On the steady compressible Navier–Stokes–Fourier system

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Abstract

We study the motion of the steady compressible heat conducting viscous fluid in a bounded three dimensional domain governed by the compressible Navier-Stokes-Fourier system. Our main result is the existence of a weak solution to these equations for arbitrarily large data. A key element of the proof is a special approximation of the original system guaranteeing pointwise uniform boundedness of the density as well as the positiveness of the temperature. Therefore the passage to the limit omits tedious technical tricks required by the standard theory. Basic estimates on the solutions are possible to obtain by a suitable choice of physically reasonable boundary conditions.

1 Introduction

We consider the following system of partial differential equations describing the steady flow of a compressible heat conducting Newtonian fluid in a bounded three dimensional domain Ω

$$\operatorname{div}(\varrho \boldsymbol{v}) = 0, \tag{1.1}$$

$$\operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{S}(\boldsymbol{v}) + \nabla p(\varrho, \theta) = \varrho \boldsymbol{F}, \tag{1.2}$$

$$\operatorname{div}\big(\varrho e(\varrho,\theta)\boldsymbol{v}\big) - \operatorname{div}\big(\kappa(\theta)\nabla\theta\big) = \mathbf{S}(\boldsymbol{v}) : \nabla \boldsymbol{v} - p(\varrho,\theta)\operatorname{div}\boldsymbol{v}, \tag{1.3}$$

where $\varrho : \Omega \to \mathbb{R}_0^+$ is the density of the fluid, $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ is the velocity field, $\boldsymbol{S}(\boldsymbol{v}) = 2\mu \boldsymbol{D}(\boldsymbol{v}) + \lambda(\operatorname{div}\boldsymbol{v})\boldsymbol{I}$ is the viscous part of the stress tensor, $\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$ is the

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symmetric part of the velocity gradient, $p(\cdot,\cdot) : \mathbb{R}_0^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$, a given function, is the pressure, $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ is the external force , $e(\cdot,\cdot) : \mathbb{R}_0^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$, a given function, is the internal energy. System (1.1)-(1.3) is known as the compressible Navier-Stokes-Fourier equations or the full Navier-Stokes system [6].

We assume that the constitutive equation has the form

$$p(\varrho,\theta) = a_1\varrho^\gamma + a_2\varrho\theta, \qquad a_1, a_2 > 0, \tag{1.4}$$

i.e. the pressure has one part corresponding to the ideal fluid and a so called elastic part; for more information see e.g. [6]. Even though we could consider more general pressure laws, we restrict ourselves to this simple model to avoid unnecessary technicalities in the proof. The corresponding internal energy takes the form

$$e(\varrho,\theta) = a_1\frac{\varrho^{\gamma-1}}{\gamma-1} + c_v\theta, \tag{1.5}$$

see e.g. [6] or [1]. Note that in the full generality, equation (1.3) should be replaced by the conservation of the total energy, instead of conservation of the internal energy only. For a sufficiently regular class of solutions, including that we are going to construct, the balance of the kinetic energy is just a consequence of the momentum equation. We further simplify (1.3). Our solutions will be such that $\varrho \in L_\infty(\Omega)$ and $\boldsymbol{v} \in W^1_p(\Omega)$ for all $p < \infty$. We get due to the fact that $\operatorname{div}(\varrho\boldsymbol{v}) = 0$ in the weak sense (see [16])

$$\operatorname{div}\Big(\frac{1}{\gamma-1}\varrho^\gamma\boldsymbol{v}\Big) = -\varrho^\gamma \operatorname{div}\boldsymbol{v},$$

again in the weak sense. Thus we write instead of (1.3) (we put $a_1 = a_2 = c_v = 1$) the energy equation (1.3) in the form

$$\operatorname{div}\big(\varrho\theta\boldsymbol{v}\big) - \operatorname{div}\big(\kappa(\theta)\nabla\theta\big) = \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - \varrho\theta\operatorname{div}\boldsymbol{v}. \tag{1.6}$$

The viscosity coefficients are, for the sake of simplicity, considered to be constant such that the conditions of the thermodynamical stability

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu \geq 0 \tag{1.7}$$

are satisfied. Finally, the heat conductivity is assumed to be temperature dependent, i.e.

$$\kappa(\theta) = a_3(1+\theta^m), \qquad a_3, m > 0. \tag{1.8}$$

This fact is important for our study, we are not able to consider a constant heat conductivity. Our domain Ω is sufficiently smooth, at least a C^2 domain. We supplement the system (1.1), (1.2) and (1.6) with the following boundary conditions at $\partial\Omega$. For the velocity, we consider the slip boundary conditions

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{\tau}_k \cdot (\boldsymbol{T}(p,\boldsymbol{v})\boldsymbol{n}) + f\boldsymbol{v}\cdot\boldsymbol{\tau}_k = 0 \quad \text{at } \partial\Omega, \tag{1.9}$$

where $\boldsymbol{\tau}_k$, $k = 1,2$ are two perpendicular tangent vectors to $\partial\Omega$, $\boldsymbol{n}$ is the outer normal vector and $\boldsymbol{T}(p,\boldsymbol{v}) = -p\boldsymbol{I} + \boldsymbol{S}(\boldsymbol{v})$ is the stress tensor. The friction coefficient f is non-negative (if $f = 0$ we assume additionally that Ω is not axially symmetric). Recall that

$f = 0$ corresponds to the perfect slip, while $f \to \infty$ leads to the homogeneous Dirichlet boundary conditions. However, we are not able to perform this limit passage.

Concerning the temperature, we assume that

$$\kappa(\theta)\frac{\partial \theta}{\partial \boldsymbol{n}} + L(\theta)(\theta - \theta_0) = 0 \quad \text{at } \partial\Omega, \tag{1.10}$$

where $\theta_0 : \partial\Omega \to \mathbb{R}^+$ is a strictly positive sufficiently smooth given function, say $\theta_0 \in C^2(\partial\Omega)$, $0 < \theta_* \le \theta_0 \le \theta^* < \infty$ with $\theta_*, \theta^* \in \mathbb{R}^+$ and

$$L(\theta) = a_4(1 + \theta^l), \qquad l \in \mathbb{R}_0^+. \tag{1.11}$$

We also add the prescribed mass of the gas

$$\int_\Omega \varrho dx = M > 0. \tag{1.12}$$

The objective of this paper is to prove the existence of weak solutions to problem (1.1)–(1.12) for arbitrarily large data. Till now only partial results have been proved (see e.g. [2], [9], [14], [15]) and only known general theorems concern weak solutions to the evolutionary version of the system [6]. One of main obstacles was to construct suitable a priori estimates. Due to properties of boundary condition (1.10) we are able to obtain a nontrivial energy bound for weak solutions, saving the thermodynamical structure of the system. In the case of the barotropic gas we do not meet such difficulties. The energy bound follows elementary from the momentum equation. However, it is not the only difference. The standard methods introduced by P.L. Lions [9] do not work successfully for the heat conducting case. However, a generalization of the technique introduced in [11], [17] gives us sufficient tools to solve the stated problem.

An approach to system (1.1)–(1.12) was considered in the book [9]. Unfortunately, this result can be viewed as conditional only, since instead of (1.12) the author assumed artificially that weak solutions satisfy $\int_\Omega \varrho^p dx = M^p$ for sufficiently large p. On the one hand, this condition is physically not acceptable, on the other hand, it simplifies considerably the mathematical analysis. Nevertheless, this result shows us what is the difference in techniques for the barotropic and heat conducting models.

Looking at results concerning the classical solutions for problems with small data, we realize that the heat conducting system has the same mathematical structure (difficulties) as the barotropic version of the model. Thus results from [2], [15] are almost immediately transformed to the case of system (1.1)–(1.12). For large data solutions the energy equation starts to play an important role, essentially changing the properties of the whole system.

The evolutionary case of system (1.1)–(1.12), under general assumptions on the pressure law was considered in [7] and [8]; the authors assumed only the situation when the fluid is thermically isolated, i.e. $\frac{\partial\theta}{\partial n} = 0$ at the boundary. However, the same technique works also for our boundary conditions (1.10). The thermically isolated situation guarantees immediately the energy bound for weak solutions, but considering the limit $t \to \infty$, the only solution which can be obtained as the limit for large times (with time independent force) is a solution with the constant temperature. This is connected to the fact that the model does not allow the heat transfer through the boundary and either the energy increases to infinity (non potential force) or the temperature approaches a constant value (potential force). The boundary condition (1.10) allows the heat transfer through the boundary, guaranteeing the balance of the total energy, and thus we are able to prove existence of solutions which are definitely nontrivial and physically acceptable.

The main result of this paper is the following.

Theorem 1 *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$ which is not axially symmetric if $f = 0$. Let $\boldsymbol{F} \in L_\infty(\Omega)$ and*

$$\gamma > 3, \qquad m = l + 1 > \frac{3\gamma - 1}{3\gamma - 7}.$$

Then there exists a weak solution to (1.1)–(1.12) such that

$$\varrho \in L_\infty(\Omega), \qquad \boldsymbol{v} \in W^1_q(\Omega), \quad \theta \in W^1_q(\Omega) \ \text{ for all } 1 \le q < \infty \ \text{ and } \ \theta > 0 \ \text{a.e.}$$

The solution constructed by Theorem 1 is meant in the following sense.

Definition 1 *The triple $(\varrho, \boldsymbol{v}, \theta)$ is a weak solution to (1.1)–(1.12), if $\varrho \in L_s(\Omega)$, $s \ge \gamma$, $\boldsymbol{v} \in W^1_2(\Omega)$, $\theta \in W^1_2(\Omega)$, $\theta^m \nabla\theta \in L_1(\Omega)$ and $\theta > 0$ a.e.; $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ at $\partial\Omega$ in the sense of traces and*

$$\int_\Omega \varrho\boldsymbol{v} \cdot \nabla\eta = 0 \qquad \forall \eta \in C^\infty(\overline{\Omega}), \tag{1.13}$$

$$\begin{gathered}\int_\Omega \big(-\varrho\boldsymbol{v} \otimes \boldsymbol{v} : \nabla\boldsymbol{\varphi} + 2\mu\boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{\varphi}) + \lambda \operatorname{div}\boldsymbol{v} \operatorname{div}\boldsymbol{\varphi} - p(\varrho,\theta)\operatorname{div}\boldsymbol{\varphi}\big) dx \\ + f\int_{\partial\Omega} (\boldsymbol{v} \odot \boldsymbol{\tau}) \cdot (\boldsymbol{\varphi} \odot \boldsymbol{\tau}) d\sigma = \int_\Omega \varrho\boldsymbol{F} \cdot \boldsymbol{\varphi} dx \qquad \forall \boldsymbol{\varphi} \in C^\infty(\overline{\Omega}); \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \text{ at } \partial\Omega\end{gathered} \tag{1.14}$$

(we denoted by $\boldsymbol{v} \odot \boldsymbol{\tau}$ the vector $\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$[1]) and finally

$$\begin{gathered}\int_\Omega \big(\kappa(\theta)\nabla\theta \cdot \nabla\psi - \varrho\theta\boldsymbol{v} \cdot \nabla\psi\big) dx + \int_{\partial\Omega} L(\theta)(\theta - \theta_0)\psi d\sigma \\ = \int_\Omega \big(2\mu|\boldsymbol{D}(\boldsymbol{v})|^2\psi dx + \lambda(\operatorname{div}\boldsymbol{v})^2\psi - \varrho\theta \operatorname{div}\boldsymbol{v}\psi\big) dx \qquad \forall \psi \in C^\infty(\overline{\Omega}).\end{gathered} \tag{1.15}$$

The proof of Theorem 1 will be based on a special approximation procedure described in the next section which is the kernel of our method. This section includes also a priori estimates for the approximation. The structure of the approximative system gives us immediately the approximative density bounded uniformly in L_∞, but we must prove refined L_∞ estimates to verify that the limit solves the original system (1.1)–(1.3). This idea has already been successfully applied in [11] and [17] in the case of barotropic flows.

The third section contains a detailed proof of existence to the approximative system. Here the main difficulty comes from the energy equation, since the required positiveness of the temperature does not follow immediately. In the next section we introduce an important quantity, the effective viscous flux and prove its main properties, i.e. the

[1] Note that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ at $\partial\Omega$ and thus $\boldsymbol{v} \odot \boldsymbol{\tau} = \boldsymbol{v}$.

compactness. This feature allows to improve information about the convergence of the density, which is the basic/fundamental fact in the theory of the compressible Navier-Stokes equations [6], [9]. The last section describes the refined L_∞ estimates for the approximative density and the passage to the limit. Then we prove that the limit is indeed our sought solution in the meaning of Definition 1.

As the reader may easily check, our method works for slightly larger class of the pressure laws. It allows to consider e.g.

$$p(\varrho,\theta) = p_b(\varrho) + \varrho\theta, \tag{1.16}$$

where $p_b(\varrho)$ is a strictly monotone function which behaves for large values as ϱ^γ. The main steps of this generalization are similar to the barotropic case and can be found in [17]; since our problem is technically enough complicated, we shall avoid such generalizations.

