Transport equation: extension of classical results for div $b \in$ BMO

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Abstract. We investigate the transport equation: $u_t + b \cdot \nabla u = 0$. Our result improves the criteria on uniqueness of weak solutions, replacing the classical condition: $\operatorname{div} b \in L_\infty$ by $\operatorname{div} b \in BMO$.

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1 Introduction

The goal of this note is to improve classical results concerning the Cauchy problem for the transport equation. The basis of our analysis is the following system

$$\begin{array}{lll} u_t + b \cdot \nabla u = 0 & \text{in} & \mathbb{R}^n \times (0,T), \\ u|_{t=0} = u_0 & \text{on} & \mathbb{R}^n, \end{array} \tag{1.1}$$

where u is an unknown scalar function, b – some given vector field and u_0 is an initial datum.

The transport equation is one of the most fundamental examples in the theory of partial differential equations. It describes the motion of matter under influence of the velocity field b. Classically, for smooth data b and u_0, (1.1) is solvable elementary by the method of characteristics. In the language of the fluid mechanics, (1.1) says that u is constant along streamlines defined by the Lagrangian coordinates. This physical interpretation gives enough reasons for (1.1) to be intensively studied from the mathematical point of view. Here we want to concentrate on the optimal/critical regularity of the vector field b to control the existence, stability and uniqueness of weak solutions. The last point seems to be the most interesting.

In order to control the uniqueness of weak solutions to (1.1) the classical theory [11] requires that the vector field b must satisfy the following conditions

$$\operatorname{div} b \in L_1(0,T,L_\infty(\mathbb{R}^n)). \tag{1.2}$$

Then thanks to the renormalized meaning of solutions for (1.1), the energy method and Gronwall lemma yield immediately the uniqueness. The main goal of analysis of the present paper is to relax condition (1.2) by

$$\operatorname{div} b \in L_1(0,T;BMO(\mathbb{R}^n)). \tag{1.3}$$

Let us observe that this "slightly" broader class than (1.2) is on the boundary of known counterexamples [11]. For any $p < \infty$ we are able to construct such $b \in W^1_p(\mathbb{R}^n)$ (time independent) to obtain an example of the loss of uniqueness to (1.1). On the other hand the BMO-space appears naturally in many considerations, since it is the limit space for the embedding $W^1_n(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$, where the L_∞-space is not reached. We are able to prove existence and uniqueness of weak solutions to (1.1) in the case of bounded solutions and improve the uniqueness criteria for L_p-solutions. Additionally we show a result concerning stability with respect to initial data. Our approach follows from techniques introduced in [16] to improve the uniqueness criteria for the Euler system in bounded domains. The main tool of our method is a logarithmic type inequality between the Hardy space $\mathcal{H}^1$ and Lebesgue space L_1, stated in Theorem D below.

Fundamental results of our issue have been proved by R.J. DiPerna and P.L. Lions in [11], where general questions concerning the well posedness of the problem found positive answer under condition (1.2). An interesting extension of the theory has been developed by L. Ambrosio [1], for the case of bounded solutions replacing the condition $b \in W^1_1(\mathbb{R}^n)$ by $b \in BV(\mathbb{R}^n)$. In the literature one can find also numerous works on generalizations of the mentioned results on broader classes of function spaces [2],[4],[6],[7],[8],[12],[13],[14], but positive answers still require condition (1.2). A step to relax the condition (1.2) has been done recently in [3] with a vector field b satisfying a Osgood type condition from ODEs.

In the present note we consider weak solutions meant in the following sense:

We say that $u \in L_\infty(0,T;L_p(\mathbb{R}^n))$ is a weak solution to (1.1) iff the following integral identity holds

$$\int_0^T \int_{\mathbb{R}^n} u\phi_t dxdt + \int_0^T \int_{\mathbb{R}^n} \text{div } b\, u\phi dx + \int_0^T \int_{\mathbb{R}^n} b \cdot \nabla\phi u dxdt = -\int_{\mathbb{R}^n} u_0 \phi(\cdot,0)dx \tag{1.4}$$

for each $\phi \in C^\infty([0,T];C_0^\infty(\mathbb{R}^n))$ such that $\phi|_{t=T} \equiv 0$.

Let us state the main results of this paper. First we start with the case of pointwise bounded solutions, in that case our technique delivers the most complete result.

