 2-small subsets of Polish groups has been studied by Banach, Lyaskovska and Repovš [3] in the context of packing indices and earlier by Balcerzak [1] in the context of property (D). The following characterization of compact perfectly 2-small sets (equivalently: compact members of $\mathcal{F}_0(G)$) is implicit in [3] (see [3, Lemma 1] and the proof of [3, Theorem 1]; see also [1, Proposition 3.1])

Remark 3.3. For a compact subset C of a Polish abelian group G the following are equivalent:

- (1) C is perfectly 2-small,
- (2) $C - C$ does not contain a neighbourhood of the neutral element of G ,
- (3) $C - C$ has empty interior.

Leaving aside interrelations between κ -small and perfectly κ -small sets (see, however, Proposition 3.13), in the next two results we show that all Borel subsets of 2^{\aleph} with any of the properties under consideration, possibly apart from (\aleph_1) -smallness, are in the σ -ideal I_{ccc} . The statement that every small Borel subset of 2^{\aleph} is in I_{ccc} is independent from ZFC.

Recall that $S \subseteq 2^{\aleph}$ is a *Sierpiński set* if S is uncountable but $|S \cap B| \leq \aleph_0$ for every null-set $B \subseteq 2^{\aleph}$.

Proposition 3.4. Let B be a Borel subset of 2^{\aleph} .

- (1) If B is \aleph_0 -small, then $B \in I_{ccc}$.
- (2) Under $\text{MA} + \neg\text{CH}$, if B is small, then $B \in I_{ccc}$.
- (3) If there exists a Sierpiński set $S \subseteq 2^{\aleph}$, then every Borel null-set $B \subseteq 2^{\aleph}$ is small with a witness S . In particular, there is one not in I_{ccc} .

Proof. To prove (1) and (2), assume that B is small and let I be an invariant ccc σ -ideal on 2^{\aleph} . Suppose that $B \notin I$.

If B is, moreover, \aleph_0 -small with a witness P , $|P| \geq \aleph_1$, then, by Proposition 3.4, $\{-g + B : g \in P\}$ is a point-finite family of Borel subsets of 2^{\aleph} not in I . But, I being ccc and P being uncountable, this is impossible by a result of Fremlin (see [10, Lemma 1.E(b)]) and this contradiction ends the proof of (1).

Analogically, if B is small with a witness P , $|P| \geq \aleph_1$, then, by Lemma 3.1, $\{-g + B : g \in P\}$ is a point-countable family of Borel subsets of 2^{\aleph} not in I . But under $\text{MA} + \neg\text{CH}$ this is again prevented by [10, Lemma 1.E(b)] of Fremlin, since then, by a theorem of Martin and Solovay (see, e.g. [18, Theorem 11.1]), no Borel set not in I can be covered by fewer than \aleph_2 members of I .

The first part of (3) is clear. To prove the second one, it suffices to take a dense G_δ nullset B . □

The following result generalizes a theorem of Cichoń, Kharazishvili and Węglorz (cf. [5, Lemma 4]) stating that a perfectly small Borel subset of \mathbb{R} has Lebesgue measure zero.

Theorem 3.5. Every perfectly small Borel subset of 2^{\aleph} is in I_{ccc} .

Proof. The idea of the proof is closely related to a reasoning described at the end of [4, Section 4] by Cichoń, Jasiński, Kamburelis and Szczepaniak.

Fix an invariant ccc σ -ideal I in 2^{\aleph} . Assume that a Borel set $A \subseteq 2^{\aleph}$ is small with a perfect witness $P \subseteq 2^{\aleph}$.

Consider the set:

$$B = \{(x, y) \in 2^{\mathbb{N}} \times P : x + y \in A\}$$

The set B is Borel in $2^{\mathbb{N}} \times P$ and $B_x = P \cap (A + x)$ is countable for each $x \in 2^{\mathbb{N}}$. By a result of Reclaw and Zakrzewski (see [20, Theorem 2.1]) there is $y \in P$ such that $B^y \in I$ (actually, this holds for all but countably many points $y \in P$). But $B^y = A + y$ so $A \in I$ by the invariance of I . □

Remark 3.6. The proof above shows also that every perfectly small Borel subset of a Polish uncountable abelian group G is in the σ -ideal $I_{ccc}(G)$ generated by the Borel subsets of G which belong to every invariant ccc σ -ideal on G .

