ON BOOLEAN ALGEBRAS RELATED TO σ-IDEALS GENERATED BY COMPACT SETS

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ABSTRACT. Let μ^h, μ^g be Hausdorff measures on compact metric spaces X, Y and let $Bor(X)/J_\sigma(\mu^h)$ and $Bor(Y)/J_0(\mu^g)$ be the Boolean algebras of Borel sets modulo σ-ideals of Borel sets that can be covered by countably many compact sets of σ-finite μ^h-measure or μ^g-measure null, respectively. We shall show that if μ^h is not σ-finite, and one of the quotient Boolean algebras embeds densely in the other, then for some Borel B with $\mu^h(B) = \infty$, μ^h takes on Borel subsets of B only values 0 or ∞.

This is a particular instance of some more general results concerning Boolean algebras $Bor(X)/J$, where J is a σ-ideal of Borel sets generated by compact sets.

1. INTRODUCTION

A Borel σ-ideal I (shortly: a σ-ideal) on a compactum (i.e., a compact metrizable space) X is a collection of Borel sets in X, closed under countable unions and such that for any $A \in I$, all Borel subsets of A are in I; I is generated by compact sets if any element of I can be covered by countably many compact sets in I. We always assume that $X \notin I$.

The subject of this paper are quotient Boolean algebras $Bor(X)/I$ of Borel sets in compacta X modulo σ-ideals I generated by compact sets. Our results describe some pairs I, J of such σ-ideals, including natural examples associated with Hausdorff measures on compact metric spaces, with the property that $Bor(X)/I$ does not embed densely in $Bor(Y)/J$ (in the sense of Definition 4.8 in [11]).

In particular, given a Hausdorff measure $\mu^h : Bor(X) \to [0, +\infty]$ defined on the σ-algebra of Borel sets in a compact metric space X and determined by a continuous nondecreasing function $h : [0, +\infty) \to [0, +\infty)$ with $h(r) > 0$ for $r > 0$ and $h(0) = 0$ (cf. Rogers [16]), we shall consider σ-ideals $J_0(\mu^h)$ ($J_\sigma(\mu^h)$, respectively) of Borel sets in X that can be covered by countably many compact sets of μ^h-measure null (with σ-finite μ^h-measure, respectively, provided that μ^h is not σ-finite).

Let us recall that a Borel measure μ is *semifinite* if each Borel set in X of positive μ-measure contains a Borel set of finite positive μ-measure, cf. Edgar [3] or Fremlin [5]. By results of Larman [12] and

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Howroyd [6], if X is a subset of the Euclidean space $\mathbb{R}^n$ or h is of finite order, e.g. $h(t) = t^s$, cf. [6], then the measure μ^h is semifinite.

Our results (cf. Corollary 4.1) imply that if one of the Boolean algebras $Bor(X)/J_\sigma(\mu^h)$, $Bor(Y)/J_0(\mu^g)$ embeds densely in the other, then the measure μ^h is not semifinite.

This is also true for a wider class of pairs of σ-ideals determined by Borel measures or capacities, and in fact we shall derive this result from Theorem 3.1 concerning a broad class of σ-ideals investigated in the literature.

Our interest in this topic was strongly influenced by the work of Zapletal [21], Farah and Zapletal [4] and Sabok and Zapletal [18]. However, while that work concentrates on refined analysis of forcings associated with the algebras $Bor(X)/I$, we stay in the realm of classical descriptive set theory, as presented by Kechris [8]. This paper is a continuation of our paper [15], and a subsequent note [14].

The main results of the paper are stated and proved in Sections 3, 4 and 5. They are preceded by Section 2 where the terminology is clarified, and some results from the literature, or close to the ones in the literature, needed in the proofs, are explained. Finally, Section 6 contains some additional observations and comments.

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2. Preliminaries

Our notation is standard and mostly agrees with [8]. In particular,

- $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ are the Cantor and the Baire space, respectively,
- $\mathbb{N}^{<\mathbb{N}}$ is the family of all finite sequences of natural numbers,
- $\mathrm{diam}(A)$ is the diameter of a set A in the given metric space.

2.1. **Borel measures.** By a *Borel measure* on a compactum X we mean a countably additive measure $\mu : Bor(X) \to [0, \infty]$, defined on the σ-algebra of Borel sets in X and such that $\mu(X) > 0$. We say that μ is finite (σ-finite, respectively), if $\mu(X) < \infty$ (X is the union of a countable family of Borel sets with finite measure, respectively). Let us recall that for a Borel measure μ on a compactum, each Borel set of positive finite μ-measure contains a compact set of positive μ-measure, cf. [8]. Any σ-finite Borel measure μ is semifinite, which is equivalent to the fact that each Borel set in X of positive μ-measure contains a compact set of positive finite μ-measure.

A starting point of our work were σ-ideals associated with Hausdorff measures μ^h. Let us recall that given a metric space (X, d) and a continuous nondecreasing function $h : [0, +\infty) \to [0, +\infty)$ with $h(r) > 0$ for $r > 0$ and $h(0) = 0$, we let μ^h be the Hausdorff measure on X

determined by h, i.e., cf. Rogers [16], for $\varepsilon > 0$ we let

$$\mu^h_\varepsilon(E) = \inf\{\sum_n h(\mathrm{diam}(U_n)) : U_n \text{ open with } \mathrm{diam}(U_n) \leq \varepsilon \text{ and } E \subseteq \bigcup_n U_n\}$$

and

$$\mu^h(E) = \lim_{\varepsilon \to 0^+} \mu^h_\varepsilon(E), \quad \text{for} \quad E \in Bor(X).$$

As already noted in Section 1, in some important cases the measure μ^h is semifinite. On the other hand, Davies and Rogers [1] gave an example of a Hausdorff measure μ^h on a compactum X with $\mu^h(X) = \infty$ and containing no compact sets of finite positive μ^h-measure.

We say that a Borel measure μ on X is *G_δ-regular on σ-compact sets* if every countable union of compact μ-null subsets of X is contained in a G_δ μ-null subset of X. For any Hausdorff measure μ^h, each Borel set is contained in a G_δ-set of the same μ^h-measure, so in particular, μ^h is G_δ-regular on σ-compact sets.

2.2. **Calibration, the 1-1 or constant property, and property $(*)$ of σ-ideals.** Throughout this subsection let I be a σ-ideal on a compactum X.

