ON TWO CONSEQUENCES OF CH ESTABLISHED BY SIERPIŃSKI

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Abstract. We study the relations between two consequences of the Continuum Hypothesis discovered by Wacław Sierpiński, concerning uniform continuity of continuous functions and uniform convergence of sequences of real-valued functions, defined on subsets of the real line of cardinality continuum.

1. Introduction

In his classical treaty *Hypothèse du continu* [18] Wacław Sierpiński distinguished the following consequences of the Continuum Hypothesis (CH) (the notation is taken from [18]):

- C_8 There exists a continuous function $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}$, $|E| = \mathfrak{c}$, not uniformly continuous on any uncountable subset of E.
- C_9 There is a sequence of functions $f_n : E \to \mathbb{R}$, $E \subseteq \mathbb{R}$, $|E| = \mathfrak{c}$, converging pointwise but not converging uniformly on any uncountable subset of E.

Sierpiński established the equivalences of C_9 to several other statements, notably, to the existence of a matrix of sets of real numbers (called in [2] a *BK-matrix*), constructed under CH by Banach and Kuratowski [1] (statement C_{11} in [18]).

Bartoszyński and Halbeisen [2] (see also [5]) proved that the existence of a BK-matrix is independent of CH. They also pointed out that the existence of a BK-matrix (hence statement C_9) is equivalent to the existence of a subset of $\mathbb{N}^{\mathbb{N}}$ (where $\mathbb{N}^{\mathbb{N}}$ denotes the set of functions $f : \mathbb{N} \to \mathbb{N}$ endowed with the topology of the countable product of the discrete space of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$) of cardinality $\mathfrak{c}$, intersecting each compact set in $\mathbb{N}^{\mathbb{N}}$ in an at most countable set (following [2] we shall call such sets *K-Lusin*), see [2, Proposition 1.1 and Lemma 2.3], cf. also [15].

Sierpiński [17] noticed that C_8 implies C_9 but he did not discuss the converse implication. However, in *Topology I* by Kuratowski [8], footnote (3) on page 533 suggests that the two statements are in fact

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equivalent. We are not aware of any publication addressing the implication $C_9 \Rightarrow C_8$ and this note is the result of our pondering on this matter.

We shall consider the following stratifications of statements C_8 and C_9 for uncountable cardinals $\kappa \leq \lambda \leq \mathfrak{c}$:

$C_8(\lambda, \kappa)$ There exist a set $E \subseteq \mathbb{R}$ of cardinality λ and a continuous function $f : E \to \mathbb{R}$, which is not uniformly continuous on any subset of E of cardinality κ.

$C_9(\lambda, \kappa)$ There exists a set $E \subseteq \mathbb{R}$ of cardinality λ (equivalently: for any set $E \subseteq \mathbb{R}$ of cardinality λ) and there is a sequence of functions $f_n : E \to \mathbb{R}$, converging on E pointwise but not converging uniformly on any subset of E of cardinality κ.

Clearly, statements C_i (for $i = 8, 9$) are $C_i(\mathfrak{c}, \aleph_1)$ in our notation, and C_i implies $C_i(\lambda, \kappa)$ for all uncountable cardinals $\kappa \leq \lambda \leq \mathfrak{c}$.

In this note we prove (in ZFC) that, in particular:

- $C_8(\mathfrak{c}, \mathfrak{c}) \Leftrightarrow C_9(\mathfrak{c}, \mathfrak{c})$ provided that the cardinal $\mathfrak{c}$ is regular, and under this assumption each of these statements is equivalent to the assertion $\mathfrak{d} = \mathfrak{c}$ (cf. Corollary 3.7),
- $C_8(\aleph_1, \aleph_1) \Leftrightarrow C_9(\aleph_1, \aleph_1)$, and each of these statements is equivalent to the assertion $\mathfrak{b} = \aleph_1$ (cf. Theorem 3.10).

Here $\mathfrak{d}$ and $\mathfrak{b}$ denote, as usual, the smallest cardinality of a dominating and, respectively, an unbounded family in $\mathbb{N}^{\mathbb{N}}$ corresponding to the ordering of eventual domination $\leq^*$ (cf. [5]).

An important role in our considerations is played by the notion of a K-Lusin set which we extend (cf. [2]) declaring that an uncountable subset E of a Polish space X is a *κ-K-Lusin set in X*, $\aleph_1 \leq \kappa \leq \mathfrak{c}$, if $|E \cap K| < \kappa$ for every compact set $K \subseteq X$.

The existence of a κ-K-Lusin set of cardinality λ in $\mathbb{N}^{\mathbb{N}}$ is equivalent to $C_9(\lambda, \kappa)$ (cf. Theorem 2.3) and if $E \subseteq \mathbb{R}$, $|E| = \lambda$, is a witnessing set for $C_8(\lambda, \kappa)$, then E is a κ-K-Lusin set in some G_δ-extension of E (cf. Proposition 3.1).

However, it is not the case that every κ-K-Lusin set is a witnessing set for $C_8(\lambda, \kappa)$. In particular, assuming CH, we show that there is a K-Lusin set in the irrationals of cardinality $\mathfrak{c}$ such that every continuous function $f : E \to \mathbb{R}$ is uniformly continuous on an uncountable subset of E (cf. Theorem 3.3). Our reasoning to that effect yields also that for every continuous function $f : X \to \mathbb{R}$ defined on a G_δ-set X in the irrationals, there exists a closed copy of irrationals P in X such that f is uniformly continuous on P (cf. Theorem 3.2).

The paper is organized as follows.

In Section 2 we establish the equivalences of $C_9(\lambda, \kappa)$ to several other statements, notably, to its topological counterparts (see Theorem 2.3).

Section 3 is devoted to $C_8(\lambda, \kappa)$ and its relations to $C_9(\lambda, \kappa)$ including proofs of the equivalence $C_8(\lambda, \kappa) \Leftrightarrow C_9(\lambda, \kappa)$ for $\kappa = \lambda = \mathfrak{c}$ (if the cardinal $\mathfrak{c}$ is regular) and $\kappa = \lambda = \aleph_1$. We will end the section by listing some additional set-theoretic assumptions under which the equivalence $C_8(\mathfrak{c}, \aleph_1) \Leftrightarrow C_9(\mathfrak{c}, \aleph_1)$ is true. Although the status of the implication $C_9(\mathfrak{c}, \aleph_1) \Rightarrow C_8(\mathfrak{c}, \aleph_1)$ remains unclear, these observations point out at difficulties in refuting it.

In Section 4 we gathered some comments and additional results related to the topic without proofs – we plan to present details elsewhere.

In this note $\mathbb{P}$ always denotes the set of irrationals of the unit interval $[0, 1]$. It is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ (cf. [7]).