Our new result is closely related to the barotropic version of the system (1.1)–(1.12). Let us remind the state of the art in this theory. The steady compressible Navier–Stokes equations for arbitrarily large data were firstly successfully studied in the book [9], where, in the case of $p(\varrho) = \varrho^\gamma$ the existence of renormalized weak solutions was shown for $\gamma > 1$ ($N = 2$) and $\gamma \geq \frac{5}{3}$ ($N = 3$) for Dirichlet boundary conditions. For potential forces with a small non potential perturbation the existence was improved in [13] for $\gamma > \frac{3}{2}$ ($N = 3$). In the recent paper [5] the authors proved the existence in two space dimensions also for $\gamma = 1$. See also [3], where the authors considered the three dimensional case and got existence for certain γ–s less than $\frac{5}{3}$, however, for periodic boundary conditions. P.L. Lions also considered the existence of solutions with locally bounded density: for the case of Dirichlet boundary conditions he was able to show their existence for $\gamma > 1$ ($N = 2$) and $\gamma \geq 3$ ($N = 3$). Nevertheless, to prove Theorem 1 the above methods are not sufficient, thus we present our new approach for the heat conducting model.

Throughout the paper we use the standard notations for the Lebesgue, Sobolev, etc. spaces; generic constants are denoted by C and sequences $\epsilon \to 0$ always mean suitable chosen subsequences $\epsilon_k \to 0^+$. For the sake of simplicity we put $a_1 = a_2 = a_3 = a_4 = c_v = 1$.

2 Approximation

This section contains one of the main difficulties in the proof of Theorem 1 — to find a good approximation of problem (1.1)–(1.12). Then we shall be able to show existence and prove the corresponding a priori estimates. Here we present the approximative system as well as the proof of the fundamental a priori estimates, provided the temperature is positive and all quantities are sufficiently smooth. The next section deals then with the solvability of this system and with further a priori bounds. In particular in Section 3 the positiveness of the approximative temperature and smoothness of all quantities is proved.

Our approximative system will contain two parameters: a number $\epsilon > 0$ and an auxiliary function $K(\cdot)$ defined by a number $k > 0$ as follows:

$$K(t) = \begin{cases} 1 & \text{for} \quad t < k-1 \\ \in [0,1] & \text{for} \quad k-1 \leq t \leq k \\ 0 & \text{for} \quad t > k; \end{cases} \tag{2.1}$$

moreover, we assume that $K'(t) < 0$ for $t \in (k-1,k)$, where $k \in \mathbb{R}^+$. In the last section we pass with $\epsilon \to 0^+$ and we shall show that we may take k sufficiently large such that $K(\varrho) \equiv 1$ for our solution. The approximation of our problem (1.1)–(1.12) reads as follows

$$
\left.\begin{aligned}
\epsilon\varrho + \operatorname{div}(K(\varrho)\varrho\boldsymbol{v}) - \epsilon\Delta\varrho = \epsilon h K(\varrho)\\
\frac{1}{2}\operatorname{div}(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}) + \frac{1}{2}K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \operatorname{div}\mathbf{S}(\boldsymbol{v}) + \nabla P(\varrho,\theta) = \varrho K(\varrho)\boldsymbol{F}\\
-\operatorname{div}\Big((1+\theta^m)\frac{\epsilon+\theta}{\theta}\nabla\theta\Big) + \operatorname{div}\Big(\boldsymbol{v}\int_0^\varrho K(t)dt\Big)\theta + \operatorname{div}\Big(K(\varrho)\varrho\boldsymbol{v}\Big)\theta\\
+K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\theta - \theta K(\varrho)\boldsymbol{v}\cdot\nabla\varrho = \mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}
\end{aligned}\right\}\ \text{in } \Omega, \tag{2.2}
$$

where

$$
P(\varrho,\theta) = \int_0^\varrho \gamma t^{\gamma-1}K(t)dt + \theta\int_0^\varrho K(t)dt = P_b(\varrho) + \theta\int_0^\varrho K(t)dt \tag{2.3}
$$

and $h = \frac{M}{|\Omega|}$.

The equation $(2.2)_3$ can be reformulated in the following way being the modification of the entropy equation:

$$
\begin{aligned}
-\operatorname{div}\Big((1+\mathrm{e}^{sm})\frac{(\epsilon+\mathrm{e}^s)}{\mathrm{e}^s}\nabla s\Big) + K(\varrho)\varrho\boldsymbol{v}\cdot\nabla s - K(\varrho)\boldsymbol{v}\cdot\nabla\varrho + \operatorname{div}\Big(\boldsymbol{v}\int_0^\varrho K(t)dt\Big)\\
+\operatorname{div}\big(K(\varrho)\varrho\boldsymbol{v}\big) = \frac{\mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}}{e^s} + \frac{(1+\mathrm{e}^{sm})(\epsilon+\mathrm{e}^s)}{\mathrm{e}^s}|\nabla s|^2 \quad \text{in } \Omega,
\end{aligned} \tag{2.4}
$$

with the "entropy" s defined as follows

$$
s = \ln\theta. \tag{2.5}
$$

The solvability of (2.2)-(2.4), guaranteed by Theorem 2, gives us s integrable, even continuous for a fixed $\epsilon > 0$. Hence here the temperature $\theta := e^s$ is positive. This construction is performed in Section 3.

Additionally if $s \in W^2_q(\Omega)$ or $\theta \in W^2_q(\Omega)$ with $q > \frac{3}{2}$, so $\theta \geq c_0 > 0$ in Ω. Then $(2.2)_3$ and (2.4) are equivalent. The distinguished entropy will allow to control the positiveness of the temperature, what does not seem to be elementary working directly with an equation of type $(2.2)_3$.

This system is completed by the boundary conditions at $\partial\Omega$

$$
\begin{gathered}
(1+\theta^m)(\epsilon+\theta)\frac{\partial s}{\partial\boldsymbol{n}} + L(\theta)(\theta-\theta_0) + \epsilon s = 0,\\
\boldsymbol{v}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{\tau}_k\cdot(\boldsymbol{T}(p,\boldsymbol{v})\boldsymbol{n}) + f\boldsymbol{v}\cdot\boldsymbol{\tau}_k = 0, \qquad k=1,2,\\
\frac{\partial\varrho}{\partial\boldsymbol{n}} = 0.
\end{gathered} \tag{2.6}
$$

The key element in the limit passage from the approximative problem to the original one is the energy estimate giving information independent of the choice of function K, i.e. of the choice of the positive constant k — see (2.1):

Lemma 1 *Suppose solutions to (2.1)–(2.6) to be sufficiently smooth, i.e.* ϱ, $\boldsymbol{v}$ *and* $\theta \in W^2_q(\Omega)$ *for any* $q < \infty$, $\theta > 0$ *in* Ω. *Let assumptions of Theorem 1 be satisfied. Then*

$$(2.7)\qquad \begin{gathered} 0 \le \varrho \le k, \qquad \int_\Omega \varrho dx \le M \quad \text{and} \\ ||\boldsymbol{v}||_{H^1(\Omega)} + ||K(\varrho)\varrho||_{L_{2\gamma}(\Omega)} + ||P(\varrho,\theta)||_{L_2(\Omega)} + ||\theta||_{L_{3m}(\Omega)} + ||\nabla\theta||_{L_r(\Omega)} \\ + \int_{\partial\Omega} (e^s + e^{-s}) d\sigma + ||\nabla s||_{L_2(\Omega)} \le C(||\boldsymbol{F}||_{L_\infty(\Omega)}, M), \end{gathered}$$

where the r.h.s. of (2.7) is independent of ϵ *and* k, $s = \ln\theta$ *and* $r = \min\{2, \frac{3m}{m+1}\}$.

Proof. The nonnegativness of the density and boundedness by k follow directly from features of function K and the form of $(2.2)_1$, it suffices to integrate the equation over sets $\{x \in \Omega; \varrho(x) < 0\}$ and $\{x \in \Omega; \varrho(x) > k\}$, respectively. The integration of this equation over Ω leads to the bound on the total mass. For details we refer to [11]. Let us prove the second part of (2.7) which is definitely more complicated. We divide our chain of estimates into eight steps to underline main parts of our method.

Step I. Multiply the approximative momentum equation $(2.2)_2$ by $\boldsymbol{v}$ and integrate it over Ω:

$$(2.8)\qquad \begin{aligned} &\int_\Omega \big(2\mu \mathbf{D}^2(\boldsymbol{v}) + \lambda \operatorname{div}^2 \boldsymbol{v}\big)\, dx + \int_{\partial\Omega} f|\boldsymbol{v}\odot\boldsymbol{\tau}|^2 d\sigma + \int_\Omega \boldsymbol{v}\cdot\nabla P_b(\varrho) dx \\ &= \int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{F} dx + \int_\Omega \Big(\int_0^\varrho K(t)dt\Big)\,\theta \operatorname{div}\boldsymbol{v} dx. \end{aligned}$$

To find a good form of the last term of the l.h.s. of (2.8) we use the approximative continuity equation $(2.2)_1$

$$\begin{aligned} \int_\Omega \boldsymbol{v}\cdot\nabla P_b(\varrho)dx &= \frac{\gamma}{\gamma-1}\int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\varrho^{\gamma-1}dx \\ &= -\frac{\gamma}{\gamma-1}\int_\Omega \left[\epsilon\Delta\varrho + \epsilon h K(\varrho) - \epsilon\varrho\right]\varrho^{\gamma-1}dx \\ &= \frac{\epsilon\gamma}{\gamma-1}\int_\Omega [\varrho - hK(\varrho)]\varrho^{\gamma-1}dx + \epsilon\gamma\int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2 dx. \end{aligned}$$

Thus the momentum equation gives the following inequality

$$(2.9)\qquad \begin{aligned} &\int_\Omega \boldsymbol{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} dx + \int_{\partial\Omega} f|\boldsymbol{v}\odot\boldsymbol{\tau}|^2 d\sigma + \epsilon\gamma\int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2 dx + \frac{\epsilon\gamma}{\gamma-1}\int_\Omega \varrho^\gamma dx \\ &- \int_\Omega \Big(\int_0^\varrho K(t)dt\Big)\,\theta\operatorname{div}\boldsymbol{v} dx \le C\Big(1 + \int_\Omega |K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{F}|dx\Big). \end{aligned}$$

Step II. Integrating the energy equation $(2.2)_3$ and employing the boundary condition $(2.6)_1$ we get

$$(2.10)\qquad \int_{\partial\Omega} \big(L(\theta)(\theta-\theta_0) + \epsilon s\big) d\sigma = \int_\Omega \Big(\boldsymbol{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - \big(\int_0^\varrho K(t)dt\big)\theta\operatorname{div}\boldsymbol{v}\Big) dx,$$

since the integration by parts gives the following identity

$$\int_\Omega \Big[K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\theta - \theta K(\varrho)\boldsymbol{v}\cdot\nabla\varrho + \operatorname{div}\Big(\boldsymbol{v}\int_0^\varrho K(t)dt\Big)\theta + \operatorname{div}\big(K(\varrho)\varrho\boldsymbol{v}\big)\theta\Big]dx = \int_\Omega \Big(\int_0^\varrho K(t)dt\Big)\theta \operatorname{div}\boldsymbol{v}dx.$$

Summing up (2.9) and (2.10) we get

$$\int_{\partial\Omega}\big(L(\theta)\theta+\epsilon s^+\big)d\sigma + \epsilon\gamma\int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2dx + \frac{\epsilon\gamma}{\gamma-1}\int_\Omega\varrho^\gamma dx \leq \int_{\partial\Omega}\epsilon s^- d\sigma + C\Big(1+\int_\Omega |K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{F}|dx\Big), \tag{2.11}$$

where s^+ and s^- are the positive and negative parts of the entropy, respectively ($s = s^+ - s^-$). Note that the form of $L(\cdot)$ implies that the first term of (2.11) gives an estimate on $\int_{\partial\Omega} \mathrm{e}^s d\sigma$ independently of ϵ. We shall concentrate the attention on this term, since it controls the positive part of entropy s at the boundary.

Step III. We integrate the entropy equation (2.4) over Ω getting

$$\int_{\partial\Omega}\Big[\frac{L(\theta)(\theta-\theta_0)}{\theta}+\epsilon s e^{-s}\Big]d\sigma + \int_\Omega\Big(K(\varrho)\varrho\frac{\boldsymbol{v}\cdot\nabla\theta}{\theta} - K(\varrho)\boldsymbol{v}\cdot\nabla\varrho\Big)dx = \int_\Omega\Big[\frac{\mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}}{\theta} + \frac{(1+\theta^m)(\epsilon+\theta)}{\theta}|\nabla s|^2\Big]dx. \tag{2.12}$$

So

$$\int_\Omega\Big(\frac{\mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}}{\theta}+\frac{(1+\theta^m)(\epsilon+\theta)}{\theta}|\nabla s|^2\Big)dx + \int_{\partial\Omega}\Big(\frac{L(\theta)\theta_0}{\theta}+\epsilon|s^-|e^{|s^-|}\Big)d\sigma - \int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\nabla(s-\ln\varrho)dx \leq \int_{\partial\Omega}L(\theta)d\sigma + \int_{\partial\Omega}\epsilon s^+\mathrm{e}^{-s^+}d\sigma. \tag{2.13}$$

Here we emphasize that the first term in the second integral in the l.h.s. of (2.13) gives us a bound on $\int_{\partial\Omega} e^{-s}d\sigma$, because of the properties of $L(\cdot)$ and $\theta_0 \geq \theta_* > 0$. Hence we control the negative part of entropy s.