An immediate consequence of Theorem 3.5 is the fact that Borel perfectly small subsets of $2^{\mathbb{N}}$ generate an invariant σ -ideal on $2^{\mathbb{N}}$. This generalizes a result of Cichoń, Jasiński, Kamburelis and Szczepaniak (see [4, Proposition 4.4]) stating that if \mathbb{R} is the union of two uncountable Borel sets A and B , then $|(A + x) \cap B| = \mathfrak{c}$ for a certain $x \in \mathbb{R}$.

Corollary 3.7. $2^{\mathbb{N}}$ is not the union of any countable collection of Borel perfectly small sets. Equivalently, if A_n are Borel sets, $n \in \mathbb{N}$, and $2^{\mathbb{N}} = \bigcup_n A_n$, then there is n_0 such that every perfect set $P \subseteq 2^{\mathbb{N}}$ has a perfect subset Q a translate of which is contained in A_{n_0} .

We do not know at the moment whether the σ -ideal I_{ccc} is generated by Borel perfectly small subsets of $2^{\mathbb{N}}$. In fact, we do not even know if $I_{ccc} \neq I_0$, the σ -ideal generated by the collection \mathcal{F}_0 of Borel sets which have perfectly many pairwise disjoint translates. The next result yields a much simpler fact that $I_{ccc} \neq \mathcal{F}_0$.

Theorem 3.8. There exists a compact, perfectly 3-small subset C of $2^{\mathbb{N}}$ such that $C + C = 2^{\mathbb{N}}$. In particular, $C \in I_{ccc} \setminus \mathcal{F}_0$.

Proof. First we shall make the following observation.

Lemma 3.9. There are sets $A, S \subseteq \{0, 1\}^4$ such that:

- (1) $A + A = \{0, 1\}^4$,
- (2) $|S| = 3$ and $\bigcap_{s \in S} (s + A) = \emptyset$.

To see this, it suffices to let $A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3, 4\}\}$ and $S = \{\{1\}, \{1, 2\}, \{1, 2, 3, 4\}\}$ where we identify subsets of $\{1, 2, 3, 4\}$ with the corresponding elements of $\{0, 1\}^4$ via their characteristic functions.

Now, to prove the theorem, let $I_k = [4k, 4k + 4) \cap \mathbb{N}$, $k \in \mathbb{N}$, be consecutive 4-element intervals in \mathbb{N} .

Applying Lemma 3.9, for each $k \in \mathbb{N}$ choose sets $A_k, S_k \subseteq \{0, 1\}^{I_k}$ such that

- (1) $A_k + A_k = \{0, 1\}^{I_k}$,
- (2) $|S_k| = 3$ and $\bigcap_{s \in S_k} (s + A_k) = \emptyset$

Define $C = \prod_{k \in \mathbb{N}} A_k$ or, more precisely,

- (3) $C = \{x \in 2^{\mathbb{N}} : \forall k \in \mathbb{N} x|_{I_k} \in A_k\}$.

Then, by (1), we have

- (4) $C + C = 2^{\mathbb{N}}$.

Next consider the set

$$(5) U = \{(x_1, x_2, x_3) \in (2^{\mathbb{N}})^3 : \exists k \in \mathbb{N} \{x_1|I_k, x_2|I_k, x_3|I_k\} = S_k\}.$$

Note that U is a dense open subset of $(2^{\mathbb{N}})^3$. By the Mycielski partition theorem (see [13, Theorem 19.1]), there is a perfect set $P \subseteq 2^{\mathbb{N}}$ such that

$$(6) (x_1, x_2, x_3) \in U \text{ for any three different elements } x_i \in P.$$

In view of (4), to complete the proof it is enough to show that

$$(7) P \text{ is a witness that } C \text{ is 3-small.}$$

So let x_1, x_2, x_3 be three different elements of P . Then, by (6), $(x_1, x_2, x_3) \in U$ so there is $k \in \mathbb{N}$ with $\{x_1|I_k, x_2|I_k, x_3|I_k\} = S_k$ (see (5)).