Theorem A. *Let $T > 0$, $b \in L_1(0,T;W^1_{1(loc)}(\mathbb{R}^n))$, $u_0 \in L_\infty(\mathbb{R}^n)$, additionally we assume*

$$\text{div } b \in L_1(0,T;BMO(\mathbb{R}^n)), \qquad \frac{b}{1+|x|} \in L_1(0,T;L_1(\mathbb{R}^n)) \tag{1.5}$$

$$\textit{and} \quad \textit{supp } \text{div } b(\cdot,t) \subset B(0,R) \ \ \textit{for a fixed } R > 0, \tag{1.6}$$

where $B(0,R)$ denotes the ball centered at the origin with radius R.

Then there exists a unique weak solution to the system (1.1) such that

$$u \in L_\infty(0,T;L_\infty(\mathbb{R}^n)). \tag{1.7}$$

The above result guarantees not only the uniqueness of solutions, but also their existence. It is a consequence of a maximum principle, which is valid for the L_∞-solutions.

The main difference to the classical results [11] is that having (1.2) we are able to construct the L_p-estimates of the solutions for finite p. In our case the condition (1.3) is too weak to obtain such information. Additionally we are required to add an extra condition (1.6), which is the price of our improvement of this classical criteria. The technique of the proof of Theorem A allows us to relax this strong restriction to the class of fields b prescribed by the following conditions

$$\begin{aligned} &\operatorname{div} b = H_\infty + \sum_{k=1}^{\infty} H_k \quad \text{such that} \\ &H_\infty \in L_1(0,T;L_\infty(\mathbb{R}^n)), \qquad H_k \in L_1(0,T;BMO(\mathbb{R}^n)) \quad \text{and} \\ &\sum_{k=1}^{\infty} \|H_k\|_{L_1(0,T;BMO(\mathbb{R}^n))} < \infty \qquad \text{with} \qquad \sup_{k\in\mathbb{N}} \textit{diam supp}\, H_k < \infty \end{aligned} \tag{1.8}$$

– see the Remark at the end of this section.

The next result concerns stability of solutions obtained in Theorem A with respect to perturbations of initial data in lower spaces.

Theorem B. *Let $1 \le p < \infty$ and b fulfill assumptions of Theorem A. Let $u_0, u_0^k \in L_\infty(\mathbb{R}^n)$ and $(u_0^k - u_0) \in L_p(\mathbb{R}^n)$ such that $\sup_{k\in\mathbb{N}} \|u_0^k\|_{L_\infty(\mathbb{R}^n)} + \|u_0\|_{L_\infty(\mathbb{R}^n)} = m < \infty$ and $(u_0^k - u_0) \to 0$ in $L_p(\mathbb{R}^n)$ as $k \to \infty$. Then*

$$(u^k - u) \to 0 \quad \text{in } L_\infty(0,T;L_p(\mathbb{R}^n)) \quad \text{as} \quad k \to \infty. \tag{1.9}$$

The last result concerns the uniqueness criteria for L_p-solutions to (1.1).

Theorem C. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $b \in L_1(0,T;W^1_{p'(loc)}(\mathbb{R}^n))$ and conditions (1.5), (1.6) be fulfilled. Let u^1, u^2 be two weak solutions to (1.1) with the same initial datum and $u^1, u^2 \in L_\infty(0,T;L_p(\mathbb{R}^n))$; then $u^1 \equiv u^2$.*

The above three theorems are proved by a reduction of considerations to an ordinary differential inequality of the form

$$\dot{x} = x \ln x, \qquad x|_{t=0} = 0. \tag{1.10}$$

The Osgood lemma yields the uniqueness to (1.10). This observation forms our chain of estimations in proofs of the theorems to have a possibility to adapt information obtained by the Gronwall inequality. Due to low regularity of solutions, our analysis requires a special approach. The main tool, which enables us to show the main inequality in the form of (1.10), is the following result.

Theorem D. *Let $f \in BMO(\mathbb{R}^n)$, the support of f be bounded in $\mathbb{R}^n$ and $g \in L_1(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, then*

$$\left|\int_{\mathbb{R}^n} fg dx\right| \le C_0 \|f\|_{BMO(\mathbb{R}^n)} \|g\|_{L_1(\mathbb{R}^n)} \left[|\ln \|g\|_{L_1(\mathbb{R}^n)}| + \ln(e + \|g\|_{L_\infty(\mathbb{R}^n)}) \right], \tag{1.11}$$

where C_0 depends on the diameter of support of f.

