Highly nonlinear large-competition limits of elliptic systems

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Joint work with Norman Dancer, Sydney.
Parabolic systems of form

\[ u_t = d_1 \Delta u + f(u) - kuv, \quad x \in \Omega, \quad t \geq 0, \]
\[ v_t = d_2 \Delta v + g(v) - kuv, \quad x \in \Omega, \quad t \geq 0, \]
\[ u(x) = v(x) = 0, \quad x \in \partial \Omega \]

model populations of densities \( u, v \) that compete in \( \Omega \in \mathbb{R}^N \)

\( u = v = 0 \) on \( \partial \Omega \)

form of self-interaction functions \( f, g \)
e.g. \( f(u) = u(1 - u) \)

\[ M = 1 \]
Elliptic systems of form

\[
0 = \Delta u + f(u) - kuv, \quad x \in \Omega,
\]
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0 = \Delta v + g(v) - kuv, \quad x \in \Omega,
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u(x) = v(x) = 0, \quad x \in \partial \Omega,
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model steady states of populations \(u, v\) that compete in \(\Omega \in \mathbb{R}^N\)

\(u = v = 0\) on \(\partial \Omega\)

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e.g. \(f(u) = u(1 - u)\)

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model steady states of populations \(u, v\) that compete in \(\Omega \in \mathbb{R}^N\)

\(u, v\) compete inside \(\Omega \in \mathbb{R}^N\)

\(u = v = 0\) on \(\partial \Omega\)

densities non-negative \(\Rightarrow u \geq 0, v \geq 0\)
• Elliptic systems of form

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\begin{align*}
0 &= \Delta u + f(u) - kuv, \quad x \in \Omega, \\
0 &= \Delta v + g(v) - kuv, \quad x \in \Omega, \\
u(x) &= v(x) = 0, \quad x \in \partial \Omega,
\end{align*}
\]

model steady states of populations \(u, v\) that compete in \(\Omega \in \mathbb{R}^N\)

\(\Omega \in \mathbb{R}^N\)

\(u, v\) compete inside

\(u = v = 0\) on \(\partial \Omega\)

• densities non-negative \(\Rightarrow u \geq 0, v \geq 0\)

• competition parameter \(k > 0\)
Interest in the large-competition \((k \to \infty)\) limit comes from:

(i) the \(k\)-dependent system is difficult to analyse; for example, it is not in general the Euler-Lagrange equations of an energy functional variational, whereas the limit problem is a scalar equation.

(ii) the \(k \to \infty\) limit is linked to:

- spatial segregation in population dynamics
- phase separation in, for example, Bose-Einstein condensates

both of which are of importance in applications.
Large-competition limit $k \to \infty$ of solutions $(u^k, v^k)$


- $(u^k, v^k)$ converge to the positive and negative parts resp. of a limit function $w$ satisfying the scalar equation

$$\Delta w + f(w^+) - g(-w^-) = 0, \quad x \in \Omega,$$

$$w(x) = 0, \quad x \in \partial\Omega$$
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w(x) = 0, \quad x \in \partial \Omega
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• Key ingredients:

(i) the linear combination $w^k := u^k - v^k$ satisfies

$$
\Delta w^k + f(u^k) - g(v^k) = 0, \quad x \in \Omega
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which does not depend explicitly on $k \Rightarrow$ good bounds for $w^k$ independent of $k$
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which does not depend explicitly on $k$ $\Rightarrow$ good bounds for $w_k$ independent of $k$

(ii) $u_k^k, v_k^k$ converge in some sense as $k \to \infty$

(iii) $u_k^k$ and $v_k^k$ segregate, since $k u_k^k v_k^k$ bounded $\Rightarrow$ $u_k^k v_k^k \to 0$ as $k \to \infty$

\begin{align*}
&uv = 0 \quad a.e. \\
&\text{and} \\
&u, v \geq 0 \\
&w = u - v
\end{align*}

$\Rightarrow$

$$u = w^+ \quad a.e.$$ $$v = -w^-$$
• Note: there are two aspects to large-interaction limit problem