This implies (see (2)) that $(x_1|I_k + A_k) \cap (x_2|I_k + A_k) \cap (x_3|I_k + A_k) = \emptyset$ and hence (see (3)) $(x_1 + C) \cap (x_2 + C) \cap (x_3 + C) = \emptyset$.

This however, by Lemma 3.1, completes the proof of (7).

The last statement of Theorem 3.8 follows from Remark 3.3. \square

Remark 3.10. A related example is provided by the standard middle 1/3 Cantor set $C \subseteq \mathbb{R}$. By [6, Theorem 2.5], C is perfectly small so by Theorem 3.5, C is in the σ -ideal $I_{ccc}(\mathbb{R})$ (see Remark 3.6). On the other hand, C is not perfectly 2-small since, by a well-known theorem of Steinhaus, $C - C = [-1, 1]$.

From Theorem 3.5 we may obtain a strengthening (for $2^{\mathbb{N}}$) of a theorem of Elekes and Steprans [8] who proved (see [8, Theorem 1.2]), answering a question of Darji and Keleti [6], that there exists a compact subset of \mathbb{R} of Lebesgue measure zero which is not perfectly small.

Proposition 3.11. There exists a compact nullset $C \subseteq 2^{\mathbb{N}}$ which is not in I_{ccc} hence, in particular, it is not perfectly small.

Proof. Any invariant ccc σ -ideal I on $2^{\mathbb{N}}$ of the form $\mathcal{M}_{\mathcal{G}_n^H}$ constructed by Rosłanowski and Shelah [21, Conclusion 4.7] has the property that there is a compact nullset not in I . \square

If we do not require A to be both compact and not in I_{ccc} then we can get more.

Proposition 3.12. In every invariant, ccc σ -ideal in $2^{\mathbb{N}}$:

- (1) there is a Borel set which is not in I_{ccc} ,
- (2) there is a compact set which is not perfectly small.

Proof. To prove (1), note that no invariant, ccc σ -ideal in $2^{\mathbb{N}}$ is contained in I_{ccc} , since the latter is not ccc by Theorem 2.1.

To prove (2), we shall use the following lemma due to Balcerzak, Rosłanowski and Shelah [2, Lemma 2.2]: for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ and a set $A \subseteq \{0, 1\}^m$ such that every n translates of A have non-empty intersection and likewise for $A' = \{0, 1\}^m \setminus A$.

Applying the lemma inductively, choose a strictly increasing sequence $n_k \in \mathbb{N}$, $n_0 = 0$, and associated sequence of sets $A_k \subseteq \{0, 1\}^{I_k}$, where $I_k = [n_k, n_{k+1}) \cap \mathbb{N}$ for $k \in \mathbb{N}$, such that

- (1) every 2^{k+1} translates of A_k by elements of $\{0, 1\}^{I_k}$ have non-empty intersection and likewise for A'_k .

Next, following the proof of [2, Theorem 2.1], define $f : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by $f(x)(k) = 1$ if $x \upharpoonright I_k \in A_k$ and $f(x)(i) = 0$ if $x \upharpoonright I_k \in A'_k$.

Fix an invariant, ccc σ -ideal I on $2^{\mathbb{N}}$. Clearly, each of the sets $f^{-1}(y)$, where $y \in 2^{\mathbb{N}}$, is compact and at least one of them is in I . Without loss of generality assume that the set

$$(2) \ C = \{x \in 2^{\mathbb{N}} : \forall k \in \mathbb{N} \ x \upharpoonright I_k \in A_k\}$$

is in I . We shall prove that C is not perfectly small.

To that end fix an arbitrary perfect set $P \subseteq 2^{\mathbb{N}}$. We will show (cf. [8, Theorem 1.2]) that there exists a perfect set $Q \subseteq P$ and $y \in 2^{\mathbb{N}}$ such that $y + Q \subseteq C$.

Let $T_P \subseteq \{0, 1\}^{<\mathbb{N}}$ be the perfect pruned tree on \mathbb{N} (see [13]) such that $P = [T_P]$, the set of all infinite branches of T_P . We shall find a perfect subtree T_Q of T_P and then define $Q = [T_Q]$.