We say that

- I is *generated by compact sets*, if each element in I can be covered by countably many compacta in I,
- I is *coanalytic*, if $I \cap K(X)$ is a coanalytic set in the space $K(X)$ of compact sets in X, equipped with the Vietoris topology, cf. Kechris [8],
- I is *calibrated* if it is generated by compact sets and $I \cap K(X)$ is calibrated in the sense of Kechris, Louveau and Woodin [10], i.e., for any $F \in K(X) \setminus I$ and countably many compact sets $K_n \in I$, $n \in \mathbb{N}$, there is a compactum $K \subseteq F \setminus \bigcup_{n\in\mathbb{N}} K_n$ not in I,
- I has the *1-1 or constant property*, introduced by Sabok and Zapletal [18] (cf. also [17]), if for every Borel set $B \subseteq X$ not in I and every Borel function f from B into a Polish space Y with all fibers in I, there is a Borel set $G \subseteq B$ not in I on which f is injective,
- I has *property* $(*)$, distinguished by Solecki [20], if whenever $K_n \in I$, $n \in \mathbb{N}$, there is a G_δ-set G in X containing $\bigcup_{n\in\mathbb{N}} K_n$ such that all compact subsets of G are in I.

By [15, Theorem 5.1], the σ-ideals generated by compact sets that are both coanalytic and calibrated have the 1-1 or constant property. This class includes σ-ideals $J_0(\mu^h)$ (which follows from regularity properties of Hausdorff measures, cf. [16]), and also σ-ideals $J_\sigma(\mu^h)$, provided that the measure μ^h is non-σ-finite and semifinite (in fact, a more general fact holds true, cf. Proposition 6.1, which however, is not needed for our main result).

2.3. **I-thin sets and the σ-ideals $J_t(I)$.** We say that a set $A \subseteq X$ is *I-thin* if there is no uncountable disjoint family of compact subsets of A which are not in I, cf. [10]. If X is I-thin, we say that *I is thin*. Assuming that I is not thin, we shall denote by $J_t(I)$ the σ-ideal of Borel subsets of X that can be covered by countably many compact I-thin sets.

Given a Borel measure μ on a compactum X, let $J_0(\mu)$ ($J_\sigma(\mu)$, respectively) be the σ-ideal of Borel sets in X that can be covered by countably many compact sets of μ-measure null (with σ-finite μ-measure, respectively, provided that the measure μ is not σ-finite).

Note that if μ is semifinite, $J_\sigma(\mu) = J_t(J_0(\mu))$, as in this case compact sets of σ-finite μ-measure are exactly compact $J_0(\mu)$-thin sets; this is also true for arbitrary Hausdorff measure μ^h, cf. [16, Ch.2, §6, Corollary 2].

By [10, Corollary 4 in Section 3], if I is coanalytic and calibrated, then so is $J_t(I)$ (cf. Lemma 2.1 and the remark preceding its statement). Consequently, this is true for $J_\sigma(\mu)$, whenever μ is a semifinite (non-σ-finite) measure on a compactum X with coanalytic $J_0(\mu) \cap K(X)$, and also for $J_\sigma(\mu^h)$, where μ^h is an arbitrary (non-σ-finite) Hausdorff measure.

We shall need a more refined version of the fact that for coanalytic I, $J_t(I)$ is also coanalytic. Let us recall (cf. [9], page 265) that a family $\mathscr{D} \subseteq K(X) \setminus I$ is a *co-basis* of I if every compact set not in I contains a subset from $\mathscr{D}$. Note that if μ is a semifinite Borel measure on X, then the family of compact subsets of X of finite positive μ-measure is a co-basis of $J_0(\mu)$.

Let

$$\mathscr{E} = \{\phi \in C(2^{\mathbb{N}}, K(X)) : \ \phi(t) \in \mathscr{D} \text{ and } \phi(s) \cap \phi(t) = \emptyset \text{ for } s \neq t\}.$$

Then, if A is a G_δ subset of X and $\mathscr{D}$ is analytic, A is not I-thin if and only if there is $\phi \in \mathscr{E}$ with $\bigcup \phi(2^{\mathbb{N}}) \subseteq A$. Indeed, if A is not I-thin, then since the set $K(A) \cap \mathscr{D}$ is analytic in $K(X)$ and contains an uncountable disjoint family, it contains a perfect disjoint family as well (cf. [10, Theorem 2 in Section 3]). We need a refinement of this argument leading to the next lemma. This is in fact Corollary 4 in [10, Section 3], but since the proof in [10] omits some essential details, we include a justification for readers convenience.

Lemma 2.1. *If there is an analytic co-basis $\mathscr{D}$ of I consisting of I-thin sets, then the σ-ideal $J_t(I)$ is coanalytic. If, moreover, I is calibrated, then $J_t(I)$ is calibrated and has the 1-1 or constant property.*

Proof. Let us fix a continuous surjection $\varphi : \mathbb{N}^{\mathbb{N}} \to \mathscr{D}$ and let $\mathcal{F}$ be the collection of Cantor sets C in $\mathbb{N}^{\mathbb{N}}$ such that for any distinct $s, t \in C$, $\varphi(s) \cap \varphi(t) = \emptyset$.

Notice that $\mathcal{F}$ is a G_δ-set in $K(\mathbb{N}^{\mathbb{N}})$ and for any $C \in \mathcal{F}$, $\bigcup \phi(C) \notin J_t(I)$. We shall verify that, moreover, $K \in K(X)$ is not I-thin if and only if there is $C \in \mathcal{F}$ with $\bigcup \phi(C) \subseteq K$. This implies that the set $K(X) \setminus J_t(I)$ is analytic.

So assume that $K \in K(X) \setminus J_t(I)$ and let $T = \{t \in \mathbb{N}^{\mathbb{N}} : \varphi(t) \subseteq K\}$. Then T is closed in $\mathbb{N}^{\mathbb{N}}$ and since $K \notin J_t(I)$, there is an uncountable set $A \subseteq T$ such that $\varphi(a) \cap \varphi(b) = \emptyset$ whenever $a, b \in A$ are distinct. We can assume that each point in A is an accumulation point in A. Let

$$R = \{(s,t) \in \overline{A} \times \overline{A} : \varphi(s) \cap \varphi(t) \neq \emptyset\}.$$

Then the relation R is closed with empty interior in $\overline{A} \times \overline{A}$, hence Mycielski's theorem (see [8]) provides an R-independent Cantor set $C \subseteq \overline{A}$. Clearly, $\bigcup \phi(C) \subseteq K$ which completes the proof of the main part of the assertion. The "moreover" part follows from [10, Corollary 4 in Section 3] and [15, Theorem 5.1]. □

The 1-1 or constant property of ideals is needed in this paper for the following version of Sikorski's theorem [8, 15.C], stated in [14, Lemma 2.5.1].