(i) to show that \((u^k, v^k)\) converges as \(k \to \infty\) to a solution of the limit problem

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\Delta w + f(w^+) - g(-w^-) = 0, \quad x \in \Omega,
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(ii) conversely, to show that given a solution \(w\) of the limit problem, there exists a sequence of solutions of the \(k\)-dependent system \((u^k, v^k)\) that converge to \(w\) as \(k \to \infty\)
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methods: e.g. degree theory in cones

not continuously differentiable

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Focus on (i) here

methods: e.g. degree theory in cones

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limit nonlinearity
• **Key property** that allows cancellation of competition terms "\( kuv \)" is that the same term occurs in both equations

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0 = \Delta u + f(u) - kuv, \\
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so **cancel** in equation for \( w^k := u^k - v^k \)
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• Similarly, the competition terms in the more general system cancel

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in the equation for \(\hat{w}^k = \alpha u^k - v^k\)

• Question: to what types of system with different competition terms in the two equations can this \("cancellation\) approach be extended?
Our two prototype classes of system

1. Non-autonomous system

\[ \Delta u + f(u) - \alpha_1(x)kuv = 0, \quad x \in \Omega, \]
\[ \Delta v + g(v) - \alpha_2(x)kuv = 0, \quad x \in \Omega, \]
\[ u(x) = v(x) = 0, \quad x \in \partial \Omega \]

where \( \alpha_1, \alpha_2 \in C^2(\Omega, [\alpha_0, \infty)) \) for some constant \( \alpha_0 > 0 \)

2. “Nonlinear” competition system

\[ \Delta u + f(u) - kuv = 0, \quad x \in \Omega, \]
\[ \Delta v + g(v) - k(1 + u^2)uv = 0, \quad x \in \Omega, \]
\[ u(x) = v(x) = 0, \quad x \in \partial \Omega \]
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Key feature: each competition term is of form “\( kuv \)” multiplied by a positive function that is bounded below by a strictly positive constant
systems (1.) and (2.) are special cases of the general system

\[ \Delta u + f(u) - k\alpha_1(x)\gamma_1(v)uv = 0, \quad x \in \Omega, \]
\[ \Delta v + g(v) - k\alpha_2(x)\gamma_2(u)uv = 0, \quad x \in \Omega \]

where \( \gamma_1, \gamma_2 \geq \gamma_0 \) and \( \alpha_1, \alpha_2 \geq \alpha_0 \) for some constants \( \alpha_0, \gamma_0 > 0 \).
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the system

\[ \Delta u + f(u) - kuv^2 = 0, \quad x \in \Omega, \]
\[ \Delta v + g(v) - ku^2v = 0, \quad x \in \Omega \]

is unfortunately excluded from our framework; it

- arises in modelling phase separation in Bose-Einstein condensates
- is variational, being the Euler-Lagrange equations of a functional of form

\[ J(u, v) = \int_{\Omega} \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2) - F(u) - G(v) + \frac{1}{2}ku^2v^2 \, dx \]

(references: Conti, Terracini, Verzini, Squassina, ...)
Preliminary “cancellation” calculations

System 1.

Given a solution \((u^k, v^k)\) of system 1,

\[
\Delta u + f(u) - \alpha_1(x)kuv = 0, \quad x \in \Omega,
\]

\[
\Delta v + g(v) - \alpha_2(x)kuv = 0, \quad x \in \Omega,
\]

\[
u(x) = v(x) = 0, \quad x \in \partial\Omega,
\]

define

\[
w^k = \alpha_2 u^k - \alpha_1 v^k.
\]

Then \(w^k\) satisfies the equation

\[
\Delta w^k = 2\nabla \alpha_2 \cdot \nabla u^k - 2\nabla \alpha_1 \cdot \nabla v^k
\]

\[
+ u^k \Delta \alpha_2 - v^k \Delta \alpha_1 - \alpha_2 f(u^k) + \alpha_1 g(v^k) \quad \text{in} \quad \Omega,
\]

\[
w^k = 0 \quad \text{on} \quad \partial\Omega,
\]

because

\[
\Delta w^k = \alpha_2 \Delta u^k - \alpha_1 \Delta v^k + 2\nabla \alpha_2 \cdot \nabla u^k - 2\nabla \alpha_1 \cdot \nabla v^k + u^k \Delta \alpha_2 - v^k \Delta \alpha_1
\]
System 2.