2. Uniform convergence of pointwise convergent sequences of functions and statement $C_9(\lambda, \kappa)$

The following result is based on Sierpiński's reasoning [16], cf. Remark 2.2(1) (an extension of this result is formulated in Section 4.3).

Theorem 2.1. *For any Polish space X there is a sequence $f_1 \geq f_2 \ldots$ of continuous functions $f_n : X \to [0, 1]$ which converges to zero pointwise but does not converge uniformly on any set with non-compact closure in X.*

Proof. Since X embeds as a closed subspace in $[1, +\infty)^{\mathbb{N}_+}$ (cf. [7, Theorem 4.17]), where $\mathbb{N}_+ = \{n \in \mathbb{N} : \ n > 0\}$, it is enough to construct desired functions on $[1, +\infty)^{\mathbb{N}_+}$. So, with no loss of generality, we simply assume that $X = [1, +\infty)^{\mathbb{N}_+}$.

We begin with the Sierpiński functions $s_n : X \to \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $n = 1, 2, \ldots$, defined by (cf. Remark 2.2)

$$(1)\quad s_n(x) = \begin{cases} \frac{1}{\min\{i:\ x(i) \geq n\}} & \text{if} \quad x(\mathbb{N}_+) \cap [n, +\infty) \neq \emptyset, \\ 0 & \text{if} \quad x(\mathbb{N}_+) \subseteq [0, n). \end{cases}$$

We shall check that

(2) $s_1 \geq s_2 \geq \ldots$ and $\lim_{n\to\infty} s_n(x) = 0$ for every $x \in X$,

(3) for any $A \subseteq X$, if the sequence $(s_n)_{n=1}^{\infty}$ converges uniformly on A, then the closure $\bar{A}$ is compact.

The monotonicity in (2) is clear. If $x \in X$ and $p \in \mathbb{N}_+$ is given, then for any $n > \max\{x(i) :\ i \leq p\}$ we have $s_n(x) < \frac{1}{p}$, which gives the second part of (2).

To make sure that (3) is true, we shall follow closely Sierpiński [16]. Let $A \subseteq X$ and assume that the sequence $(s_n)_{n=1}^{\infty}$ converges uniformly on A. This means that for each $i \in \mathbb{N}_+$ there is a $\varphi(i) \in \mathbb{N}_+$ such that $s_m(x) < \frac{1}{i}$, whenever $m \geq \varphi(i)$ and $x \in A$.

By (1), for any $x \in X$ and $i \in \mathbb{N}_+$ we have $s_{\lfloor x(i) \rfloor}(x) \geq \frac{1}{i}$, and hence, $x(i) \leq \varphi(i)$, for any $x \in A$. Therefore, A is contained in the compact set $\prod_{i=1}^{\infty}[1, \varphi(i)] \subseteq X$, and hence its closure $\bar{A}$ in X is compact.

Let us verify that for each $n \in \mathbb{N}_+$

(4) the function s_n is upper-semicontinuous,

i.e., for any $r > 0$ the set $\{x \in X : \ s_n(x) < r\}$ is open in X. Since s_n is bounded by 1, it is enough to consider $r \leq 1$.

So let us fix $n \in \mathbb{N}_+$, $r \leq 1$ and $a \in X$ with $s_n(a) < r$, and for any $p \in \mathbb{N}_+$, let us consider the open set V_p defined by

(5) $V_p = \{x \in X : \ x(i) < n \quad \text{for all} \quad i \leq p\}$.

We shall show that we can always find p such that V_p is a neighbourhood of a contained in the set $\{x \in X : \ s_n(x) < r\}$.

If $s_n(a) = 0$, i.e., $a(i) < n$ for all $i \in \mathbb{N}_+$, cf. (1), then taking p such that $\frac{1}{p} < r$, we have $a \in V_p$ and $s_n(x) < \frac{1}{p} < r$ for every $x \in V_p$, cf. (1) and (5).

If $s_n(a) = \frac{1}{m}$, where $m = \min\{i : \ a(i) \geq n\}$, then since $s_n(a) < 1$, we have $a(1) < n$. Hence $m \geq 2$ and let $p = m - 1$. Then $p \geq 1$, $a \in V_p$ and for any $x \in V_p$, $s_n(x) \leq \frac{1}{m} < r$.

Having checked (4), we apply a classical theorem of Hahn (cf. [4, 1.7.15(c)]) to get, for each n, continuous functions $f_{n,i} : X \to [0, 1]$, $i = 1, 2, \ldots$, such that

(6) $f_{n,1} \geq f_{n,2} \geq \ldots$ and $\lim_{i\to\infty} f_{n,i}(x) = s_n(x)$ for every $x \in X$.

Finally, we define

(7) $f_n(x) = \min_{i,j \leq n} f_{i,j}(x)$ for $x \in X$.

Clearly, the sequence $f_1 \geq f_2 \geq \ldots$ consists of continuous functions and converges pointwise to zero. Moreover, $f_n(x) \geq s_n(x)$ for any $n \in \mathbb{N}_+$ and $x \in X$. Consequently, for any $A \subseteq X$, if the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly on A, then so does the sequence $(s_n)_{n=1}^{\infty}$ and hence by (3), $\bar{A}$ is compact.

□

Remark 2.2.

(1) *The original Sierpiński functions were defined on* $\mathbb{N}_+^{\mathbb{N}_+}$ *by the formula:*

$$s_n(x) = \begin{cases} \frac{1}{\min\{i: \ x(i)=n\}} & \textit{if} \quad n \in x(\mathbb{N}_+), \\ 0 & \textit{if} \quad n \notin x(\mathbb{N}_+). \end{cases}$$

Sierpiński was interested in neither regularity of the functions (in fact, s_n *are continuous on* $\mathbb{N}_+^{\mathbb{N}_+}$*) nor the monotonicity of the function sequence.*

(2) *An approach similar to Sierpiński's idea, in a different setting, was rediscovered by Pincirolli* [11, Lemma 2 and Proposition 7].

Let us turn back to statement $C_9(\lambda,\kappa)$. The following result provides some topological counterparts to $C_9(\lambda,\kappa)$.

Theorem 2.3. *For any uncountable cardinals $\kappa \le \lambda \le \mathfrak{c}$ the following are equivalent:*

(1) *$C_9(\lambda,\kappa)$,*
(2) *there is a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality λ and a sequence $g_1 \ge g_2 \dots$ of continuous functions $g_n : A \to \mathbb{R}$, which converges to zero pointwise but does not converge uniformly on any set of cardinality κ,*
(3) *there is a κ-K-Lusin set of cardinality λ in $\mathbb{N}^{\mathbb{N}}$,*
(4) *there is a Polish space X and a κ-K-Lusin set of cardinality λ in X.*

Proof.
(1) $\Rightarrow$ (2). Subtracting from each function in $C_9(\lambda,\kappa)$ the limit function, we get a sequence $f_n : E \to \mathbb{R}$, $|E| = \lambda$, which converges to zero pointwise but does not converge uniformly on any set of cardinality κ. For every $n \in \mathbb{N}_+$ and $x \in E$ let $u_n(x) = \max\{|f_i(x)| : \ i \ge n\}$ (recall that $\lim\limits_{i\to\infty} f_i(x) = 0$, hence the maximum is attained). Let us note that $u_1 \ge u_2 \dots$ and $0 \le |f_n| \le u_n$ for each n. The properties of the sequence $(f_n)_{n=1}^{\infty}$ yield readily that the sequence $(u_n)_{n=1}^{\infty}$ converges to zero pointwise on E but it does not converge uniformly on any subset of E of cardinality κ.

