Critical functional framework and maximal regularity in action on systems of incompressible flows

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Abstract. This memoir is mainly devoted to the statement and the proof of new maximal regularity results involving Besov spaces for the evolutionary Stokes system in bounded or exterior domains of $\mathbb{R}^n$. We strive for *time independent* a priori estimates with L^1 time integrability and values in homogeneous Besov spaces. By way of application at the end of the memoir, we solve two systems of fluid mechanics.

Results of this type are known for the whole space case and have proved to be spectacularly powerful to investigate the well-posedness issue in critical spaces for a number of PDEs related to fluid mechanics. They have been extended recently to the half-space setting in [**15**]. In the more involved exterior domain case that we investigate here, we lack a control on the 'low frequencies' of the velocity. To overcome this, we adopt the method introduced by P. Maremonti and V.A. Solonnikov in [**43**], suitably modified to match the Besov space framework. As a consequence, our time-independent estimates hold true only in some suitable intersection of Besov spaces. Let us emphasize that no duality tricks are likely to be applied, because of the use of L_1 space.

As a first and important application of our work, we solve locally for large data or globally for small data, the (slightly) inhomogeneous incompressible Navier-Stokes equations in critical Besov spaces, in an exterior domain. After observing that the L_1 time integrability allows to determine globally the stream lines of the flow, the whole system is recast in the Lagrangian coordinates setting. This, in particular, enables us to consider discontinuous densities, as in [**17**], [**20**].

The second application concerns the proof of a global existence result (for small critical data) for a low Mach number system that has been studied recently in the whole space setting by the first author and X. Liao in [**14**]. The system here is investigated in the Eulerian coordinates.

Contents

CHAPTER 1

Introduction

Describing the motion of a fluid is closely related to the physical and thermodynamical laws governing the conservation of its momentum, energy and mass. We expect in general the information concerning these quantities to be enough to find out the velocity field at each point of the fluid region and, at least, on some time interval $[0, T]$ if the initial time is $t = 0$. This is the Eulerian description of the fluid.

Assuming that the velocity field is known and sufficiently smooth, one may next solve the following Ordinary Differential Equation:

$$\frac{dX}{dt} = v(t, X), \qquad X|_{t=0} = y. \tag{1.1}$$

Here y is the initial position of a particle of the fluid and $X(t, y)$ denotes the position of that particle at time t under the action of the velocity field v. Knowing X, one may thus look at the evolution of an infinitesimal fluid parcel *labelled by its initial position* y as it moves through space and time. This is the Lagrangian description of the fluid under consideration. Equation (1.1) gives the relationship between the two main descriptions of flows, that is the Eulerian and Lagrangian ones. The Eulerian coordinates system (t, x) uses the position x of the material at time t, while the Lagrangian coordinates system (t, y) uses the initial position y of a point of the medium. The change of coordinates is governed by the following identity which is the 'integrated' counterpart of (4.55):

$$x = X(t, y) \quad \text{with} \quad X(t, y) = y + \int_0^t v(\tau, X(\tau, y))\, d\tau. \tag{1.2}$$

At this point we meet mathematics. The basic question is whether those two descriptions are indeed equivalent. We would like to find out conditions under which one may go from one system of coordinates to the other without any loss of information about the motion of the fluid. From the classical theory of Ordinary Differential Equations, we know that, roughly, the minimal assumption is that

$$v \in L_1(0, T; C^1(\Omega)), \tag{1.3}$$

where Ω is the fluid domain. This in particular ensures (1.2) to have a unique solution X that is C^0 in time and C^1 in space (see [**4, 23, 48**] for more general results concerning the flow and transport equations).

In the present work, we would like to find a functional framework – possibly the most general one – ensuring the velocity field to satisfy (1.3) and the system we are looking at, to be well-posed. We would like to focus on models describing the evolution of incompressible flows with nonconstant densities. If we believe those systems to be good at describing mixtures of incompressible homogeneous fluids then the basic question is whether the following initial configuration:

$$\rho_0 = 1 + \sigma\chi_A, \tag{1.4}$$

where σ denotes some constant, and χ_A the characteristic function of some subset A of Ω, has any stability during the evolution. According to the Lagrangian description introduced above, we expect the density to be transported by the velocity field and thus to read

$$\rho(t) = 1 + \sigma\chi_{A(t)} \quad \text{with} \quad A(t) = X(t, A). \tag{1.5}$$

Note that if (1.3) is fulfilled then the flow X is C^1, which ensures for free that $A(t)$ remains uniformly C^1 during the evolution, if it is C^1 initially.

More precisely, the present work originates from investigations of the inhomogeneous incompressible Navier-Stokes system that reads

$$\begin{array}{lll} \rho_t + u\cdot\nabla\rho = 0 & \text{in} & (0,T)\times\Omega, \\ \rho(u_t + u\cdot\nabla u) - \nu\Delta u + \nabla P = 0 & \text{in} & (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in} & (0,T)\times\Omega, \\ u = 0 & \text{at} & (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0, \qquad \rho|_{t=0} = \rho_0 & \text{on} & \Omega, \end{array} \tag{1.6}$$

where $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}^n$ and $P = P(t,x) \in \mathbb{R}$ stand for the density, velocity field and pressure of the fluid, respectively. For simplicity, the given positive viscosity coefficient ν is assumed to be constant.