Lemma 2.2. *Let I and J be σ-ideals on compacta X and Y, respectively. Let $Bor(Y)/J$ embed onto a dense subalgebra of $Bor(X)/I$ and assume that I has the 1-1 or constant property and J contains all singletons. Then there exist sets $A \in Bor(X) \setminus I$ and $B \in Bor(Y) \setminus J$, and a Borel isomorphism $\varphi : A \to B$ between A and B such that $C \in I$ if and only if $\varphi(C) \in J$, whenever $C \subseteq A$.*

More precisely, to get φ as in Lemma 2.2, one first uses [14, Lemma 2.5.1] to get a Borel map $\psi : E \to Y$ on $E \in Bor(X) \setminus I$ such that, for $C \in Bor(E)$, $C \in I$ if and only if $\psi(C) \in J$, and than, appealing to the 1-1 or constant property of I, one further shrinks E to its Borel subset A, not in I, on which $\varphi = \psi|A$ is injective.

Note, that if μ is G_δ-regular on σ-compact sets (see Section 2.1), then the σ-ideal $I_0(\mu)$ has property $(*)$ and is calibrated.

2.4. **Generalized Hurewicz systems.** Given a G_δ-set G in a compactum X with a fixed complete metric on G bounded by 1, by a *generalized Hurewicz system* we shall mean a pair $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of families of subsets of X with the following properties (the closure is taken in X):

- $U_s \subseteq G$ is relatively open, non-empty and $\mathrm{diam}(U_s) \leq 2^{-length(s)}$,
- $\overline{U_s} \cap \overline{U_t} = \emptyset$ for distinct s, t of the same length,
- $\overline{U_{s\hat{\ }i}} \cap G \subseteq U_s$,
- $L_s \subseteq \overline{U_s}$ is compact,
- $L_s \cap \overline{U_{s\hat{\ }i}} = \emptyset$,
- each neighbourhood of L_s contains all but finitely many $U_{s\hat{\ }i}$.

If $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ is a generalized Hurewicz system, then

$$P = \bigcap_n \bigcup\{U_s : \text{length(s)} = n\}$$

is the G_δ-subset of G *determined by the system* and

$$\overline{P} \subseteq P \cup \bigcup\{L_s : s \in \mathbb{N}^{<\mathbb{N}}\}.$$

If, additionally, $L_s = \bigcap_j \overline{\bigcup_{i>j} U_{s\hat{}i}}$ and $\lim_i \operatorname{diam}(U_{s\hat{}i}) = 0$, then $\overline{P} = P \cup \bigcup\{L_s : s \in \mathbb{N}^{<\mathbb{N}}\}$.

Moreover, if V is a non-empty relatively open subset of P, then $\overline{V}$ contains L_s with arbitrarily long $s \in \mathbb{N}^{<\mathbb{N}}$.

Such systems of sets, with L_s being singletons, were introduced by Hurewicz [7]. Solecki showed that generalized Hurewicz systems are very useful to determine G_δ-sets not belonging to a given σ-ideal I generated by compact sets, cf. [19, proof of Theorem 1].

The following lemma describes some additional essential features of these systems.

Lemma 2.3. *Let G be a G_δ-set in a compactum X. Let $\mathcal{F}$ and $\mathcal{F}_s$, $s \in \mathbb{N}^{<\mathbb{N}}$, be families of compact subsets of X.*

If for every relatively open, non-empty subset U of G and each $s \in \mathbb{N}^{<\mathbb{N}}$ there is a compact set $L \subseteq \overline{U} \setminus G$ in $\mathcal{F} \setminus \mathcal{F}_s$, then there is a generalized Hurewicz system $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ with each $L_s \in \mathcal{F} \setminus \mathcal{F}_s$.

Consequently, if $P \subseteq G$ is the G_δ set determined by the system, then:

1. $\overline{P} \setminus P \subseteq \bigcup\{L_s : s \in \mathbb{N}^{<\mathbb{N}}\}$. *In particular, if I is the σ-ideal generated $\bigcup_s(\mathcal{F} \setminus \mathcal{F}_s)$, then $\overline{P} \setminus P \in I$,*
2. *if each $\mathcal{F}_s$ is hereditary (i.e., whenever D is a closed subset of $F \in \mathcal{F}_s$, then $D \in \mathcal{F}_s$) and length$(s) <$ length(t) implies $\mathcal{F}_s \subseteq \mathcal{F}_t$ for every $s, t \in \mathbb{N}^{<\mathbb{N}}$, then V is not in the σ-ideal J generated by $\bigcup_s \mathcal{F}_s$, for any relatively open, non-empty subset V of P; in particular, $P \notin J$.*

Proof. The details of the construction are similar to those in [19, proof of Theorem 1].

Since $\overline{P} \setminus P \subseteq \bigcup\{L_s : s \in \mathbb{N}^{<\mathbb{N}}\}$, conclusion (1) is clear.

Conclusion (2) follows from a Baire category argument using the fact that the closure of any non-empty relatively open subset of P contains L_s with arbitrary long $s \in \mathbb{N}^{<\mathbb{N}}$. □

Following [14, Section 3, (3)] we shall use the generalized Hurewicz systems in the following situation.

Let $f : G \to Y$ be a continuous function from a non-empty, G_δ-set G in a compactum X into a compactum Y and let J be a σ-ideal on Y, generated by compact sets. Let also $\mathcal{F} \subseteq K(X)$.

We shall denote by $B(x,\frac{1}{n})$ the $\frac{1}{n}$-ball about x with respect to a fixed metric on X.

For any non-empty relatively open $U \subseteq G$, we define $\hat{f}_U : \overline{U} \to K(Y)$ by

$$\hat{f}_U(x) = \bigcap_n \overline{f(B(x, \frac{1}{n}) \cap U)}.$$

and let

$$\hat{f}_U[A] = \bigcup \{\hat{f}(x) : \ x \in A\} \quad \text{for} \quad A \subseteq \overline{U}.$$

Then $\hat{f}_U : \overline{U} \to K(Y)$ is upper-semicontinuous and if $A \subseteq U$, $\hat{f}_U[A] = f(A)$.

In this setting we have the following

Lemma 2.4. *If for every relatively open, non-empty subset U of G there is a compact set $L \subseteq \overline{U} \setminus G$ in $\mathcal{F}$ with $\hat{f}_U[L] \notin J$, then there is a generalized Hurewicz system $\overline{(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ with $L_s \in \mathcal{F}$, $\hat{f}_{U_s}[L_s] \notin J$ and $\hat{f}_{U_s}[L_s] \subseteq \overline{\{f(x_i) : i \in \mathbb{N}\}}$ for each $s \in \mathbb{N}^{<\mathbb{N}}$ and any sequence $x_i \in U_{s\hat{\ }i}$, $i \in \mathbb{N}$.*

Consequently, if $P \subseteq G$ is the G_δ set determined by the system, then $f(P) \notin J$.