Let $h : E \to A$ be a bijection between E and a set $A \subseteq 2^{\mathbb{N}}$ such that all the functions $g_n = u_n \circ h^{-1}$ are continuous (for example, define h as the Marczewski characteristic function (cf. [10]) of a countable family $\{E_n : \ n \in \mathbb{N}\}$ of subsets of E, separating the points of E and containing all sets of the form $u_n^{-1}\big((p,q)\big)$, where $n \in \mathbb{N}_+$ and $p < q$ are rationals). Then the sequence $g_1 \ge g_2 \dots$ of continuous functions $g_n : A \to \mathbb{R}$ is as required.

(2) $\Rightarrow$ (3). Let us fix a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality λ and a sequence $g_1 \ge g_2 \dots$ of continuous functions $g_n : A \to \mathbb{R}$, which converges to zero pointwise but does not converge uniformly on any subset of A of cardinality κ. Let H be a G_δ-set in $\mathbb{N}^{\mathbb{N}}$ with $A \subseteq H \subseteq \bar{A}$ and such that each g_n extends to a continuous function $\tilde{g}_n : H \to \mathbb{R}$. Since $\tilde{g}_1 \ge \tilde{g}_2 \ge \dots$, for any $x \in H$ we have

$$\lim_{n\to\infty} \tilde{g}_n(x) = 0 \ \Leftrightarrow\ \forall p \in \mathbb{N}_+ \exists n \in \mathbb{N}_+ \ \tilde{g}_n(x) < \frac{1}{p},$$

so the set $G = \{x \in H : \ \lim\limits_{n\to\infty} \tilde{g}_n(x) = 0\}$ is a G_δ-set in $\mathbb{N}^{\mathbb{N}}$ containing A. Now, if $K \subseteq G$ is compact, then by the Dini theorem (see [4, Lemma

3.2.18]), the sequence $(\tilde{g}_n)_{n=1}^\infty$ converges uniformly on K, hence also $(g_n)_{n=1}^\infty$ converges uniformly on $A \cap K$, and therefore $|A \cap K| < \kappa$.

Finally, let $w : G \to \mathbb{N}^\mathbb{N}$ embed G onto a closed subspace of $\mathbb{N}^\mathbb{N}$ (see [7, Theorem 7.8]) and let $E = w(A)$. Then $|E \cap K| < \kappa$ for every compact set $K \subseteq \mathbb{N}^\mathbb{N}$, as required.

(3) $\Rightarrow$ (4). This implication is trivial.

(4) $\Rightarrow$ (1). Let E be a subset of cardinality λ of a Polish space X such that $|E \cap K| < \kappa$ for every compact set $K \subseteq X$. Theorem 2.1 provides us with a sequence $f_1 \geq f_2 \ldots$ of continuous functions $f_n : X \to [0, 1]$ which converges to zero pointwise but does not converge uniformly on any set with non-compact closure in X. Consequently, the sequence $(f_n)_{n=1}^\infty$ converges to zero pointwise on E but any set $M \subseteq E$ of cardinality κ has a non-compact closure in X, so $(f_n)_{n=1}^\infty$ does not converge uniformly on M. Clearly, this completes the proof of (1).

□

In two important cases, namely $\kappa = \lambda = \aleph_1$ and $\kappa = \lambda = \mathfrak{c}$, statement $C_9(\kappa, \lambda)$ is characterized in terms of basic cardinal characteristics of the continuum, cf. [3].

Corollary 2.4.

(1) $C_9(\aleph_1, \aleph_1) \Leftrightarrow \mathfrak{b} = \aleph_1$,

(2) $C_9(\mathfrak{c}, \mathfrak{c}) \Leftrightarrow \mathfrak{d} = \mathfrak{c}$*, provided that the cardinal* $\mathfrak{c}$ *is regular (more precisely,* $\mathfrak{d} = \mathfrak{c}$ *implies* $C_9(\mathfrak{c}, \mathfrak{c})$ *under no additional assumptions on* $\mathfrak{c}$*).*

Proof. We shall repeatedly make use of Theorem 2.3.

(1). Assume $C_9(\aleph_1, \aleph_1)$ and let $E \subseteq \mathbb{N}^\mathbb{N}$ be a set of cardinality $\aleph_1$ whose intersection with every compact set $K \subseteq \mathbb{N}^\mathbb{N}$ is countable. Clearly, E is unbounded in $(\mathbb{N}^\mathbb{N}, \leq^*)$, hence $\mathfrak{b} = \aleph_1$.

Conversely, if $\mathfrak{b} = \aleph_1$, then any subset of $\mathbb{N}^\mathbb{N}$ of the form $\{f_\alpha : \alpha < \mathfrak{b}\}$, where

- $\alpha < \beta < \mathfrak{b}$ implies $f_\alpha <^* f_\beta$,
- for every $f \in \mathbb{N}^\mathbb{N}$ there is $\alpha < \mathfrak{b}$ with $f_\alpha \not\leq^* f$,

has countable intersection with every compact $K \subseteq \mathbb{N}^\mathbb{N}$.

(2). Assume $C_9(\mathfrak{c}, \mathfrak{c})$ and let $E \subseteq \mathbb{N}^\mathbb{N}$ be a set of cardinality $\mathfrak{c}$ such that $|E \cap K| < \mathfrak{c}$ for every compact set $K \subseteq \mathbb{N}^\mathbb{N}$. Let $\{g_\alpha : \alpha < \mathfrak{d}\}$ be a dominating set in $\mathbb{N}^\mathbb{N}$. In particular $E = \bigcup_{\alpha<\mathfrak{d}}\{f \in E : \ f <^* g_\alpha\}$ and the regularity of $\mathfrak{c}$ implies that $\mathfrak{d} = \mathfrak{c}$.

Conversely, if $\mathfrak{d} = \mathfrak{c}$, then any subset of $\mathbb{N}^\mathbb{N}$ of the form $\{g_\alpha : \alpha < \mathfrak{c}\}$, where

- $\alpha < \beta < \mathfrak{c}$ implies $g_\beta \not\leq^* g_\alpha$,
- for every $f \in \mathbb{N}^\mathbb{N}$ there is $\alpha < \mathfrak{c}$ with $f <^* g_\alpha$,

has the property that $|E \cap K| < \mathfrak{c}$ for every compact $K \subseteq \mathbb{N}^{\mathbb{N}}$.

