Abstract. Let $\mu^h$, $\mu^g$ be Hausdorff measures on compact metric spaces $X$, $Y$ and let $\text{Bor}(X)/\mathcal{J}_\sigma(\mu^h)$ and $\text{Bor}(Y)/\mathcal{J}_0(\mu^g)$ be the Boolean algebras of Borel sets modulo $\sigma$-ideals of Borel sets that can be covered by countably many compact sets of $\sigma$-finite $\mu^h$-measure or $\mu^g$-measure null, respectively. We shall show that if $\mu^h$ is not $\sigma$-finite, and one of the quotient Boolean algebras embeds densely in the other, then for some Borel $B$ with $\mu^h(B) = \infty$, $\mu^h$ takes on Borel subsets of $B$ only values 0 or $\infty$.

This is a particular instance of some more general results concerning Boolean algebras $\text{Bor}(X)/\mathcal{J}$, where $\mathcal{J}$ is a $\sigma$-ideal of Borel sets generated by compact sets.

1. Introduction

A Borel $\sigma$-ideal $I$ (shortly: a $\sigma$-ideal) on a compactum (i.e., a compact metrizable space) $X$ is a collection of Borel sets in $X$, closed under countable unions and such that for any $A \in I$, all Borel subsets of $A$ are in $I$; $I$ is generated by compact sets if any element of $I$ can be covered by countably many compact sets in $I$. We always assume that $X \notin I$.

The subject of this paper are quotient Boolean algebras $\text{Bor}(X)/\mathcal{J}$ of Borel sets in compacta $X$ modulo $\sigma$-ideals $\mathcal{J}$ generated by compact sets. Our results describe some pairs $I$, $J$ of such $\sigma$-ideals, including natural examples associated with Hausdorff measures on compact metric spaces, with the property that $\text{Bor}(X)/\mathcal{J}$ does not embed densely in $\text{Bor}(Y)/\mathcal{J}$ (in the sense of Definition 4.8 in [11]).

In particular, given a Hausdorff measure $\mu^h : \text{Bor}(X) \rightarrow [0, +\infty]$ defined on the $\sigma$-algebra of Borel sets in a compact metric space $X$ and determined by a continuous nondecreasing function $h : [0, +\infty) \rightarrow [0, +\infty)$ with $h(r) > 0$ for $r > 0$ and $h(0) = 0$ (cf. Rogers [16]), we shall consider $\sigma$-ideals $\mathcal{J}_0(\mu^h)$ ($\mathcal{J}_\sigma(\mu^h)$, respectively) of Borel sets in $X$ that can be covered by countably many compact sets of $\mu^h$-measure null (with $\sigma$-finite $\mu^h$-measure, respectively, provided that $\mu^h$ is not $\sigma$-finite).

Let us recall that a Borel measure $\mu$ is semifinite if each Borel set in $X$ of positive $\mu$-measure contains a Borel set of finite positive $\mu$-measure, cf. Edgar [3] or Fremlin [5]. By results of Larman [12] and

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Howroyd [6], if \( X \) is a subset of the Euclidean space \( \mathbb{R}^n \) or \( h \) is of finite order, e.g. \( h(t) = t^r \), cf. [6], then the measure \( \mu^h \) is semifinite.

Our results (cf. Corollary 4.1) imply that if one of the Boolean algebras \( \text{Bor}(X)/J_\sigma(\mu^h), \text{Bor}(Y)/J_\delta(\mu^g) \) embeds densely in the other, then the measure \( \mu^h \) is not semifinite.

This is also true for a wider class of pairs of \( \sigma \)-ideals determined by Borel measures or capacities, and in fact we shall derive this result from Theorem 3.1 concerning a broad class of \( \sigma \)-ideals investigated in the literature.

Our interest in this topic was strongly influenced by the work of Zapletal [21], Farah and Zapletal [4] and Sabok and Zapletal [18]. However, while that work concentrates on refined analysis of forcings associated with the algebras \( \text{Bor}(X)/I \), we stay in the realm of classical descriptive set theory, as presented by Kechris [8]. This paper is a continuation of our paper [15], and a subsequent note [14].

The main results of the paper are stated and proved in Sections 3, 4 and 5. They are preceded by Section 2 where the terminology is clarified, and some results from the literature, or close to the ones in the literature, needed in the proofs, are explained. Finally, Section 6 contains some additional observations and comments.

The authors would like to thank the referee for suggestions that improved the exposition of the material.

2. Preliminaries

Our notation is standard and mostly agrees with [8]. In particular,

- \( 2^\mathbb{N} \) and \( \mathbb{N}^\mathbb{N} \) are the Cantor and the Baire space, respectively,
- \( \mathbb{N}^{<\mathbb{N}} \) is the family of all finite sequences of natural numbers,
- \( \text{diam}(A) \) is the diameter of a set \( A \) in the given metric space.

2.1. Borel measures. By a Borel measure on a compactum \( X \) we mean a countably additive measure \( \mu : \text{Bor}(X) \rightarrow [0, \infty] \), defined on the \( \sigma \)-algebra of Borel sets in \( X \) and such that \( \mu(X) > 0 \). We say that \( \mu \) is finite (\( \sigma \)-finite, respectively), if \( \mu(X) < \infty \) (\( X \) is the union of a countable family of Borel sets with finite measure, respectively). Let us recall that for a Borel measure \( \mu \) on a compactum, each Borel set of positive finite \( \mu \)-measure contains a compact set of positive \( \mu \)-measure, cf. [8]. Any \( \sigma \)-finite Borel measure \( \mu \) is semifinite, which is equivalent to the fact that each Borel set in \( X \) of positive \( \mu \)-measure contains a compact set of positive finite \( \mu \)-measure.

A starting point of our work were \( \sigma \)-ideals associated with Hausdorff measures \( \mu^h \). Let us recall that given a metric space \((X,d)\) and a continuous nondecreasing function \( h : [0, +\infty) \rightarrow [0, +\infty) \) with \( h(r) > 0 \) for \( r > 0 \) and \( h(0) = 0 \), we let \( \mu^h \) be the Hausdorff measure on \( X \).
determined by \( h \), i.e., cf. Rogers [16], for \( \varepsilon > 0 \) we let 
\[
\mu^h(E) = \inf \{ \sum_n h(\text{diam}(U_n)) : U_n \text{ open with diam}(U_n) \leq \varepsilon \text{ and } E \subseteq \bigcup_n U_n \}
\]
and
\[
\mu^h(E) = \lim_{\varepsilon \to 0^+} \mu^h_{\varepsilon}(E), \quad \text{for } E \in \text{Bor}(X).
\]

As already noted in Section 1, in some important cases the measure \( \mu^h \) is semifinite. On the other hand, Davies and Rogers [1] gave an example of a Hausdorff measure \( \mu^h \) on a compactum \( X \) with \( \mu^h(X) = \infty \) and containing no compact sets of finite positive \( \mu^h \)-measure.

We say that a Borel measure \( \mu \) on \( X \) is \( G_\delta \)-regular on \( \sigma \)-compact sets if every countable union of compact \( \mu \)-null subsets of \( X \) is contained in a \( G_\delta \mu \)-null subset of \( X \), equipped with the Vietoris topology, cf. Kechris [8].

We say that a Borel measure \( \mu \) on \( X \) is \( \sigma \)-compact sets.