Let us look closer at the last term in the l.h.s. of (2.13). We have

$$-\int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\nabla(s-\ln\varrho)dx = \int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\ln\varrho dx - \int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\nabla s dx = I_1 + I_2 \tag{2.14}$$

and employing $(2.2)_1$ we get

$$I_1 = \int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\ln\varrho dx = -\int_\Omega \operatorname{div}(K(\varrho)\varrho\boldsymbol{v})\ln\varrho dx = \int_\Omega\big(-\epsilon\Delta\varrho+\epsilon\varrho-\epsilon hK(\varrho)\big)\ln\varrho dx = \int_\Omega\Big(\epsilon\frac{|\nabla\varrho|^2}{\varrho} - \epsilon hK(\varrho)\ln\varrho + \epsilon\varrho\ln\varrho\Big)dx. \tag{2.15}$$

The first term has a good sign (we shall keep in mind this term), the second term has a good sign for $\varrho \leq 1$, too, and for $\varrho \geq 1$ is easily bounded by $\epsilon h\varrho$. Similarly, the last term can be controlled by the term $\epsilon(1+\int_\Omega \varrho^\gamma dx)$. The proof was rather formal, as we do not know whether $\varrho>0$ in Ω. However, we may write $K(\varrho)\boldsymbol{v}\cdot\nabla(\varrho+\delta)$ in (2.12) with $\delta>0$ and find an analogue of (2.15) with $\ln(\varrho+\delta)$. Finally we pass with $\delta\to 0^+$ and get precisely the same information as above.

Next

$$
\begin{aligned}
I_2 = -\int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\nabla s dx &= \int \big(\epsilon\Delta\varrho - \epsilon\varrho + \epsilon h K(\varrho)\big) s dx \\
&= \int_\Omega \big(-\epsilon\nabla\varrho\nabla s - \epsilon\varrho\ln\theta + \epsilon h K(\varrho)\ln\theta\big)dx.
\end{aligned}
\tag{2.16}
$$

Considering the r.h.s. of (2.16), we have

$$
\begin{aligned}
&\left|\epsilon\int \nabla\varrho\nabla s dx\right| \le \epsilon\|\nabla\varrho\|_{L_2(\Omega)}\|\nabla s\|_{L_2(\Omega)} \\
&\le \frac{1}{4}\epsilon\Big(\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}dx + \int_\Omega |\nabla\varrho|^2\varrho^{\gamma-2}dx\Big) + \frac{1}{4}\|\nabla s\|^2_{L_2(\Omega)}.
\end{aligned}
\tag{2.17}
$$

Moreover, $\int_\Omega -\epsilon\varrho\ln\theta dx$ has a good sign for $\theta\le 1$ and for $\theta>1$

$$
\int_\Omega -\epsilon\varrho(\ln\theta)^+dx \le \epsilon\|\varrho\|_{L_2(\Omega)}\|s^+\|_{L_2(\Omega)} \le \frac{\epsilon}{4}\big(\|s^+\|_{L_1(\partial\Omega)}+\|\nabla s\|_{L_2(\Omega)}\big)+\frac{\epsilon}{4}\|\varrho^\gamma\|_{L_1(\Omega)}+C.
\tag{2.18}
$$

The last term of (2.16) can be treated as follows (one part has again a good sign, so we consider only $\theta\in(0,1]$, i.e. $s\le 0$)

$$
\int_\Omega \epsilon h K(\varrho)|(\ln\theta)^-|dx \le C\epsilon\int_\Omega |s^-|dx \le C+\frac{1}{2}\int_{\partial\Omega}\epsilon|s^-|d\sigma + \frac{1}{4}||\nabla s||_{L_2(\Omega)}.
\tag{2.19}
$$

Here we applied a Poincaré type inequality yielding

$$
\|u\|_{L_1(\Omega)} \le c(\Omega)(\|u\|_{L_1(\partial\Omega)}+\|\nabla u\|_{L_2(\Omega)}).
\tag{2.20}
$$

Then combining (2.13) with inequality (2.11) and with (2.15)–(2.19) we obtain

$$
\int_\Omega \left(\frac{\mathbf{S}(\boldsymbol{v}):\nabla\boldsymbol{v}}{\theta}+\frac{1+\theta^m}{\theta^2}|\nabla\theta|^2\right)dx + \int_{\partial\Omega}\left(L(\theta)\theta+\frac{L(\theta)\theta_0}{\theta}+\epsilon|s|\right)d\sigma \le H,
\tag{2.21}
$$

where

$$
H = C\Big(1+\int_\Omega |K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{F}|dx\Big).
$$

The form of the l.h.s. of (2.21) implies that we control also $\|s\|_{L_1(\partial\Omega)}$ and $\|\nabla s\|_{L_2(\Omega)}$; evidently $\|s\|_{L_1(\partial\Omega)}$ is controlled by $\int_{\partial\Omega}(\mathrm{e}^s+\mathrm{e}^{-s})d\sigma$ and $\int_\Omega \frac{|\nabla\theta|^2}{\theta^2}dx = \int_\Omega|\nabla s|^2dx$, which are estimated by the l.h.s. of (2.21).

Step IV. From the growth conditions and (2.21) we deduce the following "homogeneous" estimates:

$$
\begin{aligned}
C\left(\int_{\partial\Omega}\theta^{l+1}d\sigma\right)^{1/(l+1)} &\le \int_{\partial\Omega}L(\theta)\theta d\sigma \le H^{1/(l+1)}, \\
C\left(\int_\Omega|\nabla\theta^{m/2}|^2\right)^{1/m} &\le \int_\Omega\frac{1+\theta^m}{\theta^2}|\nabla\theta|^2dx \le H^{1/m}.
\end{aligned}
$$

We use the following Poincaré type inequality (analogical to (2.20))

$$\left(\int_\Omega |\theta^{m/2}|^2 dx\right)^{1/m} \le C(\Omega)\left(\left(\int_\Omega |\nabla\theta^{m/2}|^2 dx\right)^{1/m} + \left(\int_{\partial\Omega} \theta^{l+1} d\sigma\right)^{1/(l+1)}\right).$$

Then the imbedding theorem $W^1_2(\Omega) \hookrightarrow L_6(\Omega)$ (for $N = 3$) applied to the function $\theta^{m/2}$ leads to the bound

$$\left(\int_\Omega \theta^{3m} dx\right)^{1/3m} \le H^{1/m} + H^{1/(l+1)}. \tag{2.22}$$

To simplify further calculations, we set $l + 1 = m$. Note that we may allow also different values of l, however, for the prize that the further calculations become more technical which we try to avoid.

Step V. We return to (2.9). Hölder's inequality yields[2]

$$\begin{aligned} &\|\boldsymbol{v}\|^2_{H^1(\Omega)} + \epsilon\gamma \int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2 dx + \frac{\epsilon\gamma}{\gamma-1}\int_\Omega \varrho^\gamma dx \\ &\le C\left(1 + \int_\Omega |K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{F}|dx + \int_\Omega |\theta\int_0^\varrho K(t)dt|^2 dx\right). \end{aligned} \tag{2.23}$$

The next step of our estimation is the bound on $P_b(\varrho)$ which is necessary to estimate the r.h.s. of (2.23). We just repeat the method for the barotropic case, but here we shall obtain an extra term related to the temperature.

Introduce $\mathbf{\Phi} : \Omega \to \mathbb{R}^3$ defined as a solution to the following problem

$$\begin{aligned} \operatorname{div}\mathbf{\Phi} &= P_b(\varrho) - \{P_b(\varrho)\} \quad \text{in} \quad \Omega, \\ \mathbf{\Phi} &= \mathbf{0} \quad \text{at} \quad \partial\Omega, \end{aligned} \qquad \text{with} \quad \{P_b(\varrho)\} = \frac{1}{|\Omega|}\int_\Omega P_b(\varrho)dx. \tag{2.24}$$

The basic theory to the stationary Stokes system gives the existence of a vector field satisfying (2.24) with the following estimate for a solution to (2.24) (for another possible proof, using directly estimates of special solutions to system (2.24), see [16])

$$\|\mathbf{\Phi}\|_{H^1_0(\Omega)} \le C\|P_b\|_{L_2(\Omega)}. \tag{2.25}$$

From the structure of $P_b(\varrho)$ and information that $\int_\Omega \varrho_\epsilon dx \le M$ we easily get applying the interpolation inequality

$$\{P_b(\varrho)\} \le \delta\|P_b(\varrho)\|_{L_2(\Omega)} + C(\delta, M) \qquad \text{for any } \delta > 0.$$

Multiplying the momentum equation $(2.2)_2$ by $\mathbf{\Phi}$, employing (2.23) and (2.25), we conclude after standard estimates of the r.h.s to $(2.2)_2$

$$\|P_b(\varrho)\|^2_{L_2(\Omega)} \le C\left(1 + \int_\Omega |K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}|^2 dx + \int_\Omega |\theta\int_0^\varrho K(t)dt|^2 dx\right). \tag{2.26}$$

[2]Note that we used Korn's inequality; for $f = 0$ we therefore require that Ω is not axially symmetric, for more details see [16].

As

$$(2.27)\qquad ||P_b(\varrho)||^2_{L_2(\Omega)} \geq C\left(\int_\Omega (K(\varrho)\varrho)^{2\gamma}dx + \int_\Omega\left(\int_0^\varrho K(t)dt\right)^{2\gamma}dx\right),$$

recalling that $2\gamma > 6$, we get a bound for the first integral in the r.h.s. of (2.26)
(2.28)

$$\int_\Omega |K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}|^2dx \leq c||\boldsymbol{v}||^4_{H^1(\Omega)}||K(\varrho)\varrho||^2_{L_6(\Omega)}$$
$$\leq c||\boldsymbol{v}||^4_{H^1(\Omega)}||K(\varrho)\varrho||^{\frac{2(\gamma-3)}{3(2\gamma-1)}}_{L_1(\Omega)}||K(\varrho)\varrho||^{\frac{10\gamma}{3(2\gamma-1)}}_{L_{2\gamma}(\Omega)} \leq \delta||P_b(\varrho)||^2_{L_2(\Omega)} + C(\delta,M)||\boldsymbol{v}||^{\frac{6(2\gamma-1)}{3\gamma-4}}_{H^1(\Omega)}.$$

Hence a suitable choice of δ in (2.28) simplifies (2.26) to

$$(2.29)\qquad ||P_b(\varrho)||^2_{L_2(\Omega)} \leq C\left(1 + ||\boldsymbol{v}||^{\frac{6(2\gamma-1)}{3\gamma-4}}_{H^1(\Omega)} + \int_\Omega|\theta\int_0^\varrho K(t)dt|^2dx\right).$$

The last estimate can be viewed by (2.27) in the form
(2.30)

$$||\int_0^\varrho K(t)dt||_{L_{2\gamma}(\Omega)} + ||K(\varrho)\varrho||_{L_{2\gamma}(\Omega)} \leq C\left(1 + ||\boldsymbol{v}||^{\frac{3}{\gamma}\frac{2\gamma-1}{3\gamma-4}}_{H^1(\Omega)} + \left(\int_\Omega|\theta\int_0^\varrho K(t)dt|^2dx\right)^{\frac{1}{2\gamma}}\right).$$

Within our estimation we concentrate on a precise specification of powers of norms. Then, due to our growth conditions we shall be able to construct the desired bound (2.7).

Step VI. The last integral in (2.30) can be treated as follows (we need $m > \frac{2}{3}$ and $m > \frac{2\gamma}{3(\gamma-1)}$)

$$(2.31)\qquad \begin{aligned} ||\theta\int_0^\varrho K(t)dt||^{1/\gamma}_{L_2(\Omega)} &\leq ||\theta||^{1/\gamma}_{L_{3m}(\Omega)}||\int_0^\varrho K(t)dt||^{1/\gamma}_{L_{\frac{6m}{3m-2}}(\Omega)} \\ &\leq \|\theta\|^{\frac{1}{\gamma}}_{L_{3m}(\Omega)}||\int_0^\varrho K(t)dt||^{\frac{(3m-2)\gamma-3m}{3m\gamma(2\gamma-1)}}_{L_1(\Omega)}||\int_0^\varrho K(t)dt||^{\frac{3m+2}{3m(2\gamma-1)}}_{L_{2\gamma}(\Omega)}, \end{aligned}$$

so (2.30) and (2.31) with the Young inequality imply

$$||\int_0^\varrho K(t)dt||_{L_{2\gamma}(\Omega)} + ||K(\varrho)\varrho||_{L_{2\gamma}(\Omega)} \leq C\Big(1 + ||\boldsymbol{v}||^{\frac{3}{\gamma}\frac{2\gamma-1}{3\gamma-4}}_{H^1(\Omega)} + ||\theta||^{\frac{3m}{\gamma}\frac{2\gamma-1}{6m(\gamma-1)-2}}_{L_{3m}(\Omega)}\Big).$$

Applying the inequality for the temperature — (2.22) — we obtain (recall that we put $l+1=m$)

$$(2.32)\qquad ||\int_0^\varrho K(t)dt||_{L_{2\gamma}(\Omega)} + ||K(\varrho)\varrho||_{L_{2\gamma}(\Omega)} \leq C\left(1 + ||\boldsymbol{v}||^{\frac{3}{\gamma}\frac{2\gamma-1}{3\gamma-4}}_{H^1(\Omega)} + H^{\frac{3}{\gamma}\frac{2\gamma-1}{6m(\gamma-1)-2}}\right).$$

We have to estimate H; it holds

$$\int_\Omega|K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{F}|dx \leq ||\boldsymbol{v}||_{L_6(\Omega)}||K(\varrho)\varrho||_{L_{6/5}(\Omega)}||\boldsymbol{F}||_{L_\infty(\Omega)}.$$

Using the interpolation between 1 and 2γ as above leads to the following bound

$$(2.33)\qquad \int_\Omega|K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{F}|dx \leq C(M)||\boldsymbol{v}||_{H^1(\Omega)}||K(\varrho)\varrho||^{\frac{\gamma}{3(2\gamma-1)}}_{L_{2\gamma}(\Omega)}.$$