The above inequality can be viewed as a representative of the family of logarithmic Sobolev inequalities [5],[9],[10],[15], however there is one important difference between this one and others. Here an extra information about derivatives of the function is not required, in contrast to $L_\infty - BMO$ inequalities. The crucial assumption is the boundedness of the support of the function f, it is a consequence of results of the classical theory [19],[20]. Unfortunately, it is not expected that it could be possible to omit this restriction in Theorem D. Methods of proving (1.11) distinguish this result from others, too. They base on relations between the Zygmund space $L \ln L$ and Riesz operators. Theorem D has been proved in [16], applied to the evolutionary Euler system. Outlines of the proof of Theorem D one can find in the Appendix – subsection 5.1.

The below remark shows us a possible generalization of stated theorems.

Remark. *The results stated in Theorems A, B and C can be extended on the following linear system*

$$\begin{array}{lll} u_t + b\cdot\nabla u = cu + f & in & \Omega\times(0,T), \\ u|_{t=0} = u_0 & on & \Omega \end{array} \tag{1.12}$$

in an arbitrary domain $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, enough to allow integration by parts, and with given

$$c, f \in L_1(0,T;L_\infty(\mathbb{R}^n)) \quad and \quad b\cdot n = 0 \quad on \quad \partial\Omega\times(0,T),$$

where n is the normal vector to the boundary $\partial\Omega$.

Additionally, we find a natural generalization of (1.5)-(1.6)

$$\begin{array}{c} \operatorname{div} b = H_\infty + \sum_{k=1}^\infty H_k \quad such\ that \\ H_\infty \in L_1(0,T;L_\infty(\Omega)), \qquad H_k \in L_1(0,T;BMO(\Omega)) \quad and \\ \sum_{k=1}^\infty \|H_k\|_{L_1(0,T;BMO(\Omega))} < \infty \quad with \quad \sup_{k\in\mathbb{N}} diam\ supp\ H_k < \infty. \end{array} \tag{1.13}$$

In the case of bounded Ω condition (1.13) simplifies itself and (1.6) is automatically satisfied. We leave the proof of Remark to a kind reader, it is almost the same as for (1.1), the estimations are just more technical, but the core of the problem is the same.

Throughout the paper we use the standard notation. $L_p(\mathbb{R}^n)$ denotes the common Lebesgue space, generic constant are denoted by C. Let us recall only the definition of the BMO-space. We say that $f \in BMO(\mathbb{R}^n)$, if f is locally integrable and the corresponding semi-norm

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - \{f\}_{B(x,r)}|dy \tag{1.14}$$

is finite, where $\{f\}_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$ and $B(x,r)$ is a ball with radius r centered at x – see [18]. The above definition implies that (1.14) is a semi-norm only, however in our case from assumptions on $\operatorname{div} b$ follows

$$\|\operatorname{div} b\|_{L_1(\mathbb{R}^n)} \le |diam\ supp \operatorname{div} b|^n \|\operatorname{div} b\|_{BMO(\mathbb{R}^n)}$$

which is a consequence of the properties of the support restricted by (1.6).

2 Proof of Theorem A

The proof of existence of solutions in our case is standard, we present a sketch in the Appendix – subsection 5.2. Thus, we claim that there exists a weak solution fulfilling the definition (1.4) with (1.7). The high regularity of test functions required in (1.4) does not allow us to obtain any information concerning the uniqueness of solutions to (1.4) in a direct way. To solve this issue we start with an application of the standard procedure. We introduce

$$S_\epsilon(f) = m_\epsilon * f = \int_{\mathbb{R}^n} m_\epsilon(\cdot - y) f(y) dy, \tag{2.1}$$

where m_ϵ is a smooth function with suitable properties tending weakly to the Dirac delta – see (5.12) in the Appendix. Applying the above operator to (1.1) we get

$$\partial_t S_\epsilon(u) + S_\epsilon(b \cdot \nabla u) = 0, \tag{2.2}$$

where $b \cdot \nabla u = \operatorname{div}(bu) - u \operatorname{div} b$ and the r.h.s. is well defined as a distribution. In fact (2.2) implies that $\partial_t S_\epsilon(u)$ is well defined as a Lebesgue function, too.