2.2. Calibration, the 1-1 or constant property, and property \((\ast)\) of \( \sigma \)-ideals. Throughout this subsection let \( I \) be a \( \sigma \)-ideal on a compactum \( X \).

We say that
- \( I \) is generated by compact sets, if each element in \( I \) can be covered by countably many compacta in \( I \),
- \( I \) is coanalytic, if \( I \cap K(X) \) is a coanalytic set in the space \( K(X) \) of compact sets in \( X \), equipped with the Vietoris topology, cf. Kechris [8],
- \( I \) is calibrated if it is generated by compact sets and \( I \cap K(X) \) is calibrated in the sense of Kechris, Louveau and Woodin [10], i.e., for any \( F \in K(X) \setminus I \) and countably many compact sets \( K_n \in I, n \in \mathbb{N} \), there is a compactum \( K \subseteq F \setminus \bigcup_{n \in \mathbb{N}} K_n \) not in \( I \),
- \( I \) has the 1-1 or constant property, introduced by Sabok and Zapletal [18] (cf. also [17]), if for every Borel set \( B \subseteq X \) not in \( I \) and every Borel function \( f \) from \( B \) into a Polish space \( Y \) with all fibers in \( I \), there is a Borel set \( G \subseteq B \) not in \( I \) on which \( f \) is injective,
- \( I \) has property \((\ast)\), distinguished by Solecki [20], if whenever \( K_n \in I, n \in \mathbb{N} \), there is a \( G_\delta \)-set \( G \) in \( X \) containing \( \bigcup_{n \in \mathbb{N}} K_n \) such that all compact subsets of \( G \) are in \( I \).

By [15, Theorem 5.1], the \( \sigma \)-ideals generated by compact sets that are both coanalytic and calibrated have the 1-1 or constant property. This class includes \( \sigma \)-ideals \( J_0(\mu^h) \) (which follows from regularity properties of Hausdorff measures, cf. [16]), and also \( \sigma \)-ideals \( J_\sigma(\mu^h) \), provided that the measure \( \mu^h \) is non-\( \sigma \)-finite and semifinite (in fact, a more general fact holds true, cf. Proposition 6.1, which however, is not needed for our main result).
2.3. \textit{I-thin sets and the $\sigma$-ideals $J_I(I)$}. We say that a set $A \subseteq X$ is \textit{$I$-thin} if there is no uncountable disjoint family of compact subsets of $A$ which are not in $I$, cf. [10]. If $X$ is $I$-thin, we say that \textit{$I$ is thin}. Assuming that $I$ is not thin, we shall denote by $J_I(I)$ the $\sigma$-ideal of Borel subsets of $X$ that can be covered by countably many compact $I$-thin sets.

Given a Borel measure $\mu$ on a compactum $X$, let $J_0(\mu)$ ($J_0(\mu)$, respectively) be the $\sigma$-ideal of Borel sets in $X$ that can be covered by countably many compact sets of $\mu$-measure null (with $\sigma$-finite $\mu$-measure, respectively, provided that the measure $\mu$ is not $\sigma$-finite).

Note that if $\mu$ is semifinite, $J_\sigma(\mu) = J_I(J_0(\mu))$, as in this case compact sets of $\sigma$-finite $\mu$-measure are exactly compact $J_0(\mu)$-thin sets; this is also true for arbitrary Hausdorff measure $\mu^h$, cf. [16, Ch.2, §6, Corollary 2].

By [10, Corollary 4 in Section 3], if $I$ is coanalytic and calibrated, then so is $J_I(I)$ (cf. Lemma 2.1 and the remark preceding its statement). Consequently, this is true for $J_\sigma(\mu)$, whenever $\mu$ is a semifinite (non-$\sigma$-finite) measure on a compactum $X$ with coanalytic $J_0(\mu)$-thin $K(X)$, and also for $J_\sigma(\mu^h)$, where $\mu^h$ is an arbitrary (non-$\sigma$-finite) Hausdorff measure.

We shall need a more refined version of the fact that for coanalytic $I$, $J_I(I)$ is also coanalytic. Let us recall (cf. [9], page 265) that a family $\mathcal{D} \subseteq K(X) \setminus I$ is a \textit{co-basis} of $I$ if every compact set not in $I$ contains a subset from $\mathcal{D}$. Note that if $\mu$ is a semifinite Borel measure on $X$, then the family of compact subsets of $X$ of finite positive $\mu$-measure is a co-basis of $J_0(\mu)$.

Let
\[ \mathcal{E} = \{ \phi \in C(2^\mathbb{N}, K(X)) : \phi(t) \in \mathcal{D} \text{ and } \phi(s) \cap \phi(t) = \emptyset \text{ for } s \neq t \} \text{.} \]

Then, if $A$ is a $G_\delta$ subset of $X$ and $\mathcal{D}$ is analytic, $A$ is not $I$-thin if and only if there is $\phi \in \mathcal{E}$ with $\bigcup \phi(2^\mathbb{N}) \subseteq A$. Indeed, if $A$ is not $I$-thin, then since the set $K(A) \cap \mathcal{D}$ is analytic in $K(X)$ and contains an uncountable disjoint family, it contains a perfect disjoint family as well (cf. [10, Theorem 2 in Section 3]). We need a refinement of this argument leading to the next lemma. This is in fact Corollary 4 in [10, Section 3], but since the proof in [10] omits some essential details, we include a justification for readers convenience.

\textbf{Lemma 2.1.} If there is an analytic co-basis $\mathcal{D}$ of $I$ consisting of $I$-thin sets, then the $\sigma$-ideal $J_I(I)$ is coanalytic. If, moreover, $I$ is calibrated, then $J_I(I)$ is calibrated and has the 1-1 or constant property.

\textit{Proof.} Let us fix a continuous surjection $\varphi : \mathbb{N}^\mathbb{N} \to \mathcal{D}$ and let $\mathcal{F}$ be the collection of Cantor sets $C$ in $\mathbb{N}^\mathbb{N}$ such that for any distinct $s, t \in C$,
\[ \varphi(s) \cap \varphi(t) = \emptyset. \]

Notice that $\mathcal{F}$ is a $G_\delta$-set in $K(\mathbb{N}^\mathbb{N})$ and for any $C \in \mathcal{F}$, $\bigcup \phi(C) \notin J_I(I)$. We shall verify that, moreover, $K \in K(X)$ is not $I$-thin if and
only if there is \( C \in \mathcal{F} \) with \( \bigcup \phi(C) \subseteq K \). This implies that the set \( K(X) \setminus J_t(I) \) is analytic.

So assume that \( K \in K(X) \setminus J_t(I) \) and let \( T = \{ t \in \mathbb{N}^\mathbb{N} : \phi(t) \subseteq K \} \). Then \( T \) is closed in \( \mathbb{N}^\mathbb{N} \) and since \( K \not\in J_t(I) \), there is an uncountable set \( A \subseteq T \) such that \( \phi(a) \cap \phi(b) = \emptyset \) whenever \( a, b \in A \) are distinct. We can assume that each point in \( A \) is an accumulation point in \( A \).

Let
\[
R = \{(s, t) \in \overline{A} \times \overline{A} : \phi(s) \cap \phi(t) \neq \emptyset\}.
\]
Then the relation \( R \) is closed with empty interior in \( \overline{A} \times \overline{A} \), hence Mycielski’s theorem (see [8]) provides an \( R \)-independent Cantor set \( C \subseteq A \). Clearly, \( \bigcup \phi(C) \subseteq K \) which completes the proof of the main part of the assertion. The “moreover” part follows from [10, Corollary 4 in Section 3] and [15, Theorem 5.1]. \( \square \)

The 1-1 or constant property of ideals is needed in this paper for the following version of Sikorski’s theorem [8, 15.C], stated in [14, Lemma 2.5.1].