Inserting this inequality to the r.h.s. of (2.32), recalling that $m \geq \frac{1}{4}$ and applying the standard Hölder inequality we obtain from (2.32) estimate on the density

$$(2.34) \qquad \|\int_0^\varrho K(t)dt\|_{L_{2\gamma}(\Omega)} + \|K(\varrho)\varrho\|_{L_{2\gamma}(\Omega)} \leq C\Big(1 + \|\boldsymbol{v}\|_{H^1(\Omega)}^{\frac{3}{\gamma}\frac{2\gamma-1}{3\gamma-4}} + \|\boldsymbol{v}\|_{H^1(\Omega)}^{\frac{1}{\gamma}\frac{2\gamma-1}{2m(\gamma-1)-1}}\Big).$$

Step VII. As we can see later, the first term is the most restrictive. So by (2.33) and (2.34) we conclude (for $m > \frac{3\gamma-1}{6\gamma-6}$)

$$(2.35) \qquad \int_\Omega |K(\varrho)\varrho\boldsymbol{v} \cdot \boldsymbol{F}|dx \leq C\Big(1 + \|\boldsymbol{v}\|_{H^1(\Omega)}^{\frac{3\gamma-3}{3\gamma-4}}\Big).$$

Hence we obtain from (2.22)

$$(2.36) \qquad \|\theta\|_{L_{3m}(\Omega)} \leq C\left(1 + \|\boldsymbol{v}\|_{H^1(\Omega)}^{\frac{1}{m}\frac{3\gamma-3}{3\gamma-4}}\right).$$

From (2.31) we easily see that

$$(2.37) \qquad \|\theta \int_0^\varrho K(t)dt\|_{L_2(\Omega)} \leq C\|\theta\|_{L_{3m}(\Omega)} \|\int_0^\varrho K(t)dt\|_{L_{2\gamma}(\Omega)}^{\frac{3m+2}{3m}\frac{\gamma}{2\gamma-1}}.$$

Step VIII. Summing up inequalities (2.23), (2.35) and (2.37) we obtain the main bound on the norm of the velocity

$$(2.38) \qquad \|\boldsymbol{v}\|_{H^1(\Omega)}^2 \leq C\left(1 + \|\boldsymbol{v}\|_{H^1(\Omega)}^{\frac{3\gamma-3}{3\gamma-4}} + \|\boldsymbol{v}\|_{H^1(\Omega)}^{\frac{2}{m}\frac{3\gamma-3}{3\gamma-4}+\frac{2}{m}\frac{3m+2}{3\gamma-4}}\right).$$

The above bound implies the a priori bound

$$(2.39) \qquad \|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\|\boldsymbol{F}\|_{L_\infty}, M),$$

provided a suitable dependence between γ and m holds. The estimate (2.39) holds as the powers in the r.h.s. of (2.38) are less than 2. It can be described by the sufficient condition ($\gamma > 3$)

$$(2.40) \qquad m > \frac{3\gamma-1}{3\gamma-7}.$$

Note that as we take γ near 3 then $m > 4$ and for $\gamma = 4$ we have $m > \frac{11}{5}$. Moreover, the above needed conditions $m > \frac{3\gamma-1}{6\gamma-1}$, $m > \frac{2}{3}$ and $m > \frac{2\gamma}{3(\gamma-1)}$ are clearly less restrictive than (2.40).

Bound (2.39) implies immediately the a priori estimate (2.7), since it follows from (2.21) with (1.11), (2.29), (2.34)–(2.37), together with (3.7) necessary in the next section. □

3 Existence for the approximative system

The aim of this section is to show that for any $\epsilon > 0$ and $k > 0$ there is a solution to the approximative system (2.2)–(2.6). In particular we ensure the positiveness of the temperature. We prove

Theorem 2 *Let the assumptions of Theorem 1 be satisfied. Moreover, let $\epsilon > 0$ and $k > 0$. Then there exists a strong solution $(\varrho, \boldsymbol{v}, s)$ to (2.2)–(2.6) such that*

$$\varrho \in W^2_p(\Omega), \quad \boldsymbol{v} \in W^2_p(\Omega) \quad \textit{and} \quad s \in W^2_p(\Omega) \quad \textit{for all} \quad 1 \le p < \infty.$$

Moreover $0 \le \varrho \le k$ in Ω, $\int_\Omega \varrho dx \le M$ and

$$||\boldsymbol{v}||_{W^1_{3m}(\Omega)} + \sqrt{\epsilon}||\nabla \varrho||_{L_2(\Omega)} + \|\nabla\theta\|_{L_r(\Omega)} + \|\theta\|_{L_{3m}(\Omega)} \le C(k), \tag{3.1}$$

where $\theta = e^s > 0$, $r = \min\{2, \frac{3m}{m+1}\}$ and the r.h.s. of (3.1) is independent of the parameter ϵ.

The proof of the existence to the approximative system (2.2) will follow from the standard application of the Leray-Schauder fixed point theorem. It will be split into several lemmas. First we consider the continuity equation. We denote for $p \in [1, \infty]$

$$M_p = \{\boldsymbol{w} \in W^2_p(\Omega); \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ at } \partial\Omega\}.$$

We have

Lemma 2 *Let $q > 3$. Then the operator*

$$S : M_q \to W^2_p(\Omega) \qquad \textit{for} \ \ 1 < p < \infty$$

such that $S(\boldsymbol{v}) = \varrho$, where ϱ is the solution to the following problem

$$\begin{aligned} &\epsilon\varrho - \epsilon\Delta\varrho = \epsilon h K(\varrho) - \operatorname{div}(K(\varrho)\varrho\boldsymbol{v}) \quad in \quad \Omega, \\ &\frac{\partial\varrho}{\partial\boldsymbol{n}} = 0 \qquad\qquad\qquad\qquad\qquad\qquad at \quad \partial\Omega \end{aligned} \tag{3.2}$$

is a well defined continuous compact operator from M_q to $W^2_p(\Omega)$, $1 < p < \infty$. In particular, the solution to (3.2) is unique. Moreover

$$\begin{aligned} \|\varrho\|_{W^1_p(\Omega)} &\le C(k,\epsilon)(\|\boldsymbol{v}\|_{L_p(\Omega)} + 1) \qquad 1 < p < \infty, \\ \|\varrho\|_{W^2_p(\Omega)} &\le \begin{cases} C(k,\epsilon)\big(1 + \|\boldsymbol{v}\|_{W^1_p(\Omega)}(1 + \|\boldsymbol{v}\|_{L_3(\Omega)})\big) & 1 < p < 3, \\ C(k,\epsilon)(1 + \|\boldsymbol{v}\|^2_{W^1_p(\Omega)}) & 3 \le p < \infty. \end{cases} \end{aligned} \tag{3.3}$$

Proof. The well possedness of the operator S was proved in [16] for $K \equiv 1$, see also [11], Proposition 3.1 (there the two dimensional case with our function K was considered). The estimates are the direct consequence of the standard elliptic theory, together with the fact that $\|\varrho\|_{L_\infty(\Omega)} \le k$. □

Next, we define the operator

$$\mathcal{T} : M_p \times W^2_p(\Omega) \to M_p \times W^2_p(\Omega) \quad \text{such that} \quad \mathcal{T}(\boldsymbol{v}, s) = (\boldsymbol{w}, z),$$

where $(\boldsymbol{w}, z)$ is the solution to the following system

$$\left.\begin{aligned} -\operatorname{div} \mathbf{S}(\boldsymbol{w}) &= -\frac{1}{2}\operatorname{div}(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}) - \frac{1}{2}K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla P(\varrho, e^s) + K(\varrho)\varrho\boldsymbol{F} \\ -\operatorname{div}\left((1+\mathrm{e}^{ms})(\epsilon+\mathrm{e}^s)\nabla z\right) &= \mathbf{S}(\boldsymbol{v}) : \nabla\boldsymbol{v} - \operatorname{div}\Big(\boldsymbol{v}\int_0^\varrho K(t)dt\Big)\mathrm{e}^s \\ &\quad - \operatorname{div}\big(K(\varrho)\varrho\boldsymbol{v}\big)\mathrm{e}^s - \mathrm{e}^s K(\varrho)\varrho\boldsymbol{v}\cdot\nabla s + \mathrm{e}^s K(\varrho)\boldsymbol{v}\cdot\nabla\varrho \end{aligned}\right\} \text{ in } \Omega, \tag{3.4}$$

$$\left.\begin{aligned} \boldsymbol{w}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{n}\cdot\mathbf{S}(\boldsymbol{w})\cdot\boldsymbol{\tau}_l + f\boldsymbol{w}\cdot\boldsymbol{\tau}_l = 0 \quad \text{for } l = 1,2 \\ (1+\mathrm{e}^{ms})(\epsilon+\mathrm{e}^s)\nabla z + \epsilon z = -L(\mathrm{e}^s)(\mathrm{e}^s - \theta_0) \end{aligned}\right\} \text{ at } \partial\Omega,$$

where $\varrho = \mathcal{S}(\boldsymbol{w})$ is given by Lemma 2. The above procedure guarantees us that the temperature obtained in this way, $\theta = e^z$, will be strictly positive for fixed $\epsilon > 0$.

Our aim is to apply the Leray–Schauder fixed point theorem. Thus we need to verify that $\mathcal{T}$ is a continuous and compact mapping from $M_p \times W^2_p(\Omega)$ to $M_p \times W^2_p(\Omega)$ and that all solutions satisfying

$$t\mathcal{T}(\boldsymbol{w}, z) = (\boldsymbol{w}, z), \qquad t \in [0,1] \quad \text{are bounded in } M_p \times W^2_p(\Omega). \tag{3.5}$$

First we easily have

Lemma 3 *Let $p > 3$ and all assumptions of Theorem 2 be satisfied. Then $\mathcal{T}$ is a continuous and compact operator from $M_p \times W^2_p(\Omega)$ to $M_p \times W^2_p(\Omega)$.*

Proof. Note that for $\epsilon > 0$ system (3.4) is strictly elliptic. Since $p > 3$, the $W^1_p(\Omega)$–space is algebra, thus the r.h.s. of (3.4) belongs to the L_p–space (the boundary term belongs to $W^{1-1/p}_p(\partial\Omega)$). The coefficients in the operator in the l.h.s. of $(3.4)_2$ are of the $C^{1+\alpha}(\overline{\Omega})$–class. Hence the standard theory for elliptic systems gives us the existence of the solution to (3.4) in $M_p \times W^2_p(\Omega)$ with the following bound

$$\begin{aligned} ||\boldsymbol{w}||_{W^2_p(\Omega)} + ||z||_{W^2_p(\Omega)} \le C(||\mathrm{e}^s||_{C^{1+\alpha}(\overline{\Omega})})\Big(&||\text{the r.h.s. of } (3.4)_1||_{L_p(\Omega)} \\ &+||\text{the r.h.s. of } (3.4)_2||_{L_p(\Omega)} + ||\text{the r.h.s. of } (3.4)_4||_{W^{1-1/p}_p(\partial\Omega)}\Big) \end{aligned}$$

which guarantees us the uniqueness and the continuous dependence on the data. Moreover, the r.h.s. of (3.4) is at most of the first order derivative of sought functions. Thus this structure implies the compactness for the map $\mathcal{T}$. □

Next we consider a priori bounds for solutions to (3.5).

Lemma 4 *All solutions to problem (3.5) in the class $M_p \times W^2_p(\Omega)$ satisfy the following bounds*

$$0 \le \varrho \le k, \qquad ||\boldsymbol{w}||_{H^1(\Omega)} + ||\theta||_{L_{3m}(\Omega)} + ||\nabla\theta||_{L_r(\Omega)} + \sqrt{\varepsilon}||\nabla\varrho||_{L_2(\Omega)} \le C(k), \tag{3.6}$$

where $r = \min\{\frac{3m}{m+1}, 2\}$, $\theta = e^z$ and the constant C(k) is independent of ϵ and $t \in [0,1]$.

Proof. We may basically repeat estimates of Lemma 1 from the previous section. Here we may use that $\varrho \leq k$ in Ω, on the other hand we must control the behaviour of all norms with respect to t. Thus, repeating steps (2.8)–(2.13) for the case $t = 1$ (the corresponding terms are only multiplied by t) we finally get

$$
\begin{aligned}
(1-t)\int_\Omega \mathbf{S}(\boldsymbol{w}) : \nabla \boldsymbol{w} dx + \int_{\partial\Omega} f(\boldsymbol{w}\odot\boldsymbol{\tau})^2 d\sigma + \int_\Omega \frac{(1+\theta^m)(\epsilon+\theta)}{\theta}|\nabla z|^2 dx \\
+t\int_\Omega \Big(\frac{\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}}{\theta} + \epsilon\gamma\varrho^{\gamma-2}|\nabla\varrho|^2 + \frac{\epsilon\gamma}{\gamma-1}\varrho^\gamma\Big)dx \\
+\epsilon\int_{\partial\Omega}\big[z_+(1-\mathrm{e}^{-z_+}) + |z_-|(\mathrm{e}^{|z_-|}-1)\big]d\sigma + t\int_{\partial\Omega}\big[L(\theta)\theta - L(\theta)\theta_0 + \frac{L(\theta)\theta_0}{\theta} - L(\theta)\big]d\sigma \\
\leq t\Big|\int_\Omega\Big(K(\varrho)\varrho\boldsymbol{w}\cdot\nabla z - K(\varrho)\boldsymbol{w}\cdot\nabla\varrho\Big)dx\Big| + tC\Big(1+\int_\Omega |K(\varrho)\varrho\boldsymbol{w}\cdot\boldsymbol{F}|dx\Big),
\end{aligned}
$$

where $\varrho = S(\boldsymbol{w})$.