□

The proof of (4) $\Rightarrow$ (1) in Theorem 2.3 yields also the following result.

Corollary 2.5. *For any uncountable cardinals $\kappa \leq \lambda \leq \mathfrak{c}$, if E is a κ-K-Lusin set of cardinality λ in a Polish space X, then there exists a sequence $f_1 \geq f_2 \dots$ of continuous functions $f_n : E \to \mathbb{R}$, which converges to zero pointwise but does not converge uniformly on any subset of E of cardinality κ.*

Remark 2.6. *As was mentioned in the introduction, the notion of a K-Lusin set was introduced by Bartoszyński and Halbeisen* [2]*, where it was pointed out that a reasoning of Banach and Kuratowski* [1]*, establishing under CH the existence of a BK-Matrix, actually shows that the existence of a BK-matrix is equivalent to the existence of a K-Lusin set of cardinality $\mathfrak{c}$. Earlier, Sierpiński* [16] *proved that a BK-Matrix exists if and only if C_9 holds. Combining these two results, we get the equivalence "$C_9 \Leftrightarrow$ there exists a K-Lusin set of cardinality $\mathfrak{c}$" which was obtained in Theorem 2.3 by a different reasoning.*

3. Uniform continuity of continuous functions and statement $C_8(\lambda, \kappa)$

Let us now turn our attention to statement $C_8(\lambda, \kappa)$. Sierpiński [17] proved that C_8 implies C_9 and his argument can be easily adapted to establish a more general implication concerning $C_i(\lambda, \kappa)$ (for $i = 8, 9$). Instead of repeating the argument of Sierpiński we present a proof based on Theorem 2.3 which gives some additional information about the involved sets.

Proposition 3.1. *For any uncountable cardinals $\kappa \leq \lambda \leq \mathfrak{c}$:*

$$C_8(\lambda, \kappa) \Rightarrow C_9(\lambda, \kappa).$$

Moreover, if a set $E \subseteq \mathbb{R}$, $|E| = \lambda$, together with a continuous function $f : E \to \mathbb{R}$ witness $C_8(\lambda, \kappa)$, then there is a G_δ-set G in $\mathbb{R}$ such that $E \subseteq G$ and E is a κ-K-Lusin set in G.

Proof. Let us extend f to a continuous function $\tilde{f} : G \to \mathbb{R}$ over a G_δ-set $G \subseteq \mathbb{R}$. Now, if $K \subseteq G$ is compact, then the extension $\tilde{f}$ is uniformly continuous on K, hence so is f on $E \cap K$. Therefore, $|E \cap K| < \kappa$, as f is not uniformly continuous on any set of cardinality κ. This shows that the equivalent to $C_9(\lambda, \kappa)$ statement, formulated in Theorem 2.3(4), is true, completing the proof. □

In the rest of this note we investigate the possibility of reversing the above implication, at least for some pairs of uncountable cardinals $\kappa \leq \lambda \leq \mathfrak{c}$.

In view of Proposition 3.1, a related question is whether a κ-K-Lusin set E in $\mathbb{P}$ always carries a continuous function $f : E \to \mathbb{R}$, which is not uniformly continuous on any set of cardinality κ. The negative answer (cf. Theorem 3.3) is a consequence of the following general result, closely related to the "limit systems" of Hurewicz [6].

Theorem 3.2. *Let X be a Polish non σ-compact space and let d be a compatible completely bounded metric on X. Then for every continuous function $f : X \to \mathbb{R}$ there exists a closed copy of irrationals P in X such that f is uniformly continuous on P in the metric d.*

Proof. Let $(\hat{X}, \hat{d})$ be the completion of (X, d); then $\hat{X}$ is compact, d being totally bounded. Since X is not σ-compact, by a theorem of Hurewicz (see [7, Theorem 7.10]), X contains a closed in X copy of the irrationals G. Let ρ be a complete metric on G.

We shall use generalized Hurewicz systems in the setting considered in [12, Section 2.4] and [13, Section 2]. Namely, we shall define a pair of families: $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of G, and $(x_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of points in $\hat{X}$ with the following properties (the closures are taken in $\hat{X}$, $B_{\hat{d}}(x_s, \varepsilon) = \{x \in \hat{X} : \hat{d}(x_s, x) < \varepsilon\}$ and for $A \subseteq G$, $\mathrm{diam}_\rho(A)$ or $\mathrm{diam}_d(A)$ stand for the diameter with respect to ρ or d):

(1) U_s is relatively open in G, $U_s \neq \emptyset$,

(2) $\mathrm{diam}_\rho(U_s) \leq 2^{-length(s)}$,

(3) $\overline{U_s} \cap \overline{U_t} = \emptyset$ for distinct $s,\ t$ of the same length,

(4) $\overline{U_{s\hat{\ }i}} \cap G \subseteq U_s$,

(5) $x_s \in \overline{U_s} \setminus G$,

(6) $x_s \notin \overline{U_{s\hat{\ }i}}$ for any $i \in \mathbb{N}$,

(7) each neighbourhood of x_s in $\hat{X}$ contains all but finitely many $U_{s\hat{\ }i}$; in particular, $\lim_i \mathrm{diam}_d(U_{s\hat{\ }i}) = 0$,

(8) $\mathrm{diam}(f(U_{s\hat{\ }i})) \leq 2^i$ for any $i \in \mathbb{N}$,

(9) if $c_i \in U_{s\hat{\ }i}$ for each $i \in \mathbb{N}$, then the sequence $(f(c_i))_{i\in\mathbb{N}}$ is convergent in $\mathbb{R}$.

To define sets U_s and points x_s we proceed as follows.

Let $U_\emptyset$ be a non-empty relatively open set in G such that f is bounded on $U_\emptyset$ and $\mathrm{diam}_\rho(U_\emptyset) \leq 1$.

At the inductive step let $n \geq 0$ and assume that we have already defined U_s and x_t for $s \in [\mathbb{N}]^{\leq n}$ and $t \in [\mathbb{N}]^{<n}$ satisfying the required conditions. Fix s with $length(s) = n$ and pick $x_s \in \overline{U_s} \setminus G$ arbitrarily (this is possible since G does not contain compact sets with non-empty interior). Let us choose points $a_n \in U_s$ such that $\lim_n a_n = x_s$ and the sequence $(f(a_n))_n$ is convergent (first, we choose $b_n \in U_s$ so that $\lim_n b_n = x_s$ and then, using the fact that the sequence $(f(b_n))_n$ is

bounded, we choose its convergent subsequence). Next, using the continuity of f on G, let us enlarge each a_n to its open neighbourhood $U_{s\hat{}n}$ in G so that relevant instances of conditions (1)–(8) are satisfied. Then (8) and the fact that the sequence $(f(a_n))_n$ is convergent readily yield (9).

