THE SURFACE DIFFUSION FLOW ON ROUGH PHASE SPACES

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Abstract. The surface diffusion flow is the gradient flow of the surface functional of compact hypersurfaces with respect to the inner product of H^{-1} and leads to a nonlinear evolution equation of fourth order. This is an intrinsically difficult problem, due to the lack of an maximum principle and it is known that this flow may drive smoothly embedded uniformly convex initial surfaces in finite time into non-convex surfaces before developing a singularity [12, 13]. On the other hand it also known that singularities may occur in finite time for solutions emerging from non-convex initial data, cf. [8].

Combining tools from harmonic analysis, such as Besov spaces, multiplier results with abstract results from the theory of maximal regularity we present an analytic framework in which we can investigate weak solutions to the original evolution equation. This approach allows us to prove well-posedness on a large (Besov) space of initial data which is in general larger than C^2 (and which is in the distributional sense almost optimal). Our second main result shows that the set of all compact embedded equilibria, i.e. the set of all spheres, is an invariant manifold in this phase space which attracts all solutions which are close enough (which respect to the norm of the phase space) to this manifold. As a consequence we are able to construct non-convex initial data which generate global solutions, converging finally to a sphere.

Key words. Surface diffusion flow, geometric evolution equation, Besov spaces, center manifold, maximal regularity, stability, global existence, free boundary problem.

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1. Introduction. In this paper we study the geometric evolution equation

$$V_n = \Delta_{M(t)} H_{M(t)} \quad \text{on} \quad M(t),\ t > 0, \tag{1.1}$$

subject to the initial condition

$$M|_{t=0} = M_0. \tag{1.2}$$

Here $M(t)$ denotes an unknown surface, V_n is the normal velocity of $M(t)$, $\Delta_{M(t)}$ and $H_{M(t)}$ are the Laplace-Beltarmi operator and the mean curvature, respectively, of the surface $M(t)$. The normal vector and the mean curvature depend on the choice of the orientation of M_0, however we are free to choose whichever we like.

Equation (1.1) was first proposed by Mullins [19] to describe thermal grooving in material sciences. It arises also in modelling viscous sintering [15] and J. W. Cahn et al. derived it as a singular limit of the Cahn-Hilliard equation with a concentration dependent mobility [3]. From the point of view of geometric analysis it appears naturally as the gradient flow of the surface functional with respect to the inner product of H^{-1}, cf. [21]. Local well-posedness in the classical sense, as well as intersting dynamic properties of this flow such as loss of convexity, loss of embeddedness, and formation of singularities have been discussed in [4, 6, 7, 8, 9, 11, 12]. In this paper we offer a further insight into the dynamic picture of (1.1) by constructing an as large as possible phase space on which weak solutions in the sense of distribution still make sense. This effort will be rewarded by the fact that we have to control a priori less than two derivatives of solutions to (1.1) in order to guarantee global existence. In

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order to give a precise statement of our results, let $B^s_{p,2}$, with $p > 1$ and $s > 0$, denote the Besov spaces as introduced in Section 2 below. Using Marcinkiewicz's multiplier theorem [5], we first prove a priori estimates in

$$C([0,T]; B^{5/2-4/p}_{p,2}(S)) \quad \text{and} \quad B^{5/2,5/8}_{p,2}(S \times (0,T)) \quad \text{with} \quad p > 8/3, \tag{1.3}$$

of solutions to a linear problem associated with (1.1) against natural norms of the data, cf. Theorem 5.1. Here S denotes a reference surface over which the initial datum M_0 is parametrized as a graph in normal direction. Restrictions on p and the number of derivatives are limited by features of the nonlinear system. Using theses estimates and Banach's fixed point theorem, we get

Theorem 1.1. *Let $p > \frac{2n+8}{3}$ and*

$$M_0 \in B^{5/2-4/p}_{p,2} \quad \textit{as a submanifold in } \mathbb{R}^{n+1},$$

then there exists $T > 0$ such that there exists unique weak solution to system (1.1) on the time interval $[0,T)$ fulfilling the following relation

$$\bigcup_{0 \le t < T} (M(t) \times \{t\}) \in B^{5/2,5/8}_{p,2} \quad \textit{as a submanifold in } \mathbb{R}^{n+2}.$$

The choice of $B^s_{p,2}$ spaces is required by our analysis of the nonlinear terms. In fact the particular properties of these Besov spaces allow us to control products of functions in spaces of negative order – see the Appendix. The condition $p > (2n+8)/3$ implies that the field of normal vectors is Hölder continuous in space and time, hence the solutions constructed in Theorem 1.1 consists of C^{1+a}-surfaces. Observe furthermore that Theorem 1.1 guarantees us local in time solutions for arbitrary initial surfaces. To obtain a global result we have to restrict the set of initial data.

Using Alexandroff's characterization of embedded surfaces of constant mean curvature [1], it is not difficult to see that spheres are the only embedded steady states of (1.1). Note that this implies that steady states are not isolated, so that the principle of linearized stability cannot be used to analyze the stability properties of spheres. Instead we shall use an abstract approach which is based on the theory of maximal regularity and spectral properties of an elliptic operator defined by (1.1). Our analysis will lead us finally to a construction of a center manfold for (1.1), which consists only in equilibria. Moreover, it attracts all solutions at an exponential rate which are close enough to it. It is worthwhile to emphasize that our techniques are not based on the abstact methods from [20], but also use maximal regularity results, however in anisotropic Besov spaces. This approach is more direct and allows us to essentially improve the results of [9]. More precisely, we have

Theorem 1.2. *Let the initial surface M_0 be sufficiently close to a sphere in the $B^{5/2-4/p}_{p,2}$-topology, then the solution to system (1.1) exists globally in time and stays in a neighborhood of the sphere. Additionally it convergences exponentially fast to a sphere with a radius determined by the volume of a set bounded by M_0.*

Recall that Theorem 1.1 holds true provided $p > (2n+8)/3$. Hence if $n < 8$ there is a $p > (2n+8)/3$ such that $5/2-4/p < 2$, which means that $C^2(\mathbb{S}^n) \hookrightarrow B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$. This implies that there is a sequence of functions over $\mathbb{S}^n$ in $B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$, say $(r_m)_{m\in\mathbb{N}}$, such that

$$\|r_m\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \to 0 \quad \text{and} \quad \|r_m\|_{C^2(\mathbb{S}^n)} \to \infty \quad \text{as} \quad m \to \infty. \tag{1.4}$$

This in turn implies that there are exist arbitrarily small non-convex perturbations in $B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$ of a given sphere. If $n \geq 8$ the embedding $C^2(\mathbb{S}^n) \hookrightarrow B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$ can no longer be guaranteed. However, a different scaling argument shows, that (1.4) nevertheless is true. Combing the above reasoning with Theorem 1.2, we get

Theorem 1.3. *There exists a smooth non-convex surface M_0 generating a global solution to (1.1), which converges eventually to a sphere.*

The paper is organized as follows. First we recall the definitions of Besov spaces. In Section 3 a parametrization of solutions to (1.1) is given. It enables us to transfrom (1.1) into a system on a rigid domain. Next, we prove Theorem 1.3, based on Theorems 1.1 and 1.2. In Section 5 we introduce a linear system related to (1.1) and give a proof of the main a priori estimate (1.3). Based on the result for the linear system Theorem 1.1 is proved in Section 6. Subsequently we examine the stability of the unit sphere building a local center manifold proving finally Theorem 1.2. At the end in the Appendix we recall some facts concerning Besov spaces. Generic constants are denoted by the same letter C.

2. Besov spaces. Our analysis is described mostly in Besov spaces of $B^s_{p,2}$-type [2], [22],[23]. We briefly recall the corresponding definitions. First we consider the isotropic spaces defined on $\mathbb{R}^n$. Given $s > 0$ and $p \geq 1$, we introduce the Banach space $B^s_{p,2}(\mathbb{R}^n)$ defined by the norm

$$\|u\|_{B^s_{p,2}(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \sum_{|\alpha|=[s]_-} < \partial_x^\alpha u >_{B^{\{s\}}_{p,2}(\mathbb{R}^n)}, \tag{2.1}$$

where $[s]_-$ is the largest integer less than s and $\{s\} = s - [s]_-$. Moreover $\alpha \in \mathbb{N}^n$ is a standard multi-index. The main seminorm of $B^s_{p,2}$ – the last term on the r.h.s. of (2.1) – is defined by

$$< u >_{B^s_{p,2}(\mathbb{R}^n)} = \left(\int_{\mathbf{R}^n} \frac{1}{|h|^{n+2s}} \|u(\cdot + h) - h(\cdot)\|^2_{L_p(\mathbb{R}^n)} dh \right)^{1/2}, \tag{2.2}$$

provided $s \in (0, s)$. For the case $s = 1$ of (2.2), we refere to [2].