From the viewpoint of hydrodynamics, the first equation is the mass conservation, the second one is the momentum conservation, and the third equation implies incompressibility of the flow. Given some initial data ρ_0, u_0, we here want to determine $(\rho, u, \nabla P)$ in the case where the fluid domain Ω is a smooth bounded or exterior domain of $\mathbb{R}^n$.

As we plan to investigate the well-posedness issue of (1.6) for possibly discontinuous initial densities such as in (1.4), the classical strong solution theory developed in e.g. [**13, 38**] is too restrictive: some Hölder or at least continuity assumptions on ρ_0 are needed therein. In order to understand what should be the relevant functional framework and tools for our analysis, a crucial point is to investigate the linearization of the momentum equation, namely the following evolutionary Stokes system with Dirichlet boundary conditions:

$$\begin{array}{lll} u_t - \nu\Delta u + \nabla P = f & \text{in} & (0,T)\times\Omega, \\ \operatorname{div} u = g & \text{in} & (0,T)\times\Omega, \\ u = 0 & \text{at} & (0,T)\times\partial\Omega, \\ u = u_0 & \text{on} & \Omega. \end{array} \tag{1.7}$$

Roughly speaking, we have to find out a functional framework so that plugging the obtained solutions in the momentum equation of (1.6) yields (1.7) with source terms f in the same class of regularity as the data we started with. Schauder or L_p estimates for the Laplace operator are exactly in that spirit: they ensure that if $f \in C^\alpha$ (resp. $f \in L^p$) then the solution to $\Delta u = f$, with, say, homogeneous Dirichlet boundary data satisfies $\nabla^2 u \in C^\alpha$ (resp. $\nabla^2 u \in L_p$), see [**33**].

In the setting of (1.7), we expect u_t, $\nabla^2 u$ and ∇P to have the same regularity as f (at least if $g \equiv 0$ and $u_0 \equiv 0$). In the most common cases (Ω stands for $\mathbb{R}^n$, $\mathbb{R}^n_+$, a bounded or exterior domain of $\mathbb{R}^n$), this fact – *the maximal regularity property* – is now classical if $f \in L_q(0,T; L_p(\Omega))$ with $1 < p, q < \infty$. For instance, if $g \equiv 0$ then there exists some constant C *independent of* T so that any solution to (1.7) satisfies (see [**31, 39, 42**]):

$$\|u_t, \nu\nabla^2 u, \nabla P\|_{L_q(0,T;L_p(\Omega))} \leq C\big(\|u_0\|_{\dot{B}^{2-\frac{2}{q}}_{p,q}(\Omega)} + \|f\|_{L_q(0,T;L_p(\Omega))}\big). \tag{1.8}$$

Those inequalities are based on Calderon-Zygmund theory for singular integrals (see [**24, 55**]) and related to the analyticity properties of the semi-group of the Stokes operator [**3**]. Therefore, unsurprisingly, they fail in the endpoint cases where one of the exponents p or q is 1 or ∞.

Compared to those results, the natural regularity required on the velocity for (1.2) to be uniquely solvable, is rather exotic: we need an L_1 in time bound. As we aim at working in the critical regularity framework, we cannot afford any loss in the estimates: this precisely means that we need an extension of (1.8) to the case $p = 1$, with a gain of two full spatial derivatives with respect to the data. In other words, we look for a Banach space X with the property that any smooth enough solution $(u,\nabla P)$ to (1.7) satisfies (if $g\equiv 0$ for simplicity):

$$\|u_t,\nu\nabla^2u,\nabla P\|_{L_1(0,T;X)}\leq C\big(\|u_0\|_X+\|f\|_{L_1(0,T;X)}\big)\quad\text{for all}\quad T \tag{1.9}$$

with a constant C *independent of* T.

On the one hand, Inequality (1.9) fails whenever X is a reflexive Banach space. On the other hand, it is known to be true in the whole space setting if X is a homogeneous Besov space *with third index* 1, namely $\dot B^s_{p,1}(\mathbb{R}^n)$ (see e.g. [**5**]). The proof relies on a very simple argument based on Fourier analysis, which is recalled at the end of the proof of Theorem 4.1.1. In our recent work [**15**], we extended that inequality to the half-space $\mathbb{R}^n_+$ setting in the case $g\equiv 0$. There, we had to restrict ourselves to values of s close to 0 (namely $-1+1/p<s<1/p$), a limitation corresponding to the case where functions of $\dot B^s_{p,1}(\mathbb{R}^n_+)$ do not have a trace at the boundary and may thus be extended by 0 over $\mathbb{R}^n$, with no loss of regularity. One of the difficulties that we had to face is that, in contrast with the $\mathbb{R}^n$ case, the half-space case is not amenable to the heat equation by projection, and cannot be reduced to the $\mathbb{R}^n$ case by a suitable symmetric/antisymmetric extension either. A great deal of the analysis was related to the use of the Fourier transform *with respect to tangential variables.*