To construct T_Q , inductively define integers $0 = k_0 < k_1 < k_2 < \dots$ and families $R_i \subseteq \{s \in T_P : \text{length}(s) = n_{k_{i+1}}\}$ such that for each $i \in \mathbb{N}$:

$$(3) \ |R_0| = 2, \forall s \in R_i \ |\{t \in R_{i+1} : s \subseteq t\}| = 2 \text{ and } \forall t \in R_{i+1} \ t \upharpoonright n_{k_{i+1}} \in R_i.$$

Consequently, letting $S_i = \{s \upharpoonright [n_{k_i}, n_{k_{i+1}}) : s \in R_i\}$, we have $|S_i| \leq 2^{i+1}$ hence

$$(4) \ |\{s \upharpoonright I_k : s \in S_i\}| \leq 2^{k+1} \text{ for each } i \in \mathbb{N} \text{ and } k \in [k_i, k_{i+1}).$$

It then follows from (1) and (4) that

$$(5) \ \bigcap_{s \in S_i} ((s \upharpoonright I_k) + A_k) \neq \emptyset \text{ for each } i \in \mathbb{N} \text{ and } k \in [k_i, k_{i+1}).$$

With the help of (5) define $y \in 2^{\mathbb{N}}$ in such a way that for each $i \in \mathbb{N}$ and $k \in [k_i, k_{i+1})$

$$(6) \ y \upharpoonright I_k \in \bigcap_{s \in S_i} ((s \upharpoonright I_k) + A_k),$$

which in turn is equivalent to

$$(7) \ y \upharpoonright I_k + s \upharpoonright I_k \subseteq A_k \text{ for every } s \in S_i.$$

Letting T_Q be the smallest tree on \mathbb{N} containing $\bigcup_{i \in \mathbb{N}} R_i$ and $Q = [T_Q]$ we easily conclude from (2) and (7) that $y + Q \subseteq C$. □

As another immediate corollary of Theorem 3.5 we get the consistency of the existence of a small set (even one with a witness of cardinality \mathfrak{c}) which is not perfectly small (cf. Proposition 3.4(3)).

Corollary 3.13. If I is an invariant ccc σ -ideal on $2^{\mathbb{N}}$ and there exists an I -Lusin $S \subseteq 2^{\mathbb{N}}$, then every Borel set from I is small with a witness S . In particular, there is one not in I_{ccc} hence not perfectly small.

With the help of Corollary 3.13 and Proposition 3.11 we get a result related to a theorem of Mátrai stating that it is consistent with ZFC that even for open sets $C \subseteq (2^{\mathbb{N}})^{\aleph_0}$ the existence of a C -homogeneous set of cardinality \mathfrak{c} does not imply the existence of a perfect one (cf. [16, Theorem 1.1]).

Proposition 3.14. If there exists a Sierpiński set $S \subseteq 2^{\mathbb{N}}$, then there exists an open subset C of $(2^{\mathbb{N}})^{\aleph_1}$ with the property that S is C -homogeneous but no perfect C -homogeneous set exists.

Proof. Let $A \subseteq 2^{\mathbb{N}}$ be a compact nullset which is not perfectly small (cf. Proposition 3.11) and let $C = (2^{\mathbb{N}})^{\aleph_1} \setminus F_{\aleph_1}[2^{\mathbb{N}} \times A^{\aleph_1}]$. The function $F_{\aleph_1} : 2^{\mathbb{N}} \times (2^{\mathbb{N}})^{\aleph_1} \rightarrow (2^{\mathbb{N}})^{\aleph_1}$ being continuous and the set A being compact, C is an open subset of $(2^{\mathbb{N}})^{\aleph_1}$.

By Corollary 3.13 and Lemma 3.1(3), S is C -homogeneous but no perfect C -homogeneous set exists since there is no perfect witness of smallness of A . \square

4. COVERING 2^{\aleph_1} BY \aleph_1 MANY DISJOINT TRANSLATES OF A COMPACT SET

The purpose of this section is to point out some links between the subject of this paper and that of covering 2^{\aleph_1} (or, more generally, an uncountable Polish group) by less than \mathfrak{c} many translates of its compact subset. The latter has been recently dealt with by many authors (see [6], [7], [8], [9], [19], [15]).