The above definitions can be extended on anisotropic spaces, where the time direction is distinguished. Given $T > 0$, we define $B^{s,s'}_{p,2}(\mathbb{R}^n \times (0,T))$ by the norm

$$\begin{aligned} \|u\|_{B^{s,s'}_{p,2}(\mathbb{R}^n\times(0,T))} &= \|u\|_{L_p(\mathbb{R}^n\times(0,T))} \\ + < \partial_t^{[s']_-} u >_{B^{0,\{s'\}}_{p,2}(\mathbb{R}^n\times(0,T))} &+ \sum_{|\alpha|=[s]_-} < \partial_x^\alpha u >_{B^{\{s\},0}_{p,2}(\mathbb{R}^n\times(0,T))}, \end{aligned} \tag{2.3}$$

where the sum of the last two terms of (2.3) is the main seminorm of $B^{s,s'}_{p,2}$, given by

$$\begin{aligned} < u >_{B^{s,0}_{p,2}(\mathbb{R}^n\times(0,T))} &= \left(\int_{\mathbb{R}^n} \frac{1}{|h|^{n+2s}} \|u(\cdot + h, \cdot) - u(\cdot,\cdot)\|^2_{L_p(\mathbb{R}^n\times(0,T))} dh \right)^{1/2}, \\ < u >_{B^{0,s'}_{p,2}(\mathbb{R}^n\times(0,T))} &= \left(\int_0^T \frac{1}{|h|^{1+s'2}} \|u(\cdot, \cdot + h) - u(\cdot,\cdot)\|^2_{L_p(\mathbb{R}^n\times(0,T-h))} dh \right)^{1/2}, \end{aligned} \tag{2.4}$$

provided $s, s' \in (0, 1)$.

Definitions (2.1)-(2.2) can be reformulated by using the Fourier transform [22]. For this we introduce the Paley-Littlewood decomposition $\{p_k\}_{k\in\mathbb{N}}$ in the following way: Let $p_k : \mathbb{R}^n \to [0,1]$ be a sequence of smooth functions such that

$$\text{supp } \ p_k \subset \{\xi \in \mathbb{R}^n : 2^{k-1} \le |\xi| \le 2^{k+1}\} \quad \text{ for } \ k \ge 1 \tag{2.5}$$

and

$$\text{supp } p_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\} \quad \text{ and } \quad \sum_{k=0}^{\infty} p_k \equiv 1. \tag{2.6}$$

Then for $s \ge 0$, the norm given by (2.1)-(2.2) is equivalent to

$$\|u\|_{B^s_{p,2}(\mathbb{R}^n)} = \left(\sum_{k=0}^{\infty} 2^{ks2}\|u^k\|^2_{L_p(\mathbb{R}^n)}\right)^{1/2}, \tag{2.7}$$

where $u^k = \mathcal{F}_x^{-1}[p_k(\xi)\mathcal{F}_x[u]]$ and $\mathcal{F}_z$ denotes the Fourier transform with respect to z. Observe that in this definition s may be any real number.

In the case of the spacetime $\mathbb{R}^n \times \mathbb{R}$ the norm (2.3) reads

$$\|u\|_{B^{s,s'}_{p,2}(\mathbb{R}^n\times\mathbb{R})} = \left(\sum_{k=0}^{\infty} 2^{ks2}\|u^k_x\|^2_{L_p(\mathbb{R}^n\times\mathbb{R})}\right)^{1/2} + \left(\sum_{k=0}^{\infty} 2^{ks2}\|u^k_t\|^2_{L_p(\mathbb{R}^n\times\mathbb{R})}\right)^{1/2}, \tag{2.8}$$

where $u^k_x = \mathcal{F}_x^{-1}[p^x_k(\xi)\mathcal{F}_x[u]]$ and $u^k_t = \mathcal{F}_t^{-1}[p^t_k(\xi_0)\mathcal{F}_t[u]]$, where p^x_k, p^t_k denote the Paley-Littlewood decomposition for $\mathbb{R}^n$ and $\mathbb{R}$, respectively.

The above Besov spaces can be defined in a natural way for functions on smooth submanifolds. Using an atlas of the manifold and a suitable partition of unity we define the norm as the sum taken over all maps from the atlas.

A basic tool in analysis of linear system in the presented framework is Marcinkiewicz's theorem for Fourier multipliers. A version of this result [5] reads as follows

Theorem 2.1. *Let $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ such that*

$$|\xi^\alpha||\partial^\alpha m| \le C_\alpha M, \quad \textit{where } \ 0 \le |\alpha| \le n,$$

$\alpha \in (\mathbb{N})^n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$, $\partial^\alpha = \partial_{\xi_1}^{\alpha_1} \cdot \ldots \cdot \partial_{\xi_n}^{\alpha_n}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

If $g \in L_p(\mathbb{R}^n)$, then $Tg = \mathcal{F}^{-1}[m\mathcal{F}g]$ is a linear bounded operator from $L_p(\mathbb{R}^n)$ into itself and

$$\|Tg\|_{L_p(\mathbb{R}^n)} \le A_{p,n} M \|g\|_{L_p(\mathbb{R}^n)}.$$

In [22],[23] one can find versions of this theorem for Besov spaces, too.

3. Parametrization. The goal of this section is to transform system (1.1) into a system on a rigid domain and to parametrize the solution by a scalar function.

For a given compact initial surface M_0 we can find a compact smooth hypersurface S satisfying the following conditions. First we fix two finite coverings $\{s^k\}$ and $\{S^k\}$ such that $s^k \subset S^k \subset S, \bigcup_k s^k = \bigcup_k S^k = S$. For each k we define a smooth function $\zeta^k : S \to [0,1]$ such that

$$\zeta^k(x) = \begin{cases} 1 & \text{for} \quad x \in s^k \\ \in [0,1] & \text{for} \quad x \in S^k \setminus s^k \\ 0 & \text{for} \quad x \in S \setminus S^k \end{cases}$$

Then we set

$$\pi^k(x) = \frac{\zeta^k(x)}{\sum_l (\zeta^l(x))^2} \quad \text{and} \quad \eta^k = \pi^k \zeta^k, \quad \text{so that} \quad \sum_l \eta^l \equiv 1 \text{ on } S. \tag{3.1}$$

The family $\{\eta^k\}$ is a partition of unity on S. Further we choose an atlas of maps (U^l, Z_l) such that $U^l \subset \mathbb{R}^n$ and $Z_l : U^l \to \mathbb{R}^{n+1}$ with $Z_l(U^l) = S^l$ and

$$\max_l (diam\ U^l + diam\ S^l) \leq \lambda, \tag{3.2}$$

where parameter λ will be specified later. Condition (3.2) yields a bound for the derivatives of the functions

$$|\partial^\alpha \zeta^k| + |\partial^\alpha \pi^k| + |\partial^\alpha \eta^l| \leq \frac{C}{\lambda^{|\alpha|}}. \tag{3.3}$$

Given a function $\phi : S \times [0,T) \to \mathbb{R}^{n+1}$ we describe a family of surface by

$$M(t) = \{x(y,t) = y + \phi(y,t)\vec{n} \ \text{ for all } \ y \in S \text{ and } t \in [0,T)\}, \tag{3.4}$$

where $\vec{n}$ is the normal vector to S. The initial datum has the following parametrization

$$M_0 = \{x(y) = y + \phi(y,0)\vec{n} \ \text{ such that } \ y \in S\}. \tag{3.5}$$

We require only that the parametrization of (3.4) is valid at least for small $T > 0$.

The regularity of M_0 enables to choose the surface S such that

$$||\phi(\cdot,0)||_{B^{5/2-4/p}_{p,2}(S)} \leq \varepsilon, \tag{3.6}$$

where $\varepsilon > 0$ is an arbitrary small number. Requirement (3.6) implies that higher norms of S will depend essentially on smallness of ε, however we will overcome this obstacle by choosing $T > 0$ small enough.

The function ϕ describes a diffeomorphism between the surfaces $M(t)$ and S. Hence we are able to transform (1.1) into a problem on the fixed reference domain S in the following way

$$\begin{array}{lll} v = \tilde{\Delta}_{M(t)} \tilde{H}_{M(t)} & \text{on} & S \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & S, \end{array} \tag{3.7}$$

where $v(y,t) = V_n(x(y,t),t)$ and $v = \phi_t \vec{n} \cdot \vec{n}^t$, with $\vec{n}^t$ being the unit outward normal vector to $M(t)$ in the coordinates (y,t). The Laplace-Beltrami and mean curvature operator $\tilde{\Delta}_{M(t)}$ and $\tilde{H}_{M(t)}$ have the following form

$$\tilde{\Delta}_{M(t)} \sim g^{ij} \frac{\partial^2}{\partial y^i \partial y^j} \quad \text{and} \quad \tilde{H}_{M(t)} \sim \frac{1}{n} \tilde{\Delta}_{M(t)} \phi + \text{ lower terms as } H_0(\nabla\phi, \phi), \tag{3.8}$$

where g_{ij} is the metric induced on $M(t)$ by the coordinates y, and where $g = det\{g_{ij}\}$, locally in each map of the atlas of the manifold S. For the explicit form of (3.8) we refer to [9],[10]. They can be determined in terms of the metric g_{ij}. However, using similar perturbation results as presented in [9], we may concentrate on the principal parts only.

The construction of the function ϕ allows us to reformulate Theorem 1.1 as follows.

Theorem 1.1*. *If $\phi_0 \in B^{5/2-4/p}_{p,2}(S)$, then there exists $T > 0$ such that there exists a unique weak solution to problem (3.7) on time interval $[0,T)$ such that*

$$\phi \in C([0,T]; B^{5/2-4/p}_{p,2}(S)) \quad \textit{and} \quad \phi \in B^{5/2,5/8}_{p,2}(S \times (0,T)).$$

The proof of Theorem 1.1* will be given in section 6.