Given a solution \((u^k, v^k)\) of system 2,

\[
\Delta u + f(u) - ku^k v = 0, \quad x \in \Omega, \\
\Delta v + g(v) - k(1 + u^2)v^k = 0, \quad x \in \Omega, \\
u(x) = v(x) = 0, \quad x \in \partial \Omega,
\]

define

\[
y^k = u^k + \frac{(u^k)^3}{3} - v^k.
\]

Then \(y^k\) satisfies the equation

\[
\Delta y^k = 2u^k |\nabla u^k|^2 - (1 + (u^k)^2)f(u^k) + g(v^k) \quad \text{in} \quad \Omega, \\
y^k = 0 \quad \text{on} \quad \partial \Omega.
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\[
y^k = 0 \quad \text{on } \partial\Omega.
\]

Note:

(i) no terms involving second derivatives of \(u^k\) or \(v^k\) in eqn for \(y^k\)

(ii) \(u \mapsto u + \frac{u^3}{3}\) is invertible, since \(\frac{d}{du} \left(u + \frac{u^3}{3}\right) = 1 + u^2 \geq 1\) for all \(u\)
Why? - form of system is

$$\Delta u + f(u) - kuv = 0, \ x \in \Omega,$$
$$\Delta v + g(v) - k \gamma(u) uv = 0, \ x \in \Omega$$

where

$$\gamma(u) = 1 + u^2 \geq \gamma_0 > 0$$
Why? - form of system is

\[ \Delta u + f(u) - kuv = 0, \quad x \in \Omega, \]
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where

\[ \gamma(u) = 1 + u^2 \geq \gamma_0 > 0 \]

Define

\[ y^k := \Gamma(u^k) - v^k, \quad \text{where } \Gamma(u) := \int_0^u \gamma(s) \, ds \]
Why? - form of system is

\[ \Delta u + f(u) - kuv = 0, \quad x \in \Omega, \]
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• Define

\[ y^k := \Gamma(u^k) - v^k, \quad \text{where} \quad \Gamma(u) := \int_0^u \gamma(s) \, ds \]

Then

\[ \Gamma'(u) = \gamma(u) \geq \gamma_0 > 0 \quad \text{for all} \quad u \quad \Rightarrow \quad \Gamma \text{ is invertible} \]
Why? - form of system is
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\Delta u + f(u) - kuv = 0, \quad x \in \Omega,
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where
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- Define
\[
y^k := \Gamma(u^k) - v^k, \quad \text{where } \Gamma(u) := \int_0^u \gamma(s) \, ds
\]
Then
\[
\Gamma'(u) = \gamma(u) \geq \gamma_0 > 0 \quad \text{for all } u \quad \Rightarrow \quad \Gamma \text{ is invertible}
\]
and
\[
\nabla y^k = \gamma(u^k) \nabla u^k - \nabla v^k
\]
\[
\Rightarrow \quad \Delta y^k = \gamma'(u^k) \|\nabla u^k\|^2 + \gamma(u^k) \Delta u^k - \Delta v^k
\]
\[
= \gamma'(u^k) \|\nabla u^k\|^2 - \gamma(u^k) f(u^k) + g(v^k)
\]
• Also,

\[ \Gamma(u) = \int_0^u \gamma(s) \, ds > 0 \quad \text{if} \quad u > 0 \]

and

\[ \Gamma(0) = 0 \]

• thus

\[ uv = 0 \quad a.e. \quad \quad \begin{cases} u, v \geq 0 \\ y = \Gamma(u) - v \end{cases} \Rightarrow \begin{cases} y^+ = \Gamma(u) \\ y^- = -v \end{cases} \Rightarrow \Gamma^{-1}(y^+) = u \quad y^- = v \]