For details related to Lemma 2.4 we refer the reader to [14, parts (B) and (C) in Section 3].

3. Main results

Let us recall that, given a non-thin σ-ideal I on a compactum X, we denote by $J_t(I)$ the σ-ideal generated by compact I-thin sets, cf. Section 2.3; the notion of a co-basis of I was also explained there.

Theorem 3.1. *Let I and J be calibrated σ-ideals on compacta X and Y, respectively. If there is an analytic co-basis $\mathscr{D}$ of I consisting of I-thin sets and J has property $(*)$ and contains all singletons, then $Bor(Y)/J$ does not embed onto a dense subalgebra of $Bor(X)/J_t(I)$. If, moreover, J has the 1-1 or constant property (in particular, if J is coanalytic), then there is also no embedding of $Bor(X)/J_t(I)$ onto a dense subalgebra of $Bor(Y)/J$.*

Proof. (A) Aiming at a contradiction, assume that either $Bor(Y)/J$ embeds onto a dense subalgebra of $Bor(X)/J_t(I)$ or J has the 1-1 or constant property and $Bor(X)/J_t(I)$ embeds onto a dense subalgebra of $Bor(Y)/J$. In either case Lemma 2.2 gives us a Borel injection $f : B \to Y$, $B \in Bor(X) \setminus J_t(I)$, such that

(1) for any Borel set $C \subseteq B$ we have $C \in J_t(I)$ if and only if $f(C) \in J$.

One can moreover assume, using Solecki's theorem [19, Theorem 1], that f is continuous on B and B is a G_δ-set such that no non-empty relatively open subset of B is in $J_t(I)$.

A key element of our proof is the following observation concerning the maps $\hat{f}_U$ introduced in Section 2.4 (in the part preceding Lemma 2.4).

Claim. For any non-empty relatively open set $U \subseteq B$, each compact set $D \subseteq \overline{U}$ with $D \notin J_t(I)$ either contains a compact subset $L \in J_t(I)$ with $\hat{f}_U[L] \notin J$ or a compact subset $L \notin J_t(I)$ with $\hat{f}_U[L] \in J$.

To prove the claim, let

$$\mathscr{E} = \{\phi \in C(2^{\mathbb{N}}, K(X)) : \ \phi(t) \in \mathscr{D} \text{ and } \phi(s) \cap \phi(t) = \emptyset \text{ for } s \neq t\}.$$

Since D is not I-thin, there is (cf. remarks preceding Lemma 2.1) $\phi \in \mathscr{E}$ with $\bigcup \phi(2^{\mathbb{N}}) \subseteq D$.

Let $\mathbb{Q}$ be a countable set dense in $2^{\mathbb{N}}$.

We have $\phi(q) \in \mathscr{D} \subseteq J_t(I)$ for each $q \in \mathbb{Q}$, so if $\hat{f}_U[\phi(q)] \notin J$ for at least one q, we are done.

So assume that $\hat{f}_U[\phi(q)] \in J$ for every $q \in \mathbb{Q}$.

Since the σ-ideal J has property $(*)$, there is a G_δ subset S of Y such that $\bigcup_{q \in \mathbb{Q}} \hat{f}_U[\phi(q)] \subseteq S$ and all compact subsets of S are in J.

The set $\{F \in K(D) : \hat{f}_U[F] \subseteq S\}$ is G_δ in the hyperspace, $\hat{f}_U$ being upper-semicontinuous, and since ϕ is continuous we infer that $E = \{t \in 2^{\mathbb{N}} : \ \hat{f}_U[\phi(t)] \subseteq S\}$ is a G_δ-set containing $\mathbb{Q}$. In effect, for any homeomorphic embedding $\gamma : 2^{\mathbb{N}} \to E$, letting $\psi = \phi \circ \gamma$ and $L = \bigcup \psi(2^{\mathbb{N}})$ we obtain $L \notin J_t(I)$ with $\hat{f}_U[L] \subseteq S$ hence $\hat{f}_U[L] \in J$, as required. This completes the proof of the claim.

As an immediate consequence of Claim we have

(2) all compact subsets of B are in $J_t(I)$.

Indeed, since $\hat{f}_B|B = f$, (1) implies that for any compact $L \subseteq B$, $\hat{f}_B[L] = f(L) \in J$ if and only if $L \in J_t(I)$. This, by Claim, implies (2).

(B) We shall consider now two cases.

Case 1. There is a relatively open non-empty set $U \subseteq B$ such that for each compact set $L \subseteq \overline{U} \setminus B$ with $L \in J_t(I)$ we have $\hat{f}_U[L] \in J$.

We shall check that in this case

(3) for each non-empty relatively open $V \subseteq U$ there is a compact set $L \subseteq \overline{V} \setminus B$ with $L \notin J_t(I)$ but $\hat{f}_U[L] \in J$.

So let V be a non-empty relatively open subset of U. Note that V being relatively open in B, $\overline{V} \notin J_t(I)$.

Since $\overline{V} \setminus B = \bigcup_n K_n$, $K_n \in K(X)$, and all compact subsets of $\overline{V} \setminus \bigcup_n K_n = \overline{V} \cap B$ are in $J_t(I)$, calibration of $J_t(I)$ provides n_0 with $K_{n_0} \notin J_t(I)$. Applying Claim to $D = K_{n_0}$, we get in this case a compact set $L \subseteq K_{n_0} \subseteq \overline{V} \setminus B$ with $L \notin J_t(I)$ but $\hat{f}_U[L] \in J$.

Now, equipped with (3), one can apply Lemma 2.3(2) (for $\mathcal{F} = \{L \in K(X) : \hat{f}_U[L] \in J\}$ and $\mathcal{F}_s = J_t(I)$ for each $s \in \mathbb{N}^{<\mathbb{N}}$) to get a G_δ-set $P \notin J_t(I)$ in U such that $\overline{P} \setminus P \subseteq \bigcup_n L_n$, where $L_n \notin J_t(I)$ but $\hat{f}_U[L_n] \in J$, for every $n \in \mathbb{N}$.

Since $P \subseteq B$ and $P \notin J_t(I)$, we have $\hat{f}_U[P] = f(P) \notin J$, cf. (1).

However, $\hat{f}_U[L_n] \in J$ for each n, so using the fact that J is calibrated, we get a compact set $T \subseteq \hat{f}_U[\overline{P}] \setminus \bigcup_n \hat{f}_U[L_n]$ which is not in J. Note that $T \subseteq f(P)$, since $\overline{P} \setminus P \subseteq \bigcup_n L_n$. Let $K = f^{-1}(T)$. Since f is injective, $K \subseteq P$. As for any $x \in \overline{P} \setminus P$, $\hat{f}_U(x) \cap T = \emptyset$, it follows that $K = \{x \in \overline{P} : \hat{f}_U(x) \cap T \neq \emptyset\}$. Hence, $\hat{f}_U$ being upper-semicontinuous, K is compact.