4. Proof of Theorem 1.3. We distinguish three cases: $n = 1$, $1 < n < 8$ and $n \geq 8$. We will construct a smooth surface M_0 close to the unit sphere such that M_0 will be non-convex and it will fulfill the assumptions of Theorem 1.2, namely, M_0 will be sufficiently close to the sphere in the $B^{5/2-4/p}_{p,2}$-topology, generating in this way a global in time solution to (3.7).

In the first case $n = 1$, we introduce

$$M_0 = \{y + \varepsilon \sin my\, \vec{n} : y \in \mathbb{S}^1\}, \tag{4.1}$$

where $\varepsilon > 0$, $m \in \mathbb{N}$ and $\vec{n}$ is the normal vector to $\mathbb{S}^1$, and where $\mathbb{S}^1$ is the unit circle on the plane. Then we see that the "distance" between M_0 and the sphere is measured by the function $\varepsilon \sin my$. A simple calculation leads us to

$$< \varepsilon \sin my >_{B^{5/2-4/p}_{p,2}(\mathbb{S}^1)} = \varepsilon\, m^{5/2-4/p} < \sin y >_{B^{5/2-4/p}_{p,2}(\mathbb{S}^1)} . \tag{4.2}$$

Since $n = 1$, we find $p > \frac{10}{3} (= \frac{2n+8}{3}$ with $n = 1)$ and $\delta > 0$ such that $5/2 - 4/p + \delta < 2$. Taking $\varepsilon = m^{-2+\delta}$ in (4.2), we conclude that

$$\|m^{-2+\delta} \sin my\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^1)} \to 0 \quad \text{as} \quad m \to \infty. \tag{4.3}$$

On the other hand the curvature of M_0 is dominated by the second derivative of this function, for which we have

$$\|m^{-2+\delta} \sin my\|_{C^2(\mathbb{S}^1)} = Cm^{\delta} \to +\infty \quad \text{as} \quad m \to \infty. \tag{4.4}$$

The simple form of M_0 allows us even to compute the curvature of M_0 explicitly. Parametrization by angle gives the following formula for the curvature

$$\kappa = 1 + \frac{R'^2 - R''R}{R^2 + R'^2} = 1 + \frac{\varepsilon m^2 \sin mt + \varepsilon^2 m^2}{(1 + \varepsilon \sin mt)^2 + \varepsilon^2 m^2 \cos^2 mt}, \tag{4.5}$$

with $R(t) = 1 + \varepsilon \sin mt$. Choosing $\varepsilon = m^{-2+\delta}$, we get from (4.5):

$$\kappa = 1 + \varepsilon m^2 \sin mt + o(1) \quad \text{for} \quad m \to \infty. \tag{4.6}$$

Now relation (4.5) shows that M_0 as given by (4.1) becomes non-convex provided m is large enough. This proves Theorem 1.3 in the first case.

In the second case, i.e. for $1 < n < \infty$, we proceed similarly. First we introduce spherical coordinates on $\mathbb{S}^n$, i.e. we represent $x \in \mathbb{S}^n$ as

$$\begin{aligned} &x_1 = \cos \phi_1, \\ &x_2 = \sin \phi_1 \cos \phi_2, \\ &\ldots \\ &x_n = \sin \phi_1 \ldots \sin \phi_{n-1} \cos \phi_n, \\ &x_{n+1} = \sin \phi_1 \ldots \sin \phi_{n-1} \sin \phi_n, \end{aligned} \tag{4.7}$$

where $\phi_1, \phi_2, ..., \phi_{n-1} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\phi_n \in [0, 2\pi) = \mathbb{S}^1$. Then we define

$$M_0 = \{x = y(\phi_1, ..., \phi_n) + s(\phi_1, ..., \phi_n)\vec{n} : y(\phi_1, ..., \phi_n) \in \mathbb{S}^n\}, \tag{4.8}$$

where $s(\cdot)$ depends on two parameters ε and m in the following way

$$s(\phi_1, ..., \phi_n) = \varepsilon\, l(\phi_1, ..., \phi_{n-1}) \sin m\phi_n, \tag{4.9}$$

where l is a smooth function such that supp $l \subset\subset (-\frac{\pi}{2}, \frac{\pi}{2})^{n-1}$. Because of $n < 8$ we then find $\delta > 0$ and $p > \frac{2n+8}{3}$ such that $5/2 - 4/p + \delta < 2$. Next putting $\varepsilon = m^{-2+\delta}$ in (4.9), we conclude as for (4.3) that

$$\|s\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \to 0 \quad \text{as} \quad m \to \infty. \tag{4.10}$$

Repeating the considerations for (4.4), we also get that

$$\|s\|_{C^2(\mathbb{S}^n)} \sim Cm^{\delta} \to +\infty \quad \text{as} \quad m \to \infty. \tag{4.11}$$

So we are able to find $m > 0$ so large that M_0, defined by (4.8), will be non-convex.

In the case $n \geq 8$ we proceed in a different way. Consider

$$M_0 = \{x = y + s(y)\vec{n} : y \in \mathbb{S}^n\}, \text{ where } \; s(\phi_1, ..., \phi_n) = \varepsilon\, f(m(\phi_1, ..., \phi_n)) \tag{4.12}$$

with $f : \mathbb{R}^n \to [0, 1]$ such that supp $f \subset B(0, 1)$ and $f(0) = 1$, and with constants $\varepsilon > 0$ and $m \in \mathbb{N}$. To compute the semi-norm of s given by (4.12), we observe that for $n \geq 8$ we have $5/2 - 4/p > 2$. We have

$$\begin{aligned} &< \varepsilon f(mx) >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)} \\ &= \varepsilon m^2 \left(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+(1/2-4/p)2}} \|\nabla^2 f(mx + mh) - \nabla^2 f(mx)\|^2_{L_p(\mathbb{R}^n)} \right)^{1/2}; \end{aligned}$$

and changing coordinates $z = mx$ we get:

$$= \varepsilon\, m^2\, m^{-n/p} \left(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+(1/2-4/p)2}} \|\nabla^2 f(z + mh) - \nabla^2 f(z)\|^2_{L_p(\mathbb{R}^n)} \right)^{1/2}.$$

Taking further $w = mh$, we obtain

$$< \varepsilon f(mx) >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)} = \varepsilon\, m^2\, m^{-n/p}\, m^{1/2-4/p} < f >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}.$$

So we find the following estimate for the semi-norm of the function s

$$< s >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)} \leq C\varepsilon\, m^{5/2-4/p-n/p}. \tag{4.13}$$

Now observe that we are able to find $p > \frac{2n+8}{3}$ and $\delta > 0$ such that

$$\frac{5}{2} - \frac{4}{p} - \frac{n}{p} + \delta < 2, \quad \text{since} \quad \frac{5}{2} - \frac{12}{2n+8} - \frac{3n}{2n+8} = 1 < 2. \tag{4.14}$$

Inserting $\varepsilon = m^{-2+\delta}$ into (4.12), we see that (4.10) and (4.11) hold true also for $n \geq 8$. This completes the proof of Theorem 1.3.

5. The linear system. Our next task is to find a suitable linearization of system (3.7) and to prove an existence result with an appropriate Schauder-type estimates. We propose the following linear problem

$$\begin{aligned} &\phi_t + \Lambda^*\Lambda\phi = f && \text{on} \quad S \times (0, T), \\ &\phi|_{t=0} = \phi_0 && \text{on} \quad S, \end{aligned} \tag{5.1}$$

where

$$\Lambda^*\Lambda = \sum_l Z_l[\Delta^2 Z_l^{-1}(\eta^l \phi)], \tag{5.2}$$

with Δ being the Laplacian on $\mathbb{R}^n$, and $\{\eta^l\}$, $\{Z_l\}$ are given by (3.1) – see Section 3.

The parabolic character of the system of order four implies that (5.1) should be solved in $B^{4s,s}_{p,2}$-type spaces. From the analysis of the nonlinear system we obtain that the optimal factor is $4s = 5/2$. In this part we show the key result in proving Theorem 1.1.

Theorem 5.1. *Let $p > \frac{8}{3}$ and $\phi_0 \in B^{5/2-4/p}_{p,2}(S)$ and $f \in B^{-3/2,-3/8}_{p,2}(S \times (0,T))$, i.e. $f \in (B^{3/2,3/8}_{q,2}(S \times (0,T)))^*$, then there exists a unique solution to the system (5.1) such that*

$$\phi \in C([0,T]; B^{5/2-4/p}_{p,2}(S)) \quad \textit{and} \quad \phi \in B^{5/2,5/8}_{p,2}(S \times (0,T))$$

with the estimate

$$\begin{aligned} &\|\phi\|_{C([0,T];B^{5/2-4/p}_{p,2}(S))} + \|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} \\ &\le C(T,\lambda)\left(\|f\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T))} + \|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)}\right). \end{aligned} \tag{5.3}$$

First we prove a model version of Theorem 3.1 in the case $S = \mathbb{R}^n$.

Lemma 5.1. *Let $H \in B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n \times (0,T))$ and $\phi_0 \in B^{5/2-4/p}_{p,2}(\mathbb{R}^n)$ be given and assume that these data are compactly supported. Then the problem*

$$\begin{array}{lll} \phi_t + \Delta^2\phi = H & \textit{in} & \mathbb{R}^n \times (0,T), \\ \phi|_{t=0} = \phi_0 & \textit{on} & \mathbb{R}^n \end{array} \tag{5.4}$$

has a unique solution in the class $B^{5/2,5/8}_{p,2}(\mathbb{R}^n \times (0,T))$ and the following estimate

$$\begin{aligned} &\|\phi\|_{C([0,T];B^{5/2-4/p}_{p,2}(\mathbb{R}^n))} + \|\phi\|_{B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times(0,T))} \\ &\le C(T)\left(\|H\|_{B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n\times(0,T))} + \|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}\right), \end{aligned} \tag{5.5}$$

holds true, where C may depend on diameter of supports of the data.