\textit{i.e. segregation} of \( u \) and \( v \) implies that \( u \) and \( v \) can be written in terms of the positive and negative parts of \( y \).
Theorem Given a sequence of non-negative solutions \((u^k, v^k)\) of either system 1 or 2, there exist subsequences \(\{u^{k_n}\}, \{v^{k_n}\}\) and non-negative functions \(u, v \in L^\infty(\Omega) \cap W^{1,2}_0(\Omega)\) such that

- \(u^{k_n} \to u, \ v^{k_n} \to v\) in \(W^{1,2}_0(\Omega)\) as \(k_n \to \infty\);
- \(uv = 0\) a.e. in \(\Omega\).

- In the case of system 1, the function \(w := \alpha_2 u - \alpha_1 v\) is such that \(w^+ = \alpha_2 u, \ w^- = -\alpha_1 v\) and \(w\) is a weak solution of the equation

\[
\Delta w = 2\nabla \alpha_2 \cdot \nabla (\alpha_2^{-1}w^+) - 2\nabla \alpha_1 \cdot \nabla (-\alpha_1^{-1}w^-) \\
+ \alpha_2^{-1}w^+\Delta \alpha_2 - \alpha_2 f(\alpha_2^{-1}w^+) + \alpha_1^{-1}w^-\Delta \alpha_1 + \alpha_1 g(-\alpha_1 w^-) \quad \text{in} \quad \Omega,
\]

\(w = 0\) on \(\partial \Omega\).

- In the case of system 2, the function \(y := \Gamma(u) - v\), where \(\Gamma(u) := u + \frac{u^3}{3}\), is such that \(y^+ = \Gamma(u), \ y^- = -v\) and \(y\) is a weak solution of the equation

\[
\Delta y = \frac{2\Gamma^{-1}(y^+)}{(1 + \Gamma^{-1}(y^+)^2)^2}|\nabla y^+|^2 + (1 + \Gamma^{-1}(y^+)^2) f(\Gamma^{-1}(y^+)) + g(-y^-) \quad \text{in} \quad \Omega,
\]

\(y = 0\) on \(\partial \Omega\).
Basic estimates on solutions \((u^k, v^k)\) of system 1 or 2

(i) \(L^\infty\)-bound

\[
0 \leq u^k, v^k \leq M \quad \text{for all } x \in \Omega, \ k > 0
\]

by maximum principle, since \(f(u), g(v) < 0\) when \(u, v > M\) and so if, say, \(u^k\) attains a maximum value \(u^k(x_0) > M\), then

\[
-\Delta u^k(x_0) \leq f(u^k(x_0)) < 0,
\]

which is impossible

(ii) \(L^2\)-gradient bound there exists \(K_1 > 0\) such that

\[
\int_{\Omega} |\nabla u^k(x)|^2 \, dx, \int_{\Omega} |\nabla v^k(x)|^2 \, dx \leq K_1 \quad \text{for all } k > 0
\]

since, e.g., multiplication of \(u^k\) equation by \(u^k\) and integration over \(\Omega\) gives

\[
-\int_{\Omega} |\nabla u^k|^2 \, dx + \int_{\Omega} u^k f(u^k) \, dx \geq 0
\]
(iii) normal derivative bound there exists $K_2 > 0$ such that

$$\left| \frac{\partial u^k}{\partial \nu} \right|(x), \left| \frac{\partial v^k}{\partial \nu} \right|(x) \leq K_2 \text{ for all } x \in \partial \Omega, k > 0$$

since

$$-\Delta u^k \leq f(u^k), \; \; x \in \Omega, \; \; u^k = 0 \; \text{ on } \partial \Omega,$$

and so $0 \leq u^k \leq \bar{u}$, where $\bar{u}$ is the maximal solution in $[0, M]$ of

$$-\Delta u = f(u), \; \; x \in \Omega, \; \; u = 0 \; \text{ on } \partial \Omega$$