**Lemma 2.2.** Let \( I \) and \( J \) be \( \sigma \)-ideals on compacta \( X \) and \( Y \), respectively. Let \( \text{Bor}(Y)/J \) embed onto a dense subalgebra of \( \text{Bor}(X)/I \) and assume that \( I \) has the 1-1 or constant property and \( J \) contains all singletons. Then there exist sets \( A \in \text{Bor}(X) \setminus I \) and \( B \in \text{Bor}(Y) \setminus J \), and a Borel isomorphism \( \phi : A \rightarrow B \) between \( A \) and \( B \) such that \( C \in I \) if and only if \( \phi(C) \in J \), whenever \( C \subseteq A \).

More precisely, to get \( \phi \) as in Lemma 2.2, one first uses [14, Lemma 2.5.1] to get a Borel map \( \psi : E \rightarrow Y \) on \( E \in \text{Bor}(X) \setminus I \) such that, for \( C \in \text{Bor}(E) \), \( C \in I \) if and only if \( \psi(C) \in J \), and then, appealing to the 1-1 or constant property of \( I \), one further shrinks \( E \) to its Borel subset \( A \), not in \( I \), on which \( \psi = \phi|A \) is injective.

Note, that if \( \mu \) is \( G_\delta \)-regular on \( \sigma \)-compact sets (see Section 2.1), then the \( \sigma \)-ideal \( I_0(\mu) \) has property (\( * \)) and is calibrated.

### 2.4. Generalized Hurewicz systems.

Given a \( G_\delta \)-set \( G \) in a compactum \( X \) with a fixed complete metric on \( G \) bounded by 1, by a generalized Hurewicz system we shall mean a pair \((U_s)_{s \in \mathbb{N}^\mathbb{N}}, (L_s)_{s \in \mathbb{N}^\mathbb{N}}\) of families of subsets of \( X \) with the following properties (the closure is taken in \( X \)):

- \( U_s \subseteq G \) is relatively open, non-empty and \( \text{diam}(U_s) \leq 2^{-\text{length}(s)} \),
- \( \overline{U_s} \cap \overline{U_t} = \emptyset \) for distinct \( s, t \) of the same length,
- \( \overline{U_{s,t}} \cap G \subseteq U_s \)
- \( L_s \subseteq \overline{U_s} \) is compact,
- \( L_s \cap \overline{U_{s,t}} = \emptyset \),
- each neighbourhood of \( L_s \) contains all but finitely many \( U_{s,t} \).
If \((U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}\), \((L_s)_{s \in \mathbb{N}^{<\mathbb{N}}}\) is a generalized Hurewicz system, then
\[
P = \bigcap_n \bigcup \{ U_s : \text{length}(s) = n \}
\]
is the \(G_\delta\)-subset of \(G\) determined by the system and
\[
\mathcal{P} \subseteq P \cup \bigcup \{ L_s : s \in \mathbb{N}^{<\mathbb{N}} \}.
\]

If, additionally, \(L_s = \bigcap_j \bigcup_{i>j} U_{s_i}^j\) and \(\lim_i \text{diam}(U_{s_i}^j) = 0\), then
\[
P = \mathcal{P} \cup \bigcup \{ L_s : s \in \mathbb{N}^{<\mathbb{N}} \}.
\]

Moreover, if \(V\) is a non-empty relatively open subset of \(P\), then \(V\) contains \(L_s\) with arbitrarily long \(s \in \mathbb{N}^{<\mathbb{N}}\).

Such systems of sets, with \(L_s\) being singletons, were introduced by Hurewicz [7]. Solecki showed that generalized Hurewicz systems are very useful to determine \(G_\delta\)-sets not belonging to a given \(\sigma\)-ideal \(I\) generated by compact sets, cf. [19, proof of Theorem 1].

The following lemma describes some additional essential features of these systems.

**Lemma 2.3.** Let \(G\) be a \(G_\delta\)-set in a compactum \(X\). Let \(F\) and \(F_s, s \in \mathbb{N}^{<\mathbb{N}}\), be families of compact subsets of \(X\).

If for every relatively open, non-empty subset \(U\) of \(G\) and each \(s \in \mathbb{N}^{<\mathbb{N}}\) there is a compact set \(L \subseteq U \setminus G\) in \(F \setminus F_s\), then there is a generalized Hurewicz system \((U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}, (L_s)_{s \in \mathbb{N}^{<\mathbb{N}}}\) with each \(L_s \in F \setminus F_s\).

Consequently, if \(P \subseteq G\) is the \(G_\delta\) set determined by the system, then:

1. \(\overline{P} \setminus P \subseteq \bigcup \{ L_s : s \in \mathbb{N}^{<\mathbb{N}} \}\). In particular, if \(I\) is the \(\sigma\)-ideal generated \(\bigcup_s (F \setminus F_s)\), then \(\overline{P} \setminus P \in I\),

2. if each \(F_s\) is hereditary (i.e., whenever \(D\) is a closed subset of \(F \in F_s\), then \(D \in F_s\)) and \(\text{length}(s) < \text{length}(t)\) implies \(F_s \subseteq F_t\) for every \(s, t \in \mathbb{N}^{<\mathbb{N}}\), then \(V\) is not in the \(\sigma\)-ideal \(J\) generated by \(\bigcup_s F_s\), for any relatively open, non-empty subset \(V\) of \(P\); in particular, \(P \notin J\).

**Proof.** The details of the construction are similar to those in [19, proof of Theorem 1].

Since \(\overline{P} \setminus P \subseteq \bigcup \{ L_s : s \in \mathbb{N}^{<\mathbb{N}} \}\), conclusion (1) is clear.

Conclusion (2) follows from a Baire category argument using the fact that the closure of any non-empty relatively open subset of \(P\) contains \(L_s\) with arbitrary long \(s \in \mathbb{N}^{<\mathbb{N}}\).

Following [14, Section 3, (3)] we shall use the generalized Hurewicz systems in the following situation.

Let \(f : G \to Y\) be a continuous function from a non-empty, \(G_\delta\)-set \(G\) in a compactum \(X\) into a compactum \(Y\) and let \(J\) be a \(\sigma\)-ideal on \(Y\), generated by compact sets. Let also \(F \subseteq K(X)\).
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We shall denote by $B(x, \frac{1}{n})$ the $\frac{1}{n}$-ball about $x$ with respect to a fixed metric on $X$.

For any non-empty relatively open $U \subseteq G$, we define $\hat{f}_U : \overline{U} \rightarrow K(Y)$ by

$$\hat{f}_U(x) = \bigcap_n f(B(x, \frac{1}{n}) \cap U).$$

and let

$$\hat{f}_U[A] = \bigcup \{ \hat{f}(x) : x \in A \} \text{ for } A \subseteq \overline{U}.$$

Then $\hat{f}_U : \overline{U} \rightarrow K(Y)$ is upper-semicontinuous and if $A \subseteq U$, $\hat{f}_U[A] = f(A)$.

