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ON BOREL MAPS, CALIBRATED σ-IDEALS AND HOMOGENEITY

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ABSTRACT. Let μ be a Borel measure on a compactum X. The main objects in this paper are σ-ideals $I(dim)$, $J_0(\mu)$, $J_f(\mu)$ of Borel sets in X that can be covered by countably many compacta which are finite-dimensional, or of μ-measure null, or of finite μ-measure, respectively. Answering a question of J. Zapletal, we shall show that for the Hilbert cube, the σ-ideal $I(dim)$ is not homogeneous in a strong way. We shall also show that in some natural instances of measures μ with non-homogeneous σ-ideals $J_0(\mu)$ or $J_f(\mu)$, the completions of the quotient Boolean algebras $Borel(X)/J_0(\mu)$ or $Borel(X)/J_f(\mu)$ may be homogeneous.

We discuss the topic in a more general setting, involving calibrated σ-ideals.

1. Introduction

The results of this paper provide more information on the topic investigated in our articles [12], [11], [13], which were strongly influenced by the work of Zapletal [19], [21], Farah and Zapletal [3] and Sabok and Zapletal [17].

Given a subset E of a compactum (i.e., a compact metrizable space) or, more generally, of a Polish (i.e., a separable completely metrizable) space X, we denote by $Bor(E)$ the σ-algebra of Borel sets in E, and $K(E)$ is the collection of compact subsets of E.

A *σ-ideal on X* is a collection $I \subseteq Bor(X)$, closed under taking Borel subsets and countable unions of elements of I; it is *generated by compact sets* if any element of I can be enlarged to a σ-compact set in I. We usually assume that $X \notin I$.

A σ-ideal I generated by compact sets in X is *calibrated* if for any $K \in K(X)\setminus I$ and $K_n \in I \cap K(X)$, $n \in \mathbb{N}$, there is a compact set $L \subseteq K\setminus \bigcup_{n\in\mathbb{N}} K_n$ not in I, cf. Kechris, Louveau and Woodin [8].

Let us recall that a compactum is *countable-dimensional* if it is a union of countably many zero-dimensional sets, cf. [2].

One of the main results in this paper is the following theorem.

Theorem 1.1. *Let I be a calibrated σ-ideal on a compactum X without isolated points, containing all singletons, and let $f : B \to Y$ be a Borel map from $B \in Bor(X)\setminus I$ to a compactum Y without isolated points. Then*

(i) *there exists a compact meager set $C \subseteq Y$ with $f^{-1}(C) \notin I$,*
(ii) *if Y is countable-dimensional, there is a zero-dimensional compactum C in Y with $f^{-1}(C) \notin I$,*

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(iii) *for any σ-finite nonatomic Borel measure μ on Y, there is a compact set C in Y with $\mu(C) < \infty$ and $f^{-1}(C) \notin I$.*

The statement in (i) strengthens a result in [13] concerning the "1-1 or constant" property of Sabok and Zapletal [17], [16] (some deep refinements of this result, in another direction, are given in the book by Kanovei, Sabok and Zapletal [6, Section 6.1.1]), cf. Section 7.2.

To comment on (ii), let us recall the notion of homogeneity of σ-ideals introduced by Zapletal [19], [20]: a σ-ideal I on a Polish space X is *homogeneous*, if for each $E \in Bor(X)\backslash I$ there exists a Borel map $f : X \to E$ such that $f^{-1}(A) \in I$, whenever $A \in I$.

Now, (ii) implies that the σ-ideal $I(dim)$ of Borel sets in the Hilbert cube $[0,1]^{\mathbb{N}}$ that can be covered by countably many finite-dimensional compacta is not homogeneous in a strong way: there are compacta X, Y in $[0,1]^{\mathbb{N}}$ not in $I(dim)$ such that for any Borel map $f : B \to Y$ on $B \in Bor(X)\backslash I(dim)$ there is a zero-dimensional compactum C in Y with $f^{-1}(C) \notin I(dim)$. Combined with a theory developed by Zapletal [19], it shows that the forcings associated with the collections $Bor(X)\backslash I(dim)$ and $Bor(Y)\backslash I(dim)$, partially ordered by inclusion, are not equivalent. This provides an answer to a question by Zapletal [21], cf. Section 6.1 for more details.

In the context of homogeneity we shall discuss also the σ-ideals $J_f(\mu)$ and $J_0(\mu)$ associated with Borel measures μ on compacta X: $J_f(\mu)$ ($J_0(\mu)$) is the collection of Borel sets in X that can be covered by countably many compact sets of finite μ-measure (of μ-measure zero, respectively, cf. [1] and [16]).

The σ-ideal $J_0(\mu)$ is calibrated, and hence, if μ is σ-finite and nonatomic, (iii) shows that for any Borel map $f : B \to X$ on $B \in Bor(X)\backslash J_0(\mu)$, there is a compact set C in X with $\mu(C) < \infty$ and $f^{-1}(C) \notin J_0(\mu)$, cf. Section 7.6(A) for additional information.

The classical Lusin theorem shows that $J_0(\lambda)$ is not homogeneous for the Lebesgue measure λ on $[0,1]$, and a refinement of the Lusin theorem, cf. [13, Proposition 6.2], provides non-homogeneity of the σ-ideal $J_f(\mathscr{H}^1)$ associated with the 1-dimensional Hausdorff measure $\mathscr{H}^1$ on the Euclidean square $[0,1]^2$ (cf. Corollary 6.2.2).

Shifting our attention from the σ-ideals $J_0(\mu)$ and $J_f(\mu)$ to the collections $Bor(X)\backslash J_0(\mu)$ and $Bor(X)\backslash J_f(\mu)$, we get a different picture concerning homogeneity.

Let us recall that a Borel measure μ on a compactum X is *semifinite* if each Borel set of positive μ-measure contains a Borel set of finite positive μ-measure (σ-finite Borel measures and Hausdorff measures on Euclidean cubes are semifinite, cf. [15]).

Theorem 1.2. *Let μ be a nonatomic Borel measure on a compactum X.*

(i) *Assume that $X \notin J_f(\mu)$ and every Borel set $B \notin J_f(\mu)$ contains a Borel set $C \notin J_f(\mu)$ with $\mu(C) < \infty$. Then the completion of the quotient Boolean algebra $Bor(X)/J_f(\mu)$ is homogeneous and isomorphic to the completion of the quotient Boolean algebra $Bor([0,1]^2)/J_f(\mathscr{H}^1)$.*

(ii) *If μ is semifinite, then the completion of the quotient Boolean algebra $Bor(X)/J_0(\mu)$ is homogeneous and isomorphic to the completion of the quotient Boolean algebra $Bor([0,1])/J_0(\lambda)$.*

In particular, the partial order $Bor([0,1]^2)\backslash J_f(\mathscr{H}^1)$ is forcing homogeneous, while the σ-ideal $J_f(\mathscr{H}^1)$ is not homogeneous, and the same is true if $J_f(\mathscr{H}^1)$ is replaced by $J_0(\lambda)$. It seems that examples illustrating this phenomenon did not appear in the literature, (cf. [19], comments following Definition 2.3.7).

Let us however remark that if μ is a σ-finite nonatomic Borel measure on a compactum X not in $J_f(\mu)$, then the σ-ideal $J_f(\mu)$ can be homogeneous, cf. Proposition 7.1.

The proof of Theorem 1.1 is presented in Sections 3, 4 and 5. They are preceded by Section 2 containing some preliminaries. Our approach is similar to that in [11] and [13], an essential difference being that we shall analyze compact-valued functions $\check{f}_U : Y \to K(X)$ associated with f^{-1} rather than functions $\widehat{f}_U : \overline{U} \to K(Y)$ considered in [11] or [13], associated with f.

Theorem 1.2 is based on results of Oxtoby [10] and its proof is presented in Section 6.

2. Preliminaries

Our notation is standard and mostly agrees with [7]. In particular,

- $\mathbb{N} = \{0, 1, \ldots\}$,
- $\mathbb{N}^{<\mathbb{N}}$ is the family of all finite sequences of natural numbers,
- in a given metric space: $\mathrm{diam}(A)$ is the diameter of A, $B(x, r)$ is the open r-ball centered at x and $B(A, \varepsilon)$ is the open ε-ball around A.

The terminology concerning Boolean algebras agrees with [9].

As in our earlier work on this topic, the key element of our reasonings are generalized Hurewicz systems, cf. [13, 2.4]; such systems were introduced in some special cases by W. Hurewicz [5], and significantly developed by S. Solecki [18] in connection with σ-ideals generated by closed sets.