which, as $u^k = \bar{u} = 0$ on $\partial \Omega$, then implies

$$\left| \frac{\partial u^k}{\partial \nu} \right|(x) \leq \left| \frac{\partial \bar{u}}{\partial \nu} \right|(x) \text{ for all } x \in \partial \Omega$$
(iii) normal derivative bound there exists $K_2 > 0$ such that
\[
\left| \frac{\partial u^k}{\partial \nu} \right|(x), \left| \frac{\partial v^k}{\partial \nu} \right|(x) \leq K_2 \text{ for all } x \in \partial \Omega, k > 0
\]
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\[-\Delta u^k \leq f(u^k), \quad x \in \Omega, \quad u^k = 0 \text{ on } \partial \Omega,
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\left| \frac{\partial u^k}{\partial \nu} \right|(x) \leq \left| \frac{\partial \bar{u}}{\partial \nu} \right|(x) \text{ for all } x \in \partial \Omega
\]

(i), (ii) and (iii) use the sign of the competition term
(iv) \textbf{basic segregation bound} there exists $K_3 > 0$ such that

$$
\int_{\Omega} ku^k v^k \, dx \leq K_3
$$

since

$$
0 \leq \min\{1, \alpha_0\} \int_{\Omega} ku^k v^k \, dx \leq \int_{\Omega} \Delta u^k + f(u^k) \, dx
$$

$$
= \int_{\partial \Omega} \frac{\partial u^k}{\partial \nu} \, dx + \int_{\Omega} f(u^k) \, dx
$$

$$
\leq C
$$

\textbf{Note:} (iv) uses key feature that $\alpha, etc$ are bounded below by a \textbf{positive const}
Key lemma \( \nabla u^{k_n} \rightarrow \nabla u, \; \nabla v^{k_n} \rightarrow \nabla v \) in \( L^2(\Omega) \) as \( k_n \rightarrow \infty \).

Idea of proof for system 1

- have to prove that

\[
\limsup_{k_n \rightarrow \infty} \int_{\Omega} |\nabla u^{k_n}|^2 \, dx \leq \int_{\Omega} |\nabla u|^2 \, dx
\]

- multiplication of \( v^{k_n} \) equation by limit \( u \) and integration over \( \Omega \) yields

\[
- \int_{\Omega} \nabla u \cdot \nabla v^{k_n} \, dx + \int_{\Omega} u g(v^{k_n}) \, dx - k_n \int_{\Omega} u u^{k_n} v^{k_n} \alpha_2 \, dx = 0
\]

- then as \( k_n \rightarrow \infty \),

\[
\int_{\Omega} \nabla u \cdot \nabla v^{k_n} \, dx \rightarrow \int_{\Omega} \nabla u \cdot \nabla v \, dx = 0, \quad \int_{\Omega} u g(v^{k_n}) \, dx \rightarrow \int_{\Omega} u g(v) \, dx = 0
\]

so that

\[
k_n \int_{\Omega} u u^{k_n} v^{k_n} \alpha_2 \, dx \rightarrow 0 \quad \text{as} \quad k_n \rightarrow \infty
\]

\[
\Rightarrow
\]

\[
k_n \int_{\Omega} u u^{k_n} v^{k_n} \alpha_1 \, dx \rightarrow 0 \quad \text{as} \quad k_n \rightarrow \infty
\]
Idea of proof contd....

- now by multiplication of $u^{kn}$ equation by the limit $u$ and integration over $\Omega$,\
  \[- \int_{\Omega} \nabla u^{kn} \cdot \nabla u \, dx + \int_{\Omega} uf(u^{kn}) \, dx - k_n \int_{\Omega} uu^{kn}v^{kn} \alpha_1 \, dx = 0,\]
  and then letting $k_n \to \infty$ gives\
  \[\int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} uf(u) \, dx\]

- multiplication of $u^{kn}$ equation by $u^{kn}$ and integration over $\Omega$ gives\
  \[- \int_{\Omega} |\nabla u^{kn}|^2 \, dx + \int_{\Omega} u^{kn}f(u^{kn}) \, dx - k_n \int_{\Omega} (u^{kn})^2 v^{kn} \alpha_1 \, dx = 0,\]
  which, since $\alpha_1$ and $v^{kn}$ are non-negative, implies that\
  \[\int_{\Omega} |\nabla u^{kn}|^2 \, dx \leq \int_{\Omega} u^{kn}f(u^{kn}) \, dx\]
  \[\to \int_{\Omega} uf(u) \, dx\]
  \[= \int_{\Omega} |\nabla u|^2 \, dx\]
Remark: improved segregation