In this setting we have the following

**Lemma 2.4.** If for every relatively open, non-empty subset $U$ of $G$ there is a compact set $L \subseteq U \setminus G$ in $F$ with $\hat{f}_U[L] \notin J$, then there is a generalized Hurewicz system $(U_s)_{s \in \mathbb{N}^\mathbb{N}}$, $(L_s)_{s \in \mathbb{N}^\mathbb{N}}$ with $L_s \in F$, $\hat{f}_{U_s}[L_s] \notin J$ and $\hat{f}_{U_s}[L_s] \subseteq \{ f(x_i) : i \in \mathbb{N}\}$ for each $s \in \mathbb{N}^\mathbb{N}$ and any sequence $x_i \in U_{s \cdot i}$, $i \in \mathbb{N}$.

Consequently, if $P \subseteq G$ is the $G_\delta$ set determined by the system, then $f(P) \notin J$.

For details related to Lemma 2.4 we refer the reader to [14, parts (B) and (C) in Section 3].

3. Main results

Let us recall that, given a non-thin $\sigma$-ideal $I$ on a compactum $X$, we denote by $J_t(I)$ the $\sigma$-ideal generated by compact $I$-thin sets, cf. Section 2.3; the notion of a co-basis of $I$ was also explained there.

**Theorem 3.1.** Let $I$ and $J$ be calibrated $\sigma$-ideals on compacta $X$ and $Y$, respectively. If there is an analytic co-basis $D$ of $I$ consisting of $I$-thin sets and $J$ has property ($\ast$) and contains all singletons, then $Bor(Y)/J$ does not embed onto a dense subalgebra of $Bor(X)/J_t(I)$.

If, moreover, $J$ has the 1-1 or constant property (in particular, if $J$ is coanalytic), then there is also no embedding of $Bor(X)/J_t(I)$ onto a dense subalgebra of $Bor(Y)/J$.

**Proof.** (A) Aiming at a contradiction, assume that either $Bor(Y)/J$ embeds onto a dense subalgebra of $Bor(X)/J_t(I)$ or $J$ has the 1-1 or constant property and $Bor(X)/J_t(I)$ embeds onto a dense subalgebra of $Bor(Y)/J$. In either case Lemma 2.2 gives us a Borel injection $f : B \rightarrow Y$, $B \in Bor(X) \setminus J_t(I)$, such that

1. for any Borel set $C \subseteq B$ we have $C \in J_t(I)$ if and only if $f(C) \in J$.

One can moreover assume, using Solecki’s theorem [19, Theorem 1], that $f$ is continuous on $B$ and $B$ is a $G_\delta$-set such that no non-empty relatively open subset of $B$ is in $J_t(I)$. 
A key element of our proof is the following observation concerning the maps \( \hat{f}_U \) introduced in Section 2.4 (in the part preceding Lemma 2.4).

**Claim.** For any non-empty relatively open set \( U \subseteq B \), each compact set \( D \subseteq \overline{U} \) with \( D \notin J_t(I) \) either contains a compact subset \( L \in J_t(I) \) with \( \hat{f}_U[L] \notin J \) or a compact subset \( L \notin J_t(I) \) with \( \hat{f}_U[L] \in J \).

To prove the claim, let

\[
\mathcal{E} = \{ \phi \in C(2^N, K(X)) : \phi(t) \in \mathcal{D} \text{ and } \phi(s) \cap \phi(t) = \emptyset \text{ for } s \neq t \}.
\]

Since \( D \) is not \( I \)-thin, there is (cf. remarks preceding Lemma 2.1) \( \phi \in \mathcal{E} \) with \( \bigcup \phi(2^N) \subseteq D \).

Let \( Q \) be a countable set dense in \( 2^N \).

We have \( \phi(q) \in \mathcal{D} \subseteq J_t(I) \) for each \( q \in Q \), so if \( \hat{f}_U[\phi(q)] \notin J \) for at least one \( q \), we are done.

So assume that \( \hat{f}_U[\phi(q)] \in J \) for every \( q \in Q \).

Since the \( \sigma \)-ideal \( J \) has property \((*)\), there is a \( G_\delta \) subset \( S \) of \( Y \) such that \( \bigcup_{q \in Q} \hat{f}_U[\phi(q)] \subseteq S \) and all compact subsets of \( S \) are in \( J \).

The set \( \{ F \in K(D) : \hat{f}_U[F] \subseteq S \} \) is \( G_\delta \) in the hyperspace, \( \hat{f}_U \) being upper-semicontinuous, and since \( \phi \) is continuous we infer that \( E = \{ t \in 2^N : \hat{f}_U[\phi(t)] \subseteq S \} \) is a \( G_\delta \)-set containing \( Q \). In effect, for any homeomorphic embedding \( \gamma : 2^N \to E \), letting \( \psi = \phi \circ \gamma \) and \( L = \bigcup \psi(2^N) \) we obtain \( L \notin J_t(I) \) with \( \hat{f}_U[L] \subseteq S \) hence \( \hat{f}_U[L] \in J \), as required. This completes the proof of the claim.

As an immediate consequence of Claim we have

(2) all compact subsets of \( B \) are in \( J_t(I) \).

Indeed, since \( \hat{f}_B | B = f \), (1) implies that for any compact \( L \subseteq B \), \( \hat{f}_B[L] = f(L) \in J \) if and only if \( L \in J_t(I) \). This, by Claim, implies (2).

(B) We shall consider now two cases.

**Case 1.** There is a relatively open non-empty set \( U \subseteq B \) such that for each compact set \( L \subseteq \overline{U} \setminus B \) with \( L \in J_t(I) \) we have \( \hat{f}_U[L] \in J \).

We shall check that in this case

(3) for each non-empty relatively open \( V \subseteq U \) there is a compact set \( L \subseteq \overline{V} \setminus B \) with \( L \notin J_t(I) \) but \( \hat{f}_U[L] \in J \).

So let \( V \) be a non-empty relatively open subset of \( U \). Note that \( V \) being relatively open in \( B \), \( V \notin J_t(I) \).

Since \( \overline{V} \setminus B = \bigcup K_n \), \( K_n \in K(X) \), and all compact subsets of \( \overline{V} \setminus \bigcup K_n \) are in \( J_t(I) \), calibration of \( J_t(I) \) provides \( n_0 \) with \( K_{n_0} \notin J_t(I) \). Applying Claim to \( D = K_{n_0} \), we get in this case a compact set \( L \subseteq K_{n_0} \subseteq \overline{V} \setminus B \) with \( L \notin J_t(I) \) but \( \hat{f}_U[L] \in J \).
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Now, equipped with (3), one can apply Lemma 2.3(2) (for \( \mathcal{F} = \{ L \in K(X) : \hat{f}_U[L] \in J \} \) and \( \mathcal{F}_s = J_t(I) \) for each \( s \in \mathbb{N}^{<\mathbb{N}} \)) to get a \( G_0 \)-set \( P \notin J_t(I) \) in \( U \) such that \( \overline{P} \setminus P \subseteq \bigcup_n L_n \), where \( L_n \notin J_t(I) \) but \( \hat{f}_U[L_n] \in J \), for every \( n \in \mathbb{N} \).

Since \( P \subseteq B \) and \( P \notin J_t(I) \), we have \( \hat{f}_U[P] = f(P) \notin J \), cf. (1).