We may now repeat the arguments between (2.14)–(2.21) (all the corresponding terms are only multiplied by t) and we finally get

$$
\begin{aligned}
\int_\Omega \frac{1+\theta^m}{\theta^2}|\nabla\theta|^2 dx + t\int_\Omega \frac{\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w}}{\theta}dx + \int_{\partial\Omega}\Big(tL(\theta)\theta + t\frac{L(\theta)\theta_0}{\theta} + \epsilon|z|\Big)d\sigma \\
\leq tC\Big(1+\int_\Omega |K(\varrho)\varrho\boldsymbol{w}\cdot\boldsymbol{F}|dx\Big).
\end{aligned}
$$

As $0 \leq \varrho \leq k$, we easily get (the Poincaré inequality is just the same as in the previous section), after dividing by t (the case $t = 0$ is clear; recall also $m = l + 1$)

$$
||\theta||_{L_{3m}(\Omega)} \leq C(1 + ||\boldsymbol{w}||_{L_2(\Omega)})^{1/m}
$$

and from an analogue to (2.23) also $||\boldsymbol{w}||^2_{H^1(\Omega)} \leq C(1 + ||\theta||^2_{L_2(\Omega)})$. As $m > 1$, it implies

$$
||\boldsymbol{w}||_{H^1(\Omega)} + ||\theta||_{L_{3m}(\Omega)} \leq C(k).
$$

Further, if $m \geq 2$ then due to the control of $\frac{|\nabla\theta|}{\theta}$ and $|\nabla\theta|\theta^{\frac{m-2}{2}}$ in $L_2(\Omega)$ we have also $\nabla\theta$ bounded in the same space. For $1 < m < 2$, we only get

$$
\|\nabla\theta\|_{L_{\frac{3m}{m+1}}(\Omega)} \leq \||\nabla\theta|\theta^{\frac{m-2}{2}}\|_{L_2(\Omega)}\|\theta\|^{\frac{2-m}{2}}_{L_{3m}(\Omega)} \leq C(k). \tag{3.7}
$$

Finally, multiplying the approximative continuity equation by ϱ and integrating by parts we get

$$
\epsilon\int_\Omega(|\nabla\varrho|^2+\varrho^2)dx \leq \epsilon\int_\Omega hK(\varrho)\varrho dx + \int_\Omega\Big(\int_0^\varrho K(t)t dt\Big)|\operatorname{div}\boldsymbol{w}|dx,
$$

from where we deduce the bound for $\sqrt{\epsilon}\|\nabla\varrho\|_{L_2(\Omega)}$. □

We continue the proof of Theorem 2. To conclude, we verify the bound on $(\boldsymbol{w}, z)$ in $W^2_p(\Omega) \times W^2_p(\Omega)$, $p < \infty$, independently of t. We apply the bootstrap method to system

$$
\left.\begin{aligned}
-\operatorname{div} \mathbf{S}(\boldsymbol{w}) = t\Big[-\frac{1}{2}\operatorname{div}(K(\varrho)\varrho\boldsymbol{w}\otimes\boldsymbol{w}) - \frac{1}{2}K(\varrho)\varrho\boldsymbol{w}\cdot\nabla\boldsymbol{w} \\
-\nabla P(\varrho, \mathrm{e}^z) + K(\varrho)\varrho\boldsymbol{F}\Big] \\
-\operatorname{div}\big((1+\mathrm{e}^{mz})(\epsilon+\mathrm{e}^z)\nabla z\big) = t\Big[\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w} - \operatorname{div}\Big(\boldsymbol{w}\int_0^{\varrho} K(t)dt\Big)\mathrm{e}^z \\
-\operatorname{div}\big(K(\varrho)\varrho\boldsymbol{w}\big)\mathrm{e}^z - \mathrm{e}^z K(\varrho)\varrho\boldsymbol{w}\cdot\nabla z + \mathrm{e}^z K(\varrho)\boldsymbol{w}\cdot\nabla\varrho\Big]
\end{aligned}\right\} \text{ in } \Omega, \tag{3.8}
$$

$$
\left.\begin{aligned}
\boldsymbol{w}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{n}\cdot\mathbf{S}(\boldsymbol{w})\cdot\boldsymbol{\tau}_l + f\boldsymbol{w}\cdot\boldsymbol{\tau}_l = 0 \quad \text{for } l = 1,2 \\
(1+\mathrm{e}^{mz})(\epsilon+\mathrm{e}^z)\nabla z + \epsilon z = -tL(\mathrm{e}^z)(\mathrm{e}^z - \theta_0)
\end{aligned}\right\} \text{ at } \partial\Omega,
$$

where $\varrho = \mathcal{S}(\boldsymbol{w})$ given by Lemma 2. Note first that due to bounds from Lemma 4 we have

$$\|\boldsymbol{w}\|_{W^1_3(\Omega)} \leq C$$

as $K(\varrho)\varrho\boldsymbol{w}\otimes\boldsymbol{w}$ is bounded in $L_3(\Omega)$. Thus $\boldsymbol{w}$ is bounded in any $L_q(\Omega)$, $q < \infty$ and the most restrictive term is $\nabla P(\varrho, \mathrm{e}^z)$. As $\mathrm{e}^z = \theta$ is bounded in $L_{3m}(\Omega)$, ϱ in $L_\infty(\Omega)$, we deduce the bound

$$\|\boldsymbol{w}\|_{W^1_{3m}(\Omega)} \leq C \quad \text{and consequently also} \quad \|\varrho\|_{W^2_{3m}(\Omega)} \leq C.$$

Note that the constant in the estimate for $\boldsymbol{w}$ is independent of ϵ.

Next, we rewrite equation $(3.8)_2$ as follows

$$
\begin{aligned}
-\Delta\Phi(z) = t\Big[\mathbf{S}(\boldsymbol{w}):\nabla\boldsymbol{w} - \mathrm{e}^z K(\varrho)\varrho\boldsymbol{w}\cdot\nabla z + \mathrm{e}^z K(\varrho)\boldsymbol{w}\cdot\nabla\varrho \\
-\operatorname{div}\Big(\boldsymbol{w}\int_0^{\varrho} K(t)dt\Big)\mathrm{e}^z - \operatorname{div}\big(K(\varrho)\varrho\boldsymbol{w}\big)\mathrm{e}^z\Big] \quad \text{in } \Omega, \\
\frac{\partial\Phi(z)}{\partial\boldsymbol{n}} = -\epsilon z - tL(\mathrm{e}^z)(\mathrm{e}^z - \theta_0) \quad \text{at } \partial\Omega
\end{aligned} \tag{3.9}
$$

with

$$\Phi(z) = \int_0^z (1+\mathrm{e}^{m\tau})(\epsilon+\mathrm{e}^\tau)d\tau. \tag{3.10}$$

We multiply $(3.9)_1$ by Φ and integrate over Ω. It leads to

$$||\nabla\Phi||^2_{L_2(\Omega)} + \int_{\partial\Omega}\big(tL(\mathrm{e}^z)(\mathrm{e}^z-\theta_0)\Phi + \epsilon z\Phi\big)d\sigma \leq C||\text{the r.h.s. of } (3.9)_1||_{L_{6/5}(\Omega)}||\Phi||_{L_6(\Omega)}.$$

It is not difficult to realize that the most restrictive term on the r.h.s is $\mathrm{e}^z K(\varrho)\varrho\boldsymbol{w}\cdot\nabla z \in L_{\frac{3m}{m+1}}(\Omega)$, where $\frac{3m}{m+1} > \frac{6}{5}$ for $m > 1$.

Let us look at the boundary terms. Note that $\Phi(s) \sim \epsilon s$ for $s \to -\infty$ and $\Phi(s) \sim \mathrm{e}^{(m+1)s}$ for $s \to +\infty$. Thus

$$\int_{\partial\Omega}\big[tL(\mathrm{e}^s)(\mathrm{e}^s-\theta_0)\Phi + \epsilon s\Phi\big]I_{\{\Phi\leq 0\}}d\sigma \geq C_1\epsilon^2||\Phi I_{\{\Phi\leq 0\}}||_{L_2(\partial\Omega)} - C_2$$

and

$$\int_{\partial\Omega}\big[tL(\mathrm{e}^s)(\mathrm{e}^s-\theta_0)\Phi+\epsilon s\Phi\big]I_{\{\Phi\geq 0\}}d\sigma\geq C_1\epsilon||\Phi I_{\{\Phi\geq 0\}}||_{L_1(\partial\Omega)}-C_2.$$

Thus, the estimates above yield $\|\Phi\|_{W^1_2(\Omega)}\leq C$ with C independent of t which implies

$$\|\theta^{m+1}\|_{L_6(\Omega)}=\|\mathrm{e}^{(m+1)z}\|_{L_6(\Omega)}\leq C\quad\text{and also}\quad\|\nabla\theta\|_{L_2(\Omega)}=\|\mathrm{e}^z\nabla z\|_{L_2(\Omega)}\leq C.$$

Now, it is not difficult to verify that from (3.9) we get $\|\Phi\|_{W^2_{p^*}(\Omega)}\leq C$ with $p^*=\min\{\frac{3m}{2},2\}$ (as $\mathrm{e}^z\nabla z\in L_2(\Omega)$ and $\nabla\boldsymbol{w}\in L_{3m}(\Omega)$). In particular,

$$\|z\|_{L_\infty(\Omega)}+\|\theta\|_{L_\infty(\Omega)}\leq C,\qquad\|\nabla z\|_{L_q(\Omega)}+\|\nabla\theta\|_{L_q(\Omega)}\leq C$$

for $1\leq q\leq q^*=\frac{3p^*}{3-p^*}>3$. Thus from the approximative momentum equation we get (recall $\nabla(\varrho\theta)\in L_{q^*}(\Omega)$) the bound $\|\boldsymbol{w}\|_{W^2_{q^*}(\Omega)}\leq C$ and from the energy/entropy equation also

$$\|z\|_{W^2_{q^*}(\Omega)}+\|\theta\|_{W^2_{q^*}(\Omega)}\leq C.$$

The imbedding theorem yields $\|\nabla z\|_{L_\infty(\Omega)}+\|\nabla\theta\|_{L_\infty(\Omega)}\leq C$ which finally gives as above

$$\|\varrho\|_{W^2_r(\Omega)}+\|\boldsymbol{w}\|_{W^2_r(\Omega)}+\|z\|_{W^2_r(\Omega)}+\|\theta\|_{W^2_r(\Omega)}\leq C,\quad 1\leq r<\infty$$

with C independent of t. This finishes the proof of Theorem 2.

4 Effective viscous flux

In this part we investigate the properties of the effective viscous flux. Estimates (3.1) from Theorem 2 guarantee us existence of a subsequence $\epsilon\to 0^+$ such that

$$\begin{array}{ll}\boldsymbol{v}_\epsilon\rightharpoonup\boldsymbol{v}\quad\text{in } W^1_{3m}(\Omega), & \boldsymbol{v}_\epsilon\to\boldsymbol{v}\quad\text{in } L_\infty(\Omega),\\ \varrho_\epsilon\rightharpoonup^*\varrho\quad\text{in } L_\infty(\Omega), & P_b(\varrho_\epsilon)\rightharpoonup^*\overline{P_b(\varrho)}\quad\text{in } L_\infty(\Omega),\\ K(\varrho_\epsilon)\varrho_\epsilon\rightharpoonup^*\overline{K(\varrho)\varrho}\quad\text{in } L_\infty(\Omega), & K(\varrho_\epsilon)\rightharpoonup^*\overline{K(\varrho)}\quad\text{in } L_\infty(\Omega),\end{array}$$

$$\int_0^{\varrho_\epsilon}K(t)dt\rightharpoonup^*\overline{\int_0^{\varrho}K(t)dt}\quad\text{in } L_\infty(\Omega),\tag{4.1}$$

$$\theta_\epsilon\rightharpoonup\theta\quad\text{in } W^1_r(\Omega)\text{ with } r=\min\{2,\tfrac{3m}{m+1}\},$$

$$\theta_\epsilon\to\theta\quad\text{in } L_q(\Omega)\quad\text{for } q<3m.$$

Here we follow the notation that a weak limit of a sequence $\{A(a_\epsilon)\}_\epsilon$ is denoted by $\overline{A(a)}$ (for a fixed subsequence $\epsilon\to 0^+$).

Passing to the limit in the weak formulation of our problem (2.2) we get

$$\operatorname{div}(\overline{K(\varrho)\varrho}\boldsymbol{v})=0,\tag{4.2}$$

$$\overline{K(\varrho)\varrho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}-\operatorname{div}\Big(2\mu\boldsymbol{D}(\boldsymbol{v})+\nu(\operatorname{div}\boldsymbol{v})\boldsymbol{I}-\overline{P_b(\varrho)}\boldsymbol{I}-\theta\big(\overline{\int_0^{\varrho}K(t)dt}\big)\boldsymbol{I}\Big)=\overline{K(\varrho)\varrho}\boldsymbol{F},\tag{4.3}$$

$$(4.4)\quad -\operatorname{div}((1+\theta^m)\nabla\theta) + \theta\overline{\big(\operatorname{div}\boldsymbol{v}\int_0^{\varrho} K(t)dt\big)} + \operatorname{div}(\overline{K(\varrho)\varrho}\theta\boldsymbol{v}) = 2\mu\overline{|D(\boldsymbol{v})|^2} + \nu\overline{(\operatorname{div}\boldsymbol{v})^2}$$

together with the boundary conditions (1.9)–(1.10). Recall that (4.2)–(4.4) is satisfied in the weak sense, similar to Definition 1.