Let

(10) $P = \bigcap_n \bigcup \{U_s : \text{length(s)} = n\}$.

be the copy of the irrationals determined by the generalized Hurewicz system $(U_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$, $(L_s)_{s\in[\mathbb{N}]^{<\mathbb{N}}}$ ([13, Section 2]), where $L_s = \{x_s\}$ for each $s \in \mathbb{N}^{<\mathbb{N}}$. In particular, $\overline{P} = P \cup \{x_s : s \in \mathbb{N}^{<\mathbb{N}}\}$, so $P = \overline{P} \cap G$ is closed in G, and hence also in X.

We claim that for each $s \in \mathbb{N}^{<\mathbb{N}}$

(11) $\inf_{\varepsilon>0} \operatorname{diam}\Big(f\big(B_{\hat{d}}(x_s,\varepsilon)\cap P\big)\Big) = 0.$

To justify the claim, let us fix $s \in \mathbb{N}^{<\mathbb{N}}$ and for each $i \in \mathbb{N}$ let us pick $c_i \in U_{s\hat{}i}$. By (9), $\lim_{i\to\infty} f(c_i) = r$, and let J be an arbitrary open interval containing r. From (7) and (8) we get i_0 such that $f(U_{s\hat{}i}) \subseteq J$, whenever $i > i_0$.

Now, we can find an $\varepsilon > 0$ such that $B_{\hat{d}}(x_s, \varepsilon)$ is disjoint from any U_t with $t \neq s$ and $length(t) = length(s)$, i.e., cf. (10),

(12) $W = B_{\hat{d}}(x_s,\varepsilon) \cap P = B_{\hat{d}}(x_s,\varepsilon) \cap U_s$.

For suppose that for every $\varepsilon > 0$, (12) is false. This allows us to define a sequence $(z)_{n\in\mathbb{N}}$ converging to x_s such that the set $Z = \{z_n : n \in \mathbb{N}\}$ is disjoint from U_s.

By (3) and (5), $Z \cap U_t$ is finite for any $t \in \mathbb{N}^{<\mathbb{N}}$ with $t \neq s$ and $length(t) = length(s)$. Then, since $Z \subseteq U_\emptyset$, it follows that we can find $t \in \mathbb{N}^{<\mathbb{N}}$ such that $length(t) < length(s)$, $Z \cap U_t$ is infinite but $Z \cap U_{t\hat{}i}$ is finite for each $i \in \mathbb{N}$. By (9), $\lim_{n\to\infty} z_n = x_t$, however, by (3) and (6), $x_t \neq x_s$, and this contradiction completes the justification of (12).

Next, by appealing to (6), we can make ε still smaller to ensure that $B_{\hat{d}}(x_s,\varepsilon)$ omits also all $U_{s\hat{}i}$ with $i \leq i_0$. Consequently, cf. (10) and (12), $W \subseteq \bigcup_{i>i_0} U_{s\hat{}i}$, hence $f(W) \subseteq J$.

This completes the justification of the claim, and let us note that (11) means exactly that the oscillation of f at any point of $\overline{P} \setminus P = \{x_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ is zero. This guarantees that f can be extended continuously over $\overline{P}$. By compactness of $\overline{P}$ this extension is uniformly continuous, and in effect we get uniform continuity of f on P.

□

As a corollary we obtain that not every κ-K-Lusin set of cardinality λ in the irrationals is a witnessing set for $C_8(\lambda,\kappa)$ (cf. [17, a remark at the end of the paper]).

Theorem 3.3. *If $\mathfrak{d} = \mathfrak{c}$, then there is a $\mathfrak{c}$-K-Lusin set E in $\mathbb{P}$ of cardinality $\mathfrak{c}$ such that every continuous function $f : E \to \mathbb{R}$ is uniformly continuous on a subset of E of cardinality $\mathfrak{c}$.*

In particular, assuming CH, there is a K-Lusin set $E \subseteq \mathbb{P}$ of cardinality $\mathfrak{c}$ such that every continuous function $f : E \to \mathbb{R}$ is uniformly continuous on an uncountable subset of E.

Proof. We list all compact sets in $\mathbb{P}$ as $(K_\alpha : \ \alpha < \mathfrak{c})$, and all closed copies of irrationals in $\mathbb{P}$ as $(P_\alpha : \ \alpha < \mathfrak{c})$, where each closed copy of irrationals P in $\mathbb{P}$ appears in this transfinite sequence $\mathfrak{c}$-many times.

Then we inductively pick

$$x_\alpha \in P_\alpha \setminus \Big(\bigcup_{\beta<\alpha} K_\beta \cup \{x_\beta : \beta < \alpha\} \Big),$$

the choice being made possible by the assumption $\mathfrak{d} = \mathfrak{c}$ which means that $\mathbb{P}$ is not covered by any collection of less that $\mathfrak{c}$-many its compact subsets (cf. [3, Theorem 2.8]).

We let $E = \{x_\alpha : \alpha < \mathfrak{c}\}$. Let us notice that E is a $\mathfrak{c}$-K-Lusin set E in $\mathbb{P}$ and

(1) E intersects each closed copy of irrationals in $\mathbb{P}$ in a set of cardinality $\mathfrak{c}$.

To see that E is a set we are looking for, let $f : E \to \mathbb{R}$ be a continuous function, and let X be a G_δ-set in $\mathbb{P}$ containing E such that f extends to the (uniquely defined) continuous function $\hat{f} : X \to \mathbb{R}$.

Now, E being a $\mathfrak{c}$-K-Lusin set in $\mathbb{P}$, it cannot be covered by countably many compact sets in $\mathbb{P}$. Consequently, X is a non σ-compact Polish space contained in $[0, 1]$, so by Theorem 3.2 there exists a closed copy of irrationals P in X such that $\hat{f}$ is uniformly continuous on P (in the metric inherited from $[0, 1]$). Shrinking P, if necessary, we may assume that P is closed also in $\mathbb{P}$. By (1), $|E \cap P| = \mathfrak{c}$ and we conclude that f is uniformly continuous on $E \cap P$.

□

On the other hand, we have the following result.

Theorem 3.4. *If $\mathfrak{d} = \mathfrak{c}$, then there is a $\mathfrak{c}$-K-Lusin set E in $\mathbb{P}$ of cardinality $\mathfrak{c}$ and a continuous function $f : E \to \mathbb{R}$, which is not uniformly continuous on any set of cardinality $\mathfrak{c}$.*

In particular, we have $\mathfrak{d} = \mathfrak{c} \Rightarrow C_8(\mathfrak{c}, \mathfrak{c})$.

Our proof will be based on the following proposition.