We state equation (2.2) as follows

$$\partial_t S_\epsilon(u) + b \cdot \nabla S_\epsilon(u) = R_\epsilon, \qquad \text{where} \qquad R_\epsilon = b \cdot \nabla S_\epsilon(u) - S_\epsilon(b \cdot \nabla u). \tag{2.3}$$

Standard facts, known from the DiPerna-Lions theory [11], follow (see (5.13) in Appendix – subsection 5.3) the remainder is controlled in the limit: $R_\epsilon \to 0$ in $L_1(0,T; L_{1(loc)}(\mathbb{R}^n))$. Since R_ϵ convergences locally in space, only, we introduce a smooth function $\pi_r : \mathbb{R}^n \to [0,1]$ such that $\pi_r(x) = \pi_1(\frac{x}{r})$ and

$$\pi_1(x) = \begin{cases} 1 & |x| < 1 \\ \in [0,1] & 1 \le |x| \le 2 \\ 0 & |x| > 2 \end{cases} \qquad \text{with} \qquad |\nabla \pi_r| \le \frac{C}{r}. \tag{2.4}$$

In order to prove the uniqueness for our system it is enough to consider (2.3) with zero initial data (due to its linearity). Since we are forced to localize the problem, we multiply (2.3) by $S_\epsilon(u)\pi_r$ and integrate over the space, getting

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} (S_\epsilon(u))^2 \pi_r dx - \frac{1}{2}\int_{\mathbb{R}^n} \operatorname{div} b\,(S_\epsilon(u))^2 \pi_r dx - \frac{1}{2}\int_{\mathbb{R}^n} b \cdot \nabla \pi_r\,(S_\epsilon(u))^2 dx \\ = \int_{\mathbb{R}^n} R_\epsilon\, S_\epsilon(u) \pi_r dx. \end{aligned} \tag{2.5}$$

Then integrating over time, using properties of S_ϵ and letting $\epsilon \to 0$, next differentiating with respect t we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} u^2 \pi_r dx - \frac{1}{2}\int_{\mathbb{R}^n} \operatorname{div} b\, u^2 \pi_r dx = \frac{1}{2}\int_{\mathbb{R}^n} b \cdot \nabla \pi_r u^2 dx \quad \text{for} \quad r > 0. \tag{2.6}$$

The r.h.s. of (2.6) is estimated as follows

$$|\int_{\mathbb{R}^n} b\cdot\nabla\pi_r\, u^2 dx| \le C\|u\|^2_{L_\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n\setminus B(0,r)} \frac{|b|}{1+|x|}(1+|x|)|\nabla\pi_r| dx \to 0 \quad \text{as} \quad r \to \infty. \tag{2.7}$$

By definition $(1+|x|)|\nabla\pi_r| \leq C$, because the support of $\nabla\pi_k$ is a subset of the set: $\{r \leq |x| \leq 2r\}$. By (5.9) the norm $\|u\|_{L_\infty}$ is controlled, too. Then letting $r \to \infty$ in (2.7), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} u^2 dx - \frac{1}{2}\int_{\mathbb{R}^n} \operatorname{div} b\, u^2 dx = 0 \qquad \text{with} \qquad u|_{t=0} \equiv 0. \tag{2.8}$$

The function u can be viewed as a difference of two solutions to (1.1) with the same initial datum, hence (1.7) holds, so there exists $m > 0$ such that

$$\|u\|_{L_\infty(0,T;L_\infty(\mathbb{R}^n))} \leq m. \tag{2.9}$$

Let us observe that the second term in (2.8) is well defined. By (1.6) the support of $\operatorname{div} b$ is bounded. Hence by (2.9) we conclude that $\int_{\mathbb{R}^n} \operatorname{div} b(x,\cdot)u^2(x,\cdot)dx \in L_1(0,T)$. It follows that (2.8) guarantees that

$$u \in C([0,T]; L_2(\mathbb{R}^n)). \tag{2.10}$$

Now we are prepared to the application of Theorem D to the second integral in (2.8). Inequality (1.11) yields

$$\frac{d}{dt}\|u^2\|_{L_1(\mathbb{R}^n)} \leq C_0\|\operatorname{div} u\|_{BMO(\mathbb{R}^n)}\|u^2\|_{L_1(\mathbb{R}^n)}\left[|\ln\|u^2\|_{L_1(\mathbb{R}^n)}| + \ln(e+m)\right] \tag{2.11}$$

with the initial datum $\|u^2|_{t=0}\|_{L_1} = 0$ and m from (2.9). The Osgood lemma applied to (2.11), together with (2.10) yield the uniqueness to (1.1). Theorem A is proved.

Note that (2.11) has the form of (1.10) mentioned in Introduction. A direct analysis for an inequality of type (2.11) is presented in the next section.

3 Proof of Theorem B

The next result concerns the stability of solutions from Theorem A. We start with the mollified equation (2.2) for reasons same as previously, testing it now by $|S_\epsilon(u)|^{p-2}S_\epsilon(u)\pi_r$ with p as in Theorem B. Repeating the considerations for (2.3)-(2.8) we deduce

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u^k - u|^p dx \leq \int_{\mathbb{R}^n} |\operatorname{div} b|\,|u^k - u|^p dx. \tag{3.1}$$

The r.h.s. of (3.1) is controlled due to the boundedness of the support of $\operatorname{div} b$.