In what follows we must carefully study the dependence of the a priori bounds on k. We have

Lemma 5 *Under the assumptions of Theorems 1 and 2, we have*

$$(4.5)\qquad ||\varrho_\epsilon||_{L_\infty(\Omega)} \leq k \quad \text{and} \quad ||\boldsymbol{v}_\epsilon||_{W^1_{3m}(\Omega)} \leq C(1 + k^{\frac{\gamma}{3}\frac{3m-2}{m}}).$$

Proof. The bound on the density follows directly from Theorem 2. We therefore estimate the velocity. If we write $(2.2)_2$ in the form

$$-\operatorname{div}\boldsymbol{S}(\boldsymbol{v}) = -\nabla\Big(P_b(\varrho_\epsilon) + \theta_\epsilon\big(\int_0^{\varrho_\epsilon} K(t)dt\big)\Big) + K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{F}$$
$$-\frac{1}{2}\operatorname{div}[K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\otimes\boldsymbol{v}_\epsilon] - \frac{1}{2}K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\cdot\nabla\boldsymbol{v}_\epsilon,$$

we immediately see that

$$\|\boldsymbol{v}_\epsilon\|_{W^1_{3m}(\Omega)} \leq C\big(\|K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\otimes\boldsymbol{v}_\epsilon\|_{L_{3m}(\Omega)} + \|K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\cdot\nabla\boldsymbol{v}_\epsilon\|_{L_{\frac{3m}{m+1}}(\Omega)}$$
$$+\|P_b(\varrho_\epsilon)\|_{L_{3m}(\Omega)} + \|\theta_\epsilon\big(\int_0^{\varrho_\epsilon} K(t)dt\big)\|_{L_{3m}(\Omega)} + \|K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{F}\|_{L_{\frac{3m}{m+1}}(\Omega)}\big).$$

Note that due to the bound of the temperature we cannot expect an ϵ–independent estimate for $q > 3m$. The bounds on the density and temperature yield

$$\|P_b(\varrho_\epsilon)\|_{L_{3m}(\Omega)} \leq \|P_b(\varrho_\epsilon)\|^{\frac{2}{3m}}_{L_2(\Omega)}\|P_b(\varrho_\epsilon)\|^{\frac{3m-2}{3m}}_{L_\infty(\Omega)} \leq Ck^{\gamma\frac{3m-2}{3m}},$$

while

$$\|\theta_\epsilon\big(\int_0^{\varrho_\epsilon} K(t)dt\big)\|_{L_{3m}(\Omega)} \leq Ck.$$

Note that for m and γ satisfying assumptions of Theorem 1, $\gamma\frac{3m-2}{3m} > 1$. It remains to estimate the convective terms $(C.T.)$

$$C.T. \leq \|K(\varrho_\epsilon)\varrho_\epsilon|\boldsymbol{v}_\epsilon|^2\|_{L_{3m}(\Omega)} + \|K(\varrho_\epsilon)\varrho_\epsilon|\boldsymbol{v}_\epsilon||\nabla\boldsymbol{v}_\epsilon|\|_{L_{\frac{3m}{m+1}}(\Omega)}$$
$$\leq C\|\varrho_\epsilon\|_{L_\infty(\Omega)}\big(\|\boldsymbol{v}_\epsilon\|^2_{L_{6m}(\Omega)} + \|\nabla\boldsymbol{v}_\epsilon\|_{L_{\frac{3m}{m+1}}(\Omega)}\|\boldsymbol{v}_\epsilon\|_{L_\infty(\Omega)}\big)$$

for $m \geq 2$, while for $m < 2$ the last term is replaced by $\|\nabla\boldsymbol{v}_\epsilon\|_{L_2(\Omega)}\|\boldsymbol{v}_\epsilon\|_{L_{\frac{6m}{2-m}}(\Omega)}$. Using the fact that for $6 < q \leq \infty$

$$\|\boldsymbol{v}_\epsilon\|_{L_q(\Omega)} \leq C\|\boldsymbol{v}_\epsilon\|^\alpha_{L_6(\Omega)}\|\boldsymbol{v}_\epsilon\|^{1-\alpha}_{W^1_{3m}(\Omega)} \quad \text{with} \qquad \frac{1}{q} = \frac{\alpha}{6} + (1-\alpha)\big(\frac{1}{3m} - \frac{1}{3}\big)$$

and for $2 < r < 3m$

$$\|\nabla\boldsymbol{v}_\epsilon\|_{L_r(\Omega)} \leq \|\boldsymbol{v}_\epsilon\|^\alpha_{L_2(\Omega)}\|\nabla\boldsymbol{v}_\epsilon\|^{1-\alpha}_{L_{3m}(\Omega)} \quad \text{with} \qquad \frac{1}{r} = \frac{\alpha}{2} + \frac{1-\alpha}{3m},$$

we end up with

$$C.T. \leq C\|\varrho_\epsilon\|_{L_\infty(\Omega)}\|\boldsymbol{v}_\epsilon\|^{2\frac{2m-1}{3m-2}}_{W^1_2(\Omega)}\|\boldsymbol{v}_\epsilon\|^{2\frac{m-1}{3m-2}}_{W^1_{3m}(\Omega)}.$$

Note that $\frac{2(m-1)}{3m-2} < 1$. Thus we may use the bound on ϱ_ϵ and Young's inequality yields

$$\|\boldsymbol{v}_\epsilon\|_{W^1_{3m}(\Omega)} \leq C(1 + k^{\frac{\gamma}{3}\frac{3m-2}{m}}) + Ck^{\frac{3m-2}{m}} + \frac{1}{2}\|\boldsymbol{v}_\epsilon\|_{W^1_{3m}(\Omega)}.$$

As $\gamma > 3$, the lemma is proved. □

Before using the bounds proved above, we show one useful result which in particular implies that the limit temperature is positive.

Lemma 6 *There exists a subsequence* $\{s_\epsilon\}$ *such that*

$$s_\epsilon \to s \text{ in } L_2(\Omega),$$

subsequently,

$$\theta_\epsilon \to \theta \text{ in } L_q(\Omega), \quad q < 3m \quad \text{with} \quad \theta > 0 \quad \text{a.e. in } \Omega.$$

Proof. Recall that from the energy bound we have the following information

$$\int_\Omega |\nabla s_\epsilon|^2 dx + \int_{\partial\Omega} (\mathrm{e}^{s_\epsilon} + \mathrm{e}^{-s_\epsilon}) d\sigma \leq C$$

which in particular gives

$$\int_\Omega |\nabla s_\epsilon|^2 dx + \int_{\partial\Omega} s_\epsilon^2 d\sigma \leq C.$$

Thus, remembering that Ω is bounded, we are allowed to choose a subsequence $s_\epsilon \to s$ in $L_2(\Omega)$. Recall also that $\theta_\epsilon = \mathrm{e}^{s_\epsilon}$ and $\theta_\epsilon \to \theta$ strongly in $L_r(\Omega)$, $r < 3m$. Hence by Vitali's theorem (for a subsequence, if necessary)

$$\mathrm{e}^{s_\epsilon} \to \mathrm{e}^s \qquad \text{in } L_r(\Omega) \qquad \text{and} \qquad \theta = \mathrm{e}^s \qquad \text{with } s \in L_2(\Omega).$$

Thus $\theta > 0$ a.e. in Ω, since $s > -\infty$ a.e. in Ω. □

A crucial role in the proof of the strong convergence of the density is played by a quantity called the effective viscous flux. To define it, the Helmholtz decomposition of the velocity is needed

$$\boldsymbol{v} = \nabla\phi + \mathrm{rot}\ \boldsymbol{A}, \tag{4.6}$$

where the divergence-free part of the velocity is given as a solution to the following elliptic problem

$$\begin{aligned} \mathrm{rot\ rot}\ \boldsymbol{A} = \mathrm{rot}\ \boldsymbol{v} = \boldsymbol{\omega} \quad & \text{in } \Omega, \\ \mathrm{div\ rot}\ \boldsymbol{A} = 0 \quad & \text{in } \Omega, \\ \mathrm{rot}\ \boldsymbol{A}\cdot\boldsymbol{n} = 0 \quad & \text{at } \partial\Omega. \end{aligned} \tag{4.7}$$

The potential part of the velocity is given by the solution to

$$
\begin{array}{c} \Delta\phi = \operatorname{div} \boldsymbol{v} \quad \text{in } \Omega, \\ \frac{\partial \phi}{\partial \boldsymbol{v}} = 0 \quad \text{at } \partial\Omega, \end{array} \qquad \int_\Omega \phi dx = 0. \tag{4.8}
$$

The classical theory for elliptic equations [18], [19] gives us for $1 < q < \infty$

$$
||\nabla \operatorname{rot} \boldsymbol{A}||_{L_q(\Omega)} \leq C||\boldsymbol{\omega}||_{L_q(\Omega)}, \qquad ||\nabla^2 \operatorname{rot} \boldsymbol{A}||_{L_q(\Omega)} \leq C||\boldsymbol{\omega}||_{W^1_q(\Omega)},
$$

$$
||\nabla^2\phi||_{L_q(\Omega)} \leq C||\operatorname{div} \boldsymbol{v}||_{L_q(\Omega)}, \qquad ||\nabla^3\phi||_{L_q(\Omega)} \leq C||\operatorname{div} \boldsymbol{v}||_{W^1_q(\Omega)}.
$$

The properties of the slip boundary condition enable us to state the following problem

$$
\begin{gathered}
-\mu\Delta\boldsymbol{\omega}_\epsilon = \operatorname{rot}\big(K(\varrho_\epsilon)\varrho_\epsilon \boldsymbol{F} - K(\varrho_\epsilon)\varrho_\epsilon \boldsymbol{v}_\epsilon \cdot \nabla \boldsymbol{v}_\epsilon \\
-\frac{1}{2}\epsilon h K(\varrho_\epsilon)\boldsymbol{v}_\epsilon + \frac{1}{2}\epsilon\varrho_\epsilon \boldsymbol{v}_\epsilon\big) - \operatorname{rot}(\frac{1}{2}\epsilon\Delta\varrho_\epsilon \boldsymbol{v}_\epsilon) := \boldsymbol{H}_1 + \boldsymbol{H}_2 \quad \text{in } \Omega, \\
\boldsymbol{\omega}_\epsilon \cdot \boldsymbol{\tau}_1 = -(2\chi_2 - f/\mu)\boldsymbol{v}_\epsilon \cdot \boldsymbol{\tau}_2 \quad \text{at } \partial\Omega, \\
\boldsymbol{\omega}_\epsilon \cdot \boldsymbol{\tau}_2 = (2\chi_1 - f/\mu)\boldsymbol{v}_\epsilon \cdot \boldsymbol{\tau}_1 \quad \text{at } \partial\Omega, \\
\operatorname{div} \boldsymbol{\omega}_\epsilon = 0 \quad \text{at } \partial\Omega,
\end{gathered} \tag{4.9}
$$

where χ_k are curvatures related with directions $\boldsymbol{\tau}_k$. For the proof of relations $(4.9)_{2,3}$ – see [12] (also [10]).

The structure of $\boldsymbol{\omega}_\epsilon$ gives us a hint to consider it as a sum of three components

$$
\boldsymbol{\omega}_\epsilon = \boldsymbol{\omega}^0_\epsilon + \boldsymbol{\omega}^1_\epsilon + \boldsymbol{\omega}^2_\epsilon, \tag{4.10}
$$

where they are determined by the following systems

$$
\begin{array}{cccl}
-\mu\Delta\boldsymbol{\omega}^0_\epsilon = 0, & -\mu\Delta\boldsymbol{\omega}^1_\epsilon = \boldsymbol{H}_1, & -\mu\Delta\boldsymbol{\omega}^2_\epsilon = \boldsymbol{H}_2 & \text{in } \Omega, \\
\boldsymbol{\omega}^0_\epsilon \cdot \boldsymbol{\tau}_1 = -(2\chi_2 - f/\mu)\boldsymbol{v}_\epsilon \cdot \boldsymbol{\tau}_2, & \boldsymbol{\omega}^1_\epsilon \cdot \boldsymbol{\tau}_1 = 0, & \boldsymbol{\omega}^2_\epsilon \cdot \boldsymbol{\tau}_1 = 0 & \text{at } \partial\Omega, \\
\boldsymbol{\omega}^0_\epsilon \cdot \boldsymbol{\tau}_2 = (2\chi_1 - f/\mu)\boldsymbol{v}_\epsilon \cdot \boldsymbol{\tau}_1, & \boldsymbol{\omega}^1_\epsilon \cdot \boldsymbol{\tau}_2 = 0, & \boldsymbol{\omega}^2_\epsilon \cdot \boldsymbol{\tau}_2 = 0 & \text{at } \partial\Omega, \\
\operatorname{div} \boldsymbol{\omega}^0_\epsilon = 0, & \operatorname{div} \boldsymbol{\omega}^1_\epsilon = 0, & \operatorname{div} \boldsymbol{\omega}^2_\epsilon = 0 & \text{at } \partial\Omega.
\end{array} \tag{4.11}
$$

Lemma 7 *For the vorticity $\boldsymbol{\omega}_\epsilon$ written in the form (4.10) we have:*[3]

$$
\begin{gathered}
||\boldsymbol{\omega}^2_\epsilon||_{L_r(\Omega)} \leq C(k)\epsilon^{1/2} \quad \textit{for } 1 \leq r \leq 2, \\
||\boldsymbol{\omega}^0_\epsilon||_{W^1_q(\Omega)} + ||\boldsymbol{\omega}^1_\epsilon||_{W^1_q(\Omega)} \leq C(1 + k^{1+\gamma(\frac{4}{3}-\frac{2}{q})}) \quad \textit{for } 2 \leq q \leq 3m.
\end{gathered} \tag{4.12}
$$

Proof. First, let us consider $\boldsymbol{\omega}^0_\epsilon$. Take $\boldsymbol{\alpha}_0$ any divergence–free extension of the boundary data to $\boldsymbol{\omega}_\epsilon$, e.g. in the form of a solution to the following Stokes problem

$$
\begin{array}{cl}
-\mu\Delta\boldsymbol{\alpha}_0 + \nabla p_0 = 0 & \text{in } \Omega, \\
\operatorname{div} \boldsymbol{\alpha}_0 = 0 & \text{in } \Omega, \\
\boldsymbol{\alpha}_0 \cdot \boldsymbol{\tau}_1 = -(2\chi_2 - f/\mu)\boldsymbol{v}_\epsilon \cdot \boldsymbol{\tau}_2 & \text{at } \partial\Omega, \\
\boldsymbol{\alpha}_0 \cdot \boldsymbol{\tau}_2 = (2\chi_1 - f/\mu)\boldsymbol{v}_\epsilon \cdot \boldsymbol{\tau}_1 & \text{at } \partial\Omega, \\
\boldsymbol{\alpha}_0 \cdot \boldsymbol{n} = 0 & \text{at } \partial\Omega.
\end{array} \tag{4.13}
$$

[3]Note that we can prove that $\|\boldsymbol{\omega}^2_\epsilon\|_{L_r(\Omega)} = o(\epsilon)$ for $\epsilon \to 0^+$ for any $r < 3m$. As we do not need it and the proof of the rate is slightly more complicated, we skip it. Analogously we may consider the other inequality also for $q < 2$, with different powers of k.