Let X be a compactum without isolated points. In this paper by a *generalized Hurewicz system* we shall mean (adopting a slightly more restrictive definition than in [13, 2.4]) a pair $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of families of subsets of X with the following properties, where G is a given non-empty G_δ-set in X, the diameters are with respect to a fixed complete metric on G and the closures are taken in X:

- $U_s \subseteq G$ is relatively open, non-empty and $\mathrm{diam}(U_s) \leqslant 2^{-length(s)}$,
- $\overline{U_s} \cap \overline{U_t} = \varnothing$ for distinct $s,\ t$ of the same length,
- $\overline{U_{s\hat{}i}} \cap G \subseteq U_s$,
- $L_s \subseteq \overline{U_s}$ is compact,
- $L_s \cap \overline{U_{s\hat{}i}} = \varnothing$,
- $L_s = \bigcap_j \overline{\bigcup_{i>j} U_{s\hat{}i}}$,
- $\lim_i \mathrm{diam}(U_{s\hat{}i}) = 0$.

If a pair $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ is a generalized Hurewicz system, then

$$P = \bigcap_n \bigcup \{U_s : \mathrm{length(s)} = n\}$$

is the G_δ-subset of G (actually, a copy of the irrationals) *determined by the system* and we have, cf. [13, 2.4],

(1) $\overline{P} = P \cup \bigcup\{L_s : s \in \mathbb{N}^{<\mathbb{N}}\}$.

Moreover,

(2) if V is a non-empty relatively open subset of P, then $\overline{V}$ contains L_s with arbitrarily long $s \in \mathbb{N}^{<\mathbb{N}}$.

We shall use the generalized Hurewicz systems in the following situation.

Let I be a calibrated σ-ideal on a compactum X without isolated points, containing all singletons, and let $f : B \to Y$ be a Borel map from $B \in Bor(X)\backslash I$ to a compactum Y without isolated points.

Moreover, suppose that $G \subseteq B$ is a non-empty, G_δ-set in X such that

(3) $V \notin I$ for any non-empty relatively open set V in G,

(4) $f|G : G \to Y$ is continuous.

Such a set G can always be found by a theorem of Solecki [18].

Given a non-empty relatively open set U in G we shall consider the map $\check{f}_U$: $Y \to K(\overline{U})$ defined by

(5) $\check{f}_U(y) = \bigcap_n \overline{f^{-1}(B(y, \frac{1}{n})) \cap U}$,

$B(y, r)$ being the open r-ball centered at y, with respect to a fixed metric on Y.

In other words, $x \in \check{f}_U(y)$ if and only if there is a sequence (x_n) of elements of U such that $\lim_n x_n = x$ and $\lim_n f(x_n) = y$.

Notice that for $y \notin \overline{f(U)}$, $\check{f}_U(y) = \emptyset$. The map $\check{f}_U$ is upper-semicontinuous so, in particular, the set $\check{f}_U[E]$ defined by

$$\check{f}_U[E] = \bigcup\{\check{f}_U(y) : y \in E\}$$

is compact, whenever E is compact. Let us also notice that $x \in \check{f}_U(f(x))$ for any $x \in U$ and hence, $\check{f}_U[Y]$ being compact,

(6) $\overline{U} = \check{f}_U[Y]$.

Functions $\check{f}_U$, associated with f, G fixed and U varying over non-empty open subsets of G, will be used to define generalized Hurewicz systems providing some control simultaneously over sets determined by the systems and their images under f. This is explained by the following lemma where we gathered some observations vital for the proof of Theorem 1.1.

Lemma 2.1. *Assume that I is a calibrated σ-ideal on a compactum X without isolated points, containing all singletons, and let $f : B \to Y$ be a Borel map from $B \in Bor(X)\backslash I$ to a compactum Y without isolated points. Moreover, suppose that $G \subseteq B$ is a non-empty, G_δ-set in X satisfying conditions (3) and (4).*

(A) *Let U be a non-empty relatively open set in G and assume that $L \subseteq \overline{U}$ and $M \subseteq \overline{f(U)}$ are compacta such that*

(A1) *L is boundary in $\overline{U}$ and $L \notin I$,*

(A2) *$f^{-1}(M) \in I$,*

(A3) *$L \subseteq \check{f}_U[M]$.*

Then there exist nonempty relatively open subsets V_i of U, $i \in \mathbb{N}$, such that

(A4) *$\overline{V_i} \cap G \subseteq U$,*

(A5) *$\overline{V_i}$ are pairwise disjoint and disjoint from L,*

(A6) *$L = \bigcap_n \overline{\bigcup_{i \geqslant n} V_i}$,*

(A7) *$\overline{f(V_i)}$ are pairwise disjoint and disjoint from M,*

(A8) *$\lim_{i\to\infty} \operatorname{diam}(V_i) = 0$ and $\lim_{i\to\infty} \operatorname{diam}(f(V_i)) = 0$ with respect to fixed metrics on X and Y, respectively,* (A9) $\bigcap_n \overline{\bigcup_{i\geqslant n} f(V_i)} \subseteq M$.

(B) *Let $(J_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ be a family of hereditary collections of closed subsets of Y. Assume that for every non-empty relatively open set U in G and each s, there exist compacta L and $M \in J_s$ with properties (A1)–(A3). Then there exists a generalized Hurewicz system $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ with an associated family $(M_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ such that the following additional conditions are satisfied for each $s \in \mathbb{N}^{<\mathbb{N}}$:*

(B1) $L_s \notin I$,
(B2) $M_s \in K(\overline{f(U_s)}) \cap J_s$,
(B3) $\overline{f(U_{s\frown i})}$ *are pairwise disjoint and disjoint from* M_s,
(B4) $\lim_{i\to\infty} \operatorname{diam}(U_{s\frown i}) = 0$ *and* $\lim_{i\to\infty} \operatorname{diam}(f(U_{s\frown i})) = 0$ *with respect to fixed metrics on X and Y, respectively,*
(B5) $M_s = \bigcap_n \overline{\bigcup_{i\geqslant n} f(U_{s\frown i})}$,

(C) *If $P \subseteq G$ is the set determined by the system from part (B), then*

(C1) $\overline{P} = P \cup \bigcup\{L_s : s \in \mathbb{N}^{<\mathbb{N}}\}$,
(C2) *each non-empty relatively open subset of $\overline{P}$ contains some L_s (with arbitrarily long $s \in \mathbb{N}^{<\mathbb{N}}$),*
(C3) $P \notin I$,
(C4) $\overline{f(P)} = f(P) \cup \bigcup\{M_s : s \in \mathbb{N}^{<\mathbb{N}}\}$,
(C5) *each non-empty relatively open subset of $\overline{f(P)}$ contains some M_s (with arbitrarily long $s \in \mathbb{N}^{<\mathbb{N}}$).*

Proof. In order to prove part (A), let us fix a countable dense set in L and list its elements, repeating each point infinitely many times, as $a_0, a_1, \ldots$.

We shall choose inductively non-empty relatively open sets V_i in U such that,

(7) $V_i \subseteq B(a_i, \frac{1}{i+1})$, $\overline{V_i} \cap G \subseteq U$, $diam f(V_i) \leqslant \frac{1}{i+1}$, $f(V_i) \subseteq B(M, \frac{1}{i+1})$,
(8) $\overline{V_i} \subseteq \overline{U}\backslash(L \cup \bigcup_{j<i} \overline{V_j})$, $\overline{f(V_i)} \subseteq \overline{f(U)}\backslash(M \cup \bigcup_{j<i} \overline{f(V_j)})$.

Suppose that V_j, $j < i$, are already defined, where $V_0 = \varnothing$.

Since $a_i \in L$, by (A3) we have $a_i \in \check{f}_U(b_i)$ for some $b_i \in M$. Let $\delta_i < \frac{1}{i+1}$ be such that

(9) $B(a_i, \delta_i) \cap \bigcup_{j<i} \overline{V_j} = \varnothing$, $B(b_i, \delta_i) \cap \bigcup_{j<i} \overline{f(V_j)} = \varnothing$.

From (5),

(10) $W = B(a_i, \delta_i) \cap f^{-1}(B(b_i, \delta_i)) \cap U \neq \varnothing$,

and by (4), W is relatively open in U. Since L is boundary in $\overline{U}$, $W\backslash L \neq \varnothing$ and by (3), $W\backslash L \notin I$. Since, cf. (A2), $f^{-1}(M) \in I$, we can pick $c \in W\backslash(L \cup f^{-1}(M))$. Then $f(c) \in B(b_i, \delta_i)\backslash M$, cf. (10), and appealing again to continuity of f, we get a relatively open neighbourhood V_i of c in U with $\overline{V_i} \subseteq B(a_i, \delta_i)\backslash L$, $\overline{V_i} \cap G \subseteq U$ and $\overline{f(V_i)} \subseteq B(b_i, \delta_i)\backslash M$. By (9), V_i satisfies (8).

This completes the inductive construction. It is now easy to see that requirements (A4)–(A9) of part (A) are met.

Having checked part (A), we can use it subsequently to define inductively a generalized Hurewicz system in X with properties (B1)–(B4) and property (B5) replaced by, cf. (A9),

$$\bigcap_n \overline{\bigcup_{i\geqslant n} f(U_{s\frown i})} \subseteq M_s.$$

Taking into account that J_s is hereditary, to secure (B5), it suffices to replace M_s by $\bigcap_n \overline{\bigcup_{i \geqslant n} f(U_{s^\frown i})}$. This completes the proof of part (B).