**Lemma** Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ and $(u^k, v^k)$ is a non-negative solution of

\[
\begin{align*}
\Delta u + f(u) - k\alpha_1(x)\gamma_1(v)uv &= 0, \quad x \in \Omega, \\
\Delta v + g(v) - k\alpha_2(x)\gamma_2(u)uv &= 0, \quad x \in \Omega,
\end{align*}
\]

\[
u = v = 0, \quad x \in \partial \Omega,
\]

then given $x \in \Omega$,

\[
u^k(x) \leq \varepsilon_0 \quad \text{or} \quad v^k(x) \leq \varepsilon_0
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---

**Idea of proof**: Suppose not. Then there exist $\varepsilon_0 > 0$ and sequences $k_j \to \infty$ and $x_j \in \Omega$ such that

\[u^{k_j}(x_j) \geq \varepsilon_0 \quad \text{and} \quad v^{k_j}(x_j) \geq \varepsilon_0.\]
Rescale

\[(U^{kj}, V^{kj})(\sqrt{k_j}(x - x_j)) = (u^{kj}, v^{kj})(x), \; x \in \Omega\]

satisfies

\[\Delta U^{kj} + k_j^{-1}f(U^{kj}) - \alpha_1(x_j + \frac{x'}{\sqrt{k_j}})\gamma_1(V^{kj})U^{kj}V^{kj} = 0 \; \text{in} \; \Omega_j,\]

\[\Delta V^{kj} + k_j^{-1}g(V^{kj}) - \alpha_2(x_j + \frac{x'}{\sqrt{k_j}})\gamma_2(U^{kj})U^{kj}V^{kj} = 0 \; \text{in} \; \Omega_j,\]

\[U^{kj} = V^{kj} = 0 \; \text{on} \; \partial \Omega_j\]

and

\[0 \leq U^{kj}, V^{kj} \leq M, \; 0 \in \Omega_j, \; U^{kj}(0) \geq \varepsilon_0 \; \text{and} \; V^{kj}(0) \geq \varepsilon_0\]
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\]

\[
\Delta V^{k_j} + k_j^{-1} g(V^{k_j}) - \alpha_2(x_j + \frac{x'}{\sqrt{k_j}})\gamma_2(U^{k_j})U^{k_j}V^{k_j} = 0 \quad \text{in } \Omega_j,
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\]

\[k_j \to \infty \text{ limit system}\]

\[
\Delta U = \alpha_1(\bar{x})\gamma_1(V)UV, \quad \text{in } \mathbb{R}^N
\]

\[
\Delta V = \alpha_2(\bar{x})\gamma_2(U)UV,
\]

and

\[
0 \leq U, V \leq M, \quad U(0) \geq \varepsilon_0 \quad \text{and} \quad V(0) \geq \varepsilon_0
\]
Then on the one hand...

\[ \Delta U \geq 0 \quad \text{and} \quad U \text{ is bounded on } \mathbb{R}^N; \]
\[ \Delta V \geq 0 \quad \text{and} \quad V \text{ is bounded on } \mathbb{R}^N, \]

\[ \Rightarrow \exists \text{ direction } \left\{ \lambda \xi : \xi \in S^{n-1}, \ \lambda \geq 0 \right\} \text{ along which} \]
\[ U(x) \to \sup U \quad \text{and} \quad V(x) \to \sup V \text{ as } |x| \to \infty, \]

by properties of subharmonic functions

\[ \therefore \text{ limit } (\tilde{U}, \tilde{V}) \text{ of translates } U(\cdot + x_n), V(\cdot + x_n) \text{ along this direction satisfies} \]
\[ \tilde{U}(0) = \sup \tilde{U}, \quad \Delta \tilde{U}(0) \leq 0 \quad \text{and} \quad \tilde{V}(0) = \sup \tilde{V}, \quad \Delta \tilde{V}(0) \leq 0 \]
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But on the other hand....