However, \( \hat{f}_U[L_n] \in J \) for each \( n \), so using the fact that \( J \) is calibrated, we get a compact set \( T \subseteq \hat{f}_U[\overline{P}] \setminus \bigcup_n \hat{f}_U[L_n] \) which is not in \( J \). Note that \( T \subseteq f(P) \), since \( \overline{P} \setminus P \subseteq \bigcup_n L_n \). Let \( K = f^{-1}(T) \). Since \( f \) is injective, \( K \subseteq P \). As for any \( x \in \overline{P} \setminus P \), \( \hat{f}_U(x) \cap T = \emptyset \), it follows that \( K = \{ x \in \overline{P} : \hat{f}_U(x) \cap T \neq \emptyset \} \). Hence, \( \hat{f}_U \) being upper-semicontinuous, \( K \) is compact.

Thus we obtained a compact set \( K \subseteq P \) with \( f(K) \notin J \). But on the other hand, \( K \in J_t(I) \), as \( K \subseteq B \), cf. (2). This contradicts (1), completing the proof of the theorem in Case 1.

Case 2. For every relatively open, non-empty set \( U \subseteq B \) there is a compact set \( L \subseteq U \setminus B \) such that \( L \in J_t(I) \) but \( \hat{f}_U[L] \notin J \).

In this case one can apply Lemma 2.4 to get a \( G_0 \)-set \( P \subseteq B \) such that \( f(P) \notin J \) and \( \overline{P} \setminus P \subseteq \bigcup_n L_n \), where \( L_n \in J_t(I) \), for every \( n \in \mathbb{N} \). Again, a contradiction with (1) will be reached as soon as we prove that \( P \in J_t(I) \). But if \( P \notin J_t(I) \), then since \( L_n \in J_t(I) \) for each \( n \), using the fact that \( J \) is calibrated, we get a compact set \( K \subseteq \overline{P} \setminus \bigcup_n L_n \subseteq B \) which is not in \( J_t(I) \). This, however, again contradicts (2), completing the proof of the theorem. \( \square \)

4. SOME COROLLARIES

In this section we shall combine Theorem 3.1 with observations made in Section 2 to get, in particular, results stated in the introduction.

Throughout this section we shall assume that

- \( \mu \) is a semifinite but not \( \sigma \)-finite Borel measure on a compactum \( X \) and the collection of compact sets of finite, positive \( \mu \)-measure is analytic in \( K(X) \).

**Corollary 4.1.** With \( \mu \) as above, for any calibrated \( \sigma \)-ideal \( J \) on a compactum \( Y \) which has property \( \ast \) and contains all singletons, \( \text{Bor}(Y) / J \) does not embed onto a dense subalgebra of \( \text{Bor}(X) / J_{\sigma}(\mu) \). If, moreover, \( J \) has the 1-1 or constant property (in particular, if \( J \) is coanalytic), then there is also no embedding of \( \text{Bor}(X) / J_{\sigma}(\mu) \) onto a dense subalgebra of \( \text{Bor}(Y) / J \).

In particular, none of the embeddings exist if \( \mu \) is a semifinite but not \( \sigma \)-finite Hausdorff measure \( \mu^h \) and \( J = J_0(\mu^h) \) for a Hausdorff measure \( \mu^h \).
The next corollary shows that Boolean algebras $\text{Bor}(X)/J_\sigma(\mu)$ may differ essentially, already for various non-$\sigma$-finite Hausdorff measures $\mu = \mu^h$. Let us recall that Davies and Rogers [1] constructed a Hausdorff measure $\mu^g$ on a compactum $Y$ with no compact set of finite positive $\mu^g$-measure. Let us notice that Zapletal [21, 4.4.2] exhibited substantial differences between $\sigma$-ideals generated by Borel sets of $\sigma$-finite measure for the specific $\mu^h$ constructed by Davies and Rogers and any semifinite, non-$\sigma$-finite $\mu^g$, respectively.

Corollary 4.2. Let $\mu$ be as above and let $\nu$ be a Borel measure on a compactum $X$ which is non-$\sigma$-finite, $G_\delta$-regular on $\sigma$-compact sets and vanishes on singletons. Then, if $\text{Bor}(Y)/J_\sigma(\nu)$ embeds densely in $\text{Bor}(X)/J_\sigma(\mu)$, $Y$ must contain a compact set of finite positive $\nu$-measure.

In particular, this applies to non-$\sigma$-finite Hausdorff measures $\mu = \mu^h$ and $\nu = \mu^g$ with $\mu^h$ semifinite.

Proof. Assume that $\text{Bor}(Y)/J_\sigma(\nu)$ embeds densely in $\text{Bor}(X)/J_\sigma(\mu)$ and, aiming at a contradiction, suppose that $Y$ contains no compact set of finite positive $\nu$-measure. In particular, $K \in K(Y)$ has $\sigma$-finite $\nu$-measure if and only if $\nu(K) = 0$. Consequently, $J_\sigma(\nu) = J_0(\nu)$ is calibrated, has property ($\ast$) and contains all singletons, so putting $J = J_\sigma(\nu)$ we get a contradiction with Corollary 4.1.

5. The $\sigma$-ideals generated by compact sets of finite measure

Given a Borel non-$\sigma$-finite measure $\mu$ on a compactum $X$, one can consider yet another natural $\sigma$-ideal related to $\mu$, generated by compact sets, lying between the $\sigma$-ideals $J_0(\mu)$ and $J_\sigma(\mu)$ – the $\sigma$-ideal $J_f(\mu)$ of Borel sets in $X$ that can be covered by countably many compact sets with finite $\mu$-measure.

We shall show that if $\mu^h$ is a semifinite non-$\sigma$-finite Hausdorff measure on $X$, then $\text{Bor}(X)/J_f(\mu^h)$ embeds densely into neither $\text{Bor}(X)/J_0(\mu^h)$ nor $\text{Bor}(X)/J_\sigma(\mu^h)$. This is a consequence of the following, more general result.

Theorem 5.1. Let $\mu$ be a Borel measure on a compactum $X \not\in J_f(\mu)$. Assume that $\mu$ is semifinite, $G_\delta$-regular on $\sigma$-compact sets and vanishes on singletons. Let $J$ be a calibrated $\sigma$-ideal with the 1-1 or constant property on a compactum $Y$. Then $\text{Bor}(X)/J_f(\mu)$ does not embed onto a dense subalgebra of $\text{Bor}(Y)/J$. In particular, $\text{Bor}(X)/J_f(\mu)$ embeds densely into neither $\text{Bor}(X)/J_0(\mu)$ nor $\text{Bor}(X)/J_\sigma(\mu)$ (provided that $\mu$ is not $\sigma$-finite).

Proof. (A) Aiming at a contradiction, assume that $\text{Bor}(X)/J_f(\mu)$ embeds onto a dense subalgebra of $\text{Bor}(Y)/J$. Since $J$ has the 1-1 or
constant property and \( J_f(\mu) \) contains all singletons, Lemma 2.2 gives
us a Borel injection \( f : A \to Y, A \in Bor(X) \setminus J_f(\mu), \) such that

(1) for any Borel set \( C \subseteq A \) we have \( C \in J_f(\mu) \) if and only if
\( f(C) \in J. \)

Moreover, one can assume, using Solecki’s theorem [19, Theorem 1] and the fact that \( \mu \) vanishes on singletons and is \( G_\delta \)-regular on \( \sigma \)-compact sets, that \( A \) is a \( G_\delta \)-set of \( \mu \)-measure zero, \( f \) is continuous on \( A \) and no non-empty relatively open subset of \( A \) is in \( J_f(\mu). \) Consequently, for every relatively open, non-empty subset \( U \) of \( A \), we have \( \mu(U \setminus A) = \infty \) which implies, \( \mu \) being semifinite, that for each \( n \in \mathbb{N} \) there is a compact set \( L \subseteq U \setminus A \) with \( n < \mu(L) < \infty. \) It follows that one can apply Lemma 2.3(2) (for \( F = K(X) \) and \( F_s = \{ L \in K(X) : \mu(L) \leq \text{length}(s) \} \) to get a non-empty \( G_\delta \)-set \( P \) in \( A \) with \( V \notin J_f(\mu) \) for any non-empty relatively open subset \( V \) of \( P \) and such that \( P \setminus P \subseteq \bigcup_n L_n, \)
where \( L_n \) are compacta of finite \( \mu \)-measure.