To prove part (C), first note that properties (C1), (C2) hold for any generalized Hurewicz system considered in this paper and (C3) follows from (B1) and (C2) by a Baire category argument.

We proceed to the proof of (C4). The inclusion

$$f(P) \cup \bigcup\{M_s : s \in \mathbb{N}^{<\mathbb{N}}\} \subseteq \overline{f(P)}$$

can be easily justified with the help of (B4) and (B5) combined with the observation that $U_s \cap P \neq \varnothing$ for each s.

To prove the opposite inclusion, first note that, by (B5), for each s we have:

$$\overline{f(P \cap U_s)} = \overline{\bigcup_i f(P \cap U_{s^\frown i})} \subseteq \bigcup_i \overline{f(P \cap U_{s^\frown i})} \cup M_s.$$

So assume that $y \in \overline{f(P)} \setminus \bigcup\{M_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ and notice that for each s, if $y \in \overline{f(P \cap U_s)}$, then there is (precisely one, cf. (B3)) i such that $y \in \overline{f(P \cap U_{s^\frown i})}$. Using the fact that $y \in \overline{f(P \cap U_\varnothing)}$ (recall that $P \subseteq U_\varnothing$) this allows us to construct inductively a sequence $z \in \mathbb{N}^{\mathbb{N}}$ with

$$y \in \overline{f(P \cap U_{z|n})} \text{ for each } n \in \mathbb{N}.$$

It follows, by the continuity of f, that if $x \in P$ is the unique element of $\bigcap_n U_{z|n}$, then $y = f(x)$, which shows that $y \in f(P)$ completing the proof of (C4).

Finally, (C5) can be easily justified with the help of (B5) and (B4).

□

3. Proof of Theorem 1.1 (i)

Striving for a contradiction, let us assume that for any meager set C in Y, $f^{-1}(C) \in I$. In particular, since Y has no isolated points, it follows that $f^{-1}(y) \in I$ for any $y \in Y$.

Using a theorem of Solecki [18], we can find a non-empty G_δ-set G in X such that $G \subseteq B$,

(1) $V \notin I$ for any non-empty relatively open V in G,
(2) $f|G : G \to Y$ is continuous.

We shall apply Lemma 2.1, and to that end, we shall first establish the following fact.

Claim 3.1. *Let U be a non-empty relatively open set in G. Then there exist compacta $L \subseteq \overline{U}$ and $M \subseteq \overline{f(U)}$ such that*
(3) L is boundary in $\overline{U}$ and $L \notin I$,
(4) M is boundary in $\overline{f(U)}$,
(5) $L \subseteq \check{f}_U[M]$.

To prove the claim, first note that $\overline{f(U)}$ has no isolated points. For suppose that y is an isolated point in $\overline{f(U)}$. Then, by the continuity of f, $f^{-1}(y)$ contains a non-empty relatively open subset of G which, by (1), implies that $f^{-1}(y) \notin I$, contradicting our assumptions.

Now let us fix a countable set D dense in $\overline{f(U)}$. We shall consider two cases.

Case 1. There exists $d \in D$ with $\check{f}_U(d) \notin I$.

Then, since all singletons of $\check{f}_U(d)$ are in I and I is calibrated, there exists a boundary in $\overline{U}$ compactum $L \subseteq \check{f}_U(d)$ not in I, and we let $M = \{d\}$.

Case 2. For all $d \in D$, $\check{f}_U(d) \in I$.

Then, I being calibrated and containing all singletons of X, we have a boundary compactum $L \subseteq \overline{U} \backslash \bigcup_{d \in D} \check{f}_U(d)$, $L \notin I$. Let

$$M = \{y \in Y : \check{f}_U(y) \cap L \neq \varnothing\}.$$

The compactum $M \subseteq \overline{f(U)}$ is disjoint from D, hence boundary in $\overline{f(U)}$, and we have, cf. (6) in Section 2, $L \subseteq \check{f}_U[M]$, which completes the proof of the claim.

Having verified the claim, we shall modify the proof of Lemma 2.1 to get for any non-empty relatively open set U in G a sequence (V_i) of non-empty relatively open subsets of U with properties (A4)–(A9) and the following additional property

(6) $M \subseteq \overline{\overline{f(U)} \backslash \overline{\bigcup_i f(V_i)}}$,

Namely, since M is boundary in $\overline{f(U)}$ and $\overline{f(U)}$ has no isolated points, we can enlarge M to a compactum $M^* \subseteq \overline{f(U)}$ such that

(7) M^* is boundary in $\overline{f(U)}$ and $M \subseteq \overline{M^* \backslash M}$.

To get M^*, we fix a countable set C dense in M, and then we pick subsequently points d_n in $\overline{f(U)} \backslash M$ so that $d_n \in B(M, \frac{1}{n+1})$ and each point in C is the limit of a subsequence of $(d_n)_{n \in \mathbb{N}}$. Then we let $M^* = M \cup \{d_n : n \in \mathbb{N}\}$.

Having defined M^* satisfying (7), we proceed as in the proof of part (A) of Lemma 2.1 and using the fact that our assumptions yield $f^{-1}(M^*) \in I$ we can choose inductively non-empty relatively open sets V_i in U such that

(8) $V_i \subseteq B(a_i, \frac{1}{i+1})$, $\overline{V_i} \cap G \subseteq U$, $diam f(V_i) \leqslant \frac{1}{i+1}$, $f(V_i) \subseteq B(M, \frac{1}{i+1})$,

(9) $\overline{V_i} \subseteq \overline{U} \backslash (L \cup \bigcup_{j<i} \overline{V_j})$, $\overline{f(V_i)} \subseteq \overline{f(U)} \backslash (M^* \cup \bigcup_{j<i} \overline{f(V_j)})$.

Then requirements (A4)–(A9) of Lemma 2.1 are still met. In particular, cf. (8), $\bigcap_n \overline{\bigcup_{i \geqslant n} f(V_i)} \subseteq M$ which, combined with (9), guarantees that $\overline{\bigcup_i f(V_i)}$ is disjoint from $M^* \backslash M$. However, by (7), the latter set contains M in its closure which justifies (6).

We can now define a generalized Hurewicz system $(U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ with the associated family $(M_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ satisfying for each $s \in \mathbb{N}^{<\mathbb{N}}$ conditions (B1)–(B5) (with J_s being the the collection of meager compacta in Y, see part (B) of Lemma 2.1) and the following additional condition

(10) $M_s \subseteq \overline{\overline{f(U_s)} \backslash \overline{\bigcup_i f(U_{s \frown i})}}$.

These conditions guarantee that the set P determined by this system not only has properties (C1)–(C5) (see part (C) of Lemma 2.1) but satisfies the following one as well

(11) For each s, $M_s \subseteq \overline{Y \backslash \overline{f(P)}}$.

To see this, it suffices to prove, by (10) and (C4), that for each s

$$(\overline{f(U_s)} \backslash \overline{\bigcup_i f(U_{s \frown i})}) \cap (f(P) \cup \bigcup \{M_s : s \in \mathbb{N}^{<\mathbb{N}}\}) = \varnothing.$$

So fix $s \in \mathbb{N}^{<\mathbb{N}}$ and let $y \in \overline{f(U_s)} \setminus \overline{\bigcup_i f(U_{s\frown i})}$.

Striving for a contradiction suppose first that $y \in f(P)$. Let $k = length(s)$. Since

$$f(P) \subseteq \bigcup \{f(U_t) : \ length(t) = k\}$$

where the sets $f(U_t)$ have pairwise disjoint closures, the fact that $y \in \overline{f(U_s)} \cap f(P)$ implies that $y \in f(U_s)$. Consequently, since

$$f(P) \cap f(U_s) \subseteq f(P \cap U_s) \subseteq \bigcup_i f(U_{s\frown i}),$$

we conclude that $y \in \bigcup_i f(U_{s\frown i})$, contrary to the assumption that $y \notin \overline{\bigcup_i f(U_{s\frown i})}$.

Next, if $y \in M_t$ for some $t \in \mathbb{N}^{<\mathbb{N}}$, then a contradiction can be easily reached by considering the four mutual positions of t and s (namely, $s = t$, $s \subsetneq t$, $t \subsetneq s$, s and t are incompatible).

Having justified (11), let us note that combined with (C5) it implies that $\overline{f(P)}$ has empty interior in Y. But on the other hand, $P \notin I$ cf. (C3). In effect, for the meager compactum $C = \overline{f(P)}$ we have $f^{-1}(C) \notin I$ and this contradiction with our assumptions completes the proof of part (i) of Theorem 1.1. □

4. Proof of Theorem 1.1 (ii)

The reasoning in this case goes along similar lines as for Theorem 1.1(i).

Striving for a contradiction, suppose that for any zero-dimensional compactum C in Y, $f^{-1}(C) \in I$.