Thus we obtained a compact set $K \subseteq P$ with $f(K) \notin J$. But on the other hand, $K \in J_t(I)$, as $K \subseteq B$, cf. (2). This contradicts (1), completing the proof of the theorem in Case 1.

Case 2. For every relatively open, non-empty set $U \subseteq B$ there is a compact set $L \subseteq \overline{U} \setminus B$ such that $L \in J_t(I)$ but $\hat{f}_U[L] \notin J$.

In this case one can apply Lemma 2.4 to get a G_δ-set $P \subseteq B$ such that $f(P) \notin J$ and $\overline{P} \setminus P \subseteq \bigcup_n L_n$, where $L_n \in J_t(I)$, for every $n \in \mathbb{N}$. Again, a contradiction with (1) will be reached as soon as we prove that $P \in J_t(I)$. But if $P \notin J_t(I)$, then since $L_n \in J_t(I)$ for each n, using the fact that J is calibrated, we get a compact set $K \subseteq \overline{P} \setminus \bigcup_n L_n \subseteq B$ which is not in $J_t(I)$. This, however, again contradicts (2), completing the proof of the theorem. □

4. Some corollaries

In this section we shall combine Theorem 3.1 with observations made in Section 2 to get, in particular, results stated in the introduction.

Throughout this section we shall assume that

- μ is a semifinite but not σ-finite Borel measure on a compactum X and the collection of compact sets of finite, positive μ-measure is analytic in $K(X)$.

Corollary 4.1. *With μ as above, for any calibrated σ-ideal J on a compactum Y which has property $(*)$ and contains all singletons, $Bor(Y)/J$ does not embed onto a dense subalgebra of $Bor(X)/J_\sigma(\mu)$. If, moreover, J has the 1-1 or constant property (in particular, if J is coanalytic), then there is also no embedding of $Bor(X)/J_\sigma(\mu)$ onto a dense subalgebra of $Bor(Y)/J$.*

In particular, none of the embeddings exist if μ is a semifinite but not σ-finite Hausdorff measure μ^h and $J = J_0(\mu^g)$ for a Hausdorff measure μ^g.

The next corollary shows that Boolean algebras $Bor(X)/J_\sigma(\mu)$ may differ essentially, already for various non-σ-finite Hausdorff measures $\mu = \mu^h$. Let us recall that Davies and Rogers [1] constructed a Hausdorff measure μ^g on a compactum Y with no compact set of finite positive μ^g-measure. Let us notice that Zapletal [21, 4.4.2] exhibited substantial differences between σ-ideals generated by Borel sets of σ-finite measure for the specific μ^h constructed by Davies and Rogers and any semifinite, non-σ-finite μ^g, respectively.

Corollary 4.2. *Let μ be as above and let ν be a Borel measure on a compactum Y which is non-σ-finite, G_δ-regular on σ-compact sets and vanishes on singletons. Then, if $Bor(Y)/J_\sigma(\nu)$ embeds densely in $Bor(X)/J_\sigma(\mu)$, Y must contain a compact set of finite positive ν-measure.*

In particular, this applies to non-σ-finite Hausdorff measures $\mu = \mu^h$ and $\nu = \mu^g$ with μ^h semifinite.

Proof. Assume that $Bor(Y)/J_\sigma(\nu)$ embeds densely in $Bor(X)/J_\sigma(\mu)$ and, aiming at a contradiction, suppose that Y contains no compact set of finite positive ν-measure. In particular, $K \in K(Y)$ has σ-finite ν-measure if and only if $\nu(K) = 0$. Consequently, $J_\sigma(\nu) = J_0(\nu)$ is calibrated, has property $(*)$ and contains all singletons, so putting $J = J_\sigma(\nu)$ we get a contradiction with Corollary 4.1. □

5. The σ-ideals generated by compact sets of finite measure

Given a Borel non-σ-finite measure μ on a compactum X, one can consider yet another natural σ-ideal related to μ, generated by compact sets, lying between the σ-ideals $J_0(\mu)$ and $J_\sigma(\mu)$ – the σ-ideal $J_f(\mu)$ of Borel sets in X that can be covered by countably many compact sets with finite μ-measure.

We shall show that if μ^h is a semifinite non-σ-finite Hausdorff measure on X, then $Bor(X)/J_f(\mu^h)$ embeds densely into neither $Bor(X)/J_0(\mu^h)$ nor $Bor(X)/J_\sigma(\mu^h)$. This is a consequence of the following, more general result.

Theorem 5.1. *Let μ be a Borel measure on a compactum $X \notin J_f(\mu)$. Assume that μ is semifinite, G_δ-regular on σ-compact sets and vanishes on singletons. Let J be a calibrated σ-ideal with the 1-1 or constant property on a compactum Y. Then $Bor(X)/J_f(\mu)$ does not embed onto a dense subalgebra of $Bor(Y)/J$. In particular, $Bor(X)/J_f(\mu)$ embeds densely into neither $Bor(X)/J_0(\mu)$ nor $Bor(X)/J_\sigma(\mu)$ (provided that μ is not σ-finite).*

Proof. (A) Aiming at a contradiction, assume that $Bor(X)/J_f(\mu)$ embeds onto a dense subalgebra of $Bor(Y)/J$. Since J has the 1-1 or constant property and $J_f(\mu)$ contains all singletons, Lemma 2.2 gives us a Borel injection $f : A \to Y$, $A \in Bor(X) \setminus J_f(\mu)$, such that

(1) for any Borel set $C \subseteq A$ we have $C \in J_f(\mu)$ if and only if $f(C) \in J$.

Moreover, one can assume, using Solecki's theorem [19, Theorem 1] and the fact that μ vanishes on singletons and is G_δ-regular on σ-compact sets, that A is a G_δ-set of μ-measure zero, f is continuous on A and no non-empty relatively open subset of A is in $J_f(\mu)$. Consequently, for every relatively open, non-empty subset U of A, we have $\mu(\overline{U} \setminus A) = \infty$ which implies, μ being semifinite, that for each $n \in \mathbb{N}$ there is a compact set $L \subseteq \overline{U} \setminus A$ with $n < \mu(L) < \infty$. It follows that one can apply Lemma 2.3(2) (for $\mathcal{F} = K(X)$ and $\mathcal{F}_s = \{L \in K(X) : \mu(L) \leq \text{length}(s)\}$) to get a non-empty G_δ-set P in A with $V \notin J_f(\mu)$ for any non-empty relatively open subset V of P and such that $\overline{P} \setminus P \subseteq \bigcup_n L_n$, where L_n are compacta of finite μ-measure.