In the present paper, we want to extend Inequality (1.9) with $X=\dot B^s_{p,1}(\Omega)$ to the case where Ω is a smooth bounded or exterior domain (the second case being worse from the point of view of mathematical analysis). In passing, we also treat the case

$$g\not\equiv 0$$

which is of interest for some applications that we have in mind, and that is also needed in some intermediate steps of our proof. The general strategy is the same as in our recent paper dedicated to the heat equation [**19**] but, owing to the divergence constraint, the proof is much more involved and requires a very careful preliminary analysis of the Poisson equation in domains and recent results of ours for the (generalized) divergence equation [**16**]. The main idea consists in localizing the equation by means of a resolution of unity. We then have to deal with 'interior terms', the support of which do not intersect $\partial\Omega$, and 'boundary terms' with support intersecting $\partial\Omega$. After extension by zero, the interior terms may be handled according to the maximal regularity estimates for $\mathbb{R}^n$. In contrast, the extension of the boundary terms by zero does not satisfy (1.7) on $\mathbb{R}^n$. However, performing a change of variable reduces their study to that of (1.7) in the half-space $\mathbb{R}^n_+$. Unsurprisingly, owing to the change of variables and to the localization procedure, one cannot obtain directly the desired inequality (here written in the case $g\equiv 0$ for expository purposes), namely

$$\|u_t,\nu\nabla^2u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\leq C\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)}+\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\big). \tag{1.10}$$

In fact we get it only either *up to a low order term involving* u or with a *time-dependent* constant C. In the bounded domain case however, one may take advantage of the exponential decay for the Stokes semi-group so as to remove the time dependency. It is no longer the case in exterior domains, and we have to use *mixed* Besov norms and restrict ourselves to dimension $n\geq 3$. The key to this control of the lower order term is borrowed from the work by P. Maremonti and V. Solonnikov in [**43**] based on L_p-L_q estimates for the Stokes operator, which will enable us to bound the $L_1(0,T;\dot B^s_{q,1}(\Omega))$ norm of u if $1<q<n/2$ and s close enough to 0.

As an example, let us state the result that we get in the particular case where $g\equiv 0$ (the general statement being given in Theorem 4.3.3):

THEOREM. *Let* Ω *be an exterior domain of* $\mathbb{R}^n$ *with* $n\geq 3$. *Let* $1<q\leq p<\infty$ *with* $q<n/2$, *and* $s\in(-1+1/p,1/p)$. *Assume that* $u_0\in\dot B^s_{p,1}\cap\dot B^0_{q,1}(\Omega)$ *with* $\operatorname{div}u_0=0$ *and* $u_0\cdot\vec n|_{\partial\Omega}=0$ *(here*

$\vec{n}$ is the unit outer normal vector at $\partial\Omega$) and that $f \in L_1(0,T;\dot{B}^s_{p,1} \cap \dot{B}^0_{q,1}(\Omega))$. Then System (1.7) with $g \equiv 0$ has a unique solution $(u, \nabla P)$ with

$$u \in \mathcal{C}(\mathbb{R}_+;\dot{B}^s_{p,1} \cap \dot{B}^0_{q,1}(\Omega)) \quad \textit{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_1(\mathbb{R}_+;\dot{B}^s_{p,1} \cap \dot{B}^0_{q,1}(\Omega)).$$

In addition, we have for all positive T:

$$\|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}\cap\dot{B}^0_{q,1}(\Omega))} + \|(u_t, \nu\nabla^2 u, \nabla P)\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^0_{q,1}(\Omega))} \leq C\big(\|u_0\|_{\dot{B}^s_{p,1}\cap\dot{B}^0_{q,1}(\Omega)} + \|f\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^0_{q,1}(\Omega))}\big),$$

where the above constant C is independent of T and ν.

Such time independent estimates are of importance not only for the Stokes semi-group theory but also in a number of applications related to fluid mechanics. Having a time-independent constant in (1.9) is crucial for proving the global existence of strong solutions for systems related to the incompressible Navier-Stokes equations. In effect, the fact that two full derivatives may be gained with respect to the source term allows to consider not only the Stokes operator but also small perturbations of it.

Generalizing the above theorem to the case $g \not\equiv 0$ will enable us to establish new well-posedness results for the inhomogeneous Navier-Stokes equations (1.6) in some critical functional framework related to the scaling of the equations (see Chapter 5 for more details). In the slightly inhomogeneous case, that is if ρ_0 is close enough to some positive constant, we shall prove global well-posedness for small initial velocity, and local well-posedness for large velocity. We shall see that choosing $s = -1 + n/p$ in the above statement (which corresponds to the critical regularity framework mentioned above) ensures the velocity field to be L_1-in-time with values in the set of Lipschitz functions. Hence, it admits a unique Lipschitzian flow for all time, and the system satisfied by u may thus be reformulated equivalently in Lagrangian variables as explained at the beginning of the introduction, exactly as in our recent work in [**17**] dedicated to the whole space setting. (Note that the divergence free property is obviously lost when performing the change of coordinates explained in (1.2). This, in itself, is a good motivation for considering nonzero divergence in (1.7).) This will enable us to handle discontinuous initial densities and to justify (1.5) for the solutions to (1.6) whenever ρ_0 satisfies (1.4) with ∂A uniformly C^1, and we will get for free that $\partial A(t)$ remains C^1 during the time evolution. In other words, the inhomogeneous incompressible Navier-Stokes equations may be used for describing a free boundary problem for two incompressible homogeneous fluids with different densities separated by some interface. Last but not least, our approach based on Lagrangian coordinates will enable us to recast our problem in terms of a suitable contracting mapping, and we will thus get uniqueness of the solution and continuous dependence with respect to the data, with no additional regularity assumption whatsoever.