The compactum Y being countable-dimensional, Y has defined the small inductive transfinite dimension ind Y, see [2, Theorem 7.1.9]. Let Y' be a compactum in Y such that $f^{-1}(Y') \notin I$ with minimal transfinite dimension. Replacing Y by Y' and B by $f^{-1}(Y')$, we can assume that $f^{-1}(K) \in I$ for any compactum K in Y with $\operatorname{ind} K < \operatorname{ind} Y$.

Let us choose a base for the topology of Y whose elements have boundaries $K_0, K_1, \ldots$ with $\operatorname{ind} K_i < \operatorname{ind} Y$. Then

(1) $f^{-1}(K_i) \in I$ for any i and $H = Y \setminus \bigcup_i K_i$ is zero-dimensional.

We have $B \setminus \bigcup_i f^{-1}(K_i) \notin I$ and using a theorem of Solecki [18], we can find a non-empty G_δ-set G in X, $G \subseteq B \setminus \bigcup_i f^{-1}(K_i)$, such that $V \notin I$ for any non-empty relatively open V in G and, cf. (1),

(2) $f|G : G \to H$ is continuous.

A key element of our reasoning is the following counterpart of Claim 3.1.

Claim 4.1. *Let U be a non-empty relatively open set in G. Then there exist compacta $L \subseteq \overline{U}$ and $M \subseteq \overline{f(U)}$ such that*

(3) *L is boundary in $\overline{U}$ and $L \notin I$,*
(4) *M is zero-dimensional,*
(5) *$L \subseteq \check{f}_U[M]$.*

In order to prove the claim, let, cf. (1),

(6) $Z = \overline{f(U)}$, $E_i = Z \cap K_i$

and consider two cases.

Case 1. There exists i such that $\check{f}_U[E_i] \notin I$.

Then, let S be a compactum in E_i such that $\check{f}_U[S] \notin I$ with minimal possible transfinite dimension ind.

Considering, as we did before, a base in S whose elements have boundaries $S_0, S_1, \dots$ with ind $S_i <$ ind S, we have that $\check{f}_U[S_i] \in I$ and $T = S \setminus \bigcup_i S_i$ is zero-dimensional. Since I is calibrated and $\check{f}_U[S] \notin I$, there is a compactum $L \subseteq \check{f}_U[S] \setminus \bigcup_i \check{f}_U[S_i]$ not in I and (since the sigletons of X belong to I), we may demand that L is boundary in $\overline{U}$. Then $M = \{y \in S : \check{f}_U(y) \cap L \neq \varnothing\}$ is a compact subset of T, hence zero-dimensional, and $L \subseteq \check{f}_U[M]$.

Case 2. For all i, $\check{f}_U[E_i] \in I$.

Then, as in Case 1, we can pick a boundary in $\overline{U}$ compactum $L \subseteq \overline{U} \setminus \bigcup_i \check{f}_U[E_i]$ not in I and since the compactum $M = \{y \in Z : \check{f}_U(y) \cap L \neq \varnothing\}$ is contained in H, cf. (1) and (6), M is zero-dimensional and $L \subseteq \check{f}_U[M]$.

Having justified the claim, we can use Lemma 2.1 to define a generalized Hurewicz system $(U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ with the associated family $(M_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ satisfying for each $s \in \mathbb{N}^{<\mathbb{N}}$ conditions (B1)–(B5) with J_s being the collection of zero-dimensional compacta in Y, see part (B) of Lemma 2.1.

These conditions guarantee that the set P determined by this system apart from properties (C1)–(C5) (which follow from part (C) of Lemma 2.1) satisfies also the following one

(8) The compactum $\overline{f(P)}$ contains no non-trivial continuum.

To prove this, let us first show that if $a \in f(P)$ and $b \in \overline{f(P)} \setminus f(P)$, then there exists a clopen in $\overline{f(P)}$ set containing a and missing b.

Indeed, for some $n \in \mathbb{N}$ there are sequences $s \in \mathbb{N}^n$, $t \in \mathbb{N}^{n+1}$ with $b \in M_s$ and $a \in \overline{f(U_t)}$. Let $V = \overline{f(U_t)} \cap \overline{f(P)}$.

Clearly, V is closed in $\overline{f(P)}$. To see that it is also open in $\overline{f(P)}$, let $(x_i)_{i \in \mathbb{N}}$ be a convergent sequence of elements in $\overline{f(P)} \setminus V$ with $x = \lim_{i \to \infty} x_i$. Let k be the smallest natural number (possibly 0 but clearly not grater than n) such that

$$x_i \in \bigcup_{j \neq t(k)} \overline{f(U_{t|k \frown j})}$$

for all but finitely many $i \in \mathbb{N}$ (here $t|k$ denotes $t|\{j \in \mathbb{N} : j < k\}$, in particular $t|0$ is the empty sequence).

It follows that $x \in \bigcup_{j \neq t(k)} \overline{f(U_{t|k \frown j})} \cup M_{t|k}$, cf. (B5), and the latter set being disjoint from $\overline{f(U_t)}$, we conclude that $x \notin V$.

Thus V is indeed clopen in $\overline{f(P)}$, $a \in V$ and $b \notin V$, since $M_s \cap \overline{f(U_t)} = \varnothing$.

Now, if C is any continuum in $\overline{f(P)}$, the preceding observation shows that either $C \subseteq f(P)$ or $C \subseteq \bigcup\{M_s : s \in \mathbb{N}^{<\mathbb{N}}\} = M$, cf. (C4). Since $f(P)$ is a copy of the irrationals, hence zero-dimensional, and so is M, being the countable union of closed zero dimensional sets M_s, cf. [2, Theorem 1.3.1], in both cases, C must be a singleton.

Having justified (8), we conclude that $\overline{f(P)}$ is zero-dimensional, cf. [2, Theorem 1.4.5]. On the other hand, $P \notin I$, cf. (C3). In effect, for the zero-dimensional compactum $C = \overline{f(P)}$ we have $f^{-1}(C) \notin I$ and this contradiction with our assumptions completes the proof of part (ii) of Theorem 1.1. □

5. Proof of Theorem 1.1 (iii)

Again the scheme of the proof is analogous to the ones in preceding sections.

Striving for a contradiction, suppose that $f^{-1}(C) \in I$ for any compactum C in Y with $\mu(C) < \infty$.

Since the measure μ is σ-finite, there are compact sets F_i in Y with $\mu(F_i) < \infty$ and such that if we let $H = Y \setminus \bigcup_i F_i$, then

(1) $\mu(H) = 0$.

We have $B \setminus \bigcup_i f^{-1}(F_i) \notin I$ and using a theorem of Solecki [18], we can find a non-empty G_δ-set G in X, $G \subseteq B \setminus \bigcup_i f^{-1}(F_i)$, such that $V \notin I$ for any non-empty relatively open V in G and $f|G : G \to H$ is continuous.

A key element of our reasoning is the following counterpart of Claims 3.1 and 4.1.

Claim 5.1. *Let U be a non-empty relatively open set in G and let $\varepsilon > 0$. Then there exist compacta $L \subseteq \overline{U}$ and $M \subseteq \overline{f(U)}$ such that*

(2) *L is boundary in $\overline{U}$ and $L \notin I$,*
(3) *$\mu(M) < \varepsilon$,*
(4) *$L \subseteq \check{f}_U[M]$.*

In order to prove the claim, we shall consider two cases.

Case 1. There exists i such that $\check{f}_U[F_i] \notin I$.

We can cover F_i by finitely many compacta $M_0, \ldots, M_{n-1}$ with $\mu(M_j) < \varepsilon$ for each j. Then for some j, $\check{f}_U[M_j] \notin I$. We let $M = M_j$ and pick a compactum $L \subseteq \check{f}_U[M]$, not in I and boundary in $\overline{U}$.

Case 2. For all i, $\check{f}_U[F_i] \in I$.

Then we pick a compactum $L \subseteq \overline{U} \setminus \bigcup_i \check{f}_U[F_i]$ which is boundary in $\overline{U}$ and not in I and we let $M = \{y \in \overline{f(U)} : \check{f}_U(y) \cap L \neq \varnothing\}$. Consequently, $L \subseteq \check{f}_U[M]$ and since M is contained in H, $\mu(M) = 0$, cf. (1).

Having justified the claim, we can use Lemma 2.1 to define a generalized Hurewicz system $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ with the associated family $(M_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ satisfying for each $s \in \mathbb{N}^{<\mathbb{N}}$ conditions (B1)–(B5) with J_s being the collection of compacta M in Y with $\mu(M) < \frac{1}{2^{e(s)}}$, where $e : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is a fixed bijection.