(B) Following closely part (B) of the proof of Theorem 3.1 we shall consider now two cases.

Case 1. There exist a relatively open non-empty set $U \subseteq P$ and an $\varepsilon > 0$ such that for each compact set $L \subseteq \overline{U} \setminus P$ with $\mu(L) < \varepsilon$ we have $\hat{f}_U[L] \in J$.

Since $U \subseteq P \subseteq A$ and $U \notin J_f(\mu)$, we have $\hat{f}_U[U] = f(U) \notin J$, cf. (1).

However, $\overline{U} \setminus P \subseteq \bigcup_n (\overline{U} \cap L_n)$ and $\hat{f}_U[\overline{U} \cap L_n] \in J$ for each n. This follows from the fact that, μ being nonatomic, each compactum $\overline{U} \cap L_n$ is a finite union of compacta K with $\mu(K) < \varepsilon$ for which $\hat{f}_U[K] \in J$, by the assumption.

Consequently, using the fact that J is calibrated, we reach a contradiction in exactly the same way as in the corresponding part of the proof of Theorem 3.1.

Case 2. For every relatively open, non-empty set $U \subseteq P$ and any $\varepsilon > 0$ there is a compact set $L \subseteq \overline{U} \setminus P$ such that $\mu(L) < \varepsilon$ but $\hat{f}_U[L] \notin J$.

In this case one can combine Lemma 2.3(1) (for $\mathcal{F}_s = \{L \in K(X) : \mu(L) \geq \frac{1}{2^{i(s)}}\}$, where i is a fixed bijection between $\mathbb{N}^{<\mathbb{N}}$ and $\mathbb{N}$) with Lemma 2.4, to get a G_δ-set $P \subseteq A$ such that $f(P) \notin J$ and $\mu(\overline{P} \setminus P) < \infty$. Taking into account that $P \subseteq A$ and $\mu(A) = 0$ we get $\mu(\overline{P}) < \infty$ which implies $P \in J_f(\mu)$, contradicting (1) and completing the proof of the theorem. $\square$

Corollary 5.2. *Let μ and ν be Borel measures on compacta $X \notin J_f(\mu)$ and $Y \notin J_f(\nu)$, respectively. Assume that:*

- *μ is semifinite, G_δ-regular on σ-compact sets and vanishes on singletons,*

- *the family of compact subsets of Y of ν-measure null is coanalytic.*

If $Bor(X)/J_f(\mu)$ embeds densely in $Bor(Y)/J_f(\nu)$, then Y must contain a compact set of finite positive ν-measure. In particular, this holds true, when $\mu = \mu^h$ and $\nu = \mu^g$ are Hausdorff measures.

Proof. Assume that $Bor(X)/J_f(\mu)$ embeds densely in $Bor(Y)/J_f(\nu)$ and, aiming at a contradiction, suppose that Y contains no compact set of finite positive ν-measure. In particular, any compact subset of Y has finite ν-measure if and only if it is ν-null. Consequently, $J_f(\nu) = J_0(\nu)$ is coanalytic and calibrated, hence it has the 1-1 or constant property, by [15, Theorem 5.1]. So, putting $J = J_0(\nu)$, we get a contradiction with Theorem 5.1. $\square$

Remark 5.3. *Let μ^h be a semifinite Hausdorff measure on a compactum $X \notin J_f(\mu^h)$. Then the reasoning in part (A) of the proof of Theorem 5.1 shows in particular, that the σ-ideal $J_f(\mu^h)$ is not calibrated ($P \subseteq \overline{P} \setminus \bigcup_n L_n$ has μ^h-measure zero, so every compact subset of P is in $J_f(\mu^h)$). It is not clear if this σ-ideal has the 1-1 or constant property.*

6. Additional observations and comments

6.1. **The 1-1 or constant property of $J_\sigma(\mu)$.** Recall that by [10, Corollary 4 in Section 3], if a non-thin σ-ideal I on a compactum X is coanalytic and calibrated, then so is the σ-ideal $J_t(I)$ generated by compact I-thin sets (cf. Lemma 2.1 and the remark preceding its formulation).

Given a Borel non-σ-finite measure μ on X, let $J_t(\mu) = J_t(J_0(\mu))$ and let us call $J_0(\mu)$-thin sets simply μ-thin.

Note that $J_\sigma(\mu) \subseteq J_t(\mu)$ and if the measure μ on X is semifinite, then $J_\sigma(\mu) = J_t(\mu)$. Hence, if moreover, $J_0(\mu)$ is coanalytic and calibrated, then $J_\sigma(\mu)$ has the 1-1 or constant property, by [15, Theorem 5.1]. It turns out that the assumption that μ is semifinite can be replaced by a weaker one.

Proposition 6.1. *Let μ be a non-σ-finite measure on a compactum X. If $J_0(\mu)$ is coanalytic and every Borel set of positive μ-measure contains a compact set of positive μ-measure, then $J_\sigma(\mu)$ has the 1-1 or constant property. In particular, this is the case when $\mu = \mu^h$ is a Hausdorff non-σ-finite measure on X.*

Proof. Let $B \in Bor(X) \setminus J_\sigma(\mu)$ and let $f : B \to Y$ be a Borel function into a Polish space Y with all fibers in $J_\sigma(\mu)$.

If $B \notin J_t(\mu)$, then we are done by the 1-1 or constant property of $J_t(\mu)$.

So let us assume that $B \in J_t(\mu)$, i.e., $B \subseteq Z_0 \cup Z_1 \cup \dots$ where Z_i are μ-thin compacta. Since $B \notin J_\sigma(\mu)$, there is i such that for $Z = Z_i$ we have that

(1) Z is μ-thin and $Z \cap B \notin J_\sigma(\mu)$.

We claim that

(2) the σ-ideal $Bor(Z) \cap J_\sigma(\mu)$ (on a compactum Z) is coanalytic and calibrated.

To check this, let us consider the family

(3) $\mathscr{E} = \{K \in K(Z) : \ \mu|Bor(K) : Bor(K) \to \{0, \infty\}$ and $\mu(K) = \infty\}$,

where we do not exclude that $\mathscr{E}$ is empty, and let

(4) $\mathscr{M}$ – a maximal disjoint subfamily of $\mathscr{E}$.

Let us notice, cf. (1), that

(5) $|\mathscr{M}| \leq \aleph_0$ and $Z \setminus \bigcup \mathscr{M}$ is of σ-finite μ-measure.