Another interesting application is related to the initial state $\rho_0 = 1 + Z$ in the case where the nonnegative function Z is bounded and $\operatorname{Supp} Z$ is a connected set (a model for the description of pollution in a homogeneous liquid). The uniqueness of solutions implies that, during the time evolution, the polluted area cannot be split into several components because the support of $\rho(\cdot, t) - 1$ remains connected. Besides, we get some control on the growth of the diameter and on the speed of propagation of the polluted area.

Let us mention that the local-in-time well-posedness for large jumps of the density has been proved in [**20**], by another approach that is not compatible with the optimal functional setting. Another interesting development toward this issue has been done recently in [**52**].

As a second application, we solve globally in critical spaces a low Mach number system corresponding to the case of large entropy variations for a heat conducting and viscous fluid. This model which is a nonlinear coupling between a heat equation (for the temperature) and a Stokes-like equation (for the velocity) has been investigated recently in [**14**] in the whole space (see also [**2**] and the references therein). The equations are here studied in the Eulerian coordinates, and in exterior domains. The main difficulty encountered is that the incompressibility condition is violated by the structure of the model. Nevertheless, the generalization of Theorem 1 to nonzero divergence constraints obtained in Theorem 4.3.3 turns out to be appropriate to solve the system. In passing, we have to establish new maximal regularity results (in the spirit of Theorem 1), for the heat equation with Neumann boundary conditions, involving higher order norms. This will be done by combining the methods of the present Chapter 4 and of [**19**].

We end this introduction with a short presentation of the content of the memoir. In Chapter 2, we introduce most definitions and tools that will be needed in the paper. Besov spaces (and basic properties) are presented, first on $\mathbb{R}^n$, and next, on domains. Here we also recall some results of ours concerning the divergence equation, and finally present changes of coordinates that will be useful in the analysis of the Stokes equation, and of the inhomogeneous Navier-Stokes equations. The next chapter is dedicated to the study of the Laplace equation with Neumann of Dirichlet boundary conditions. We mainly aim at proving estimates in homogeneous Besov spaces, under different kind of assumptions which are not so classical. Those estimates will be one of the keys to the proof of maximal regularity estimates for the evolutionary Stokes system (Chapter 4). In the last two chapters, we give some applications of those estimates. In Chapter 5, we establish the global well-posedness in a critical framework for (1.6), thus generalizing our recent result in [**17**]. In Chapter 6, we prove a similar result for a low Mach number limit system, but in the Eulerian framework. This model has been studied recently in e.g. [**2, 14**].

Acknowledgments. The authors are indebted to Matthias Hieber, Paolo Maremonti, François Murat, Yoshihiro Shibata and Vladimir Šverak for fruitful discussions which greatly contributed to improve our memoir. The first and second author have been supported by *Institut Universitaire de France* and *Polish Ministry of Sciences grant IdeasPLUS2011/000661*, respectively.

CHAPTER 2

Tools and spaces

In this chapter, we present basic definitions and tools that will be needed for proving our main results. We first introduce the Littlewood-Paley decomposition (a dyadic decomposition with respect to the Fourier variable) and homogeneous Besov spaces over $\mathbb{R}^n$, then state several classical and fundamental properties : density results, embedding, product estimates, and so on. In the second part of this chapter, we extend the definition of Besov spaces and some of their properties to general domains of $\mathbb{R}^n$. In the third section, we recall some results for the divergence equation, after our recent study in [**16**]. The last part of this chapter is devoted to the definition of change of coordinates that will be used a number of times in this paper to transform a PDE problem at the boundary of a domain into a problem over the whole space $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+$. In passing, we introduce the Lagrangian coordinates needed in Chapter 5, and derive related algebraic relations.

2.1. Besov spaces on $\mathbb{R}^n$

2.1.1. Definition and classical properties. Throughout we fix a smooth nonincreasing function $\chi : \mathbb{R}_+ \to [0,1]$ supported in $[0,1)$ and such that $\chi \equiv 1$ on $[0,1/2)$, and set

$$\varphi(\xi) := \chi(|\xi|/2) - \chi(|\xi|).$$

Note that this implies that φ is valued in $[0,1]$, supported in $\{1/2 \leq |\xi| \leq 2\}$ and that

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1 \quad \text{for all} \quad \xi \neq 0. \tag{2.1}$$

Then we introduce the homogeneous Littlewood-Paley decomposition $(\dot\Delta_k)_{k\in\mathbb{Z}}$ over $\mathbb{R}^n$ by setting

$$\dot\Delta_k u := \varphi(2^{-k}D)u := \mathcal{F}^{-1}\big(\varphi(2^{-k}\cdot)\mathcal{F}u\big).$$