(B) Following closely part (B) of the proof of Theorem 3.1 we shall consider now two cases.

Case 1. There exist a relatively open non-empty set \( U \subseteq P \) and an \( \varepsilon > 0 \) such that for each compact set \( L \subseteq U \setminus P \) with \( \mu(L) < \varepsilon \) we have \( \hat{f}_U[L] \notin J. \)

Since \( U \subseteq P \subseteq A \) and \( U \notin J_f(\mu) \), we have \( \hat{f}_U[U] = f(U) \notin J, \) cf. (1).

However, \( U \setminus P \subseteq \bigcup_n (U \cap L_n) \) and \( \hat{f}_U[(U \cap L_n)] \in J \) for each \( n. \) This follows from the fact that, \( \mu \) being nonatomic, each compactum \( U \cap L_n \)
is a finite union of compacta \( K \) with \( \mu(K) < \varepsilon \) for which \( \hat{f}_U[K] \in J, \)
by the assumption.

Consequently, using the fact that \( J \) is calibrated, we reach a contradiction in exactly the same way as in the corresponding part of the proof of Theorem 3.1.

Case 2. For every relatively open, non-empty set \( U \subseteq P \) and any \( \varepsilon > 0 \) there is a compact set \( L \subseteq U \setminus P \) such that \( \mu(L) < \varepsilon \) but \( \hat{f}_U[L] \notin J. \)

In this case one can combine Lemma 2.3(1) (for \( F_s = \{ L \in K(X) : \mu(L) \geq \frac{1}{2^{2^n}} \} \), where \( i \) is a fixed bijection between \( \mathbb{N}^{< \mathbb{N}} \) and \( \mathbb{N} \) with Lemma 2.4, to get a \( G_\delta \)-set \( P \subseteq A \) such that \( f(P) \notin J \) and \( \mu(P \setminus P) < \infty. \) Taking into account that \( P \subseteq A \) and \( \mu(A) = 0 \) we get \( \mu(P) < \infty \) which implies \( P \in J_f(\mu), \) contradicting (1) and completing the proof of the theorem.

\[
\text{Corollary 5.2. Let } \mu \text{ and } \nu \text{ be Borel measures on compacta } X \notin J_f(\mu) \text{ and } Y \notin J_f(\nu), \text{ respectively. Assume that:}
\]

- \( \mu \) is semifinite, \( G_\delta \)-regular on \( \sigma \)-compact sets and vanishes on singletons,
the family of compact subsets of $Y$ of $\nu$-measure null is coanalytic.

If $\text{Bor}(X)/J_f(\mu)$ embeds densely in $\text{Bor}(Y)/J_f(\nu)$, then $Y$ must contain a compact set of finite positive $\nu$-measure. In particular, this holds true, when $\mu = \mu^b$ and $\nu = \mu^a$ are Hausdorff measures.

**Proof.** Assume that $\text{Bor}(X)/J_f(\mu)$ embeds densely in $\text{Bor}(Y)/J_f(\nu)$ and, aiming at a contradiction, suppose that $Y$ contains no compact set of finite positive $\nu$-measure. In particular, any compact subset of $Y$ has finite $\nu$-measure if and only if it is $\nu$-null. Consequently, $J_f(\nu) = J_0(\nu)$ is coanalytic and calibrated, hence it has the 1-1 or constant property, by [15, Theorem 5.1]. So, putting $J = J_0(\nu)$, we get a contradiction with Theorem 5.1. □

**Remark 5.3.** Let $\mu^b$ be a semifinite Hausdorff measure on a compactum $X \notin J_f(\mu^b)$. Then the reasoning in part (A) of the proof of Theorem 5.1 shows in particular, that the $\sigma$-ideal $J_f(\mu^b)$ is not calibrated ($P \subseteq \mathcal{P} \setminus \bigcup_n L_n$ has $\mu^b$-measure zero, so every compact subset of $P$ is in $J_f(\mu^b)$). It is not clear if this $\sigma$-ideal has the 1-1 or constant property.

6. **Additional observations and comments**

6.1. **The 1-1 or constant property of $J_\sigma(\mu)$**. Recall that by [10, Corollary 4 in Section 3], if a non-thin $\sigma$-ideal $I$ on a compactum $X$ is coanalytic and calibrated, then so is the $\sigma$-ideal $J_I(I)$ generated by compact $I$-thin sets (cf. Lemma 2.1 and the remark preceding its formulation).

Given a Borel non-$\sigma$-finite measure $\mu$ on $X$, let $J_I(\mu) = J_I(J_0(\mu))$ and let us call $J_0(\mu)$-thin sets simply $\mu$-thin.

Note that $J_\sigma(\mu) \subseteq J_I(\mu)$ and if the measure $\mu$ on $X$ is semifinite, then $J_\sigma(\mu) = J_I(\mu)$. Hence, if moreover, $J_0(\mu)$ is coanalytic and calibrated, then $J_\sigma(\mu)$ has the 1-1 or constant property, by [15, Theorem 5.1]. It turns out that the assumption that $\mu$ is semifinite can be replaced by a weaker one.

**Proposition 6.1.** Let $\mu$ be a non-$\sigma$-finite measure on a compactum $X$. If $J_0(\mu)$ is coanalytic and every Borel set of positive $\mu$-measure contains a compact set of positive $\mu$-measure, then $J_\sigma(\mu)$ has the 1-1 or constant property. In particular, this is the case when $\mu = \mu^b$ is a Hausdorff non-$\sigma$-finite measure on $X$.

**Proof.** Let $B \in \text{Bor}(X) \setminus J_\sigma(\mu)$ and let $f : B \to Y$ be a Borel function into a Polish space $Y$ with all fibers in $J_\sigma(\mu)$.

If $B \notin J_I(\mu)$, then we are done by the 1-1 or constant property of $J_I(\mu)$. 
So let us assume that \( B \in J_1(\mu) \), i.e., \( B \subseteq Z_0 \cup Z_1 \cup \ldots \) where \( Z_i \) are \( \mu \)-thin compacta. Since \( B \notin J_\sigma(\mu) \), there is \( i \) such that for \( Z = Z_i \), we have that

1. \( Z \) is \( \mu \)-thin and \( Z \cap B \notin J_\sigma(\mu) \).

We claim that

2. the \( \sigma \)-ideal \( Bor(Z) \cap J_\sigma(\mu) \) (on a compactum \( Z \)) is coanalytic and calibrated.

To check this, let us consider the family

3. \( \mathcal{E} = \{ K \in K(Z) : \mu(Bor(K) : Bor(K) \to \{0, \infty\}) \mathrm{ and } \mu(K) = \infty \} \),

where we do not exclude that \( \mathcal{E} \) is empty, and let

4. \( \mathcal{M} \) – a maximal disjoint subfamily of \( \mathcal{E} \).