For a given $1 \geq \epsilon > 0$, we fix $K_\epsilon \in \mathbb{N}$ such that for all $k > K_\epsilon$

$$\|u_0^k - u_0\|_{L_p} \leq \epsilon. \tag{3.2}$$

Let $X_p = |u^k - u|^p$, then by Theorem D (m as in (2.9)) (3.1) reads

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} X dx &\leq \int_{\mathbb{R}^n} |\operatorname{div} b| X dx \\ &\leq C_0\|\operatorname{div} b\|_{BMO(\mathbb{R}^n)}\|X\|_{L_1(\mathbb{R}^n)}\left[|\ln\|X\|_{L_1(\mathbb{R}^n)}| + \ln(e+2m)\right], \end{aligned} \tag{3.3}$$

with $\int_{\mathbb{R}^n} X(x,0)dx \le \epsilon$.

By assumptions the r.h.s of (3.3) is at least locally integrable, hence $\int_{\mathbb{R}^n} X(x,\cdot)dx$ is uniformly continuous. There exits a positive time T_0 so small that

$$\sup_{t\in[0,T_0]} \int_{\mathbb{R}^n} X(x,t)dx \le e^{-1}. \tag{3.4}$$

It follows that the function $w|\ln w|$ will be considered as increasing, since $\int_{\mathbb{R}^n} X(x,\cdot)dx$ on the chosen time interval takes the values only from the interval $[0,e^{-1}]$. Monotonicity allows us to introduce a function $B : [0,T_0] \to [0,\infty)$ such that

$$\frac{d}{dt}B = C_0 \| \operatorname{div} b\|_{BMO(\mathbb{R}^n)} B[|\ln B| + \ln(e+2m)] \quad \text{and} \quad B|_{t=0} = \epsilon. \tag{3.5}$$

The definition of B guarantees that it is an increasing and continuous function, thus there exists $T_1 > 0$ such that $0 < T_1 \le T_0$ and

$$B(t) \le e^{-1} < 1 \quad \text{for} \quad t \in [0,T_1]. \tag{3.6}$$

Taking the difference between (3.5) and (3.3) we get

$$\begin{aligned} &\frac{d}{dt}[B - \int_{\mathbb{R}^n} X dx] \ge C_0 \| \operatorname{div} b\|_{BMO(\mathbb{R}^n)} \cdot \\ &\cdot \left[B|\ln B| - \int_{\mathbb{R}^n} X dx |\ln \int_{\mathbb{R}^n} X dx| + \ln(e+2m)(B - \int_{\mathbb{R}^n} X dx) \right] \end{aligned} \tag{3.7}$$

with $B(0) - \int_{\mathbb{R}^n} X(x,0)dx \ge 0$.

Since the monotonicity of the function $w|\ln w|$ on $[0,e^{-1}]$ implies

$$(B|\ln B| - \int_{\mathbb{R}^n} X dx |\ln \int_{\mathbb{R}^n} X dx|)(B - \int_{\mathbb{R}^n} X dx) \ge 0, \tag{3.8}$$

remembering that we consider $t \in [0,T_1]$, from (3.7) we get

$$0 \le \int_{\mathbb{R}^n} X(x,t)dx \le B(t) \quad \text{for} \quad t \in [0,T_1]. \tag{3.9}$$

The above fact reduces our analysis to the considerations of the function B. Additionally, by the choice of the time interval it follows that $B(t) < 1$ for $t \in [0,T_1]$, hence we can use the estimate

$$|\ln B| \le \ln \epsilon^{-1} \qquad \text{for} \quad t \in [0,T_1]. \tag{3.10}$$

Solving (3.5) we get

$$B(t) \le \epsilon \exp\left\{ C_0[\ln(e+2m) + \ln \epsilon^{-1}] \int_0^t f(s)ds \right\} \le C\epsilon\, \epsilon^{-C_0 \int_0^t f(s)ds}, \tag{3.11}$$

where $f(t) = \| \operatorname{div} b(\cdot, t)\|_{BMO(\mathbb{R}^n)}$ and C depends on data given in Theorems A and B.