Above $\mathcal{F}$ stands for the Fourier transform on $\mathbb{R}^n$. We also define the low frequency cut-off

$$\dot S_k := \chi(2^{-k}D). \tag{2.2}$$

In order to define Besov spaces on $\mathbb{R}^n$, we first introduce the following homogeneous semi-norms and nonhomogeneous Besov norms (for all $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$):

$$\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)} := \big\|2^{sk}\|\dot\Delta_k u\|_{L_p(\mathbb{R}^n)}\big\|_{\ell_r(\mathbb{Z})} \quad \text{and} \quad \|u\|_{B^s_{p,r}(\mathbb{R}^n)} := \big\|2^{sk}\|\dot\Delta_k u\|_{L_p(\mathbb{R}^n)}\big\|_{\ell_r(\mathbb{N})} + \|\dot S_0 u\|_{L_p(\mathbb{R}^n)}.$$

Note that Besov semi-norms vanish on the set of polynomials, which may be quite troublesome if dealing with solutions to PDEs. To overcome this, in order to control low frequencies, we shall adopt the following definition borrowed from [**5**]:

$$\dot B^s_{p,r}(\mathbb{R}^n) = \Big\{u \in \mathcal{S}'_h(\mathbb{R}^n) \,:\, \|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)} < \infty\Big\},$$

where $\mathcal{S}'_h(\mathbb{R}^n)$ stands for the set of tempered distributions u over $\mathbb{R}^n$ such that for all smooth compactly supported function θ over $\mathbb{R}^n$, we have

$$\lim_{\lambda\to+\infty} \theta(\lambda D)u = 0 \quad \text{in} \quad L_\infty(\mathbb{R}^n). \tag{2.3}$$

That condition is obviously satisfied whenever $\theta(D)u \in L_p(\mathbb{R}^n)$ for some $p<\infty$ and $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\theta(0)\neq 0$. Note also that any distribution in $\mathcal{S}'_h(\mathbb{R}^n)$ tends weakly to 0 at infinity. In particular, $\mathcal{S}'_h(\mathbb{R}^n)$ contains no nonzero polynomial and if $u \in \mathcal{S}'_h(\mathbb{R}^n)$ then one may write

$$u=\sum_{k\in\mathbb{Z}}\dot\Delta_k u \quad \text{in} \quad \mathcal{S}'_h(\mathbb{R}^n). \tag{2.4}$$

Conversely, if (2.4) is satisfied and $\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}<\infty$ for some index s such that $s<n/p$ (or $s\leq n/p$ if $r=1$) then u is in $\dot B^s_{p,r}(\mathbb{R}^n)$.

The following fundamental properties are proved in e.g. [**5, 10, 11**]:

PROPOSITION 2.1.1. *Basic properties.*

(1) *Completeness: the space $\dot B^s_{p,r}(\mathbb{R}^n)$ is complete whenever*

$$s\leq n/p \ \ \textit{if} \ \ r=1, \quad \textit{or} \quad s<n/p \ \ \textit{if} \ \ r>1. \tag{2.5}$$

(2) *Density: the set $\mathcal{S}_0(\mathbb{R}^n)$ of Schwartz functions with Fourier transform supported away from the origin is dense in $\dot B^s_{p,r}(\mathbb{R}^n)$ if and only if p and r are finite.*

(3) *Action of derivatives: for any $k\in\{1,\cdots,n\}$, the derivative operator ∂_k maps $\dot B^s_{p,r}(\mathbb{R}^n)$ in $\dot B^{s-1}_{p,r}(\mathbb{R}^n)$. Besides, we have for some constant $C\geq 1$ independent of u:*

$$C^{-1}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\leq\|\nabla u\|_{\dot B^{s-1}_{p,r}(\mathbb{R}^n)}\leq C\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

(4) *Embedding: if $p_1\leq p_2$ and $r_1\leq r_2$ then $\dot B^{s+n/p_1}_{p_1,r_1}(\mathbb{R}^n)$ is continuously embedded in $\dot B^{s+n/p_2}_{p_2,r_2}$.*

(5) *Comparison with Lebesgue spaces:*
   - *for any $1\leq p\leq\infty$, we have*

$$\|u\|_{\dot B^0_{p,\infty}(\mathbb{R}^n)}\lesssim\|u\|_{L_p(\mathbb{R}^n)}\lesssim\|u\|_{\dot B^0_{p,1}(\mathbb{R}^n)};$$

   - *for any $1<p<\infty$, we have*

$$\dot B^0_{p,\min(p,2)}(\mathbb{R}^n)\hookrightarrow L_p(\mathbb{R}^n)\hookrightarrow \dot B^0_{p,\max(p,2)}(\mathbb{R}^n);$$

   - *if $1\leq p<\infty$ and $0<s<n/p$ then*

$$\dot B^s_{p,p}(\mathbb{R}^n)\hookrightarrow L_{p^*}(\mathbb{R}^n) \quad \textit{with} \quad \frac{n}{s}\left(\frac1p-\frac1{p^*}\right)=1.$$

(6) *Duality: for all $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$, we have*

$$\left|\int_{\mathbb{R}^n} uv\,dx\right|\leq C\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\|v\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)}.$$

Besides the space $\dot B^{-s}_{p',r'}(\mathbb{R}^n)$ coincides with the set of $u\in\mathcal{S}'_h(\mathbb{R}^n)$ such that

$$\sup_v\left|\int_{\mathbb{R}^n} uv\,dx\right|<\infty \tag{2.6}$$

where the supremum is taken over functions v in $\mathcal{S}(\mathbb{R}^n)\cap\dot B^{-s}_{p',r'}(\mathbb{R}^n)$ with $\|v\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)}\leq 1$, and the left-hand side of (2.6) is equivalent to $\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}$.