Proof. System (5.4) can be split into the following two problems

$$\begin{array}{llll} \phi_{1,t} + \Delta^2\phi_1 = H, & \phi_{2,t} + \Delta^2\phi_2 = 0 & \text{in} & \mathbb{R}^n \times (0,T), \\ \phi_1|_{t=0} = 0, & \phi_2|_{t=0} = \phi_0 & \text{on} & \mathbb{R}^n. \end{array} \tag{5.6}$$

Clearly the solution to (5.4) is then the sum of ϕ_1 and ϕ_2.

Take the first system of (5.6). Since $H \in B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n \times (0,T))$, which is just $\left(B^{3/2,3/8}_{q,2}(\mathbb{R}^n \times (0,T))\right)^*$, and since the restriction $p > 8/3$ implies that $q < 8/5$, we can use Proposition 8.2 from the Appendix. It allows us to extend H on the whole $\mathbb{R}^{n+1}$ simple by zero. Let $\tilde{H}$ denotes this extension. Hence the first system of (5.6) can be extended in the whole $\mathbb{R}^{n+1}$ to

$$\tilde{\phi}_{1,t} + \Delta^2\tilde{\phi}_1 = \tilde{H} \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{5.7}$$

with $\tilde{H} \in B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n \times \mathbb{R})$, which follows from (8.2). Solving (5.7), we get

$$\tilde{\phi}_1 = \mathcal{F}^{-1}_{x,t}\left[\frac{1}{i\xi_0 + |\xi|^4}\mathcal{F}_{x,t}[\tilde{H}]\right]. \tag{5.8}$$

The form of the multiplier in (5.8), via Marcinkiewicz's theorem, yields the following estimate

$$< \tilde{\phi}_1 >_{B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R})} \leq C\|\tilde{H}\|_{B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n\times\mathbb{R})}. \tag{5.9}$$

Thus we conclude that the solution to $(5.6)_1$ satisfies $\phi_1 \in B^{5/2,5/8}_{p,2}(\mathbb{R}^n \times (0,T))$.

Now we want to prove uniqueness of solutions to $(5.6)_1$. By properties of the trivial extention of H, applying the standard energy approach, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\phi_1^2 dx + \int_{\mathbb{R}^n}(\Delta\phi_1)^2 dx = \int_{\mathbb{R}^n} H\phi_1 dx,$$

so

$$\sup_{t\leq T}\int_{\mathbb{R}^3}\phi_1^2 dx + \int_0^T\int_{\mathbb{R}^3}(\Delta\phi_1)^2 dxdt \leq 2|\int_0^T\int_{\mathbb{R}^3} H\phi_1 dxdt| = 2|\int_{\mathbb{R}}\int_{\mathbb{R}^3}\tilde{H}\phi_1 dxdt| \tag{5.10}$$

and by the choice of extension of H and (5.9) the r.h.s. of (5.10) is well defined. Hence we obtain an estimate in L_2. Moreover since $\tilde{\phi}_1 \to 0$ as $t \to -\infty$ inequality (5.10) for $H \equiv 0$ proves uniqueness of solutions to (5.7) in the considered class. Hence the obtained solutions must solve the first system of (5.6). In particular the initial condition $\phi_1|_{t=0} \equiv 0$ is fulfilled.

Next, we examine the second system of (5.6). To simplify the notation we write ϕ instead of ϕ_2. The Fourier transform applied to space directions yields the explicit solution

$$\phi(x,t) = \phi_2(x,t) = \mathcal{F}_x[\hat{\phi}_0 e^{-|\xi|^4 t}]. \tag{5.11}$$

First, we study the regularity with respect to time. By the definition (2.4) we have

$$< \phi >_{B^{0,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} = \left(\int_0^\infty \frac{dh}{h^{1+\frac{5}{8}2}}\|\phi(\cdot,\cdot) - \phi(\cdot,\cdot+h)\|^2_{L_p(\mathbb{R}^n\times\mathbb{R}_+)}\right)^{1/2}. \tag{5.12}$$

Since $L_p(\mathbb{R}^n) \supset B^0_{p,2}(\mathbb{R}^n)$, for $p \geq 2$ we have

$$\|\phi(\cdot,\cdot) - \phi(\cdot,\cdot+h)\|^2_{L_p(\mathbb{R}^n\times\mathbb{R}_+)} \leq C\left(\sum_{k=0}^\infty \|\phi_h^k\|^2_{L_p(\mathbb{R}^n\times\mathbb{R}_+)}\right)^{1/2}, \tag{5.13}$$

where $\hat{\phi}_h^k = p_k(\xi)\hat{\phi}_0\,(e^{-|\xi|^4 t} - e^{-|\xi|^4(t+h)})$, where p_k are the partition of unity from the Paley-Littlewood decomposition (2.5)-(2.7). So Marcinkiewicz's theorem (Theorem 2.1) gives

$$\|\phi_h^k(\cdot,t)\|_{L_p(\mathbb{R}^n)} \leq C(1 - e^{-2^{k4+4}h})\, e^{-2^{k4-4}t}\, \|\phi_0^k\|_{L_p(\mathbb{R}^n)}.$$

Taking the norm with respect to time, we get

$$\|\phi_h^k\|_{L_p(\mathbb{R}^n\times\mathbb{R}_+)} \leq C_p(1 - e^{-2^{k4+4}h})\, 2^{-\frac{k4}{p}}\, \|\phi_0^k\|_{L_p(\mathbb{R}^n)}. \tag{5.14}$$

Inserting (5.14) into (5.12) yields

$$< \phi_0 >_{B^{0,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \leq$$
$$C_p\left(\sum_{k=0}^\infty\int_0^\infty \frac{dh}{h^{1+5/4}}(1 - e^{-2^{k4+4}h})^2\, 2^{-k\frac{5}{2}2}\, 2^{k(\frac{5}{2}-\frac{4}{p})2}\, \|\phi_0^k\|^2_{L_p(\mathbb{R}^n)}\right)^{1/2} = I_1. \tag{5.15}$$

To estimate the last integral we note that, taking $l = k+1$, we have

$$\int_0^\infty \frac{dh}{h^{1+5/4}}(1-e^{-2^{k4+4}h})^2\, 2^{-k\frac{5}{2}2} = C\int_0^\infty \frac{dh}{h^{1+5/4}}(1-e^{-2^{4l}h})^2\, 2^{-5l} = I_2.$$

So putting $w = 2^{4l}h$, we obtain:

$$I_2 \le C\int_0^\infty \frac{2^{-4l}\, dw}{2^{-4l(1+5/4)}\, w^{1+5/4}}(1-e^{-w})^2\, 2^{-5l} = C\int_0^\infty \frac{dw(1-e^{-w})^2}{w^{1+5/4}} = M < \infty. \tag{5.16}$$

Hence inserting this estimate into (5.15) (I_2 to I_1), recalling definition (2.7), we conclude

$$< \phi >_{B^{0,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \le C\|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}. \tag{5.17}$$

Next, we consider regularity with respect to spatial variables. We have

$$< \phi >_{B^{5/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \le \left(\sum_{k=0}^\infty 2^{k\frac{5}{2}2}(\int_0^\infty \|\phi^k\|^p_{L_p(\mathbb{R}^n)}dt)^{2/p}\right)^{1/2} = I_3, \tag{5.18}$$

where here $\hat{\phi}^k = p_k\hat{\phi}$ – see (2.5)–(2.7). Applying Marcinkiewicz's theorem, remembering the form (5.11), we conclude

$$\|\phi^k(\cdot,t)\|_{L_p(\mathbb{R}^n)} \le Ce^{-2^{k4-4}t}\|\phi^k_0\|_{L_p(\mathbb{R}^n)}.$$

Hence

$$\left(\int_0^\infty \|\phi^k\|^p_{L_p(\mathbb{R}^n)}dt\right)^{1/p} \le C\frac{1}{(p2^{k4})^{1/p}}\|\phi^k_0\|_{L_p(\mathbb{R}^n)}. \tag{5.19}$$

Returning to (5.18) we have (see (2.7))

$$I_3 \le C_p\left(\sum_{k=0}^\infty 2^{k(5/2-4/p)2}\,\|\phi^k_0\|^2_{L_p(\mathbb{R}^n)}\right)^{1/2} = C_p\|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}. \tag{5.20}$$

Therefore it follows that

$$< \phi >_{B^{5/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \le C_p||\phi_0||_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}. \tag{5.21}$$

Summing up the estimates (5.9), (5.17) and (5.21), we obtain

$$\begin{aligned} &< \phi >_{L_\infty(\mathbb{R}_+;B^{5/2-4/p}_{p,2}(\mathbb{R}^n))} + < \phi >_{B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \\ &\le C(\|H\|_{B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} + \|\phi\|_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}), \end{aligned} \tag{5.22}$$

with a time independent constant C. But to estimate the whole norm, which is necessary in order to take the trace, we are required to use (5.10). Here extra conditions on the support of H and for ϕ_0, and finiteness of T are needed. Then we easily conclude (5.5) and the proof of Lemma 5.1 is completed.