Indeed, each Borel subset of $Z \setminus \bigcup \mathscr{M}$ of positive μ-measure contains a compact set K of positive μ-measure, and since $K \notin \mathscr{E}$, cf. (3), there is a Borel set $C \subseteq K$ with $0 < \mu(C) < 0$, and hence also a compact set $L \subseteq C$ of positive finite μ-measure. Since $Z \setminus \bigcup \mathscr{M}$ is μ-thin, cf. (1), it follows that a countable collection of compact subsets of $Z \setminus \bigcup \mathscr{M}$ with positive finite μ-measure exhausts this set up to a set of μ-measure null, which justifies (5).

Let us notice that

(6) $K(Z) \cap J_\sigma(\mu) = \{K \in K(Z) : \mu(K \cap M) = 0$ for all $M \in \mathscr{M}\}$.

Indeed, if $K \in K(Z)$ is of σ-finite μ-measure and $M \in \mathscr{M} \subseteq \mathscr{E}$, cf. (3), we get $\mu(K \cap M) = 0$. Conversely, if a compactum K hits every $M \in \mathscr{M}$ in a μ-null set, then (5) implies that $K \in J_\sigma(\mu)$.

Now, $J_0(\mu)$ being coanalytic, $K(M) \cap J_0(\mu)$ is clearly coanalytic for any compactum $M \in K(X)$. Since the map $K \mapsto K \cap M$ from $K(Z)$ to $K(M)$ is Borel, this combined with (6) confirms the first part of (2).

To check the second part of (2), let us fix $K \in K(Z) \setminus J_\sigma(\mu)$ and $K_n \in K(Z) \cap J_\sigma(\mu)$, $n \in \mathbb{N}$. By (6), there is $M \in \mathscr{M}$ with $K \cap M \notin J_0(\mu)$ but $K_n \cap M \in J_0(\mu)$ for all n. Therefore, the Borel set $(K \setminus \bigcup_n K_n) \cap M$ is of positive μ-measure, hence there exists a compactum $L \subseteq (K \setminus \bigcup_n K_n) \cap M$ with positive μ-measure. Since $M \in \mathscr{E}$, we have $L \notin J_\sigma(\mu)$, cf. (3).

Having checked (2) we conclude that the σ-ideal $Bor(Z) \cap J_\sigma(\mu)$ has the 1-1 or constant property, by [15, Theorem 5.1]. We can now complete the proof readily: by (1), $Z \cap B \notin J_\sigma(\mu)$ and the 1-1 or constant property applied to $f|(Z \cap B)$ provides a Borel subset of $Z \cap B$ not in $J_\sigma(\mu)$ on which f is injective. $\square$

6.2. **Some common features of the σ-ideals $J_\sigma(\mu)$ and $J_0(\mu)$.** Recall, cf. Corollary 4.1, that when μ^h is a semifinite, non-σ-finite Hausdorff measure on a compactum X, then none of the Boolean algebras $Bor(X)/J_\sigma(\mu^h)$ and $Bor(Y)/J_0(\mu^h)$ can be densely embedded into the other. Nonetheless, the σ-ideals $J_0(\mu^h)$ and $J_\sigma(\mu^h)$ share some essential common features. In particular, they are both coanalytic and calibrated, hence they have the 1-1 or constant property. Another similarity may be exhibited through the following general scheme, studied by Sabok and Zapletal in [18].

Given a σ-ideal I on a compactum X, let I^* be the σ-ideal consisting of Borel sets in X that can be covered by countably many compact sets from I. Sabok and Zapletal, cf. [18], obtained several results to the effect that properties of I have direct impact on properties of I^*.

Note that given a Borel, non-σ-finite measure μ on X, $J_0(\mu)$ ($J_\sigma(\mu)$, respectively) is equal to I^* for $I = I_0(\mu)$, the σ-ideal of Borel sets of μ-measure zero (for $I = I_\sigma(\mu)$, the σ-ideal of Borel sets of σ-finite μ-measure, respectively).

A σ-ideal I on X is called *polar*, cf. [21] (see also Debs [2] where this terminology was introduced), if there is a collection $\mathscr{M}$ of finite Borel measures on X such that

$$I = \bigcap_{\nu \in \mathscr{M}} I_0(\nu).$$

Note that if μ is a semifinite Borel measure on X, then the σ-ideal $I_0(\mu)$ is polar. Indeed, for each Borel set B of finite positive μ-measure, it is enough to define $\nu_B : Bor(X) \to [0, \mu(B)]$ by

$$\nu_B(E) = \mu(E \cap B),$$

and to note that the measure ν_B vanishes on $I_0(\mu)$ but is positive on B.

The following proposition is closely related to [21, Example 3.6.4] of Zapletal and [4, Corollary 5.20] of Farah and Zapletal. We decided to include a direct proof of this useful observation.

Proposition 6.2. *Let μ be a semifinite, non-σ-finite Borel measure on a compactum X such that $\mu|K(X)$ is a Borel mapping on the hyperspace $K(X)$. Then the σ-ideal $I_\sigma(\mu)$ is polar.*

Consequently, for any Borel function $f : B \to [0,1]^{\mathbb{N}}$ on a Borel set in X of non-σ-finite μ-measure, there exists a compactum $K \subseteq B$ of non-σ-finite μ-measure such that $f|K$ is continuous. In particular, this holds true if $\mu = \mu^h$ is a semifinite, non-σ-finite Hausdorff measure on X.

Proof. Let B be a Borel set in X of non-σ-finite μ-measure. We are going to find a finite Borel measure ν_B which vanishes on $I_\sigma(\mu)$ but is positive on B.

Let, cf. Section 2.3,

$$\mathscr{E} = \{\phi \in C(2^{\mathbb{N}}, K(X)) :\ 0 < \mu(\phi(t)) < \infty \text{ and } \phi(s) \cap \phi(t) = \emptyset \text{ for } s \neq t\}.$$

Claim. There is $\phi \in \mathscr{E}$ with

(1) $\bigcup \phi(2^{\mathbb{N}}) \subseteq B$.

Note that by the assumptions, the family of compacta in X of finite positive μ-measure is an analytic co-basis of $I_0(\mu)$ and, μ being semifinite, B is not $I_0(\mu)$-thin. Hence, by the remarks preceding Lemma 2.1, the claim follows readily, *provided* that B is a G_δ subset of X.