(7) *Fatou property: under Condition (2.5), there exists a constant C such that for any bounded sequence $(u_j)_{j\in\mathbb{N}}$ of $\dot B^s_{p,r}(\mathbb{R}^n)$ converging to some u in $\mathcal{S}'(\mathbb{R}^n)$, we have*

$$\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\leq C\liminf_{j\to+\infty}\|u_j\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

In some parts of the paper, we shall also use the more classical *nonhomogeneous* Besov spaces $B^s_{p,r}(\mathbb{R}^n)$ which are defined by

$$B^s_{p,r}(\mathbb{R}^n)=\Big\{u\in\mathcal{S}'(\mathbb{R}^n)\ :\ \|u\|_{B^s_{p,r}(\mathbb{R}^n)}<\infty\Big\}\cdot$$

Those spaces have the above properties with no restriction (2.5). Furthermore both the set $\mathcal{C}^\infty_c(\mathbb{R}^n)$ of smooth functions with compact support, and the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ are dense in $B^s_{p,q}(\mathbb{R}^n)$ whenever p and q are finite.

2.1.2. Product laws. We shall make an extensive use of the following inequalities, sometimes named *tame estimates* because of their linear dependence with respect to the highest norm.

PROPOSITION 2.1.2. *Let $b^s_{p,r}$ denote $\dot B^s_{p,r}(\mathbb{R}^n)$ or $B^s_{p,r}(\mathbb{R}^n)$. Then the following estimates hold true:*

- *For any $s>0$,*
$$\|uv\|_{b^s_{p,r}}\lesssim\|u\|_{L_\infty}\|v\|_{b^s_{p,r}}+\|v\|_{L_\infty}\|u\|_{b^s_{p,r}}.$$
- *For any $s>0$ and $t>0$,*
$$\|uv\|_{b^s_{p,r}}\lesssim\|u\|_{L_\infty}\|v\|_{b^s_{p,r}}+\|v\|_{b^{-t}_{\infty,r}}\|u\|_{b^{s+t}_{p,\infty}}.$$
- *For any $t>0$ and $s>-n/p'$,*
$$\|uv\|_{b^s_{p,r}}\lesssim\|u\|_{L_\infty}\|v\|_{b^s_{p,r}}+\|u\|_{b^{n/p'}_{p',\infty}}\|v\|_{b^s_{p,r}}+\|v\|_{b^{-t}_{\infty,r}}\|u\|_{b^{s+t}_{p,\infty}}.$$

Proof: The proof is based on continuity results for the paraproduct and on Bony's decomposition that has been introduced in [**8**]:

$$uv=T_uv+R(u,v)+T_vu.$$

Above, T and R stand for the paraproduct and remainder operators, respectively, that may be defined in the homogeneous case by

$$T_uv:=\sum_{j\in\mathbb{Z}}\dot S_{j-1}u\,\dot\Delta_jv\quad\text{and}\quad R(u,v)=\sum_{j\in\mathbb{Z}}\sum_{|i|\leq1}\dot\Delta_ju\,\dot\Delta_{j+i}v,$$

and in the nonhomogeneous case by

$$T_uv=\sum_{j\geq1}\dot S_{j-1}u\,\Delta_jv\quad\text{and}\quad R(u,v)=\sum_{j\geq-1}\sum_{|i|\leq1}\Delta_ju\,\Delta_{j+i}v,$$

with $\Delta_k=\dot\Delta_k$ if $k\geq0$, $\Delta_{-1}=\dot S_0$ and $\Delta_k=0$ if $k\leq-2$. Recall that $\dot S_k$ has been defined in (2.2).

So, in order to prove the above estimates, it suffices to use the classical properties of continuity for R and T, namely in the cases we are interested in:

$$\begin{aligned}
&\|T_uv\|_{b^s_{p,r}}\lesssim\|u\|_{L_\infty}\|v\|_{b^s_{p,r}}\quad\text{and}\quad\|T_uv\|_{b^s_{p,r}}\lesssim\|u\|_{b^{-t}_{\infty,r}}\|v\|_{b^{s+t}_{p,\infty}}\quad\text{if}\ \ t>0,\\
&\|R(u,v)\|_{b^s_{p,r}}\lesssim\|u\|_{L_\infty}\|v\|_{b^s_{p,r}}\quad\text{and}\quad\|R(u,v)\|_{b^s_{p,r}}\lesssim\|u\|_{b^{-t}_{\infty,r}}\|v\|_{b^{s+t}_{p,\infty}}\quad\text{if}\ \ s>0,\\
&\|R(u,v)\|_{b^s_{p,r}}\lesssim\|u\|_{b^{n/p'}_{p',\infty}}\|v\|_{b^s_{p,r}}\quad\text{if}\ \ s>-n/p'.
\end{aligned}$$

As an example, let us establish the second inequality for T_uv in the homogeneous case. The reader may refer to [**5, 53, 54**] for the proof of the other inequalities. Owing to the support properties of the function φ entering in the definition of $\dot\Delta_j$, we may write for all $j\in\mathbb{Z}$ and some large enough integer N:

$$\dot\Delta_j(T_uv)=\sum_{|k-j|\leq N}\dot\Delta_j(\dot S_{k-1}u\dot\Delta_kv).$$