Proposition 3.5. *Let $f : \mathbb{P} \to [0, 1]$ be a continuous function such that the closure $\overline{G(f)}$ in $[0, 1]^2$ of the graph $G(f)$ of f intersects each $\{q\} \times [0, 1]$, $q \in \mathbb{Q} \cap [0, 1]$, in an uncountable set. Then $\mathbb{P}$ cannot be covered by less than $\mathfrak{d}$ sets on which f is uniformly continuous.*

Proof of Proposition 3.5. Let $\mathcal{A}$ be a collection of subsets of $\mathbb{P}$ such that $|\mathcal{A}| < \mathfrak{d}$ and f is uniformly continuous on each $A \in \mathcal{A}$. We will show that $\mathbb{P} \setminus \bigcup \mathcal{A} \neq \emptyset$.

Let $A \in \mathcal{A}$. Then $f|A$ being uniformly continuous extends continuously over the closure of A in $[0,1]$, and let K_A be the graph of this extension.

For each $q \in \mathbb{Q} \cap [0,1]$, K_A intersects $\{q\} \times [0,1]$ in at most a singleton, and hence $|\bigcup_{A \in \mathcal{A}} \big(K_A \cap (\{q\} \times [0,1])\big)| < \mathfrak{d}$. If V is an open neighbourhood of $(t, f(t))$, $t \in \mathbb{P}$, in $[0,1]^2$, then there are non-empty open intervals I_1, I_2 in $[0,1]$ such that $t \in I_1$, $I_1 \times I_2 \subseteq V$ and $f(I_1 \cap \mathbb{P}) \subseteq I_2$. It follows that for every $q \in I_1 \cap \mathbb{Q}$, $\overline{G(f)} \cap (\{q\} \times \bar{I}_2) = \overline{G(f)} \cap (\{q\} \times [0,1])$, so by the properties of f, $|\overline{G(f)} \cap (\{q\} \times I_2)| = \mathfrak{c}$. Consequently, the set

$$H = \overline{G(f)} \cap (\mathbb{Q} \times [0,1]) \setminus \bigcup_{A \in \mathcal{A}} K_A$$

is dense in $\overline{G(f)}$.

Since $G(f)$ is a G_δ-set dense in $\overline{G(f)}$, by the Baire theorem, each F_σ-set covering $G(f)$ must hit H, and by the Kechris-Louveau-Woodin theorem (see [7, Theorem 21.22]), we get a Cantor set $C \subseteq G(f) \cup H$ such that $P = C \cap G(f)$ is a copy of the irrationals, closed in $G(f)$.

For each $A \in \mathcal{A}$, K_A being compact, the set $K_A \cap P = K_A \cap C$ is compact and since $|\mathcal{A}| < \mathfrak{d}$ it follows that $P \setminus \bigcup_{A \in \mathcal{A}} K_A \neq \emptyset$ (cf. [3, Theorem 2.8]). In effect, $G(f) \setminus \bigcup_{A \in \mathcal{A}} K_A \neq \emptyset$ which proves that $\mathbb{P} \not\subseteq \bigcup \mathcal{A}$. □

With Proposition 3.5 in hand, we can easily get Theorem 3.4.

Proof of Theorem 3.4. Let $f : \mathbb{P} \to [0,1]$ be as in Proposition 3.5 (see Example 3.6).

Let us list as $(F_\alpha : \alpha < \mathfrak{c})$ all closed sets in $\mathbb{P}$ on which f is uniformly continuous. Since $\mathfrak{d} = \mathfrak{c}$, by the assertion of Proposition 3.5, we can inductively pick points

$$x_\alpha \in \mathbb{P} \setminus \Big(\bigcup_{\beta < \alpha} F_\beta \cup \{x_\beta : \beta < \alpha\} \Big), \alpha < \mathfrak{c},$$

and finally let $E = \{x_\alpha : \alpha < \mathfrak{c}\}$.

Then if $A \subseteq E$ and f is uniformly continuous on A, the closure of A in $\mathbb{P}$ is listed as some F_α, and hence $|A| < \mathfrak{c}$.

Likewise, every compact set K in $\mathbb{P}$ is on the list, so $|E \cap K| < \mathfrak{c}$ which shows that E is a $\mathfrak{c}$-K-Lusin set in $\mathbb{P}$.

□

For the sake of completeness we recall an example given by Kuratowski and Sierpiński in [9], of a function satisfying the assertion of Proposition 3.5.

Example 3.6. *Let $\psi : [0,1] \to [0,1]$ be given by the formula*

$$\psi(t) = \sum_{n=1}^{\infty} \frac{\phi(t - q_n)}{2^n},$$

where $\phi(t) = |\sin(\frac{1}{t})|$ for $t \neq 0$ and $\phi(0) = 0$, and $(q_1, q_2, \ldots)$ is an injective enumeration of $\mathbb{Q} \cap [0,1]$.

Then $f = \psi|([0,1] \setminus \mathbb{Q})$ satisfies the assertion of Proposition 3.5. To see this, let us fix q_n, and let

$$\sigma(t) = \sum_{m \neq n} \frac{\phi(t - q_m)}{2^m} \quad \text{for } t \in [0,1].$$

Then σ is continuous at q_n,

$$\psi(t) = \sigma(t) + \frac{\phi(t - q_n)}{2^n},$$

and the definition of ϕ yields that

$$\overline{G(f)} \cap (\{q_n\} \times [0,1]) = \{q_n\} \times [\sigma(t), \sigma(t) + 2^{-n}].$$

By combining Theorem 3.4 with Proposition 3.1 and Corollary 2.4, we immediately get the following corollary.

Corollary 3.7. *If the cardinal $\mathfrak{c}$ is regular, then the following statements are equivalent:*

(1) $C_8(\mathfrak{c}, \mathfrak{c})$,
(2) $C_9(\mathfrak{c}, \mathfrak{c})$,
(3) $\mathfrak{d} = \mathfrak{c}$.

Without any additional set-theoretic assumptions we have (in ZFC) the following weaker result.

Theorem 3.8. *For any uncountable cardinal $\kappa < \mathfrak{c}$,*

$$C_9(\mathfrak{c}, \kappa) \Rightarrow C_8(\mathfrak{c}, \mathfrak{c}).$$

Before giving the proof of Theorem 3.8 let us recall that a subset S of a Polish space X is *κ-concentrated in X on a set $D \subseteq X$*, $\aleph_1 \leq \kappa \leq \mathfrak{c}$, if $|S \setminus U| < \kappa$ for every open in X set $U \subseteq X$ that contains D.

Let us assume that $C \subseteq \mathbb{R}$ is the Cantor set, Q is a countable dense set in C and $P = C \setminus Q$.

Let us note that if $S \subseteq P$, then S is κ-concentrated on Q in C if and only if S is a κ-K-Lusin in P. (cf. [2, Proposition 3.4]).