Note that $\boldsymbol{v}_\epsilon \in W^{1-1/(3m)}_{3m}(\partial\Omega)$, thus $\boldsymbol{\alpha}_0 \in W^1_{3m}(\Omega)$ with the estimate

$$\|\boldsymbol{\alpha}_0\|_{W^1_q(\Omega)} \le C\|\boldsymbol{v}_\epsilon\|_{W^1_q(\Omega)}, \qquad 1 < q \le 3m.$$

Thus we may transform the system for $\boldsymbol{\omega}^0_\epsilon$ to the form

$$\begin{array}{c} -\mu\Delta(\boldsymbol{\omega}^0_\epsilon - \boldsymbol{\alpha}_0) = \mu\Delta\boldsymbol{\alpha}_0 \quad \text{in } \Omega, \\ (\boldsymbol{\omega}^0_\epsilon - \boldsymbol{\alpha}_0)\cdot\boldsymbol{\tau}_1 = 0 \quad \text{at } \partial\Omega, \\ (\boldsymbol{\omega}^0_\epsilon - \boldsymbol{\alpha}_0)\cdot\boldsymbol{\tau}_2 = 0 \quad \text{at } \partial\Omega, \\ \operatorname{div}(\boldsymbol{\omega}^0_\epsilon - \boldsymbol{\alpha}_0) = 0 \quad \text{at } \partial\Omega. \end{array} \tag{4.14}$$

To find the estimates for solutions to (4.14) we consider its weak form, then the r.h.s. of $(4.14)_1$ delivers a nontrivial boundary term. It is well defined, since $\operatorname{div}\boldsymbol{\alpha}_0 = 0$. Then results from [18], [19] guarantee desired bounds. As the system for $\boldsymbol{\omega}^0_\epsilon$ has the same structure as that for $\boldsymbol{\omega}^1_\epsilon$, we get

$$||\boldsymbol{\omega}^1_\epsilon||_{W^1_q(\Omega)} \le C||\boldsymbol{H}_1||_{W^{-1}_q(\Omega)} \quad \text{and} \quad ||\boldsymbol{\omega}^0_\epsilon||_{W^1_q(\Omega)} \le C||\boldsymbol{v}_\epsilon||_{W^1_q(\Omega)}, \quad 1 < q \le 3m.$$

Analyzing the form of $\boldsymbol{H}_1$ we see that the only not elementary term is the convective one; so we obtain

$$||\boldsymbol{\omega}^1_\epsilon||_{W^1_q(\Omega)} \le C(1 + ||K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\cdot\nabla\boldsymbol{v}_\epsilon||_{L_q(\Omega)}).$$

We easily see that for $q \ge 2$

$$||K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\cdot\nabla\boldsymbol{v}_\epsilon||_{L_q(\Omega)} \le k||\boldsymbol{v}_\epsilon||_{L_\infty(\Omega)}||\nabla\boldsymbol{v}_\epsilon||_{L_q(\Omega)}.$$

Using interpolation inequalities as in Lemma 5 we prove that

$$\begin{aligned} ||K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\cdot\nabla\boldsymbol{v}_\epsilon||_{L_q(\Omega)} &\le Ck\|\boldsymbol{v}_\epsilon\|^{\frac{2(m-1)}{3m-2}}_{L_6(\Omega)}\|\nabla\boldsymbol{v}_\epsilon\|^{\frac{m}{3m-2}}_{L_{3m}(\Omega)}\|\nabla\boldsymbol{v}_\epsilon\|^{\frac{6m-2q}{(3m-2)q}}_{L_2(\Omega)}\|\nabla\boldsymbol{v}_\epsilon\|^{\frac{3m(q-2)}{(3m-2)q}}_{L_{3m}(\Omega)} \\ &\le Ck^{1+\gamma(\frac{4}{3}-\frac{2}{q})}. \end{aligned}$$

Evidently, the estimate for $\boldsymbol{\omega}^0_\epsilon$ is less restrictive.

Similarly, for $\boldsymbol{\omega}^2_\epsilon$ we have

$$||\boldsymbol{\omega}^2_\epsilon||_{L_q(\Omega)} \le C||\epsilon\Delta\varrho_\epsilon\boldsymbol{v}_\epsilon||_{W^{-1}_q(\Omega)} \le C\epsilon\sup_\phi|\int_\Omega \Delta\varrho_\epsilon\boldsymbol{v}_\epsilon\phi dx|,$$

where the sup is taken over all functions belonging to $W^1_q(\Omega)$ with $1/p + 1/q = 1$.

From the continuity equation we know that

$$\sqrt{\epsilon}||\nabla\varrho_\epsilon||_{L_2(\Omega)} \le C(k).$$

(For $q > 2$ we have only $\epsilon\|\nabla\varrho_\epsilon\|_{L_q(\Omega)} \le C$.) As $q \le 2$,

$$||\boldsymbol{\omega}^2_\epsilon||_{L_q(\Omega)} \le C\epsilon(\|\nabla\varrho_\epsilon\|_{L_2(\Omega)}\|\boldsymbol{v}_\epsilon\|_{L_\infty(\Omega)} + \|\nabla\varrho_\epsilon\|_{L_2(\Omega)}\|\nabla\boldsymbol{v}_\epsilon\|_{L_{3m}(\Omega)}) \le C(k)\epsilon^{\frac{1}{2}}.$$

The lemma is proved. □

We now introduce the fundamental quantity — the effective viscous flux — which is in fact the potential part of the momentum equation. Using the Helmholtz decomposition in the approximative momentum equation we have

$$\nabla(-(2\mu+\nu)\Delta\phi_\epsilon + P(\varrho_\epsilon,\theta_\epsilon)) = \mu\Delta \operatorname{rot} \boldsymbol{A}_\epsilon + K(\varrho_\epsilon)\varrho_\epsilon \boldsymbol{F}$$

$$-K(\varrho_\epsilon)\varrho_\epsilon \boldsymbol{v}_\epsilon \cdot \nabla \boldsymbol{v}_\epsilon - \frac{1}{2}\epsilon h K(\varrho_\epsilon)\boldsymbol{v}_\epsilon + \frac{1}{2}\epsilon\varrho_\epsilon \boldsymbol{v}_\epsilon - \frac{1}{2}\epsilon\Delta\varrho_\epsilon \boldsymbol{v}_\epsilon.$$

We define

$$G_\varepsilon = -(2\mu+\nu)\Delta\phi_\epsilon + P(\varrho_\epsilon,\theta_\epsilon) = -(2\mu+\nu)\operatorname{div}\boldsymbol{v}_\epsilon + P(\varrho_\epsilon,\theta_\epsilon) \tag{4.15}$$

and its limit version

$$G = -(2\mu+\nu)\operatorname{div}\boldsymbol{v} + \overline{P(\varrho,\theta)}. \tag{4.16}$$

Note that we are able to control integrals $\int_\Omega G_\epsilon dx = \int_\Omega P(\varrho_\epsilon,\theta_\epsilon)dx$ and $\int_\Omega G dx = \int_\Omega \overline{P(\varrho,\theta)}dx$, where $\overline{P(\varrho,\theta)} = \overline{P_b(\varrho)} + \theta\big(\overline{\int_0^\varrho K(t)dt}\big)$.

The result of the lemma below gives the most important properties of the effective viscous flux, guaranteeing the compactness of $\{G_\epsilon\}$ as well as the pointwise bound of the limit in term of the parameter k from definition (2.1).

Lemma 8 *We have, up to a subsequence $\epsilon \to 0^+$:*

$$G_\epsilon \to G \text{ strongly in } L_2(\Omega) \tag{4.17}$$

and

$$||G||_{L_\infty(\Omega)} \le C(\eta)(1+k^{1+\frac{2}{3}\gamma+\eta}) \quad \text{for any } \eta > 0. \tag{4.18}$$

Proof. The function G_ϵ can be naturally decomposed as $G_\epsilon = G^1_\epsilon + G^2_\epsilon$, where $\int_\Omega G^2_\epsilon dx = 0$ and $\nabla G^2_\epsilon = -\frac{1}{2}\epsilon\Delta\varrho_\epsilon \boldsymbol{v}_\epsilon - \mu \operatorname{rot} \boldsymbol{\omega}^2_\epsilon$. Thus

$$||G^2_\epsilon||_{L_q(\Omega)} \le C(\epsilon||\Delta\varrho_\epsilon \boldsymbol{v}_\epsilon||_{W^{-1}_q(\Omega)} + \mu\|\operatorname{rot} \boldsymbol{\omega}^2_\epsilon\|_{W^{-1}_q(\Omega)}).$$

Using Lemma 7 we see that

$$\|G^2_\epsilon\|_{L_q(\Omega)} \le C(k)\epsilon^{\frac{1}{2}}, \qquad 1 \le q \le 2.$$

Next, using again Lemma 7 and calculations in its proof, we immediately see that (recall that $|\int_\Omega G_\epsilon dx| \le C$, we control the average of the r.h.s of (4.15) from the energy bound — Lemma 1)

$$||G^1_\epsilon||_{W^1_q(\Omega)} \le C(1+k^{1+\gamma(\frac{4}{3}-\frac{2}{q})}) \qquad \text{for } 2 \le q \le 3m. \tag{4.19}$$

Thus we have, at least for a subsequence

$$G^1_\epsilon \to G^1 \qquad \text{in } L^\infty(\Omega) \quad \text{and} \quad G^2_\epsilon \to 0 \;\text{ in }\; L_2(\Omega).$$

Therefore

$$G_\epsilon = G^1_\epsilon + G^2_\epsilon \to G^1 \qquad \text{in } L^q(\Omega), \qquad 1 \le q \le 2$$

and due to the definition, $G^1 = G$. Finally, choosing $q = 3 + \tilde\eta$ in (4.19)

$$\|G\|_{L_\infty(\Omega)} \le C(q)\|G\|_{W^1_q(\Omega)} \le C(q)\sup_{\epsilon>0}\|G^1_\epsilon\|_{W^1_q(\Omega)} \le C(\eta)(1+k^{1+\frac{2}{3}\gamma+\eta})$$

with $\eta > 0$, arbitrarily small if $\tilde\eta$ is so. This finishes the proof of Lemma 8. □

5 Limit passage

In this section we apply the properties of the effective viscous flux shown in the previous part. First we prove a result characterizing the sequence of approximative densities.

Theorem 3 *There exists a sufficiently large number $k_0 > 0$ such that for $k > k_0$*

$$\frac{k-3}{k}(k-3)^\gamma - ||G||_{L_\infty(\Omega)} \geq 1 \tag{5.1}$$

and for a subsequence $\epsilon \to 0^+$ it holds

$$\lim_{\epsilon\to 0^+} |\{x \in \Omega : \varrho_\epsilon(x) > k-3\}| = 0. \tag{5.2}$$

In particular it follows: $\overline{K(\varrho)\varrho} = \varrho$ a.e. in Ω.

Proof. We define a smooth function $M : \mathbb{R}_0^+ \to [0,1]$ such that

$$M(t) = \begin{cases} 1 & \text{for} \quad t \leq k-3 \\ \in [0,1] & \text{for} \quad k-3 < t < k-2 \\ 0 & \text{for} \quad k-2 \leq t \end{cases}$$

and $M'(t) < 0$ for $t \in (k-3, k-2)$.