Hence

$$2^{js}\|\dot\Delta_j(T_uv)\|_{L_p(\mathbb{R}^n)}\leq C\sum_{|k-j|\leq N}2^{(j-k)s}\,2^{-kt}\|\dot S_{k-1}u\|_{L_\infty(\mathbb{R}^n)}\,2^{k(s+t)}\|\dot\Delta_kv\|_{L_p(\mathbb{R}^n)},$$

and one may thus assert that

$$\|T_uv\|_{\dot B^s_{p,r}}\leq C\Big\|2^{-kt}\|\dot S_{k-1}u\|_{L_\infty(\mathbb{R}^n)}\,2^{k(s+t)}\|\dot\Delta_kv\|_{L_p(\mathbb{R}^n)}\Big\|_{\ell^r(\mathbb{Z})}.$$

We may further write

$$2^{-kt}\|\dot S_{k-1}u\|_{L_\infty(\mathbb{R}^n)}\leq\sum_{k'\leq k-2}2^{(k'-k)t}\,2^{-k't}\|\dot\Delta_{k'}u\|_{L_\infty(\mathbb{R}^n)}.$$

Because $t>0$, taking the $\ell_r(\mathbb{Z})$ norm of both sides and using convolution inequalities for series completes the proof. ■

As obviously a smooth compactly supported function belongs to any space $\dot B^{n/p}_{p,1}(\mathbb{R}^n)$ with $1\leq p\leq\infty$, and to any Besov space $B^\sigma_{p,1}(\mathbb{R}^n)$, we deduce from the previous proposition and embedding the following localization properties[1]:

COROLLARY 2.1.1. *Let θ be in $\mathcal{C}^\infty_c(\mathbb{R}^n)$. Then $u\mapsto\theta\,u$ is a continuous mapping of $b^s_{p,r}(\mathbb{R}^n)$*

- *for any $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$, if $b^s_{p,r}(\mathbb{R}^n)=B^s_{p,r}(\mathbb{R}^n)$;*
- *for any $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$ satisfying $-n/p'<s<n/p$ ($-n/p<s\leq n/p$ if $r=1$ and $-n/p'\leq s<n/p$ if $r=\infty$) if $b^s_{p,r}(\mathbb{R}^n)=\dot B^s_{p,r}(\mathbb{R}^n)$.*

The above proposition will also enable us to compare the spaces $B^s_{p,r}(\mathbb{R}^n)$ and $\dot B^s_{p,r}(\mathbb{R}^n)$ for compactly supported functions[2]:

PROPOSITION 2.1.3. *Let $1\leq p,r\leq\infty$ and $s>-n/p'$ (or $s\geq-n/p'$ if $r=\infty$). Then for any* compactly supported *distribution f we have*

$$f\in B^s_{p,r}(\mathbb{R}^n)\iff f\in\dot B^s_{p,r}(\mathbb{R}^n)$$

and there exists a constant $C=C(s,p,r,n,K)$ (with $K=\operatorname{Supp}f$) such that

$$C^{-1}\|f\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\leq\|f\|_{B^s_{p,r}(\mathbb{R}^n)}\leq C\|f\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

Proof: Let us first treat the case $s>0$. Then the embedding $B^s_{p,r}(\mathbb{R}^n)\hookrightarrow\dot B^s_{p,r}(\mathbb{R}^n)$ is clear. Conversely, assume that f belongs to $\dot B^s_{p,r}(\mathbb{R}^n)$. In order to prove that $f\in B^s_{p,r}(\mathbb{R}^n)$, it suffices to establish that $f\in L_p(K)$. This is in fact obvious as one may write that

$$f=\dot S_0f+(\mathrm{Id}-\dot S_0)f.$$

The first term belongs to $L_\infty(\mathbb{R}^n)$ (as f is in $\mathcal{S}'_h$) hence to $L_p(K)$ and the Fourier transform of the second term is supported away from the origin hence belongs to $L_p(\mathbb{R}^n)$ since $s>0$. We claim that there exists some constant C depending only on p, K and s, such that

$$\|f\|_{L_p(K)}\leq C\|f\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

Let us write that $f=\dot S_jf+(\mathrm{Id}-\dot S_j)f$ for some $j\in\mathbb{Z}$ to be chosen hereafter. We have

$$\begin{aligned}\|f\|_{L_p(K)}&\leq\|\dot S_jf\|_{L_p(K)}+\|(\mathrm{Id}-\dot S_j)f\|_{L_p(\mathbb{R}^n)}\\&\leq|K|^{\frac1p}\|\dot S_jf\|_{L_\infty(\mathbb{R}^n)}+C2^{-js}\|f\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.\end{aligned}$$

[1]In the nonhomogeneous case with very negative s, we need to resort to other continuity results for R than those that have been recalled above.

[2]Without any support assumption, it is obvious that if s is positive then we have $\|\cdot\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\lesssim\|\cdot\|_{B^s_{p,r}(\mathbb{R}^n)}$, and the opposite inequality holds true if s is negative.