Next, we choose T_2 so small that $0 < T_2 \leq T_1$ and $C_0 \int_0^{T_2} f(s)ds \leq 1/2$, then (3.11) yields

$$\sup_{t\in[0,T_2]} B(t) \leq C\epsilon^{1/2}. \tag{3.12}$$

Here we shall emphasize that T_2 is independent from the smallness of ϵ – see (3.2). Thus we are able to start our analysis over from the very beginning, but for the initial time $t = T_2$. Since C_0 in (3.11) is an absolute constant we find the next interval $[T_2, T_3]$, where we obtain

$$\sup_{t\in[T_2,T_3]} \|u^k - u\|_{L_p(\mathbb{R}^n)} \leq C\epsilon^{1/4} \tag{3.13}$$

for all $k > K_\epsilon$ – see (3.2). Since T is fixed and finite and by the assumptions $f \in L_1(0,T)$, we are always able to cover the whole interval $[0,T]$ in finite steps, so finally we obtain

$$\sup_{t\in[0,T]} B(t) \leq C\epsilon^{a}. \tag{3.14}$$

with $a > 0$ defined by the properties of f and again C depending on all data, but independent from ϵ. Letting $\epsilon \to 0$ we prove (1.9). Theorem B is proved.

As a remark to considerations in this section we note that repeating analysis for function B, we are able to prove – in a direct way – the uniqueness for (2.11), keeping in mind (2.10). This way omits an indirect application of the Osgood lemma from ODEs at the end of the proof of Theorem A. However such analysis is almost the same as (3.5)-(3.14). Thus, we omitted this approach in section 2 in order to avoid unnecessary repetitions.

4 Proof of Theorem C

Our last result describes the uniqueness criteria for weak solutions, provided their existence in the $L_\infty(0,T;L_p(\mathbb{R}^n))$-class in the meaning of the definition (1.4). The problem reduces to (1.1) with zero initial data and $u \in L_\infty(0,T;L_p(\mathbb{R}^n))$. To work in optimal regularity of coefficients we consider (2.3)

$$\partial_t S_\epsilon(u) + b\cdot\nabla S_\epsilon(u) = R_\epsilon \to 0 \ \text{ in } \ L_1(0,T;L_{1(loc)}(\mathbb{R}^n)).$$

Next, we introduce the renormalized solution for (1.1) – we refer here to [11] where this approach has been developed. Take $\beta \in C^1(\mathbb{R})$, i.e. $\|\beta\|_{L_\infty(\mathbb{R})} + \|\beta'\|_{L_\infty(\mathbb{R})} < \infty$, then

$$\partial_t\beta(S_\epsilon(u)) + b\cdot\nabla\beta(S_\epsilon(u)) = R_\epsilon\beta'(S_\epsilon(u)) \tag{4.1}$$

which implies the limit for $\epsilon \to 0$

$$\partial_t\beta(u) + b\cdot\nabla\beta(u) = 0. \tag{4.2}$$

As the function β we choose $T_m : \mathbb{R} \to [0, m^p]$ such that

$$T_m(s) = \begin{cases} |s|^p & \text{for} \quad |s| < m \\ m^p & \text{for} \quad |s| \geq m \end{cases} \tag{4.3}$$

defined for fixed $m \in \mathbb{R}_+$. T_m is not a C^1-function, but a simple approximation procedure will lead us to (4.2) with $\beta = T_m$.

Since we do not control integrability of all terms in (4.2), we use the function π_r from (2.4) to localize the problem, getting

$$\frac{d}{dt}\int_{\mathbb{R}^n} T_m(u)\pi_r dx \leq \int_{\mathbb{R}^n} |\operatorname{div} b|\, T_m(u)\pi_r dx + \int_{\mathbb{R}^n} |b \cdot \nabla \pi_r|\, T_m(u) dx. \tag{4.4}$$

For fixed m and r letting to infinity the last term vanishes – see (2.7), so we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n} T_m(u) dx \leq \int_{\mathbb{R}^n} |\operatorname{div} b|\, T_m(u) dx. \tag{4.5}$$

Theorem D applied to the r.h.s. of (4.5) yields

$$\frac{d}{dt}\|T_m(u)\|_{L_1(\mathbb{R}^n)} \leq C_0 \|\operatorname{div} b\|_{BMO(\mathbb{R}^n)} \|T_m(u)\|_{L_1(\mathbb{R}^n)} \left[|\ln \|T_m(u)\|_{L_1(\mathbb{R}^n)}| + \ln(e + m^p) \right] \tag{4.6}$$

with $\|T_m(u)(\cdot, 0)\|_{L_1(\mathbb{R}^n)} = 0$.