These conditions guarantee that the set P determined by this system has properties (C1)–(C5) (granted by part (C) of Lemma 2.1) and, moreover, $\mu(\bigcup\{M_s : s \in \mathbb{N}^{<\mathbb{N}}\}) \leqslant 2$. Since $f(P) \subseteq H$ and $\mu(H) = 0$, cf. (1), it follows, cf. (C4), that $\mu(\overline{f(P)}) \leqslant 2$. On the other hand, $P \notin I$ so in effect, for the compactum $C = \overline{f(P)}$ we have $\mu(C) < \infty$ but $f^{-1}(C) \notin I$ which contradicts our assumptions and ends the proof. □

6. Homogeneity notions related to σ-ideals

Recall, cf. Section 1, that a σ-ideal I on a Polish space X is homogeneous, if for each $E \in Bor(X)\backslash I$ there exists a Borel map $f : X \to E$ such that $f^{-1}(A) \in I$, whenever $A \in I$ (cf. [19], [20]). Examples of homogeneous σ-ideals include, cf. [19]:

- the σ-ideal of countable subsets of X,
- the σ-ideal generated by compact sets in the irrationals,
- the σ-ideal of meager Borel sets in the Cantor set,
- the σ-ideal of Lebesgue-null Borel sets in the Cantor set.

6.1. **The σ-ideal $I(dim)$.** Let (cf. Section 1), $I(dim)$ be the σ-ideal of Borel sets in the Hilbert cube $[0,1]^{\mathbb{N}}$ that can be covered by countably many finite-dimensional compacta, and let, for a compactum $X \subseteq [0,1]^{\mathbb{N}}$, $I_X(dim)$ be the σ-ideal $I(dim)$ restricted to $Bor(X)$.

The σ-ideal $I(dim)$ is not homogeneous in a strong way. To see this, let X be a Henderson compactum in $[0,1]^{\mathbb{N}}$, cf. 7.1, and let $Y \subseteq [0,1]^{\mathbb{N}}$ be a countable-dimensional compactum not in $I(dim)$, cf. [2, Example 5.1.7].

Since $I_X(dim)$ is calibrated, cf. 7.1, by Theorem 1.1(ii), there is no Borel map $f : B \to Y$ with $B \in Bor(X)\backslash I_X(dim)$ such that $f^{-1}(A) \in I_X(dim)$, whenever $A \in I_Y(dim)$.

Applying a theory developed by Zapletal [19], one infers that forcings associated with the collections $Bor(X)\backslash I_X(dim)$ and $Bor(Y)\backslash I_Y(dim)$, partially ordered by inclusion, are not equivalent, cf. [19], the final part of Section 2.3.

This answers Question 3.1 of Zapletal [21] (a partial answer was given in [11]).

6.2. **The σ-ideals $J_0(\mu)$, $J_f(\mu)$.** Given a Borel measure μ on a compactum X, let (cf. Section 1) $J_0(\mu)$, $J_f(\mu)$ be the σ-ideals of Borel sets in X that can be covered by countably many compact sets of μ-measure zero, or finite μ-measure, respectively.

Proposition 6.2.1. *Let μ be a semifinite nonatomic Borel measure on a compactum X with $\mu(X) > 0$.*

(i) *The σ-ideal $J_0(\mu)$ is not homogeneous.*
(ii) *If, moreover, μ is not σ-finite (in particular, $X \notin J_f(\mu)$) and there exists a Borel set $Y \notin J_f(\mu)$ with $\mu(Y) < \infty$ and $\mu|K(X)$ is a Borel mapping on the hyperspace $K(X)$, then the σ-ideal $J_f(\mu)$ is not homogeneous.*

Proof. (i) Pick $Y \in Bor(X)\backslash J_0(\mu)$ with $\mu(Y) = 0$ and any Borel map $f : X \to Y$. By the Lusin theorem, there is a compact set K in X with $\mu(K) > 0$ such that $f|K$ is continuous. If $C = f(K)$, then $C \in J_0(\mu)$ but $f^{-1}(C) \notin J_0(\mu)$.

(ii) Pick $Y \in Bor(X)\backslash J_f(\mu)$ with $\mu(Y) < \infty$. Let $f : X \to Y$ be any Borel function. By [13, Proposition 6.2], there is a compact set K in X with $K \notin J_f(\mu)$ (even of non-σ-finite μ-measure) such that $f|K$ is continuous. If $C = f(K)$, then $C \in J_f(\mu)$ but $f^{-1}(C) \notin J_f(\mu)$.

□

In contrast to (ii) above, we shall show in Proposition 7.1 that for some σ-finite measures μ on compacta X with $X \notin J_f(\mu)$, the σ-ideal $J_f(\mu)$ can be homogeneous.

Recall that λ and $\mathscr{H}^1$ denote the Lebesgue measure on $[0,1]$ and the 1-dimensional Hausdorff measure on the Euclidean square $[0,1]^2$, respectively. It is well known that the measure $\mathscr{H}^1$ (restricted to Borel sets in $[0,1]^2$) is nonatomic, semifinite but not σ-finite and $\mathscr{H}^1|K([0,1]^2)$ is a Borel map (cf. [15]). Moreover, it is easy to construct a dense G_δ set Y in $[0,1]^2$ of $\mathscr{H}^1$-measure zero. Consequently, $Y \notin J_f(\mathscr{H}^1)$ since, non-empty open sets in $[0,1]^2$ having infinite $\mathscr{H}^1$-measure, the σ-ideal $J_f(\mathscr{H}^1)$ contains meager sets only. This leads to the following corollary of Proposition 6.2.1.

Corollary 6.2.2. *The σ-ideals $J_0(\lambda)$, $J_0(\mathscr{H}^1)$ and $J_f(\mathscr{H}^1)$ are not homogeneous.*

6.3. **The partial orders $Bor(X)\backslash J_0(\mu)$ and $Bor(X)\backslash J_f(\mu)$.** Let us now shift our attention from the σ-ideals $J_0(\mu)$ and $J_f(\mu)$ to the collections of Borel sets $Bor(X)\backslash J_0(\mu)$ and $Bor(X)\backslash J_f(\mu)$, partially ordered by inclusion. The key step in the proof of Theorem 1.2 is the following result.

Theorem 6.3.1.

(i) *There is a copy of the irrationals P in $[0,1]^2$ such that*

- $P \notin J_f(\mathscr{H}^1)$,
- *if μ is any nonatomic Borel measure on a compactum $X \notin J_f(\mu)$ such that every Borel set $B \notin J_f(\mu)$ contains a Borel set $C \notin J_f(\mu)$ with $\mu(C) < \infty$, then for each $B \in Bor(X)\backslash J_f(\mu)$ there is a homeomorphic embedding $h: P \to B$ such that, for $A \subseteq P$, $A \in J_f(\mathscr{H}^1)$ if and only if $h(A) \in J_f(\mu)$.*

(ii) *There is a copy of the irrationals P in $[0,1]$ such that*

- $P \notin J_0(\lambda)$,
- *for any semifinite nonatomic Borel measure μ on a compactum X with $\mu(X) > 0$ and for each $B \in Bor(X)\backslash J_0(\mu)$ there is a homeomorphic embedding $h: P \to B$ such that, for $A \subseteq P$, $A \in J_0(\lambda)$ if and only if $h(A) \in J_0(\mu)$.*

Proof. (i) Let G be a copy of the irrationals in $[0,1]^2$ which is $\mathscr{H}^1$-null and dense in $[0,1]^2$. Consequently, if U is a non-empty relatively open set in G, then $\overline{U} \notin J_f(\mathscr{H}^1)$ and since $\mathscr{H}^1(G) = 0$ it follows that $\mathscr{H}^1(\overline{U}\backslash G) = \infty$. Hence there is a Cantor set $L \subseteq \overline{U}\backslash G$ with $\mathscr{H}^1(L) = 1$ (cf. [15]).

This observation can be used to define a generalized Hurewicz system $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ such that, in particular, the following conditions are satisfied for each $s \in \mathbb{N}^{<\mathbb{N}}$:

(1) L_s is a Cantor set with $\mathscr{H}^1(L_s) = 1$,
(2) U_s is a non-empty relatively clopen subset of G,
(3) $\lim_{i\to\infty} \operatorname{diam}(U_{s^\frown i}) = 0$, $\operatorname{diam}(U_s) \leqslant 2^{-length(s)}$, with respect to a fixed complete metric on G.

These conditions guarantee, cf. Section 2, that the copy of the irrationals P determined by this system has the following properties:

(4) $\overline{P} = P \cup \bigcup\{L_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ and $L_s \cap L_t = \varnothing$ for $s \neq t$,
(5) each nonempty relatively open subset of $\overline{P}$ contains infinitely many sets L_s.

In particular, by a Baire category argument, $P \notin J_f(\mathscr{H}^1)$.

Let us now consider an arbitrary $B \in Bor(X)\backslash J_f(\mu)$. By the properties of μ without loss of generality we can assume that $\mu(B) < \infty$. By a theorem of Solecki [18] we can first find a G_δ in B not in $J_f(\mu)$ and then shrinking it further we can pick a copy of the irrationals G' in B with $\mu(G') = 0$ such that for each non-empty relatively open set U' in G', we have $U' \notin J_f(\mu)$ so, in particular, $\mu(\overline{U'}\backslash G') = \infty$.

A key element of our reasoning is the following observation.