Now we are prepared to consider the problem in on the manifold S. Let

$$\begin{aligned} &\mathcal{L}\phi := \phi_t + \Lambda^*\Lambda\phi = H && \text{in} \quad S\times(0,T), \\ &\phi|_{t=0} = 0 && \text{on} \quad S. \end{aligned} \tag{5.23}$$

The method of proving Theorem 5.1 via Lemma 5.1 is adapted from the classical approach to parabolic systems introduced in [14]. It is based on the regularizer operator, see the definition below, which somehow gives the solution by a procedure related to the Banach fixed point theorem. This technique have been effectively applied for similar problems to (5.23) in [16],[17].

By Lemma 3.1 and the properties of the atlas (U^l, Z_l) of S we are able to construct an extension $\tilde{\phi}_0$ of the initial data such that it belongs to $B^{5/2,5/8}_{p,2}(S\times(0,T))$, i.e.

$$\|\tilde{\phi}_0\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} \le C||\phi_0||_{B^{5/2-4/p}_{p,2}(S)} \quad \text{and} \quad \tilde{\phi}_0|_{t=0} = \phi_0. \tag{5.24}$$

Thus we obtain (5.23) with the initial condition $\phi_0 \equiv 0$ by a subtraction putting $\phi_{old} = \tilde{\phi}_0 + \phi_{new}$ and $H_{old} = \mathcal{L}\tilde{\phi}_0 + H_{new}$ into (5.1). By (5.24) the regularity of the data in (5.23) is preserved.

Definition 5.1. *Introduce a function $\phi^k = R^k(\zeta^k H)$, where $\phi^k = Z_k^{-1*}[\bar{\phi}^k]$ is the solution to the following problem*

$$\begin{array}{lll} \bar{\phi}^k_t + \Delta^2\bar{\phi} = Z^*_k[\zeta^k H] & \textit{in} & \mathbb{R}^n\times(0,T), \\ \bar{\phi}|_{t=0} = 0 & \textit{on} & \mathbb{R}^n. \end{array} \tag{5.25}$$

Then we define an operator $R : B^{-3/2,-3/8}_{p,2}(S\times(0,T)) \to B^{5/2,5/8}_{p,2}(S\times(0,T))$, called the regularizer, by

$$RH = \sum_k \pi^k\phi_k. \tag{5.26}$$

By Lemma 5.1, R is bounded and by definition it is linear. The introduction of the regularizer allows to solve system (5.23) on short time intervals. The properties of R are explained by following two lemmas.

Lemma 5.2. *Assume that $T > 0$ is small enough. Then $\mathcal{L}RH = H + \mathcal{T}H$ and $\|\mathcal{T}\| \le \frac{1}{2}$.*

Proof. From the definition of the operator R we obtain

$$\mathcal{L}RH = \sum_k \pi^k\mathcal{L}\phi^k + \sum_k \left(\mathcal{L}(\pi^k\phi^k) - \pi^k\mathcal{L}\phi^k\right). \tag{5.27}$$

Assume that supp $\eta^l \cap$ supp $\pi^k \neq \emptyset$. Then we conclude from (5.25) that

$$\bar{\eta}^l\bar{\phi}^k_t + \Delta^2(\bar{\eta}^l\bar{\phi}^k) = Z_k[\eta^l\pi^k H] + r_{lk}, \tag{5.28}$$

where the term r_{lk} has the following structure (in the highest order of derivatives): $r_{lk} \sim \lambda^{-1}\nabla\bar{\phi}^k$ supported on supp $\eta^l\pi^k$. Hence taking into account (5.28) and recalling the properties of η^l, we get

$$\sum_k \pi^k\mathcal{L}\phi^k = \sum_{k,l}\pi^k Z_k[\bar{\phi}^k_t + \Delta^2(\bar{\eta}^l\bar{\phi}^k)] = H + \sum_{k,l} Z_k[r_{lk}]. \tag{5.29}$$

Moreover the second term on the r.h.s. of (5.27) has a structure similar to r_{kl}, namely $\sum_k \mathcal{L}(\pi^k\phi^k) - \pi^k\mathcal{L}\phi^k \sim \lambda^{-1}\sum_k \nabla^3\phi^k$. To estimate these remainders, we apply the imbedding theorem (8.1)

$$\begin{aligned} \|\nabla^3\phi^k\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T))} &\le C\|\phi^k\|_{B^{3/2,3/8}_{p,2}(S\times(0,T))} \\ &\le \varepsilon\|\phi^k\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} + c(\varepsilon)\|\phi^k\|_{L_p(S\times(0,T))} \\ &\le (\varepsilon + c(\varepsilon)T^{1/p})\|H\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T))}, \end{aligned} \tag{5.30}$$

where we used the fact that

$$\sup_{t\leq T} < \phi^k(\cdot,t) >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)} \leq C < \phi^k >_{B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times(0,T))} . \tag{5.31}$$

Observe that C is independent of T, since $\phi^k|_{t=0} \equiv 0$ – see also (5.10). Thus by (5.30) and the form of the remainder there exists $T_* > 0$ such that the norm of the operator $\mathcal{T}$ is less than $1/2$ on the time interval $[0,T]$ for $T \leq T_*$. This completes the proof of Lemma 5.2

Note that the constant in (5.30) must be small enough in order to absorb the terms of order λ^{-1}. This is possible by choosing T sufficiently small.

Lemma 5.3. *Assume that $T > 0$ is small enough. Then $R\mathcal{L}\phi = \phi + \mathcal{W}\phi$ and $\|\mathcal{W}\| \leq \frac{1}{2}$.*

Proof. Repeating the considerations from the proof of Lemma 3.3, we get

$$R\mathcal{L}\phi = \sum_k \pi^k R^k[\zeta^k(\sum_l Z_k[\bar{\eta}^l\bar{\phi}_l + \Delta^2(\bar{\eta}^l\bar{\phi}^l)])] = \phi + \mathcal{W}\phi,$$

with a suitable estimate for the norm of operator $\mathcal{W}$ for small $T \leq T*$. Thus we claim that Lemma 5.3 is true. Similar results with detailed proofs can be found in [17].

To prove Theorem 5.1 we first prove the existence of solutions to system (5.23) on time interval $[0,T_*]$, where T_* is chosen such that Lemmas 5.2 and 5.3 hold true. ¿From Lemmas 5.2 and 5.3 we conclude that

$$R\mathcal{L} = Id + \mathcal{W} \quad \text{and} \quad \mathcal{L}R = Id + \mathcal{T}. \tag{5.32}$$

By the smallness of the norms of $\mathcal{T}$ and $\mathcal{W}$ we get that the r.h.s. from (5.32) are invertible on a time interval $[0,T_*]$, so $(Id+\mathcal{W})^{-1}R\mathcal{L} = Id$ and $\mathcal{L}R(Id+\mathcal{T})^{-1} = Id$. Thus, the solution to (5.23) is given by the following formula

$$\phi = (Id+\mathcal{W})^{-1}RH \quad \text{on } S\times(0,T) \quad \text{for } \ T \leq T_*. \tag{5.33}$$

The definition of R and Lemma 5.1 imply that the solution to (5.23) fulfills

$$\|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} \leq C\|H\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T))} \qquad \text{for } \ T \leq T_*. \tag{5.34}$$

Thus applying remark (5.24), we obtain the solvability of problem (5.1) on the time interval $[0,T_*]$. Moreover we have

$$\|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T_*))} \leq C\left(\|H\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T_*))} + \|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)}\right). \tag{5.35}$$

Next, by the trace theorem we prolong the existence of the solutions on $[T_*,2T_*]$, $[2T_*,3T_*]$, etc. So we finally obtain the estimate (5.3) with a constant depending on T. So Theorem 5.1 is proved.

Remark. It is worthwhile to underline that the magnitude of the constant in (5.35) is independent of λ and T_*. It depends only on the geometry of S. However we have $\lambda \to 0$ as $T_* \to 0$. On the other hand fixing C in (5.35), we can choose λ and then consider $T \in (0,T_*]$. This remark is crucial in the treating of the nonlinear problem in the next section – see in particular (6.15). The simplest solution to avoid the mentioned difficulty is just to use this 'universal' constant from (5.35), and then prescribe first λ and then T.

6. Local in time existence. Here we prove Theorem 1.1* which implies unique solvability of system (1.1) in the meaning of Theorem 1.1. The system (1.1) will be considered in the following form

$$\begin{aligned} \phi_t = L^\phi \tilde{\Delta}_{M(t)} \tilde{H}_{M(t)} \quad & \text{in } S \times (0,T), \\ \phi|_{t=0} = \phi_0 \quad & \text{on } S, \end{aligned} \tag{6.1}$$

where L^ϕ is defined by

$$L^\phi = (\vec{n} \cdot \vec{n}^t)^{-1}, \tag{6.2}$$

$\vec{n}$ and $\vec{n}^t$ are normal vectors to S and $M(t)$, respectively. Additionally

$$\|\vec{n}^t\|_{L_\infty(S\times(0,T))} \le C(\|S\|_{C^1} + \|\phi\|_{W^1_\infty(S\times(0,T))}). \tag{6.3}$$

The goal of this section is to prove existence and uniqueness of weak solutions to (6.1) in the following sense:

Definition 6.1. *We say that a function $\phi : S \times (0,T) \to \mathbf{R}$ is a weak solution to problem (3.7) iff $\phi \in B^{5/2,5/8}_{p,2}(S \times (0,T))$ with $p > \frac{2n+8}{3}$ and $\phi|_{t=0} = \phi_0$ and the following identity is valid*

$$(\phi_t, \pi)_{L_2(S)} + (\tilde{H}_{M(t)}\phi, \tilde{\Delta}_{M(t)}[L^\phi \pi])_{L_2(S)} = 0 \tag{6.4}$$

in the distributional sense on $[0,T)$ for each $\pi \in B^{3/2,3/8}_{q,2}(S \times (0,T))$ with $1/p + 1/q = 1$.