We shall need the following lemma.

Lemma 3.9. *Suppose that there is a set $S \subseteq P$ of cardinality $\mathfrak{c}$ κ-concentrated on Q in C. Then there is a set $H \subseteq S \times C$ such that for every compact set K in $C \times C$ with $K \cap (Q \times C)$ countable, we have $|H \cap K| < \mathfrak{c}$.*

Proof of Lemma 3.9. We list all compact sets in $C \times C$ with $K \cap (Q \times C)$ countable as $(K_\alpha : \alpha < \mathfrak{c})$. Then we inductively pick distinct points $s_\alpha \in S$ and $t_\alpha \in C$ such that $(s_\alpha, t_\alpha) \notin \bigcup_{\beta<\alpha} K_\beta$ for $\alpha < \mathfrak{c}$.

At stage $\alpha < \mathfrak{c}$, let us notice that $\bigcup_{\beta<\alpha} K_\beta \cap (Q \times C)$ has cardinality less than $\mathfrak{c}$, and therefore, there is $t \in C$ such that $\bigcup_{\beta<\alpha} K_\beta \cap (Q \times \{t\}) = \emptyset$. Since $S \times \{t\}$ is κ-concentrated on $Q \times \{t\}$ in $C \times \{t\}$, it follows that, for each $\beta < \alpha$, $|K_\beta \cap (S \times \{t\})| < \kappa$. Therefore,

$$|\bigcup_{\beta<\alpha} \big(K_\beta \cap (S \times \{t\})\big)| < \mathfrak{c},$$

so we can pick $s \in S$, distinct from s_β for all $\beta < \alpha$, such that $(s, t) \notin \bigcup_{\beta<\alpha} K_\beta$, and we let $(s_\alpha, t_\alpha) = (s, t)$.

Finally, we let $H = \{(s_\alpha, t_\alpha) : \alpha < \mathfrak{c}\}$. □

With Lemma 3.9 in hand, we can now prove Theorem 3.8 as follows.

Proof of Theorem 3.8. Let us assume that statement $C_9(\mathfrak{c}, \kappa)$ is true. With the help of Theorem 2.3 and the fact that P is a homeomorphic copy of $\mathbb{N}^{\mathbb{N}}$, we get a κ-K-Lusin S in P of cardinality $\mathfrak{c}$.

There is a continuous map $\phi : C \times C \to C$ such that $\phi|(P \times C)$ is a homeomorphism onto $G = \phi(P \times C)$, and the set $D = \phi(Q \times C)$ is countable and disjoint from G (a simple argument to this effect is given in [14, Lemma 4.2]).

Since S is κ-concentrated on Q in C there is, by Lemma 3.9, a set $H \subseteq S \times C$ such that for every compact set K in $C \times C$ with $K \cap (Q \times C)$ countable, we have $|H \cap K| < \mathfrak{c}$. Let $E = \phi(H)$ and $f = \phi^{-1}|E : E \to H$. Upon an embedding of $C \times C$ in $\mathbb{R}$, we can consider f as a function from a subset of $\mathbb{R}$ of cardinality $\mathfrak{c}$ into $\mathbb{R}$ and we are going to prove that it is a witness that statement $C_8(\mathfrak{c}, \mathfrak{c})$ is true.

Aiming at a contradiction, assume that $f|A$ is uniformly continuous (with respect to any metric compatible with the topology of $C \times C$) on a set $A \subseteq E$ of cardinality $\mathfrak{c}$ and let $B = f(A) = \phi^{-1}(A)$. Then, since $\phi|B : B \to A$ is also uniformly continuous, the function $f|A$ extends to a homeomorphism $\tilde{f} : \bar{A} \to \bar{B}$, where $\bar{A}$ and $\bar{B}$ are the closures of A and B in C and $C \times C$, respectively (cf. [4, Theorem 4.3.17]). Since D is countable and $\tilde{f}^{-1} = \phi|\bar{B}$ injectively maps $\bar{B} \cap (Q \times C)$ into $\bar{A} \cap D$, $\bar{B} \cap (Q \times C)$ is also countable, and we have $\bar{B} = K_\alpha$ for some $\alpha < \mathfrak{c}$. This, however, is impossible, as by Lemma 3.9, we have $|H \cap K_\alpha| < \mathfrak{c}$, but on the other hand $B \subseteq H \cap K_\alpha$ has cardinality $\mathfrak{c}$.

□

A similar reasoning shows, in particular, that also the statements $C_8(\aleph_1, \aleph_1)$ and $C_9(\aleph_1, \aleph_1)$ are equivalent.

Let us recall that E is a *λ'-set* in X if for every countable set L in X, L is a relative G_δ-set in $E \cup L$; by a theorem of Sierpiński, there is (in ZFC) an uncountable λ'-set in $\mathbb{R}$, cf. [8].

Theorem 3.10. *For any uncountable cardinal $\nu \leq \mathfrak{c}$, the existence of a λ'-set T of cardinality ν in the Cantor set $C \subseteq \mathbb{R}$ and a K-Lusin set S in $P = C \setminus Q$ of cardinality ν, where Q is a countable dense set in C, implies $C_8(\nu, \aleph_1)$. Consequently, the existence of a λ'-set of cardinality ν in the Cantor set C implies that $C_8(\nu, \aleph_1) \Leftrightarrow C_9(\nu, \aleph_1)$ and hence, we have (in ZFC) $C_8(\aleph_1, \aleph_1) \Leftrightarrow C_9(\aleph_1, \aleph_1)$, and each of the statements is equivalent to the assertion $\mathfrak{b} = \aleph_1$.*

Proof. Let H be the graph of a bijection from S onto T. Since S is concentrated in C on Q, it follows that H is concentrated in $C \times C$ on $Q \times C$. Indeed, if U is an arbitrary open set in $C \times C$ containing $Q \times C$, and $D = (C \times C) \setminus U$, then $V = C \setminus \mathrm{proj}_1(D)$ is an open set in C containing Q (where proj_1 is the projection of $C \times C$ onto the first axis). Thus V contains all but countably many points of S, and consequently, the set $H \setminus U$ is countable.

As in the proof of Theorem 3.8, let $\phi : C \times C \to C$ be a continuous map such that $\phi|(P \times C)$ is a homeomorphism onto $G = \phi(P \times C)$, and the set $D = \phi(Q \times C)$ is countable and disjoint from G.

Let $E = \phi(H)$ and $f = \phi^{-1}|E : E \to H$. Upon an embedding of $C \times C$ in $\mathbb{R}$, we consider f as a function from a subset of $\mathbb{R}$ of cardinality ν into $\mathbb{R}$ and we shall prove that it is a witness that statement $C_8(\nu, \aleph_1)$ is true.