Let us notice, cf. (1), that

5. \( |\mathcal{M}| \leq \aleph_0 \) and \( Z \setminus \bigcup \mathcal{M} \) is of \( \sigma \)-finite \( \mu \)-measure.

Indeed, each Borel subset of \( Z \setminus \bigcup \mathcal{M} \) of positive \( \mu \)-measure contains a compact set \( K \) of positive \( \mu \)-measure, and since \( K \notin \mathcal{E} \), cf. (3), there is a Borel set \( C \subseteq K \) with \( 0 < \mu(C) < \infty \), and hence also a compact set \( L \subseteq C \) of positive finite \( \mu \)-measure. Since \( Z \setminus \bigcup \mathcal{M} \) is \( \mu \)-thin, cf. (1), it follows that a countable collection of compact subsets of \( Z \setminus \bigcup \mathcal{M} \) with positive finite \( \mu \)-measure exhausts this set up to a set of \( \mu \)-measure null, which justifies (5).

Let us notice that

6. \( K(Z) \cap J_0(\mu) = \{ K \in K(Z) : \mu(K \cap M) = 0 \} \) for all \( M \in \mathcal{M} \).

Indeed, if \( K \in K(Z) \) is of \( \sigma \)-finite \( \mu \)-measure and \( M \in \mathcal{M} \subseteq \mathcal{E} \), cf. (3), we get \( \mu(K \cap M) = 0 \). Conversely, if a compactum \( K \) hits every \( M \in \mathcal{M} \) in a \( \mu \)-null set, then (5) implies that \( K \in J_\sigma(\mu) \).

Now, \( J_0(\mu) \) being coanalytic, \( K(M) \cap J_0(\mu) \) is clearly coanalytic for any compactum \( M \in K(X) \). Since the map \( K \mapsto K \cap M \) from \( K(Z) \) to \( K(M) \) is Borel, this combined with (6) confirms the first part of (2).

To check the second part of (2), let us fix \( K \in K(Z) \setminus J_\sigma(\mu) \) and \( K_n \in K(Z) \setminus J_\sigma(\mu) \), \( n \in \mathbb{N} \). By (6), there is \( M \in \mathcal{M} \) with \( K \cap M \notin J_0(\mu) \) but \( K_n \cap M \in J_0(\mu) \) for all \( n \). Therefore, the Borel set \( (K \setminus \bigcup_n K_n) \cap M \) is of positive \( \mu \)-measure, hence there exists a compactum \( L \subseteq (K \setminus \bigcup_n K_n) \cap M \) with positive \( \mu \)-measure. Since \( M \in \mathcal{E} \), we have \( L \notin J_\sigma(\mu) \), cf. (3).

Having checked (2) we conclude that the \( \sigma \)-ideal \( Bor(Z) \cap J_\sigma(\mu) \) has the 1-1 or constant property, by [15, Theorem 5.1]. We can now complete the proof readily: by (1), \( Z \cap B \notin J_\sigma(\mu) \) and the 1-1 or constant property applied to \( f(B \cap B) \) provides a Borel subset of \( Z \cap B \) not in \( J_\sigma(\mu) \) on which \( f \) is injective.

\[ \square \]

6.2. Some common features of the \( \sigma \)-ideals \( J_\sigma(\mu) \) and \( J_0(\mu) \). Recall, cf. Corollary 4.1, that when \( \mu^h \) is a semifinite, non-\( \sigma \)-finite Hausdorff measure on a compactum \( X \), then none of the Boolean algebras \( Bor(X) / J_\sigma(\mu^h) \) and \( Bor(Y) / J_0(\mu^h) \) can be densely embedded
into the other. Nonetheless, the $\sigma$-ideals $J_0(\mu^h)$ and $J_\sigma(\mu^h)$ share some essential common features. In particular, they are both coanalytic and calibrated, hence they have the 1-1 or constant property. Another similarity may be exhibited through the following general scheme, studied by Sabok and Zapletal in [18].

Given a $\sigma$-ideal $I$ on a compactum $X$, let $I^*$ be the $\sigma$-ideal consisting of Borel sets in $X$ that can be covered by countably many compact sets from $I$. Sabok and Zapletal, cf. [18], obtained several results to the effect that properties of $I$ have direct impact on properties of $I^*$.

Note that given a Borel, non-$\sigma$-finite measure $\mu$ on $X$, $J_0(\mu^h)$ (or $J_\sigma(\mu^h)$), respectively) is equal to $I^*$ for $I = I_0(\mu)$, the $\sigma$-ideal of Borel sets of $\mu$-measure zero (for $I = I_\sigma(\mu)$, the $\sigma$-ideal of Borel sets of $\sigma$-finite $\mu$-measure, respectively).

A $\sigma$-ideal $I$ on $X$ is called polar, cf. [21] (see also Debs [2] where this terminology was introduced), if there is a collection $\mathcal{M}$ of finite Borel measures on $X$ such that

$$I = \bigcap_{\nu \in \mathcal{M}} I_0(\nu).$$

Note that if $\mu$ is a semifinite Borel measure on $X$, then the $\sigma$-ideal $I_0(\mu)$ is polar. Indeed, for each Borel set $B$ of finite positive $\mu$-measure, it is enough to define $\nu_B : \text{Bor}(X) \to [0, \mu(B)]$ by

$$\nu_B(E) = \mu(E \cap B),$$

and to note that the measure $\nu_B$ vanishes on $I_0(\mu)$ but is positive on $B$.

The following proposition is closely related to [21, Example 3.6.4] of Zapletal and [4, Corollary 5.20] of Farah and Zapletal. We decided to include a direct proof of this useful observation.

**Proposition 6.2.** Let $\mu$ be a semifinite, non-$\sigma$-finite Borel measure on a compactum $X$ such that $\mu|K(X)$ is a Borel mapping on the hyperspace $K(X)$. Then the $\sigma$-ideal $I_\sigma(\mu)$ is polar.

Consequently, for any Borel function $f : B \to [0, 1]^\mathbb{N}$ on a Borel set in $X$ of non-$\sigma$-finite $\mu$-measure, there exists a compactum $K \subseteq B$ of non-$\sigma$-finite $\mu$-measure such that $f|K$ is continuous. In particular, this holds true if $\mu = \mu^h$ is a semifinite, non-$\sigma$-finite Hausdorff measure on $X$.

**Proof.** Let $B$ be a Borel set in $X$ of non-$\sigma$-finite $\mu$-measure. We are going to find a finite Borel measure $\nu_B$ which vanishes on $I_\sigma(\mu)$ but is positive on $B$.

Let, cf. Section 2.3,

$$\mathcal{E} = \{ \phi \in C(2^\mathbb{N}, K(X)) : 0 < \mu(\phi(t)) < \infty \text{ and } \phi(s) \cap \phi(t) = \emptyset \text{ for } s \neq t \}. $$

**Claim.** There is $\phi \in \mathcal{E}$ with

1. $\bigcup \phi(2^\mathbb{N}) \subseteq B$.  

Note that by the assumptions, the family of compacta in $X$ of finite positive $\mu$-measure is an analytic co-basis of $I_0(\mu)$ and, $\mu$ being semifinite, $B$ is not $I_0(\mu)$-thin. Hence, by the remarks preceding Lemma 2.1, the claim follows readily, provided that $B$ is a $G_\delta$ subset of $X$.