\[ \Delta \tilde{U}(0) = \alpha_1(\bar{x}) \gamma_1(\tilde{V}(0)) \tilde{U}(0)\tilde{V}(0) > 0 \]

\[ \therefore \text{ contradiction} \]

here also use feature that \( \alpha, etc \) are bounded below by a positive const
Remark: regularity for limit equation of System 2

- $y \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$ is a weak solution of limit equation

$$\Delta y = \frac{2\Gamma^{-1}(y^+)}{(1 + \Gamma^{-1}(y^+)^2)^2} |\nabla y^+|^2 + (1 + \Gamma^{-1}(y^+)^2) f(\Gamma^{-1}(y^+)) + g(-y^-) \text{ in } \Omega,$$

$$y = 0 \text{ on } \partial \Omega$$
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\]
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\]

- since \( \Gamma^{-1}(0) = 0 \), this can be re-written as

\[
\Delta y = h(y) |\nabla y|^2 + d(y) \quad \text{in} \quad \Omega,
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- change of variables: let \( r : \mathbb{R} \to \mathbb{R} \) be such that

\[
r'(t) = e^{H(r(t))}, \quad t \in \mathbb{R},
\]

\( r(0) = 0, \)

where \( H' = h \), and note that

\[
r''(t) - h(r(t))r'(t)^2 = 0, \quad t \in \mathbb{R}.
\]
• define \( s : \Omega \rightarrow \mathbb{R} \) by

\[
s(x) = r^{-1}(y(x)), \quad x \in \Omega,
\]

where \( y \in W^{1,2}_0(\Omega) \) is a solution of the limit equation

• then \( s \) satisfies

\[
\Delta s = \frac{d(s(x))}{r'(s(x))}, \quad x \in \Omega
\]

\[
s = 0 \quad \text{on} \quad \partial \Omega
\]
• define $s : \Omega \to \mathbb{R}$ by

$$s(x) = r^{-1}(y(x)), \quad x \in \Omega,$$

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$$\Delta s = \frac{d(s(x))}{r'(s(x))} \in L^\infty(\Omega)$$

$$s = 0 \text{ on } \partial\Omega$$

since

$$r'(s(x)) = e^{H(r(s(x)))} = e^{H(y(x))} \geq r_0 > 0$$

because $y \in L^\infty(\Omega)$
• define $s : \Omega \to \mathbb{R}$ by

$$s(x) = r^{-1}(y(x)), \quad x \in \Omega,$$

where $y \in W_0^{1,2}(\Omega)$ is a solution of the limit equation

• then $s$ satisfies

$$\Delta s = \frac{d(s(x))}{r'(s(x))} \in L^\infty(\Omega)$$

$$s = 0 \quad \text{on } \partial \Omega$$

since

$$r'(s(x)) = e^{H(r(s(x)))} = e^{H(y(x))} \geq r_0 > 0$$

because $y \in L^\infty(\Omega)$

• hence

$$\Delta s \in L^\infty(\Omega), \quad s = 0 \quad \text{on } \partial \Omega \quad \Rightarrow \quad s \in W^{2,p}(\Omega) \quad \text{for all } \quad p \in [1, \infty)$$

$$\Rightarrow \quad s \in C^{1,\mu}(\Omega) \quad \text{for all } \quad \mu \in (0, 1)$$

$$\Rightarrow \quad y \in C^{1,\mu}(\Omega) \quad \text{for all } \quad \mu \in (0, 1)$$
Main open question...

- to better understand solutions of the scalar limit problems, especially sign-changing solutions of the limit problems
- in particular, to understand which sign-changing solutions arises as the limit as $k \to \infty$ of co-existence states of the $k$-dependent system
Main open question...

- to better understand solutions of the scalar limit problems, especially sign-changing solutions of the limit problems
  - in particular, to understand which sign-changing solutions arises as the limit as $k \to \infty$ of co-existence states of the $k$-dependent system

Thank you for your attention ...