Claim 6.3.2. *Let U and U' be non-empty relatively open sets in G and G', respectively. Let $L \subseteq \overline{U}\backslash U$ be a Cantor set with $\mathscr{H}^1(L) = 1$.*

Then there exist a Cantor set $L' \subseteq \overline{U'}\backslash U'$ and a homeomorphism $g : U \cup L \to U' \cup L'$ such that $\mu(L') = 1$ and $g|L$ is a measure preserving homeomorphism between L and L'.

To prove the claim, we can first appeal to results of Oxtoby [10] to find a Cantor set $L' \subseteq \overline{U'}\backslash U'$ and a measure preserving homeomorphism $f : L \to L'$.

More precisely, let $\mathcal{N}$ denote the set of the irrationals in $[0, 1]$, and let λ be the restriction of the Lebesgue measure on $[0, 1]$ to the Borel subsets of $\mathcal{N}$. Considering the product $P_1 = L \times \mathcal{N}$, one can identify L with a subspace of P_1, a copy of the irrationals equipped with a Borel measure ν such that

(6) $\nu(P_1) < \infty$,
(7) $\nu(\{x\}) = 0$ for each $x \in P_1$,
(8) $\nu(U) > 0$ for every non-empty open set in P_1,
(9) ν coincides with $\mathscr{H}^1$ on Borel sets in L.

In effect, by a theorem of Oxtoby [10, Theorem 1], properties (6)–(8) guarantee that there is a homeomorphism $\varphi_1 : \mathcal{N} \to P_1$ such that $\nu(\varphi_1(A)) = \nu(P_1) \cdot \lambda(A)$ for any Borel set A in $\mathcal{N}$.

On the other hand, by theorems of Gelbaum [4] and Oxtoby [10, Theorem 2], there is a copy of the irrationals P_2 in $\overline{U'}\backslash U'$ with $\mu(P_2) = \nu(P_1)$ and a homeomorphism $\varphi_2 : \mathcal{N} \to P_2$ such that $\mu(\varphi_2(A)) = \nu(P_1) \cdot \lambda(A)$ for any Borel set A in $\mathcal{N}$.

Now it suffices to let $L' = (\varphi_2 \circ \varphi_1^{-1})(L)$ and $f = \varphi_2 \circ \varphi_1^{-1}|L$ to obtain a desired Cantor set L' and a measure preserving homeomorphism $f : L \to L'$.

Finally, since $U \cup L$ and $U' \cup L'$ are copies of the irrationals, a theorem of Pollard [14] provides an extension of f to a homeomorphism $g : U \cup L \to U' \cup L'$.

Having justified the claim, we can use it to define a generalized Hurewicz system $(U'_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L'_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of G' together with homeomorphisms

(10) $h_s : L_s \cup \bigcup_i U_{s\frown i} \to L'_s \cup \bigcup_i U'_{s\frown i}$,

satisfying the following conditions for each $s \in \mathbb{N}^{<\mathbb{N}}$:

(11) $h_s(U_{s\frown i}) = U'_{s\frown i}$,
(12) $\mu(L'_s) = 1$ and $h_s|L_s : L_s \to L'_s$ is measure preserving,
(13) U'_s is a non-empty relatively clopen subset of G',
(14) $\lim_{i\to\infty} \operatorname{diam}(U'_{s\frown i}) = 0$ and $\operatorname{diam}(U'_s) \leqslant 2^{-length(s)}$, with respect to a fixed complete metric on G'.

More precisely, we let $U'_\varnothing = G'$ and given U'_s, we select L'_s, $U'_{s\frown i}$ and h_s as follows. Claim 6.3.2 provides a Cantor set $L'_s \subseteq \overline{U'_s}\backslash U'_s$ and a homeomorphism $g_s : U_s \cup L_s \to U'_s \cup L'_s$ such that $\mu(L'_s) = 1$ and $g_s|L_s$ is a measure preserving homeomorphism between L_s and L'_s. For each $i \in \mathbb{N}$ let $W_i = g_s(U_{s\frown i})$ and pick a non-empty relatively clopen set $U'_{s\frown i} \subseteq W_i$ in G' such that $\operatorname{diam}(U'_{s\frown i}) \leqslant 2^{-(length(s)+i)}$ (with respect to a fixed complete metric on G'). Since W_i and $U'_{s\frown i}$ are copies of the irrationals, there are homeomorphisms $u_i : W_i \to U'_{s\frown i}$ which give rise to a homeomorphism h_s, letting $h_s|L_s = g_s|L_s$ and $h_s|U_{s\frown i} = u_i \circ g_s|U_{s\frown i}$.

Let $P' \subseteq G' \subseteq B$ be the copy of the irrationals determined by the system $(U'_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L'_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$. Then, exactly as in the case of P, we have, cf. (3), (4),

(15) $\overline{P'} = P' \cup \bigcup\{L'_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ and $L'_s \cap L'_t = \emptyset$ for $s \neq t$,
(16) $P' \notin J_f(\mu)$.

Note that if $e \in \mathbb{N}^{\mathbb{N}}$, both $\bigcap_m U_{e|m}$ and $\bigcap_m U'_{e|m}$ are singletons, which gives rise to a homeomorphism $h : P \to P'$, defined by letting

(17) $h(x) \in \bigcap_m U'_{e|m}$ for $x \in \bigcap_m U_{e|m}$.

For any $s \in \mathbb{N}^{<\mathbb{N}}$, if $x \in P \cap U_{s\frown i}$, both $h(x)$ and $h_s(x)$ belong to the set $U'_{s\frown i}$ of diameter less than $2^{(-length(s)+i)}$. This, combined with (10), leads to the following observation: for any relatively closed set C in P and $s \in \mathbb{N}^{<\mathbb{N}}$, $h_s(\overline{C} \cap L_s) = \overline{h(C)} \cap L'_s$, and in effect, $\mathscr{H}^1(\overline{C} \cap L_s) = \mu(\overline{h(C)} \cap L'_s)$.

Taking into account (4), (15) and the fact that $\mathscr{H}^1(P) = \mu(P') = 0$, we conclude that for each relatively closed set C in P, $\mathscr{H}^1(\overline{C}) = \mu(\overline{h(C)})$ and this shows that for every $A \subseteq P$, $A \in J_f(\mathscr{H}^1)$ if and only if $h(A) \in J_f(\mu)$.

(ii) We shall modify the proof of part (i) above in the following way. Let G be a copy of the irrationals in $[0, 1]$ which is λ-null and dense in $[0, 1]$. Consequently, if U is a non-empty relatively open set in G, then there is a Cantor set $L \subseteq \overline{U}\setminus G$ with $\lambda(L) > 0$. It follows that we may define a generalized Hurewicz system $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ satisfying for each $s \in \mathbb{N}^{<\mathbb{N}}$ conditions (1)–(5) with (1) replaced by $\lambda(L_s) > 0$. The copy of the irrationals P determined by this system has properties (4)–(5) and in effect, $P \notin J_0(\lambda)$.

If now $B \in Bor(X)\setminus J_0(\mu)$, then by the properties of μ and a theorem of Solecki [18] we can pick a copy of the irrationals G' in B with $\mu(G') = 0$ such that for each non-empty relatively open set U' in G', we have $U' \notin J_0(\mu)$ so, in particular, $\mu(\overline{U'}\setminus G') > 0$.

A refinement of the proof of Claim 6.3.2 leads to the following observation

Claim 6.3.3. *Let U and U' be non-empty relatively open sets in G and G', respectively. Let $L \subseteq \overline{U}\setminus U$ be a Cantor set with $\lambda(L) > 0$.*

Then there exist a Cantor set $L' \subseteq \overline{U'}\setminus U'$ and a homeomorphism $g : U \cup L \to U' \cup L'$ such that $\mu(L') > 0$ and $g|L$ is a homeomorphism between L and L' preserving measure up to a positive constant factor. In particular, for every $A \subseteq L$, $\lambda(A) = 0$ if and only if $\mu(g(A)) = 0$.

We can now use the claim to define a generalized Hurewicz system $(U'_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L'_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of G' together with homeomorphisms h_s satisfying conditions (10)–(14) with (12) replaced by the requirements that $\mu(L'_s) > 0$ and $h_s|L_s : L_s \to L'_s$ preserves measure up to a positive constant factor.

Arguing as before, we conclude that for each relatively closed set C in P, $\lambda(\overline{C}) = 0$ if and only if $\mu(\overline{h(C)}) = 0$ and this shows that for any $A \subseteq P$, $A \in J_0(\lambda)$ if and only if $h(A) \in J_0(\mu)$, completing the proof of part (ii) and the proof of the theorem. □

Let us observe that the measure $\mathscr{H}^1$ itself has the properties described in part (i) of Theorem 6.3.1.

Remark 6.3.4. *Every Borel set $B \notin J_f(\mathscr{H}^1)$ contains a Borel set $C \notin J_f(\mathscr{H}^1)$ with $\mathscr{H}^1(C) = 0$.*

Proof. By a theorem of Solecki [18] we find a non-empty G_δ set G in B such that no non-empty relatively open set U in G is in $J_f(\mathscr{H}^1)$. Consequently, every element of $J_f(\mathscr{H}^1)$ below G is meager in G so it suffices to pick a dense G_δ subset C of G with $\mathscr{H}^1(C) = 0$.