The proof of Theorem 1.1* will follow from a standard application of the Banach fixed point theorem and Theorem 5.1 applied to the weak formulation (6.4). Since we look for local in time solutions we shall always underline the dependence of T.

The solution to (3.7) will be searched in the set

$$\begin{aligned} \Xi = \{u \in B^{5/2,5/8}_{p,2}(S \times (0,T)) \quad \text{with} \quad p > \tfrac{2n+8}{3} \quad \text{and} \quad u|_{t=0} = \phi_0 \\ \text{such that} \quad \|u\|_\Xi = < u >_{B^{5/2,5/8}_{p,2}(S\times(0,T))} + \|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)} < \delta\}. \end{aligned} \tag{6.5}$$

Let us emphasize that the norm in (6.5) is equivalent to the standard norm in $B^{5/2,5/8}_{p,2}$, but this form gives us directly the trace theorem (in time) and an inequality with the constant independent of T. In other words we control all constants in following considerations uniformly in time. This is of particular importance, since T has to be small. In order to underline the domain we shall write $\Xi(S \times (0,T))$.

Choosing at the very beginning a sufficiently small ε in (3.6), our goal to show the existence of $\delta > \varepsilon > 0$ such that the mapping

$$K : \Xi \to \Xi \ \text{ is a contraction, where } \ K(\psi) = \phi, \tag{6.6}$$

is the solution to the following problem

$$\begin{aligned} (\phi_t, \pi)_{L_2(S)} + (\Lambda\phi, \Lambda\pi)_{L_2(S)} \\ = (\Lambda\psi - \tilde{H}_{N(t)}, \tilde{\Delta}_{N(t)}[L^\psi \pi])_{L_2(S)} + (\Lambda\psi, \Lambda\pi - \tilde{\Delta}_{N(t)}[L^\psi \pi])_{L_2(S)} \end{aligned} \tag{6.7}$$

in the distributional sense on $[0,T)$ for each $\pi \in B^{3/2,3/8}_{q,2}(S \times (0,T))$ with $1/p + 1/q = 1$. In (6.7) $N(t)$ is the surface generated by ψ via definition (3.4), and L^ψ

is the version of (6.3) for $N(t)$. Let us explain the notation in (6.7). Here we use $(\Lambda\phi, \Lambda\pi)_{L_2(S)} = (\Lambda^*\Lambda\phi, \pi)_{L_2(S)}$ in the sense of distributions – see (5.2). Λ denotes a second order elliptic operator, being a part of the operator $\Lambda^*\Lambda$, which is a consequence of integration by parts.

Applying Theorem 5.1 to system (6.7), we get

$$\begin{aligned}\|\phi\|_{L_\infty(0,T;B^{5/2-4/p}_{p,2}(S))} + \|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} &\le C(\|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)} \\ &+ \sup_\pi |\int_0^T (\Lambda\psi - \tilde{H}_{N(t)}, \tilde{\Delta}_{N(t)}[L^\psi\pi])_{L_2(S)}dt| \\ &+ \sup_\pi |\int_0^T (\Lambda\psi, \Lambda\pi - \tilde{\Delta}_{N(t)}[L^\psi\pi])_{L_2(S)}dt|),\end{aligned} \tag{6.8}$$

where the supremum is taken over all $\pi \in B^{3/2,3/8}_{q,2}(S\times(0,T))$ with norm 1.

To find the estimate on the r.h.s. of (6.8) we consider the terms in local maps. The highest order expression has the following form

$$\int_0^T \int_{U^l} (\Delta\psi - A(\nabla\psi) : \nabla^2\phi)(A(\nabla\phi) : \nabla^2[L^\phi\pi])\eta^l dxdt, \tag{6.9}$$

where $|L^\psi| \le C(1+|\nabla\psi|)$. The matrix A fulfills the following estimate

$$\begin{aligned}&\|Id - A(\nabla\psi)\|_{L_\infty(U^l\times(0,T))} \\ &\le C(\lambda^a\|S\|_{C^a} + \|\nabla\phi_0\|_{C^a(S\times(0,T))} + \|\nabla\psi\|_{C^a(S\times(0,T))}),\end{aligned} \tag{6.10}$$

provided that $\psi \in \Xi$. Note that by (3.6) $\|S\|_{C^a} \le \|M_0\|_{C^a} + C\varepsilon$.

Hence we are required to control integrals with the following strutcure

$$\int_0^T \int_{U^l} (\nabla^2\psi : \nabla^2[(1+\nabla\psi)\pi])\eta^l dxdt. \tag{6.11}$$

But $\nabla^2\psi \in B^{1/2,1/8}_{p,2}$, hence we have to prove that

$$\nabla^2[(1+\nabla\psi)\pi] \in B^{-1/2,-1/8}_{q,2}(S\times(0,T)), \tag{6.12}$$

which holds true because we have

$$\nabla^3\psi\,\pi \in B^{-1/2,-1/8}_{q,2}, \qquad \nabla^2\psi\cdot\nabla\pi \in B^{-1/2,-1/8}_{q,2}, \quad \text{and} \quad \nabla\psi\cdot\nabla^2\pi \in B^{-1/2,-1/8}_{q,2} \tag{6.13}$$

for $\psi \in B^{5/2,5/8}_{p,2}$, $\pi \in B^{3/2,3/8}_{q,2}$. The last inclusion is obvious, the first one follows from (8.3) and the second one from (8.4) – see the Appendix.

Remark. Here we shall explain the properties of (6.11). This highest term in (6.11) describes us the optimal regularity of the whole considerations. In order to estimate it, we need five derivative applied to ψ and π. Hence the space of solutions requires at least 5/2 derivative. Next, inclusions (6.13) and restrictions coming from the multiplication results for the Besov spaces imply that $p > \frac{2n+8}{3}$ – see Proposition 8.3. In this way we find our space $B^{5/2,5/8}_{p,2}$ with the trace in the $B^{5/2-4/p}_{p,2}$-space. It explains why we call our result optimal.

The regularity given by Ξ implies $\nabla\psi \in L_\infty$ – even the Hölder continuity in time and space. Remembering that $\nabla\psi \in C^{a,a/4}(S\times(0,T))$ which follows from (8.1), lower terms (in derivatives of functions ψ and π) from (6.8) can be estimated by

$$C(1/\lambda)T^a(\|\psi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} + \|\phi_0\|_{B^{5/2-4/p}_{p,2}})\|\pi\|_{B^{3/2,3/8}_{q,2}(S\times(0,T))}. \tag{6.14}$$

We emphasize that the estimations of the lower terms in local coordinates are connected to estimations of terms with derivatives of the localizers ζ^k, π^k and η^k. From (3.3) we know that these quantities are bounded only by $\frac{1}{\lambda}$. However thanks to (5.31) we estimate them with a constant T^a for some $a > 0$. Thus a suitable small T will absorb the influence of $\frac{1}{\lambda}$ – see (6.14) and the Remark at the end of the previous Section 5.

Our considerations lead to the following bound of the l.h.s. of (6.8)

$$\begin{aligned}\|\phi\|_{L_\infty(0,T;B^{5/2-4/p}_{p,2}(S))} + \|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} \le C\Big(\|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)} \\ +[\varepsilon + C(1/\lambda)T^a + \lambda^a\|M_0\|_{B^{5/2-4/p}_{p,2}} + \|\psi\|_\Xi]\|\psi\|_\Xi\Big).\end{aligned} \tag{6.15}$$

Thanks to the smallness of ε and λ, we find $\delta > 0$ such that $K : \Xi \to \Xi$ – see (6.5).

The proof of the contraction property of K on Ξ is almost the same as for (6.15). So repeating all steps done for (6.7), we get

$$\|K(\psi_1) - K(\psi_2)\|_\Xi \le 1/2\|\psi_1 - \psi_2\|_\Xi, \tag{6.16}$$

provided T sufficiently small.

This shows the unique solvability of system (3.7) in the meaning of definition (6.4). Theorems 1.1* and 1.1 are therefore proved.

7. Stability of spheres. In this part we establish stability of spheres in the $B^{5/2-4/p}_{p,2}$-topology. For simplicity we concentrate our analysis on the behavior of solutions to (1.1) in a neighborhood of the unit sphere. Since the required regularity guarantees that the normal vector is at least Hölder continuous we can choose $\mathbb{S}^n$ as the reference surface. So the surface $M(t)$ is parametrized by ϕ via formula (3.4) with $\mathbb{S}^n$ as S and the system (6.1) can be restated as follows

$$\phi_t + G(\phi) = 0, \qquad \phi|_{t=0} = \phi_0. \tag{7.1}$$

Our analysis is concerned with the stability of the zero solution, so we need the Fréchet derivative of G at 0. From [9] – Lemmas 3.1 and 3.2 – we have:

Lemma 7.1. *Let*

$$A = \frac{1}{n}\Delta^2_{\mathbb{S}^n} + \Delta_{\mathbb{S}^n}, \tag{7.2}$$

where $\Delta_{\mathbb{S}^n}$ is the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^n$, then

$$\partial G(0)h = Ah \quad \textit{for} \quad h \in B^{5/2-4/p}_{p,2}(\mathbb{S}^n). \tag{7.3}$$

Furthermore the spectrum A consists of a sequence of real eigenvalues.