Let us start with an obvious generalization of a simple observation of Darji and Keleti (see [6, Lemma 2]).

Lemma 4.1. Let G be an uncountable abelian group and $\aleph_1 \leq \kappa \leq \mathfrak{c}$. Then less than κ many translates of a small set with a witness of cardinality at least κ do not cover G .

Note that when $\mathfrak{c} > \aleph_1$ Lemma 4.1 prevents 2^{\aleph_1} from being covered by \aleph_1 many translates of any Borel perfectly small subset B . At the same time Theorem 3.5 implies that $B \in I_{ccc}$ and in fact, the smallness of a Borel set $B \subseteq 2^{\aleph_1}$ is, at least consistency-wise, the strongest among conditions considered in Section 3 (cf. Proposition 3.4) sufficient for this.

These remarks lead to the natural question whether in the absence of CH, at least for a compact set $A \subseteq 2^{\aleph_1}$, the properties of 2^{\aleph_1} *not* being covered by \aleph_1 many translates of A on one hand and A being a member of I_{ccc} on another, are related.

We have the following observation.

Proposition 4.2. It is consistent with the negation of CH that there is a compact null set $A \subseteq 2^{\aleph_1}$ such that $A \notin I_{ccc}$ but no \aleph_1 many translates of A cover 2^{\aleph_1} . In fact, the latter is true for any null set A provided a Sierpiński set $S \subseteq 2^{\aleph_1}$ of size greater than \aleph_1 exists and CH is false.

Proof. This is an immediate consequence of Proposition 3.11 and Lemma 4.1. \square

Miller [17] proved that it is consistent with the negation of CH that 2^{\aleph_1} (equivalently: any uncountable Polish space) can be partitioned into \aleph_1 many disjoint non-empty closed sets. In contrast to this, we have the following result.

Theorem 4.3. CH is equivalent to each of the following statements:

- (1) 2^{\aleph_1} can be partitioned into \aleph_1 many disjoint translates of a closed set,
- (2) some uncountable locally compact Polish abelian group can be,
- (3) all uncountable locally compact Polish abelian groups can be.

Moreover, \neg CH implies that if an uncountable locally compact Polish abelian group is the union of a collection \mathcal{A} of \aleph_1 many translates of a closed set then for every natural number n there is $x \in G$ such that x is a member of at least n elements of \mathcal{A} .

Proof. Let G be an uncountable locally compact Polish abelian group.

It suffices to prove that \neg CH implies that G cannot be covered by \aleph_1 many translates of a closed set in such a way that for some $n > 1$ no $x \in G$ is a member of n elements of the covering.

So assume that $\mathfrak{c} > \aleph_1$.

Let G be an uncountable locally compact Polish group. Suppose, towards a contradiction, that A a closed subset of G and for some $n > 1$ there is a covering of

G by \aleph_1 many translates of A such that no $x \in G$ is a member of n elements of the covering. Then, by Lemma 3.1, A is n -small and by Proposition 3.2, it is perfectly small. But this implies, by Lemma 4.1, that G cannot be covered by less than \mathfrak{c} many translates of A . □

5. OPEN PROBLEMS

Let us conclude with a list of some natural open questions.

- (1) **Problem 1.** *What is the relation of I_{ccc} to the intersection of all invariant ccc σ -ideals on 2^{\aleph} ? In particular, is there a non-Borel set $A \subseteq 2^{\aleph}$ which belongs to every invariant ccc σ -ideal but is not in I_{ccc} ?* (This question was pointed out to me by the referee whom I wish to thank for all his valuable remarks).
- (2) **Problem 2.** *What is the relation of I_{ccc} to the σ -ideal generated by Borel perfectly small (perfectly 2-small, respectively) subsets of 2^{\aleph} ?* (cf. Theorem 3.5).
- (3) **Problem 3.** *Is every Borel n -small, $1 < n < \aleph_0$, subset of a Polish abelian group G perfectly n -small?* (cf. Proposition 3.2; a closely related problem for $n = 2$ was formulated in [3] as Question 2).

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