The same way as in the proof of Theorem A, the Osgood lemma yields $T_m(u) \equiv 0$. Letting $m \to \infty$, by (4.3) we conclude $u \equiv 0$. Thus, $u_1 \equiv u_2$. Theorem C is proved.

5 Appendix

5.1 Sketch of the proof of Theorem D

We proceed almost as in [16]. The assumption of the boundedness of supp f allows us to consider the studied integral (the l.h.s. of (1.11)) on a torus $\mathbb{T}^n = \mathbb{R}^n/(d\mathbb{Z}^n)(= [0, d)^n)$ with sufficiently large d guaranteeing that supp f can be treated as a subset of $\mathbb{T}^n$. Consider the Hardy space on $\mathbb{T}^n$ with the following norm

$$\|g\|_{\mathcal{H}^1(\mathbb{T}^n)} = \|g\|_{L_1(\mathbb{T}^n)} + \sum_{k=1}^{n} \|R_k g\|_{L_1(\mathbb{T}^n)}, \tag{5.1}$$

where R_k are the Riesz operators – [18],[19]. Since $BMO(\mathbb{T}^n) = (\mathcal{H}^1(\mathbb{T}^n))^*$, we get

$$|\int_{\mathbb{T}^n} f g dx| \leq \|f\|_{BMO(\mathbb{T}^n)} \|g\|_{\mathcal{H}^1(\mathbb{T}^n)}. \tag{5.2}$$

Hence to control the norm (5.1) an estimate of $\|R_k g\|_{L_1(\mathbb{T}^n)}$ is required. The classical Zygmund's result [20] (we refer to [19], too) says:

$$\|R_k h\|_{L_1(\mathbb{T}^n)} \leq C + C \int_{\mathbb{T}^n} |h| \ln^+ |h| dx, \tag{5.3}$$

where $\ln^+ a = \max\{\ln a, 0\}$ and constants C depends on d, so on the diameter of supp f.

Let us observe that $\ln^+(g/\lambda) = \ln g - \ln \lambda$ for $g \geq \lambda$ and

$$(5.4)\quad |\ln g|_{g\geq\lambda}| \leq \ln(1+\|g\|_{L_\infty(\mathbb{T}^n)})+|\ln \frac{g}{1+\|g\|_{L_\infty(\mathbb{T}^n)}}|_{g\geq\lambda}| \leq 2\ln(1+\|g\|_{L_\infty(\mathbb{T}^n)})+|\ln\lambda|.$$

Taking $h = \frac{g}{\|g\|_{L_1(\mathbb{T}^n)}}$ in (5.3), employing (5.4), we conclude

$$(5.5)\qquad \|R_k g\|_{L_1(\mathbb{T}^n)} \leq C\|g\|_{L_1(\mathbb{T}^n)} + C\int_{\mathbb{T}^n} |g| \left[\ln(1+\|g\|_{L_\infty(\mathbb{T}^n)}) + |\ln \|g\|_{L_1(\mathbb{T}^n)}|\right] dx$$

Inequalities (5.2), (5.5) yields (1.11).

5.2 The proof of existence (Theorem A)

Here we prove the existence of weak solutions to (1.1). To construct them we find a sequence of approximation of the function b and initial datum u_0. We require that

$$b^\epsilon \in C^\infty(\mathbb{R}^n \times (0,T)), \qquad \textit{supp}\ \mathrm{div}\ b^\epsilon(\cdot,t) \subset B(0,2R)$$
$$\text{and} \qquad b^\epsilon \to b \ \text{ in } \ L_1(0,T;W^1_{1(loc)}(\mathbb{R}^n))$$

with suitable behavior of norms. For the given initial datum we find $u_0^\epsilon \in C_0^\infty(\mathbb{R}^n)$ with $u_0^\epsilon \rightharpoonup^* u_0$ in $L_\infty(\mathbb{R}^n)$ as $\epsilon \to 0$ and $\|u_0^\epsilon\|_{L_\infty(\mathbb{R}^n)} \leq \|u_0\|_{L_\infty(\mathbb{R}^n)}$. Then we consider the following equation with smooth coefficients b^ϵ and initial data u_0^ϵ.

$$(5.6)\qquad \begin{array}{lll} u_t^\epsilon + b^\epsilon \cdot \nabla u^\epsilon = 0 & \text{in} & \mathbb{R}^n \times (0,T), \\ u^\epsilon|_{t=0} = u_0^\epsilon & \text{on} & \mathbb{R}^n. \end{array}$$

The method of characteristic implies the existence of smooth solutions to (5.6) for $t \in (0,T)$ together with the pointwise bound

$$(5.7)\qquad \|u^\epsilon\|_{L_\infty(0,T;L_\infty(\mathbb{R}^n))} \leq \|u_0^\epsilon\|_{L_\infty(\mathbb{R}^n)} \leq \|u_0\|_{L_\infty(\mathbb{R}^n)}.$$

Note that we do not use any uniform bound on $\mathrm{div}\, b^\epsilon$.