Aiming at a contradiction, assume that $f|A$ is uniformly continuous on an uncountable set $A \subseteq E$, let $B = f(A) = \phi^{-1}(A)$ and extend the function $f|A$ to a homeomorphism $\tilde{f} : \bar{A} \to \bar{B}$, where $\bar{A}$ and $\bar{B}$ are the closures of A and B in C and $C \times C$, respectively (cf. the proof of Theorem 3.8).

Let us notice that E is concentrated on D in C and $\bar{A} \cap D \neq \emptyset$. It is easy to see that this implies that A is concentrated on $\bar{A} \cap D$ in $\bar{A}$, hence also B is concentrated on $L = \tilde{f}(\bar{A} \cap D)$ in $\bar{B}$. Clearly, L is a countable subset of $\bar{B} \subseteq C \times C$ and B is concentrated on L also in $C \times C$. Therefore, $B' = \mathrm{proj}_2(B)$ is concentrated on $L' = \mathrm{proj}_2(L)$ in C (where proj_2 is the projection of $C \times C$ onto the second axis). Since H is the graph of an injection, B' is uncountable, and it follows that L' is a countable set in C which is not a G_δ-set in $B' \cup L'$. This, however, contradicts the fact that T is a λ'-set in C and $B' \subseteq T$.

The assertion, stating the equivalence of statements $C_8(\nu, \aleph_1)$ and $C_9(\nu, \aleph_1)$ assuming the existence of a λ'-set of cardinality ν in the Cantor set follows now from Theorem 2.3 and Proposition 3.1. Indeed, $C_9(\nu, \aleph_1)$ implies that there is also a K-Lusin set in $P = C \setminus Q$ of cardinality ν, where Q is a countable dense set in C (cf. Theorem 2.3), which by what we have already proved, yields $C_8(\nu, \aleph_1)$. The converse implication is always true (see Proposition 3.1).

The final assertions follow now from Corollary 2.4(1) and the existence (in ZFC) of a λ'-set of cardinality $\aleph_1$ in the Cantor set. □

While the status of the implication $C_9 \Rightarrow C_8$, the central topic of this note, remains unclear, the following conditions, sufficient for the validity of $C_9 \Rightarrow C_8$, hint at difficulties in finding a model of ZFC where, possibly, C_9 is true but C_8 is false.

Proposition 3.11. *Any of the following three conditions implies that C_9 implies C_8 (and then C_8 and C_9 are equivalent):*

(1) no K-Lusin set in $\mathbb{N}^{\mathbb{N}}$ has cardinality $\mathfrak{c}$ (in particular, this is so if $\mathfrak{b} > \aleph_1$ or $\mathfrak{d} < \mathfrak{c}$),
(2) there exists a Lusin set in $\mathbb{R}$ of cardinality $\mathfrak{c}$,
(3) there exists a λ'-set in the Cantor set of cardinality $\mathfrak{c}$.

Proof. The non-existence of K-Lusin sets in $\mathbb{N}^{\mathbb{N}}$ of cardinality $\mathfrak{c}$ makes C_9 false by Theorem 2.3.

Similarly, the existence of a Lusin set in $\mathbb{R}$ of cardinality $\mathfrak{c}$ makes C_8 true (see [18, proof of Théorème 6 on page 45]).

The existence of a λ'-set of cardinality $\mathfrak{c}$ in the Cantor set C implies that $C_8(\mathfrak{c}, \aleph_1) \Leftrightarrow C_9(\mathfrak{c}, \aleph_1)$, by Theorem 3.10.

□

4. Comments

In this section we present some additional results related to the subject of this note; their proofs will be given in a forthcoming paper.

4.1. **Mappings into compact spaces.** One can show that statement $C_8(\lambda, \kappa)$ is equivalent to its analogue concerning functions with ranges in compact metric spaces.

Proposition 4.1. *For any uncountable cardinals $\kappa \leq \lambda \leq \mathfrak{c}$ if the cofinality of λ is uncountable, then the following are equivalent:*

(1) $C_8(\lambda, \kappa)$,
(2) there is a set $E \subseteq [0, 1]$ of cardinality λ, such that for every uncountable compact metric space Y there is a continuous function $f : E \to Y$, which is not uniformly continuous on any subset of E of cardinality κ.

Moreover, if a set E satisfies property (2) above for a continuous function $f : E \to Y$ into an uncountable compact metric space Y, then so does it for a certain real-valued continuous function on E.

However, the latter is no longer true when we replace $E \subseteq [0, 1]$ by an arbitrary separable metrizable space, as shown by the following result.

Proposition 4.2. *Assuming that no family of less than* $\mathfrak{c}$ *meager sets covers* $\mathbb{R}$*, there exists a set of positive dimension* $E \subseteq [0,1]^{\mathbb{N}}$ *such that*

(1) there is a continuous function $f : E \to [0,1]^{\mathbb{N}}$*, which is not uniformly continuous on any subset of* E *of cardinality* $\mathfrak{c}$*,*
(2) each continuous function $g : E \to \mathbb{R}$ *is constant on a subset of* E *of cardinality* $\mathfrak{c}$*.*

4.2. **Mappings into non-compact spaces.** One can show that also statement $C_9(\lambda, \kappa)$ is equivalent to an analogue of statement $C_8(\lambda, \kappa)$ but concerning functions with ranges in non-σ-compact Polish spaces.

Proposition 4.3. *For any uncountable cardinals* $\kappa \le \lambda \le \mathfrak{c}$ *the following are equivalent:*

(1) $C_9(\lambda, \kappa)$*,*
(2) there is a set $E \subseteq [0,1]$ *of cardinality* λ *such that for every non-*σ*-compact Polish space* Y *there is a continuous function on* E *which is not uniformly continuous (with respect to any complete metric on* Y*) on any subset of* E *of cardinality* κ*.*

Moreover, any κ-K-Lusin set E of cardinality λ in $\mathbb{P}$ has property (2) above. In particular, with the help of Theorem 3.3 and Proposition 4.1, it follows that, under CH, there exists a K-Lusin set E in $\mathbb{P}$ of cardinality $\mathfrak{c}$ such that E admits a continuous function $f : E \to \mathbb{R}^{\mathbb{N}}$ which is not uniformly continuous on any uncountable subset of E, but each continuous map $g : E \to [0,1]^{\mathbb{N}}$ is uniformly continuous on an uncountable subset of E.

4.3. **A characterization of completeness.** One can show that the existence of a function sequence described in Theorem 2.1 characterizes completeness of a separable metrizable space X.

In fact, the following more general result can be obtained (for terminology see [4]).

Proposition 4.4. *Let* X *be a Hausdorff space. Then* X *is a Čech-complete Lindelöf space if and only if there is a sequence* $f_1 \ge f_2 \ge \dots$ *of continuous functions* $f_n : X \to [0,1]$ *converging pointwise to zero but not converging uniformly on any closed non-compact set in* X*.*

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