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < ..., \tag{7.4}$$

where λ_0 is of multiplicity $n + 2$.

The properties of operator A allow us to introduce the following decomposition of the phase space

$$X = X_c \oplus X_s, \quad \text{where } X = B^{5/2-4/p}_{p,2}(\mathbb{S}^n), \tag{7.5}$$

$X_c = span\{Y_0, Y_1, ..., Y_{n+1}\}$ and $Y_0, ..., Y_{n+1}$ are eigenvectors for the eigenvalue $\lambda_0 = 0$, $Y_0 = 1$ and $Y_1, ..., Y_{n+1}$ are homogeneous functions of degree one. In the canonical

setting they are just $Y_l = x_l$. Furthermore we define projectors P_c and P_s on subspaces X_c and X_s, respectively. The space X_s called the stable subspace since the spectrum of A here is strictly positive, i.e. spec $A|_{X_s} \subset [\lambda_1, +\infty)$. Since X_c is finite dimensional, the decomposition (7.5) is well defined.

To apply the properties of A, we restate (7.1) in the following way:

$$\phi_t + A\phi = g(\phi), \quad \text{where} \quad g(\phi) = A\phi - G(\phi). \tag{7.6}$$

In order to analyze (7.6), we have a closer look at g. Thanks to (7.3) we have

$$G(\phi) = A\phi + \frac{1}{2}\partial^2 G(0)(\phi,\phi) + o(\|\phi\|^2), \tag{7.7}$$

so

$$g(\phi) = \frac{1}{2}\partial^2 G(0)(\phi,\phi) + o(\|\phi\|^2). \tag{7.8}$$

This point of view on the r.h.s. of (7.6) is helpful in the following considerations. Our next goal is to "parametrize" all small static solutions to (7.6). Eventually it will appear that they build a local unique center manifold for (7.1).

Let S_z be a sphere close to $\mathbb{S}^n$ and let $(z_1, ..., z_{n+1})$ be the coordinates of its center. Then z_0 is chosen such that $1 + z_0$ is the radius of S_z. By the properties of Y_l we find that

$$(1+z_0)^2 = \sum_{l=1}^{n+1}((1+\psi(z))Y_l - z_l)^2,$$

where ψ is the distance to $\mathbb{S}^n$. Straightforward calculations lead us to the following representation of the distance to the original sphere

$$\psi(z) = \sum_{l=1}^{n+1} z_l Y_l - 1 + \sqrt{(\sum_{l=1}^{n+1} z_l Y_l)^2 + (1+z_0)^2 - \sum_{l=1}^{n+1} z_l^2}. \tag{7.9}$$

We restrict our attention to a neighborhood of zero in $\mathbb{R}^{n+2}$. Then we get

$$\partial\psi(0)h = \sum_{l=0}^{n+1} h_l Y_l \qquad \text{for} \qquad h \in \mathbb{R}^{n+2}, \tag{7.10}$$

i.e. $P_c\partial\psi(0) = Id$. The identity (7.10) implies that $P_c\psi : \mathcal{O} \to \mathcal{O}'$, where $\mathcal{O}, \mathcal{O}'$ are suitable small neighborhoods of zero in $\mathbb{R}^{n+2}$. The inverse function theorem implies that there exists a smooth bijective function $m : \mathcal{O}' \to \mathcal{O}$ such that

$$P_c\psi(m(h)) = h \qquad \text{for} \qquad h \in \mathcal{O}' \subset \mathbb{R}^{n+2}. \tag{7.11}$$

So the set of all S_z sufficiently close to $\mathbb{S}^n$ is represented in X by

$$\mathcal{M} = \left\{ \sum_{l=0}^{n+1} h_l Y_l + P_s\psi(m(h)) : h \in \mathcal{O}' \right\}. \tag{7.12}$$

The set $\mathcal{M}$ is a smooth (n+2)-dimensional manifold in X, tangent to X_c at zero. Also by the properties of m, P_c and (7.9) we find

$$\|P_s\psi(m(h))\|_X \le C\|h\|_{\mathbb{R}^{n+2}}. \tag{7.13}$$

Summarizing the above analysis, we get that the set of all small solutions to (7.6) is described in the following way

$$\mathcal{M} = \{z + \sigma(z) : x \in \mathcal{O}'\} \quad \text{with} \;\; \sigma := P_s\psi \circ m. \tag{7.14}$$

Now we want to apply (7.14) for our investigations of (7.1). Using (7.5) the system (7.6) can be decomposed in the following way:

$$\begin{aligned} (P_c\phi)_t + P_cA(P_c\phi) &= P_cg(\phi), \\ (P_s\phi)_t + P_sA(P_s\phi) &= P_sg(\phi). \end{aligned} \tag{7.15}$$

Additionally from Lemma 7.1 we get

$$A\phi - g(\phi)|_{\mathcal{M}} \equiv 0, \quad \text{hence} \quad P_cg(\phi)|_{\mathcal{M}} \equiv 0. \tag{7.16}$$

It follows from (7.15) that

$$P_c\phi(t) = P_c\phi_0 \qquad \text{as} \;\; P_c\phi_0 \in P_c\mathcal{M}. \tag{7.17}$$

This means that any small solution to (7.6) has the following form

$$\phi(t) = P_s\phi(t) + P_c\phi_0, \tag{7.18}$$

i.e. the evolution of our system acts only on X_s. This particular feature allows us to describe precisely the behavior of solutions on X_s.

In order to reduce equations (7.15), we note that the set of all small equilibria (7.14) is the graph of the function $\sigma : P_c\mathcal{M} \to X_s$ in the sense that $z + \sigma(z) \in \mathcal{M}$ with $z \in P_c\mathcal{M}$. Consequently, any solution has the form $\phi(t) = P_s\phi(t) + z$, with some $z \in P_c\mathcal{M}$, where z is obtained from (7.18), i.e. $z = P_c\phi_0$.

(7.16) yields $P_sg(z+\sigma(z)) = P_sA\sigma(z)$, too. Hence we can reformulate the second equation of (7.15) as follows

$$(P_s\phi - \sigma(z))_t + P_sA[P_s\phi - \sigma(z)] = P_s[g(P_s\phi - \sigma(z) + z + \sigma(z)) - g(z + \sigma(z))], \tag{7.19}$$

The quantity $P_s\phi - \sigma(z)$ measures the distance to $\mathcal{M}$ in X_s. Let us describe the structure of the r.h.s. of (7.19). By (7.8)

$$\begin{aligned} &g(P_s\phi - \sigma(z) + z + \sigma(z)) - g(z + \sigma(z)) \\ &= (P_s\phi - \sigma(z))o(\|\phi\|) + \tfrac{1}{2}\partial^2 G(0)(P_s\phi + z, P_s\phi - \sigma(z)), \end{aligned} \tag{7.20}$$

which implies that

$$P_s[g(P_s\phi - \sigma(z) + z + \sigma(z)) - g(z + \sigma(z))] \sim P_s[(P_s\phi - \sigma(z))O(\|\phi\|)]. \tag{7.21}$$

Now, we want to analyze (7.19). Because of (7.21) we are required to return to (7.6), to control the norm of the whole ϕ. The form of (7.8) enables us to prove a so-called almost global in time existence result. We have

Proposition 7.1. *Let $T > 0$, then there exists $M > 0$, $\varepsilon_0 > 0$, such that there exists a unique solution to* (7.6) *on time interval* $[0, T]$ *fulfilling the following a priori estimate*

$$\|\phi\|_{\Xi(\mathbb{S}^n \times (0,T))} \le M\varepsilon, \qquad \textit{provided} \qquad \|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \le \varepsilon \tag{7.22}$$

for all $\varepsilon \leq \varepsilon_0$.

In fact, Proposition follows from the results and methods presented in Sections 5 and 6 as well as from Lemma 7.1 and expression (7.7). In order to avoid too much technicalities we omit the proof here.

Based on Proposition 7.1 we may express the solution to (7.19) in the in the following way

$$P_s\phi(t) - \sigma(z) = e^{-P_sAt}(P_s\phi_0 - \sigma(z)) + \int_0^t e^{-P_sA(t-s)}P_s[g(\phi) - g(z+\sigma(z)]ds. \quad (7.23)$$

Remembering that spec $P_sA \subset [\lambda_1, +\infty)$, using the results for the nonlinear system from Section 6, (7.21) and Proposition 7.1, we conclude

$$\begin{aligned} \|P_s\phi(t) - \sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} &\leq B_1 e^{-\lambda_1 t}\|P_s\phi_0 - \sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \\ &+ B_2(t)\|\phi\|_{\Xi(\mathbb{S}^n\times(0,t))}\|P_s\phi_0 - \sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)}, \end{aligned} \quad (7.24)$$

where B_1 is independent of t, but B_2 may depend on t. The inequality holds as long as the solution exists. Let us explain (7.24). First we prove existence of almost global solutions to (7.19). The estimate for the linear system implies

$$\begin{aligned} \|P_s\phi - \sigma(z)\|_{\Xi(\mathbb{S}^n\times(0,T))} &\leq B_3(T)(\|P_s\phi_0 - \sigma(z)\|_X \\ &+ \|\phi\|_{\Xi(\mathbb{S}^n\times(0,T))}\|P_s\phi - \sigma(z)\|_{\Xi(\mathbb{S}^n\times(0,T))}). \end{aligned} \quad (7.25)$$

For a sufficiently small (depending on T) initial data, we get $B_3(T)\|\phi\|_{\Xi(\mathbb{S}^n\times(0,T))} \leq 1/2$, thanks to Proposition 7.1. Hence we obtain

$$\|P_s\phi - \sigma(z)\|_{\Xi(\mathbb{S}^n\times(0,T))} \leq 2B_3(T)\|P_s\phi_0 - \sigma(z)\|_X. \quad (7.26)$$

On the other hand the last term in (7.24) is bounded by

$$\|\int_0^t e^{-P_sA(t-s)}P_s[g(\phi) - g(z+\sigma(z)]ds\|_X \leq C(t)\|\phi\|_{\Xi(\mathbb{S}^n\times(0,t))}\|P_s\phi - \sigma(z)\|_{\Xi(\mathbb{S}^n\times(0,t))}, \quad (7.27)$$

see our considerations in Section 6 (6.8)-(6.15)

So (7.26) and (7.27) give the form of the last term in (7.24). The first term in the r.h.s. of (7.24) follows from standard theory of semigroups. Inequality (7.24) is thus proved.