Now we pass to the limit with $\epsilon \to 0$ in (5.6). The solutions to (5.6) are classical, in particular it implies they fulfill the following integral identity

$$(5.8)\quad -\int_0^T\int_{\mathbb{R}^n} u^\epsilon\phi_t dxdt - \int_0^T\int_{\mathbb{R}^n} \mathrm{div}\ b^\epsilon u^\epsilon \phi dx - \int_0^T\int_{\mathbb{R}^n} b^\epsilon\cdot\nabla\phi u^\epsilon dxdt = \int_{\mathbb{R}^n} u_0^\epsilon \phi(\cdot,0)dx$$

for any $\phi \in C^\infty([0,T];C_0^\infty(\mathbb{R}^n))$ such that $\phi|_{t=T} \equiv 0$.

Estimate (5.7) implies that for a subsequence $\epsilon_k \to 0$

$$(5.9)\qquad u^{\epsilon_k} \rightharpoonup^* u \ \text{ in } L_\infty(0,T;L_\infty(\mathbb{R}^n)) \ \text{ with } \ \|u\|_{L_\infty(0,T;L_\infty(\mathbb{R}^n))} \leq \|u_0\|_{L_\infty(\mathbb{R}^n))}.$$

Then taking the limit of (5.6) for $\epsilon_k \to 0$, by the properties of sequences $\{b^\epsilon\}$ and $\{u_0^\epsilon\}$, we obtain

$$(5.10)\quad -\int_0^T\int_{\mathbb{R}^n} u\phi_t dxdt - \int_0^T\int_{\mathbb{R}^n} \mathrm{div}\ b\, u\phi dx - \int_0^T\int_{\mathbb{R}^n} b\cdot\nabla\phi u dxdt = \int_{\mathbb{R}^n} u_0\phi(\cdot,0)dx$$

for the same set of test functions as in (5.8).

5.3 The commutator estimate

Let us recall the well known facts concerning the mollification of the equation and the behavior of the commutators [11],[17]. Introduce $m_1 : \mathbb{R}^n \to [0,\infty)$ such that

$$m_1(x) = N_n \begin{cases} \exp\{-\frac{1}{1-|x|^2}\} & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases}$$

where the number N_n is determined by the constraint $\int_{\mathbb{R}^n} m_1 dx = 1$. Then for given $\epsilon > 0$ we define

$$m_\epsilon(x) := \frac{1}{\epsilon^n} m_1(\frac{x}{\epsilon}) \qquad \text{with} \qquad \int_{\mathbb{R}^n} m_\epsilon dx = 1. \tag{5.11}$$

It is clear that $m_\epsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$, where δ is the Dirac mass located at the origin of $\mathbb{R}^n$. The function m_ϵ introduces an operator $S_\epsilon : L_{1(loc)}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$

$$S_\epsilon(h) = m_\epsilon * h = \int_{\mathbb{R}^n} m_\epsilon(x-y)h(y)dy. \tag{5.12}$$

The standard theory [11] guarantees the following estimate for the commutator

$$b \cdot \nabla S_\epsilon(u) - S_\epsilon(b \cdot \nabla u) \to 0 \;\; \text{in} \;\; L_1(0,T; L_{1(loc)}(\mathbb{R}^n)), \tag{5.13}$$

provided that $u \in L_\infty(0,T; L_{p(loc)}(\mathbb{R}^n))$, $b \in L_1(0,T; W^1_{p'(loc)}(\mathbb{R}^n))$ and $1 = \frac{1}{p} + \frac{1}{p'}$ for $p \in [1,\infty]$. The convergence (5.13), known also as the Fridrichs lemma [17], allows us to test weak solutions by functions with lower regularity than it is required by the weak formulation. In our case it enables application of energy methods to obtain inequalities for L_2 and L_p norms — considerations: (2.5)-(2.6), (3.1) and (4.1)-(4.2). In other words, thanks to (5.13) we are able to test the equation by the solution, although it does not belong to class required by the definition (1.4).

The proof of (5.13) belongs to the by now classical theory, and since it is quite technical we omit it here and refer again to [11], [17].

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