In the general case, one can extend the original topology on X to a Polish topology τ (with the same Borel sets) and such that B is a closed set in (X,τ), cf. [8, 13.A]. Let $K(X,\tau)$ be the collection of compacta in the space (X,τ). Since the identity map from $K(X,\tau)$ to $K(X)$ is continuous, the measure μ is also Borel on $K(X,\tau)$. Since μ is a semifinite Borel measure on (X,τ), using regularity of finite Borel measures on Polish spaces, cf. [8, 17.C], we infer that the family of compacta in (X,τ) of finite positive μ-measure is an analytic co-basis of $I_0(\mu)$. Hence, there is $\phi \in C(2^{\mathbb{N}}, K(X,\tau))$ with $0 < \mu(\phi(t)) < \infty$, $\phi(s)\cap\phi(t) = \emptyset$ for $s \neq t$ and $\bigcup \phi(2^{\mathbb{N}}) \subseteq B$. Using again continuity of the identity map from $K(X,\tau)$ to $K(X)$, we infer that $C(2^{\mathbb{N}}, K(X,\tau)) \subseteq C(2^{\mathbb{N}}, K(X))$, which completes the proof of the claim.

Since the function $\mu \circ \phi$ is Borel for each $\phi \in \mathscr{E}$, we may further assume that

(2) $\sup\{\mu(\phi(t)) : t \in 2^{\mathbb{N}}\} = M < \infty$.

Note that for each Borel set E in X, the function

(3) $t \mapsto \mu(\phi(t) \cap E)$, $t \in 2^{\mathbb{N}}$, is Borel.

Indeed, we have $\mu(\phi(t) \cap E) = \mu(A_t)$, where $A_t = \{x \in X : (t,x) \in A\}$ is the vertical section of a Borel set in $2^{\mathbb{N}} \times X$ defined by $A = \bigcup_{t\in 2^{\mathbb{N}}} (\{t\} \times \phi(t)) \cap (2^{\mathbb{N}} \times E)$, cf. [8, 17.25].

Let λ be the Haar measure on $2^{\mathbb{N}}$ and let $\nu_B : Bor(X) \to [0, M]$, cf. (2), be defined by, cf. (3),

(4) $\nu_B(E) = \int_{2^{\mathbb{N}}} \mu(\phi(t) \cap E)\lambda(dt)$.

Then ν_B is a finite Borel measure on X. Clearly, $\nu_B(B) > 0$, cf. (1). On the other hand, ν_B vanishes on $I_\sigma(\mu)$. Indeed, if K is a compactum in X and $\nu_B(K) > 0$, then $\mu(\phi(t) \cap K) > 0$, cf. (4), for uncountably many t and therefore K is of non-σ-finite μ-measure.

$\square$

6.3. **The σ-ideals related to capacities.** Given a non-zero subadditive capacity γ (see [10, Section 3.1] and [8, 30.A]) on a compactum X, we shall denote by $J_0(\gamma)$ the σ-ideal of Borel sets in X that can be covered by countably many compact γ-null sets. Let us call $J_0(\gamma)$-thin sets simply γ-thin (cf. [10]) and assuming that X is not γ-thin let $J_t(J_0(\gamma))$ – the σ-ideal of subsets of X that can be covered by countably many compact γ-thin sets, be denoted by $J_t(\gamma)$.

Note that every countable union of γ-null compact sets in X is contained in a γ-null G_δ-subset of X. It follows that the σ-ideal $J_0(\gamma)$ has property $(*)$ and is calibrated. Moreover, it is coanalytic, in fact

$J_0(\gamma) \cap K(X)$ is a G_δ-set in the hyperspace $K(X)$ (cf. [10]). Consequently, the σ-ideal $J_t(\gamma)$ is also coanalytic and calibrated, hence both $J_0(\gamma)$ and $J_t(\gamma)$ have the 1-1 or constant property.

These observations combined with Theorem 3.1 lead to the following result (cf. Section 4).

Proposition 6.3. *Let γ_1 and γ_2 be non-zero subadditive capacities on compacta X and Y, respectively, vanishing on singletons.*

If X is not γ_1-thin and there is an analytic co-basis $\mathscr{D}$ of $J_0(\gamma_1)$ consisting of γ_1-thin sets, then

(1) *none of the Boolean algebras $Bor(X)/J_t(\gamma_1)$ and $Bor(Y)/J_0(\gamma_2)$ can be densely embedded into the other,*
(2) *if Y is not γ_2-thin and $Bor(Y)/J_t(\gamma_2)$ embeds densely in $Bor(X)/J_t(\gamma_1)$, then Y must contain a compact γ_2-thin set of positive γ_2-capacity.*

An example of a capacity γ_1 as above is the capacity $\mu^h_{diam(X)}$ (cf. Section 2.1) associated to a semifinite but not σ-finite Hausdorff measure μ^h, as in this case $J_0(\mu^h_{diam(X)}) = J_0(\mu^h)$ (cf. [8, 30.B]). On the other hand, there are examples of capacities γ for which $J_t(\gamma) = J_0(\gamma)$, i.e., the only compact γ-thin sets are the γ-null ones (cf. [10, Section 3.3]).

6.4. **The σ-ideals** $I(dim)$**.** Given a compactum X, we shall denote by $I(dim)$ the σ-ideal of Borel subsets of X that can be covered by countably many compact sets of finite covering dimension.

The σ-ideals $I(dim)$ have the 1-1 or constant property by [15, Corollary 5.6].

For the Hilbert cube $I^{\mathbb{N}}$ and any Hausdorff measure μ^h on a compactum Y, neither of the quotient Boolean algebras $Bor(I^{\mathbb{N}})/I(dim)$ and $Bor(Y)/J_\sigma(\mu^h)$ embeds onto a dense subalgebra of the other.

This can be checked by a reasoning in [14, Section 3] using the fact that σ-ideals $J_\sigma(\mu^h)$ are calibrated (by a theorem of Davies, cf. [16, Remarks following Theorem 59]) and have the 1-1 or constant property (cf. Section 2.2 and Proposition 6.1).

Zapletal [22] showed that applying to the σ-ideals $I(dim)$ the theory from his book [21], one gets a striking solution to a natural problem in the forcing theory, raised by D. H. Fremlin. The following description of $I(dim)$ avoids explicit mentioning of the dimension (but, eventually, this setting is also based on the dimension theory), cf. Zapletal [22, Question 3.2].

Given a compactum X, let $S(2^{\mathbb{N}}, X)$ be the space of all continuous surjections of the Cantor set onto X, equipped with the topology of uniform convergence, let

$$E = \{(x, f) \in X \times S(2^{\mathbb{N}}, X) : f^{-1}(x) \text{ is infinite}\}$$

and let $\pi_1,\ \pi_2$ be the projections of E onto the first and the second coordinate, respectively.

Then by [13], a Borel set $B \subseteq X$ is in $I(dim)$ if and only if $\pi_2(\pi_1^{-1}(B))$ is meager in $S(2^{\mathbb{N}}, X)$.

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