□

6.4. **The proof of Theorem 1.2.** To begin with let us make the following observation.

Proposition 6.4.1. *Let μ be a nonatomic Borel measure on a compactum X.*

(i) *Assume that every Borel set $B \notin J_f(\mu)$ contains a Borel set $C \notin J_f(\mu)$ with $\mu(C) < \infty$. If $B \in Bor(X)\backslash J_f(\mu)$ then there is a function $\varphi : 2^{\mathbb{N}} \to Bor(B)\backslash J_f(\mu)$ such that for any distinct $c, d \in 2^{\mathbb{N}}$, $\varphi(c) \cap \varphi(d) = \emptyset$.*

(ii) *Assume that μ is semifinite. If $B \in Bor(X)\backslash J_0(\mu)$, then there is a function $\varphi : 2^{\mathbb{N}} \to Bor(B)\backslash J_0(\mu)$ such that for any distinct $c, d \in 2^{\mathbb{N}}$, $\varphi(c)\cap\varphi(d) = \emptyset$.*

Proof. To prove part (i), arguing as at the beginning of the proof of Theorem 6.3.1(i) we pick a copy of the irrationals G in B with $\mu(G) = 0$ such that for each non-empty relatively open set U in G, $\overline{U} \notin J_f(\mu)$. This leads to a generalized Hurewicz system $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ that determines a homeomorphic copy of the set $P \notin J_f(\mu)$.

For each $c \in 2^{\mathbb{N}}$ we let

$$S_c = \{s \in \mathbb{N}^{<\mathbb{N}} : s(i) + c(i) \text{ is even for every } i < \mathit{length}(s)\}.$$

and

$$\varphi(c) = \bigcap_n \bigcup \{U_s : s \in S_c \text{ and length(s)} = n\}.$$

Thus $\varphi(c)$ may be viewed as the copy of the irrationals in G determined by the system $(U_s)_{s\in S_c}$, $(L_s)_{s\in S_c}$. In particular, $\varphi(c) \notin J_f(\mu)$ and, moreover, $\varphi(c)\cap\varphi(d) = \emptyset$ for any distinct $c, d \in 2^{\mathbb{N}}$.

Part (ii) can be proved analogously along the lines of the first part of the proof of Theorem 6.3.1(ii).

□

We are now ready to complete the proof of Theorem 1.2.

To prove part (i), let P be a copy of the irrationals in $[0, 1]^2$, the existence of which is guaranteed by Theorem 6.3.1(i). Let $\mathbf{A}$ ($\mathbf{B}$) be the quotient Boolean algebra $Bor(P)/(J_f(\mathscr{H}^1) \cap Bor(P))$ ($Bor(X)/J_f(\mu)$, respectively).

Let us note that if $B \in Bor(X)\backslash J_f(\mu)$ and $h : P \to B$ is a homeomorphic embedding such that, for $A \subseteq P$, $A \in J_f(\mathscr{H}^1)$ if and only if $h(A) \in J_f(\mu)$, then h induces an isomorphisms from $\mathbf{A}$ onto the quotient Boolean algebra $Bor(h(P))/(J_f(\mu) \cap Bor(h(P))$. Consequently, by Theorem 6.3.1(i), the family $\mathbf{C}$ of all non-zero elements $\mathbf{c} \in \mathbf{B}$ such that the relative algebra $\mathbf{B}\restriction\mathbf{c}$ is isomorphic to $\mathbf{A}$, is dense in $\mathbf{B}$.

Let $\overline{\mathbf{A}}$ ($\overline{\mathbf{B}}$) be the completion of $\mathbf{A}$ ($\mathbf{B}$, respectively). It follows that for each $\mathbf{c} \in \mathbf{C}$ the relative algebra $\overline{\mathbf{B}}\restriction\mathbf{c}$ is isomorphic to $\overline{\mathbf{A}}$ and $\mathbf{C}$ is dense in $\overline{\mathbf{B}}$. Moreover, Proposition 6.4.1 implies that for any non-zero element $\mathbf{b} \in \overline{\mathbf{B}}$, the algebra $\overline{\mathbf{B}}\restriction\mathbf{b}$ has cellularity continuum from which it follows, $\mathbf{C}$ being dense below $\mathbf{b}$, that $\overline{\mathbf{B}}\restriction\mathbf{b}$ is isomorphic to the product of continuum many isomorphic copies of the algebra $\overline{\mathbf{A}}$ (cf. [9, Proposition 6.4]). This shows that the algebra $\overline{\mathbf{B}}$ is homogeneous (cf. [9, Definition 9.12]). In effect, since $\overline{\mathbf{B}}\restriction\mathbf{b}$ is isomorphic to $\overline{\mathbf{A}}$ for some non-zero $\mathbf{b} \in \overline{\mathbf{B}}$, the algebras $\overline{\mathbf{B}}$ and $\overline{\mathbf{A}}$ are isomorphic. In view of Remark 6.3.4, the same applies to the completion of the quotient Boolean algebra $Bor([0,1]^2)/J_f(\mathscr{H}^1)$ which completes the proof of part (i) of Theorem 6.3.1.

To prove part (ii), we follow closely the preceding argument, appropriately applying Theorem 6.3.1(ii). Thus the proof of Theorem 6.3.1 is completed. □

In particular, the partial order $Bor([0,1]^2)\backslash J_f(\mathscr{H}^1)$ is forcing homogeneous, while the σ-ideal $J_f(\mathscr{H}^1)$ is not homogeneous, and the same is true if $J_f(\mathscr{H}^1)$ is replaced by $J_0(\lambda)$. As already observed in Section 1, it seems that examples illustrating this phenomenon did not appear in the literature, (cf. [19], comments following Definition 2.3.7).

Finally, note that while, by Theorem 6.3.1, the completion of the quotient Boolean algebra $Bor([0,1]^2)/J_f(\mathscr{H}^1)$ is homogeneous, the algebra $Bor([0,1]^2)/J_f(\mathscr{H}^1)$ itself is not, since by Sikorski's theorem [7, 15.C], this would imply the homogeneity of the σ-ideal $J_f(\mathscr{H}^1)$. The same is also true if $J_f(\mathscr{H}^1)$ is replaced by $J_0(\lambda)$.

7. Comments

7.1. **Calibrated σ-ideals.** If X is a Henderson compactum, i.e., $dim X = \infty$ but X contains no 1-dimensional subcompactum, cf. [2, Example 5.2.23], then the σ-ideal $I_X(dim)$ is calibrated, cf. [21], [11].

Also, the σ-ideal $J_\sigma(\mathscr{H}^1)$ of Borel subsets of the Euclidean square $[0,1]^2$ that can be covered by countably many compacta of σ-finite $\mathscr{H}^1$-measure is calibrated, cf. [13].

7.2. **The 1-1 or constant property of Sabok and Zapletal.** From assertion (i) in Theorem 1.1 it follows that any calibrated σ-ideal I on a compactum X has the following property: whenever $f : B \to \mathbb{N}^{\mathbb{N}}$ is a Borel map on $B \in Bor(X)\backslash I$ with all fibers in I, then there exists $C \in Bor(B)\backslash I$ on which f is injective.

Indeed, the fact that this property can be derived from (i) was established by Sabok and Zapletal [17] (the proof in [17] is based on some forcing related arguments, and a justification in the realm of the classical descriptive set theory can be found in [12]).

7.3. **Inhomogeneity of $J_f(\mathscr{H}^1)$.** As was already proved in Corollary 6.2.2, the σ-ideal $J_f(\mathscr{H}^1)$ on the Euclidean square $[0,1]^2$ is not homogeneous. Here is another proof of this fact. Let $Y \subseteq [0,1]^2$ be a compactum not in $J_f(\mathscr{H}^1)$ on which $\mathscr{H}^1$ is σ-finite, and let $f : [0,1]^2 \to Y$ be any Borel function. As was recalled in Section 7.1, the σ-ideal $J_\sigma(\mathscr{H}^1) \supseteq J_f(\mathscr{H}^1)$ is calibrated in the square, and by (iii) in Theorem 1.1, there exists a compact set C in Y with $\mathscr{H}^1(C) < \infty$ and $f^{-1}(A) \notin J_\sigma(\mathscr{H}^1)$.

7.4. **Homogeneity of $J_f(\mu)$ for σ-finite μ.** The following result shows that the requirement imposed on μ to be *non*-σ-finite cannot be dropped from the assumptions of Proposition 6.2.1(ii).

Proposition 7.1. *Let ν be a σ-finite nonatomic measure on a compactum X such that all nonempty open sets have positive ν-measure, and let μ be a nonatomic Borel measure on a compactum $Y \notin J_f(\mu)$.*

Then for any $B \in Bor(Y)\backslash J_f(\mu)$ with $\mu(B) < \infty$ there is a Borel map $f : X \to B$ such that, whenever $A \in J_f(\mu)$, $f^{-1}(A) \in J_f(\nu)$.