We follow the method introduced in [11]. First we multiply the approximative continuity equation $(2.2)_1$ by $M^l(\varrho_\epsilon)$ for $l \in \mathbb{N}$ getting

$$\int_\Omega \left(\int_0^{\varrho_\epsilon(x)} t\, l\, M^{l-1}(t) M'(t) dt \right) \operatorname{div} \boldsymbol{v}_\epsilon \geq R_\epsilon$$

with $R_\epsilon \to 0$ as $\epsilon \to 0$, as

$$\epsilon \int_\Omega M^l(\varrho_\epsilon) \Delta \varrho_\epsilon dx = -\epsilon l \int_\Omega M^{l-1}(\varrho_\epsilon) M'(\varrho_\epsilon) |\nabla \varrho_\epsilon|^2 dx \geq 0.$$

Next, recalling definitions of G_ϵ and M, we obtain

$$\begin{aligned} -(k-3) \int_\Omega \Big(\int_0^{\varrho_\epsilon(x)} l M^{l-1}(t) M'(t) dt \Big) P(\varrho_\epsilon, \theta_\epsilon) dx \\ \leq k \Big| \int_\Omega \Big(\int_0^{\varrho_\epsilon(x)} -l M^{l-1}(t) M'(t) dt \Big) G_\epsilon dx \Big| + R_\epsilon. \end{aligned}$$

Thus the properties of M lead us to the following inequality

$$\frac{k-3}{k} \int_{\{\varrho_\epsilon > k-3\}} (1 - M^l(\varrho_\epsilon)) P(\varrho_\epsilon, \theta_\epsilon) dx \leq \int_{\{\varrho_\epsilon > k-3\}} (1 - M^l(\varrho_\epsilon)) |G_\epsilon| dx + |R_\epsilon|.$$

From the explicit form of the pressure function (2.3) we find

$$\begin{aligned} \frac{k-3}{k}(k-3)^\gamma |\{\varrho_\epsilon > k-3\}| - \frac{k-3}{k} ||P(\varrho_\epsilon, \theta_\epsilon)||_{L_2(\Omega)} ||M^l(\varrho_\epsilon)||_{L_2(\Omega)} \\ \leq ||G||_{L_\infty(\Omega)} |\{\varrho_\epsilon > k-3\}| + ||G - G_\epsilon||_{L_1(\Omega)} + |R_\epsilon|. \end{aligned}$$

But by Lemma 8 – inequality (4.18) – we are able to choose k_0 so large that for all $k > k_0$ we have (5.1), since $\gamma > 3$ and $||G||_{L_\infty(\Omega)} \leq C_\eta(1 + k^{1+\frac{2}{3}\gamma+\eta})$ with $0 < \eta \leq \frac{\gamma-3}{6}$.

Hence we get

$$(5.3) \qquad |\{x \in \Omega : \varrho_\epsilon(x) > k-3\}| \leq C\left(||M^l(\varrho_\epsilon)||_{L_2(\{\varrho_\epsilon > k-3\})} + ||G - G_\epsilon||_{L_1(\Omega)} + |R_\epsilon|\right).$$

Now, let us fix $\delta > 0$. Then there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$

$$(5.4) \qquad C(||G - G_\epsilon||_{L_1(\Omega)} + |R_\epsilon|) \leq \delta/2.$$

Having ϵ fixed, we consider the sequence $\{M^l(\varrho_\epsilon)I_{\{\varrho_\epsilon > k-3\}}\}_{l\in\mathbb{N}}$, where I_A is the characteristic function of a set A. We see that it monotonely pointwise converges to zero. Thus by the Lebesgue theorem we are able to find $l = l(\epsilon, \delta)$ such that

$$(5.5) \qquad C||M^l(\varrho_\epsilon)||_{L_2(\{\varrho_\epsilon > k-3\})} \leq \delta/2.$$

From (5.3), (5.4) and (5.5) we obtain

$$(5.6) \qquad \lim_{\epsilon\to 0} |\{x \in \Omega; \varrho_\epsilon(x) > k-3\}| \leq \delta.$$

As $\delta > 0$ can be chosen arbitrarily small, Theorem 3 is proved. □

Thanks to Theorem 3 we are prepared to present the main part of the proof, i.e. the pointwise convergence of the density.

Lemma 9 *We have*

$$(5.7) \qquad \int_\Omega \overline{P(\varrho,\theta)\varrho}dx \leq \int_\Omega G\varrho dx \quad \text{and} \quad \int_\Omega \overline{P(\varrho,\theta)}\varrho dx = \int_\Omega G\varrho dx;$$

consequently, $\overline{P(\varrho,\theta)\varrho} = \overline{P(\varrho,\theta)}\varrho$ *and up to a subsequence* $\epsilon \to 0^+$

$$(5.8) \qquad \varrho_\epsilon \to \varrho \quad \text{strongly in } L_q(\Omega) \quad \text{for any} \quad q < \infty.$$

Proof. Due to Theorem 3 we are able to omit $K(\varrho)$ in the limit equation. For details we refer to [11] – Section 4, consideration for (4.16).

Examine the approximative continuity equation $(2.2)_1$. We use as test function $\ln(\varrho_\epsilon + \delta)$ and passing with $\delta \to 0^+$ we obtain

$$(5.9) \qquad \int_\Omega K(\varrho_\epsilon)\boldsymbol{v}_\epsilon \cdot \nabla\varrho_\epsilon dx \geq \epsilon C(k),$$

thus Theorem 3 implies

$$(5.10) \qquad -\int_\Omega \varrho_\epsilon \operatorname{div} \boldsymbol{v}_\epsilon dx \geq R_\epsilon.$$

Applying (4.15) to (5.10), passing with $\epsilon \to 0$, then by the strong convergence of G_ϵ – see (4.17) – we conclude that $\overline{G\varrho} = G\varrho$, so the first relation in (5.7) is proved.

Next, we consider the limit to the continuity equation, i.e. $\operatorname{div}(\varrho \boldsymbol{v}) = 0$. Testing it by $\ln \varrho$ with an application of Friedrich's lemma to have possibility to use test functions with lower regularity we obtain (for details see [11])

$$\int_\Omega \varrho \operatorname{div} \boldsymbol{v} dx = 0.$$

The definition of G – (4.16) – shows the second part of (5.7).

Due to elementary properties of weak limits we get $\varrho \overline{P(\varrho,\theta)} \leq \overline{P(\varrho,\theta)\varrho}$ a.e. in Ω, but (5.7) implies $\int_\Omega (\overline{P(\varrho,\theta)\varrho} - \overline{P(\varrho,\theta)}\, \varrho) dx \leq 0$, hence

$$\varrho \overline{P(\varrho,\theta)} = \overline{P(\varrho,\theta)\varrho} \quad \text{a.e.,} \qquad \text{i.e.} \quad \overline{\varrho^{\gamma+1}} + \overline{\varrho^2}\theta = \overline{\varrho^\gamma}\varrho + \varrho^2\theta \quad \text{a.e.}$$

However, $\overline{\varrho^{\gamma+1}} \geq \overline{\varrho^\gamma}\varrho$ and $\overline{\varrho^2}\theta \geq \varrho^2\theta$, so

$$\overline{\varrho^{\gamma+1}} = \overline{\varrho^\gamma}\varrho \text{ a.e.} \qquad \text{and} \qquad \overline{\varrho^2}\theta = \varrho^2\theta \quad \text{a.e.}$$

By Lemma 6 the temperature $\theta > 0$ a.e., we conclude $\overline{\varrho^2} = \varrho^2$ and for a suitably taken subsequence

$$\lim_{\epsilon \to 0} ||\varrho_\epsilon - \varrho||^2_{L_2(\Omega)} = \|\overline{\varrho^2} - \varrho^2\|_{L_1(\Omega)} = 0. \tag{5.11}$$

Thus the limit (5.11) implies $\varrho_\epsilon \to \varrho$ strongly in $L_2(\Omega)$ and by the pointwise boundedness of ϱ_ϵ and ϱ we conclude (5.8). □

Next, we would like to study the limit of the energy equation. The first observation concerns the velocity, we obtain the strong convergence of its gradient. Recall that from Theorem 3 and due to the strong convergence of the temperature it follows

$$P(\varrho_\epsilon, \theta_\epsilon) \to p(\varrho, \theta) \quad \text{strongly in } L_2(\Omega),$$

hence (4.17) implies

$$\operatorname{div} \boldsymbol{v}_\epsilon \to \operatorname{div} \boldsymbol{v} \qquad \text{strongly in } L_2(\Omega). \tag{5.12}$$

Additionally we already proved that

$$\operatorname{rot} \boldsymbol{v}_\epsilon \to \operatorname{rot} \boldsymbol{v} \qquad \text{strongly in } L_2(\Omega), \tag{5.13}$$

since we observed that the vorticity can be written as sum of two parts, one bounded in $W^1_q(\Omega)$ and the other one going strongly to zero in $L_2(\Omega)$.

The regularity of systems (4.7) and (4.8) and convergences (5.12) and (5.13) imply immediately that

$$\boldsymbol{v}_\epsilon \to \boldsymbol{v} \qquad \text{strongly in } H^1(\Omega).$$

In particular, we get

$$S(\boldsymbol{v}_\epsilon) : \nabla \boldsymbol{v}_\epsilon \to S(\boldsymbol{v}) : \nabla \boldsymbol{v} \quad \text{strongly at least in } L_1(\Omega). \tag{5.14}$$

This fact will be crucial in our considerations for the limit of the energy equation. Recall that

$$
(5.15)\qquad \begin{array}{ll} \varrho_\epsilon \to \varrho \text{ in } L_q(\Omega) \quad \text{for } q<\infty, & \boldsymbol{v}_\epsilon \to \boldsymbol{v} \text{ in } W^1_q(\Omega) \quad \text{for } \; q<3m, \\ \theta_\epsilon \to \theta \text{ in } L_q(\Omega) \;\; \text{for } \; q<3m, & \theta_\epsilon \rightharpoonup \theta \text{ in } W^1_{\min\{2,\frac{3m}{m+1}\}}(\Omega). \end{array}
$$

Consider the weak form of $(2.2)_3$. For a smooth function ϕ we have

$$
(5.16)\qquad \begin{aligned} &\int_\Omega (1+\theta_\epsilon^m)\frac{\epsilon+\theta_\epsilon}{\theta_\epsilon}\nabla\theta_\epsilon\cdot\nabla\phi dx + \int_{\partial\Omega} L(\theta_\epsilon)(\theta_\epsilon-\theta_0)\phi d\sigma + \int_{\partial\Omega}\epsilon\ln\theta_\epsilon\phi d\sigma \\ &-\int_\Omega \left[\left(\int_0^{\varrho_\epsilon(x)} K(t)dt\right)\boldsymbol{v}_\epsilon\cdot\nabla(\theta_\epsilon\phi) + K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\cdot\nabla(\theta_\epsilon\phi)\right]dx \\ &+\int_\Omega\left[K(\varrho_\epsilon)\varrho_\epsilon\boldsymbol{v}_\epsilon\cdot\nabla\theta_\epsilon\phi + \operatorname{div}(\theta_\epsilon\boldsymbol{v}_\epsilon\phi)\int_0^{\varrho_\epsilon(x)}K(t)dt\right]dx = \int_\Omega \boldsymbol{S}(\boldsymbol{v}_\epsilon):\nabla\boldsymbol{v}_\epsilon\phi dx. \end{aligned}
$$

Thanks to (5.15),

$$
(1+\theta_\epsilon^m)\frac{\epsilon+\theta_\epsilon}{\epsilon}\nabla\theta_\epsilon \rightharpoonup (1+\theta^m)\nabla\theta \quad \text{ in } L_1(\Omega).
$$

Passing to the limit with the last four terms of the l.h.s. of (5.16) we get

$$
(5.17)\qquad \begin{aligned} &\int_\Omega \left[-\varrho\boldsymbol{v}\nabla(\theta\phi) - \varrho\boldsymbol{v}\nabla(\theta\phi) + \varrho\phi\boldsymbol{v}\nabla\theta + \operatorname{div}(\theta\phi\boldsymbol{v})\varrho\right]dx \\ &\qquad = \int_\Omega\left[-\varrho\theta\boldsymbol{v}\cdot\nabla\phi + \varrho\theta\operatorname{div}\boldsymbol{v}\phi\right]dx. \end{aligned}
$$

In (5.17) we essentially used the strong convergence of the density.

To control the behavior of the boundary terms we note that due to $(5.15)_2$ we see that $\theta_\epsilon|_{\partial\Omega} \to \theta|_{\partial\Omega}$ strongly in $L_{l+1}(\partial\Omega)$ and by Lemma 6, $\ln\theta_\epsilon$ is bounded in $L_2(\partial\Omega)$. Thus recalling (5.14) we get at the limit

$$
(5.18)\qquad \begin{aligned} &\int_\Omega (1+\theta^m)\nabla\theta\cdot\nabla\phi dx + \int_{\partial\Omega} L(\theta)(\theta-\theta_0)d\sigma - \int_\Omega \varrho\theta\boldsymbol{v}\cdot\nabla\phi dx \\ &\qquad = \int_\Omega \boldsymbol{S}(\boldsymbol{v}):\nabla\boldsymbol{v}\phi dx - \int_\Omega \varrho\theta\operatorname{div}\boldsymbol{v}\phi dx. \end{aligned}
$$

To conclude, note that we may show that the limit functions θ and $\boldsymbol{v}$ belong to $W^1_p(\Omega)$ for any $p<\infty$. To see this, we introduce the function $\Phi(\theta) = \int_0^\theta(1+t^m)dt$, similarly as in Section 3, formula (3.10). Thus from (5.18) we immediately see that $\theta \in L_\infty(\Omega)$ and $\boldsymbol{v} \in W^1_p(\Omega)$ for any $p<\infty$. Using this fact once more in the energy equation, we observe that $\theta \in W^1_p(\Omega)$, $p<\infty$. The positiveness of θ follows from Lemma 6. Theorem 1 is proved.

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