In the general case, one can extend the original topology on $X$ to a Polish topology $\tau$ (with the same Borel sets) and such that $B$ is a closed set in $(X, \tau)$, cf. [8, 13.A]. Let $K(X, \tau)$ be the collection of compacta in the space $(X, \tau)$. Since the identity map from $K(X, \tau)$ to $K(X)$ is continuous, the measure $\mu$ is also Borel on $K(X, \tau)$. Since $\mu$ is a semifinite Borel measure on $(X, \tau)$, using regularity of finite Borel measures on Polish spaces, cf. [8, 17.C], we infer that the family of compacta in $(X, \tau)$ of finite positive $\mu$-measure is an analytic co-basis of $I_0(\mu)$. Hence, there is $\phi \in C(2^N, K(X, \tau))$ with $0 < \mu(\phi(t)) < \infty$, $\phi(s) \cap \phi(t) = \emptyset$ for $s \neq t$ and $\bigcup \phi(2^N) \subseteq B$. Using again continuity of the identity map from $K(X, \tau)$ to $K(X)$, we infer that $C(2^N, K(X, \tau)) \subseteq C(2^N, K(X))$, which completes the proof of the claim.

Since the function $\mu \circ \phi$ is Borel for each $\phi \in \mathcal{E}$, we may further assume that

$(2)$ $\sup \{\mu(\phi(t)) : t \in 2^N\} = M < \infty$.

Note that for each Borel set $E$ in $X$, the function

$(3)$ $t \mapsto \mu(\phi(t) \cap E), t \in 2^N$, is Borel.

Indeed, we have $\mu(\phi(t) \cap E) = \mu(A_t)$, where $A_t = \{x \in X : (t, x) \in A\}$ is the vertical section of a Borel set in $2^N \times X$ defined by $A = \bigcup_{t \in 2^N} \{(t) \times \phi(t)\} \cap (2^N \times E)$, cf. [8, 17.25].

Let $\lambda$ be the Haar measure on $2^N$ and let $\nu_B : Bor(X) \to [0, M]$, cf. (2), be defined by, cf. (3),

$(4)$ $\nu_B(E) = \int_{2^N} \mu(\phi(t) \cap E)\lambda(dt)$.

Then $\nu_B$ is a finite Borel measure on $X$. Clearly, $\nu_B(B) > 0$, cf. (1). On the other hand, $\nu_B$ vanishes on $I_\sigma(\mu)$. Indeed, if $K$ is a compactum in $X$ and $\nu_B(K) > 0$, then $\mu(\phi(t) \cap K) > 0$, cf. (4), for uncountably many $t$ and therefore $K$ is of non-$\sigma$-finite $\mu$-measure.

$\square$

6.3. The $\sigma$-ideals related to capacities. Given a non-zero subadditive capacity $\gamma$ (see [10, Section 3.1] and [8, 30.A]) on a compactum $X$, we shall denote by $J_0(\gamma)$ the $\sigma$-ideal of Borel sets in $X$ that can be covered by countably many compact $\gamma$-null sets. Let us call $J_0(\gamma)$-thin sets simply $\gamma$-thin (cf. [10]) and assuming that $X$ is not $\gamma$-thin let $J_\gamma(J_0(\gamma))$ – the $\sigma$-ideal of subsets of $X$ that can be covered by countably many compact $\gamma$-thin sets, be denoted by $J_\gamma(\gamma)$.

Note that every countable union of $\gamma$-null compact sets in $X$ is contained in a $\gamma$-null $G_\delta$-subset of $X$. It follows that the $\sigma$-ideal $J_0(\gamma)$ has property $(\ast)$ and is calibrated. Moreover, it is coanalytic, in fact
$J_0(\gamma) \cap K(X)$ is a $G_\delta$-set in the hyperspace $K(X)$ (cf. [10]). Consequently, the $\sigma$-ideal $J_t(\gamma)$ is also coanalytic and calibrated, hence both $J_0(\gamma)$ and $J_t(\gamma)$ have the 1-1 or constant property.

These observations combined with Theorem 3.1 lead to the following result (cf. Section 4).

**Proposition 6.3.** Let $\gamma_1$ and $\gamma_2$ be non-zero subadditive capacities on compacta $X$ and $Y$, respectively, vanishing on singletons.

If $X$ is not $\gamma_1$-thin and there is an analytic co-basis $D$ of $J_0(\gamma_1)$ consisting of $\gamma_1$-thin sets, then

1. none of the Boolean algebras $\text{Bor}(X)/J_t(\gamma_1)$ and $\text{Bor}(Y)/J_0(\gamma_2)$ can be densely embedded into the other,
2. if $Y$ is not $\gamma_2$-thin and $\text{Bor}(Y)/J_t(\gamma_2)$ embeds densely in $\text{Bor}(X)/J_t(\gamma_1)$, then $Y$ must contain a compact $\gamma_2$-thin set of positive $\gamma_2$-capacity.

An example of a capacity $\gamma_1$ as above is the capacity $\mu_{\text{diam}}(X)$ (cf. Section 2.1) associated to a semifinite but not $\sigma$-finite Hausdorff measure $\mu^h$, as in this case $J_0(\mu_{\text{diam}}(X)) = J_0(\mu^h)$ (cf. [8, 30.B]). On the other hand, there are examples of capacities $\gamma$ for which $J_t(\gamma) = J_0(\gamma)$, i.e., the only compact $\gamma$-thin sets are the $\gamma$-null ones (cf. [10, Section 3.3]).

### 6.4. The $\sigma$-ideals $I(\text{dim})$.

Given a compactum $X$, we shall denote by $I(\text{dim})$ the $\sigma$-ideal of Borel subsets of $X$ that can be covered by countably many compact sets of finite covering dimension.

The $\sigma$-ideals $I(\text{dim})$ have the 1-1 or constant property by [15, Corollary 5.6].

For the Hilbert cube $I^\mathbb{N}$ and any Hausdorff measure $\mu^h$ on a compactum $Y$, neither of the quotient Boolean algebras $\text{Bor}(I^\mathbb{N})/I(\text{dim})$ and $\text{Bor}(Y)/J_0(\mu^h)$ embeds onto a dense subalgebra of the other.

This can be checked by a reasoning in [14, Section 3] using the fact that $\sigma$-ideals $J_\sigma(\mu^h)$ are calibrated (by a theorem of Davies, cf. [16, Remarks following Theorem 59]) and have the 1-1 or constant property (cf. Section 2.2 and Proposition 6.1).

Zapletal [22] showed that applying to the $\sigma$-ideals $I(\text{dim})$ the theory from his book [21], one gets a striking solution to a natural problem in the forcing theory, raised by D. H. Fremlin. The following description of $I(\text{dim})$ avoids explicit mentioning of the dimension (but, eventually, this setting is also based on the dimension theory), cf. Zapletal [22, Question 3.2].

Given a compactum $X$, let $S(2^\mathbb{N}, X)$ be the space of all continuous surjections of the Cantor set onto $X$, equipped with the topology of uniform convergence, let

$$E = \{(x, f) \in X \times S(2^\mathbb{N}, X) : f^{-1}(x) \text{ is infinite}\}$$
and let $\pi_1, \pi_2$ be the projections of $E$ onto the first and the second coordinate, respectively.

Then by [13], a Borel set $B \subseteq X$ is in $I(dim)$ if and only if $\pi_2(\pi_1^{-1}(B))$ is meager in $S(2^N, X)$.

**References**