Proof. Let $P \subseteq B$ be a copy of the irrationals defined as in the proof of Theorem 6.3.1(i) and let us adopt the notation from that proof. In particular, $\overline{P} = P \cup \bigcup\{L_s : s \in \mathbb{N}^{<\mathbb{N}}\}$, the Cantor sets L_s are pairwise disjoint and $\mu(L_s) = 1$ for each $s \in \mathbb{N}^{<\mathbb{N}}$.

Let us note that the compactum $\overline{P}$ is zero-dimensional as it contains no non-trivial continuum (cf. the proof of (8) in Section 4). Since, moreover, $\overline{P} \backslash P$ is dense in $\overline{P}$, removing a countable dense set from $\overline{P} \backslash P$, we get a copy of the irrationals H such that

(1) $P \subseteq H \subseteq \overline{P}, \quad |\overline{P} \backslash H| \leqslant \aleph_0$.

(A) Let us assume first that X is a copy of the irrationals.

The measure ν being σ-finite and nonatomic, by a result of Gelbaum [4], there are pairwise disjoint Cantor sets $C_0, C_1, \ldots$ in X with $\nu(C_n) \leqslant 1$ for each $n \in \mathbb{N}$ such that $\nu(X \backslash \bigcup_i C_i) = 0$.

Let us fix a complete metric d on H.

We shall define inductively homeomorphisms $h_n : X \to H$ such that for each $n \in \mathbb{N}$

(2) $\nu(A) = \mu(h_n(A))$ for any Borel $A \subseteq C_n$,
(3) $h_{n+1} | (C_0 \cup \ldots \cup C_n) = h_n | (C_0 \cup \ldots \cup C_n)$,
(4) $d(h_{n+1}(x), h_n(x)) \leqslant 2^{-n}$ for any $x \in X$.

To define h_0, using results of Oxtoby [10], we fix a homeomorphism $u : C_0 \to u(C_0) \subseteq L_\varnothing \cap H$ such that $\nu(A) = \mu(u(A))$ for any Borel $A \subseteq C_0$ (cf. the proof of 6.3.2), and let h_0 be an extension of u to a homeomorphism from X onto H whose existence is guaranteed by a theorem of Pollard [14].

Assume that h_n is already defined, and let $\mathscr{U}$ be a disjoint cover of H by relatively clopen sets of d-diameter $\leqslant 2^{-n}$.

For each $U \in \mathscr{U}$, we consider $V = h_n^{-1}(U)$ and the homeomorphism $h_{n+1} | V : V \to U$ will be defined as follows.

On $T = (C_0 \cup \ldots \cup C_n) \cap V$ we let h_{n+1} coincide with h_n. Since $h_n(T)$ is compact and nowhere dense in H, $U \backslash h_n(T)$ is a nonempty relatively open subset of H, so one can find L_s such that $L_s \cap H \subseteq U \backslash h_n(T)$ (cf. Section 2). Using again results of Oxtoby [10] and Pollard [14], we first pick a homeomorphic embedding $w : C_n \cap V \to L_s \cap H$ such that $\nu(A) = \mu(w(A))$, for any Borel $A \subseteq C_n \cap V$, and then extend w to a homeomorphism $h_{n+1} | V : V \to U$.

Then, conditions (2), (3), (4) are met.

Now, (4) guarantees that

(5) the sequence (h_n) uniformly converges to a continuous function $g : X \to H$,
(6) $g | C_n = h_n | C_n$ for $n \in \mathbb{N}$.

From (2), (6) and the fact that the Cantor sets $g(C_n)$ are pairwise disjoint, cf. (3), we infer that for any Borel set $A \subseteq X$,

$$\nu\Big(A \cap \bigcup_n C_n\Big) = \sum_n \nu(A \cap C_n) = \sum_n \mu(g(A \cap C_n)) = \mu\Big(g(A \cap \bigcup_n C_n)\Big).$$

Since $\nu(X \backslash \bigcup_n C_n) = 0$, we conclude that

(7) $\nu(A) \leqslant \mu(g(A))$ for any Borel $A \subseteq X$.

Let $A \in J_f(\mu)$ and assume that $A \subseteq \overline{P}$. Then $A \subseteq \bigcup_j F_j$, where $F_j \subseteq \overline{P}$ are closed and $\mu(F_j) < \infty$ for every $j \in \mathbb{N}$. It follows, by (5) and (7), that $g^{-1}(F_j)$ are closed sets of finite ν-measure, and hence $g^{-1}(A) \in J_f(\nu)$.

Since the range of g may not be contained in P, we shall slightly correct g to get a required map $f : X \to P \subseteq B$.

Let $M = g^{-1}(\bigcup_s L_s)$. Then, as we have noticed, $M \in J_f(\nu)$. Now, we define $f : X \to P$ so that f coincides with g on $X \backslash M$ and takes M to a point in P.

(B) Now, let ν be a σ-finite nonatomic Borel measure on a compactum X such that nonempty open sets have positive ν-measure.

By a result of Gelbaum [4], there is a countable open basis (U_n) of X such that $\nu(\partial U_n) = 0$ for all $n \in \mathbb{N}$, where ∂U_n denotes the boundary of U_n. Then $L = \bigcup_n \partial U_n$ is a σ-compact set in X with $\nu(L) = 0$ such that $X \backslash L$ is a copy of the irrationals.

Let B be a Borel set in Y satisfying the assumptions.

Using (A), we define a Borel map $f|(X\backslash L) : X\backslash L \to B$ such that for any Borel $A \in J_f(\mu)$,

$(f|(X\backslash L))^{-1}(A) \in J_f(\nu)$, and we let f send L to a point in B.

□

7.5. **A calibrated σ-ideal which is not coanalytic.** If E is a subset of a compactum X, $E \neq X$, the σ-ideal $K(E)$ is calibrated but need not be coanalytic.

However, we did not find in the literature examples of calibrated, non-coanalytic σ-ideals I on compacta X with $\bigcup I = X$.

The following construction provides examples of such σ-ideals of arbitrary high complexity.

Proposition 7.2. *Let I be a calibrated σ-ideal on a compactum X. For each $A \subseteq [0, 1]$ there exists a calibrated σ-ideal J on $[0, 1] \times X$ generated by compact sets and a continuous function $\Phi : [0, 1] \to K([0, 1] \times X)$ such that $A = \Phi^{-1}(J)$.*

Proof. Let J consist of Borel sets in $[0, 1] \times X$ that can be covered by countably many compact sets K with $K_t = \{x \in X : (t, x) \in K\} \in I$, for each $t \notin A$.

Since I is calibrated one readily checks that so is J.

The function $\Phi(t) = \{t\} \times X$, $t \in [0, 1]$, is a continuous map from $[0, 1]$ to $K([0, 1] \times X)$ and it is clear that $A = \Phi^{-1}(J)$.

□

7.6. **Comparing $Bor(X)/J_0(\mu)$ and $Bor(X)/J_f(\mu)$.**

(A) Let μ be a σ-finite nonatomic measure on a compactum X with $X \notin J_f(\mu)$. Then the consequence of Theorem 1.1(iii) indicated in Section 1, combined with some results of Zapletal [19] (indicated in Section 6.1 in the context of the σ-ideal $I(dim)$), show that the forcings associated with the partial orders $Bor(X)\backslash J_0(\mu)$ and $Bor(X)\backslash J_f(\mu)$ are not equivalent. To be more specific, assume on the contrary, that these forcings are equivalent. Then the reasoning on page 32 in [19], applied to the σ-ideal I which is the direct sum of the σ-ideals $J_0(\mu)$ and $J_f(\mu)$ on $X \times \{0, 1\}$, and the Borel sets $B_i = X \times \{i\}$, $i = 0, 1$, provides a Borel set $C_0 \subseteq B_0$ not in $J_0(\mu)$ and a Borel function $f : C_0 \to B_1$ such that $f^{-1}(A) \in J_0(\mu)$, whenever $A \in J_f(\mu)$. This, however, is impossible by Theorem 1.1(iii), the σ-ideal $J_0(\mu)$ being calibrated.

In particular, neither of the quotient Boolean algebras $Bor(X)/J_0(\mu)$ and $Bor(X)/J_f(\mu)$ embeds densely into the completion of the other.

(B) Let μ^h be a semifinite but not σ-finite Hausdorff measure on a compactum, associated with a continuous nondecreasing function $h : [0, +\infty] \to [0, +\infty]$ with $h(r) > 0$ for $r > 0$ and $h(0) = 0$, cf. [15]. Then $Bor(X)/J_f(\mu^h)$ does not embed densely into $Bor(X)/J_0(\mu^h)$.

This was proved in [13] under the additional assumption that the calibrated σ-ideal $J_0(\mu^h)$ has the 1-1 or constant property (cf. Section 7.2) but this is now granted by Theorem 1.1 (i).

It is not clear, however, if $Bor(X)/J_f(\mu^h)$ can be embedded densely into the completion of $Bor(X)/J_0(\mu^h)$.

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