Proposition 7.1 implies that we can choose the initial datum ϕ_0 sufficiently small in order to guarantee that

$$B_2(T_*)\|\phi\|_{\Xi(\mathbb{S}^n\times(0,T_*))} \leq B_1 e^{-\lambda_1 T_*}, \quad (7.28)$$

where T_* taken so small that $B_1 e^{-\lambda_1 T_*} \leq \frac{1}{4}$. Then (7.24) yields

$$\begin{aligned} \|P_s\phi(T_*) - \sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} &\leq 2B_1 e^{-\lambda_1 T_*}\|P_s\phi_0 - \sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \\ &\leq \tfrac{1}{2}\|P_s\phi_0 - \sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)}. \end{aligned} \quad (7.29)$$

Without loss of generality we can require that the projection of the initial datum $P_c\phi_0 = z$ is so small that $\|\sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} < \varepsilon/2$. Note that his quantity is fixed in time for each considered solution. Then by (7.29) we obtain

$$\|\phi(T_*)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad (7.30)$$

Hence we restart our considerations for the initial time $t = T_*$ with $\phi|_{t=T_*} = \phi(T_*)$, eventually getting

$$\|P_s\phi(t)-\sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \leq Ce^{-\lambda_1 t}\|P_s\phi_0-\sigma(z)\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \quad \text{for } t \in [0,\infty). \tag{7.31}$$

Estimate (7.31) shows that each small solution tends exponentially to an element of $\mathcal{M}$, i.e. to $P_c\phi_0 + \sigma(P_c\phi_0)$.

As a result of (7.31) and the proved properties of $\mathcal{M}$ we obtain that $\mathcal{M}$ is a local unique center manifold for the dynamical system (1.1), with a basin of attraction being a neighborhood of the sphere in the $B^{5/2-4/p}_{p,2}$-topology. The proof of Theorem 1.2 is thus completed.

8. Appendix. Here we recall some basic facts concerning the Besov spaces [2],[22],[23]. A crucial role is played by the following imbedding theorems.

Proposition 8.1. *We have:*

$$\begin{aligned} &B^s_{p,2}(\mathbb{R}^n) \subset L_m(\mathbb{R}^n), \qquad \textit{provided} \quad \frac{n}{s}(\frac{1}{p}-\frac{1}{m}) \leq 1, \\ &B^{s,s'}_{p,2}(\mathbb{R}^n \times (0,T)) \subset L_m(\mathbb{R}^n \times (0,T)), \qquad \textit{provided} \quad (\frac{n}{s}+\frac{1}{s'})(\frac{1}{p}-\frac{1}{m}) \leq 1. \end{aligned} \tag{8.1}$$

The next result concerns $B^{3/2,3/8}_{q,2}(\mathbb{R}^n \times (0,T))$ for $q < 8/5$. One can check elementary that these spaces can be obtained as the closure of compactly supported functions:

Proposition 8.2. *We have*

$$B^{3/2,3/8}_{q,2}(\mathbb{R}^n \times (0,T)) = \overline{C^\infty_0(\mathbb{R}^n \times (0,T))}^{||\cdot||_{B^{3/2,3/8}_{q,2}}}. \tag{8.2}$$

The above fact allows to extend any function from this class trivially by zero. It follows that the constant in the imbedding inequality from (8.1) does not depend on smallness of T which will be important in our considerations.

Next we prove a result which controls the regularity of the product of two functions from Besov spaces. It is an important tool for analysis of the nonlinear terms appeared in Section 6.

Proposition 8.3. *Let $p > \frac{2n+8}{3}$ and $\frac{1}{p}+\frac{1}{q} = 1$.*
(i) Let $f \in B^{-1/2,-1/8}_{p,2}(S \times (0,T))$ and $g \in B^{3/2,3/8}_{q,2}(S \times (0,T))$, then $fg \in B^{-1/2,-1/8}_{q,2}(S \times (0,T))$ and

$$\|fg\|_{B^{-1/2,-1/8}_{q,2}(S\times(0,T))} \leq C\|f\|_{B^{-1/2,-1/8}_{p,2}(S\times(0,T))}\|g\|_{B^{3/2,3/8}_{q,2}(S\times(0,T))}, \tag{8.3}$$

(ii) Let $f \in B^{1/2,1/8}_{p,2}(S \times (0,T))$ and $g \in B^{1/2,1/8}_{q,2}(S \times (0,T))$, then $fg \in B^{-1/2,-1/8}_{q,2}(S \times (0,T))$ and

$$\|fg\|_{B^{-1/2,-1/8}_{q,2}(S\times(0,T))} \leq C\|f\|_{B^{1/2,1/8}_{p,2}(S\times(0,T))}\|g\|_{B^{1/2,1/8}_{q,2}(S\times(0,T))}. \tag{8.4}$$

Proof. All calculations are done in local coordinate of the surface S. By (8.2) we are able to assume that the function g is defined on the whole $S \times \mathbb{R}$. This property implies that constants in (8.3) and (8.4) are independent of T.

In order to prove (i) it is enough to show that $gk \in B^{1/2,1/8}_{q,2}$ for $k \in B^{1/2,1/8}_{p,2}$. Then $fgk \in L_1$ is controlled. Recalling the definition (2.4), we find

$$\begin{aligned} < gh >_{B^{1/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R})} \leq & \Big(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\frac{1}{2}2}} \|[k(\cdot+h,\cdot)-k(\cdot,\cdot)]g(\cdot,\cdot)\|^2_{L_q(\mathbb{R}^n\times\mathbb{R})}\Big)^{1/2} \\ & +\Big(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\frac{1}{2}2}} \|k(\cdot,\cdot)[g(\cdot+h,\cdot)-g(\cdot,\cdot)\|^2_{L_q(\mathbb{R}^n\times\mathbb{R})}\Big)^{1/2} = I_1 + I_2. \end{aligned} \tag{8.5}$$

The last two expressions can be estimated as follows

$$\begin{aligned} I_1 \leq & \Big(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\frac{1}{2}2}} \|k(\cdot+h,\cdot)-k(\cdot,\cdot)\|^2_{L_p(\mathbb{R}^n\times\mathbb{R})} \|g(\cdot,\cdot)\|^2_{L_m(\mathbb{R}^n\times\mathbb{R})}\Big)^{1/2} \\ \leq & \|k\|_{B^{1/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R})} \|g\|_{L_m(\mathbb{R}^n\times\mathbb{R})} \leq C\|k\|_{B^{1/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R})} \|g\|_{B^{3/2,3/8}_{q,2}(\mathbb{R}^n\times\mathbb{R})} \end{aligned} \tag{8.6}$$

with $\frac{1}{p}+\frac{1}{m}=\frac{1}{q}$, and by (8.1) $B^{3/2,3/8}_{q,2} \subset L_m$ as $p > \frac{2n+8}{3}$. Next,

$$I_2 \leq \Big(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\frac{1}{2}2}} \|k(\cdot,\cdot)\|^2_{L_m(\mathbb{R}^n\times\mathbb{R})} \|g(\cdot+h,\cdot)-g(\cdot,\cdot)\|^2_{L_{q_1}(\mathbb{R}^n\times\mathbb{R})}\Big)^{1/2} \tag{8.7}$$

with q_1 such that $B^{3/2,3/8}_{p,2} \subset B^{1/2,1/8}_{q_1,2}$ with $\frac{1}{q}+\frac{1}{q_1}=\frac{1}{n+4}$; and $\frac{1}{m}+\frac{1}{q_1}=\frac{1}{q}$. On the other hand m is described by the imbedding $B^{1/2,1/8}_{p,2} \subset L_m$, so $\frac{1}{p}-\frac{1}{m}=\frac{1}{2n+8}$. Remembering that $\frac{1}{q}+\frac{1}{p}=1$, we find the condition $p > \frac{2n+8}{3}$. The considerations for the time semi-norm are almost the same and the proofs are omitted therefore. Similar considerations for B^s_{pp}-spaces can be found in [18].

In order to prove (ii) we show that $fgk \in L_1$ for $k \in B^{1/2,1/8}_{p,2}$. Then a direct application of (8.1) implies the above inclusion. the proof of Proposition 8.3 is therefore completed.

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