Using Bernstein's inequalities and, again, the fact that $\operatorname{Supp} f \subset K$, we thus get

$$\begin{aligned}\|f\|_{L_p(K)} &\leq C|K|^{\frac{1}{p}} 2^{jn} \|f\|_{L_1(\mathbb{R}^n)} + C2^{-js}\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} \\ &\leq C|K| 2^{jn} \|f\|_{L_p(K)} + C2^{-js}\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)}.\end{aligned}$$

So choosing j so that $2^{-n} < 2C|K|2^{jn} \leq 1$, we get

$$\|f\|_{L^p} \leq C|K|^{\frac{s}{n}} \|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)}.$$

Let us now focus on the case $s < 0$, it is clear that any (not necessarily compactly supported) distribution in $\dot{B}^s_{p,r}(\mathbb{R}^n)$ belongs to $B^s_{p,r}(\mathbb{R}^n)$, too. So, conversely, consider some distribution $f \in B^s_{p,r}(\mathbb{R}^n)$ with compact support and fix some cut-off function $\eta \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with value 1 on $\operatorname{Supp} f$. We decompose f into

$$f = \eta \dot{S}_0 f + \eta(\mathrm{Id} - \dot{S}_0) f. \tag{2.7}$$

Note that $\mathrm{Id} - \dot{S}_0$ maps $B^s_{p,r}(\mathbb{R}^n)$ in $\dot{B}^s_{p,r}(\mathbb{R}^n)$, as there are no low frequencies. As $\eta \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, Corollary 2.1.1 implies that the last term in (2.7) belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ and that for some constant $C = C(s, p, n, \eta)$,

$$\|\eta(\mathrm{Id} - \dot{S}_0) f\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} \leq C\|f\|_{B^s_{p,r}(\mathbb{R}^n)}. \tag{2.8}$$

Next, because $\dot{S}_0 f$ is a $\mathcal{C}^\infty$ bounded function, $\eta \dot{S}_0 f$ is in $L_1(\mathbb{R}^n)$. Hence, by embedding, it is also in $\dot{B}^{-n/p'}_{p,\infty}(\mathbb{R}^n)$, and we may thus write

$$\|\eta \dot{S}_0 f\|_{\dot{B}^{-n/p'}_{p,\infty}(\mathbb{R}^n)} \leq C\|\eta \dot{S}_0 f\|_{L_1(\mathbb{R}^n)} \leq C\|f\|_{B^s_{p,r}(\mathbb{R}^n)}.$$

Of course $\eta \dot{S}_0 f$ also belongs to $L_p(\mathbb{R}^n)$ hence to all the intermediate Besov spaces (with obvious estimates) between $\dot{B}^{-n/p'}_{p,\infty}(\mathbb{R}^n)$ and $L_p(\mathbb{R}^n)$, and in particular to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if $-n/p' < s < 0$.

The limit case $s = 0$ follows by interpolation. ∎

In the applications of the product laws that we have in mind, it is not always necessary to specify exactly in which Besov space the two terms of the product belong. Typically, given u in some Banach space X, and some function ϕ, we just need to know that ϕu belongs to the same space X. This motivates the following definition of a *multiplier space*.

DEFINITION 2.1.1. *Let X be a Banach space. We designate by $\mathcal{M}(X)$ (multiplier space for X) the set of those tempered distributions ϕ so that ϕu is in X for all $u \in X$.*

The space $\mathcal{M}(X)$ is naturally endowed with a structure of Banach space if equipped with the following norm:

$$\|\phi\|_{\mathcal{M}(X)} = \sup_{\|u\|_X = 1} \|\phi u\|_X.$$

Even for the most classical spaces (e.g. Sobolev spaces), the associated multiplier space is very complicated and describing it by means of 'standard' functional spaces is not possible (see e.g. [**44**]). From the first item of Proposition 2.1.2, one may assert that $\mathcal{M}(b^s_{p,r}(\mathbb{R}^n))$ contains $L_\infty(\mathbb{R}^n) \cap b^s_{p,r}(\mathbb{R}^n)$ if $s > 0$ and $1 \leq p, r \leq \infty$, but this is far from being optimal.

A direct application of Lemma 2.2.1 below ensures that if A is a subset of $\mathbb{R}^n$ with uniformly C^1 boundary then

$$1_A \in \mathcal{M}(\dot{B}^s_{p,q}(\mathbb{R}^n)) \quad \text{for all } s \in \big(-1 + 1/p, 1/p\big) \quad \text{with } 1 < p < \infty \quad \text{and} \quad 1 \leq q \leq \infty. \tag{2.9}$$

The following lemma will be useful when transforming a problem on the boundary to a problem on the half-space and also to justify the equivalence between the eulerian and lagrangian formulation of the systems of PDEs that we shall study in the last chapter.

LEMMA 2.1.1. *Let Z be a diffeomorphism over $\mathbb{R}^n$ and $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ with $-n/p'<s<n/p$. The linear map $u\mapsto u\circ Z$ is continuous on $\dot B^s_{p,r}(\mathbb{R}^n)$ if the following additional conditions are fulfilled:*

- $0<s<1$ *and $J_{Z^{-1}}$, DZ are bounded,*
- $-1<s<0$, *J_Z, DZ^{-1} are bounded and $J_{Z^{-1}}\in\mathcal{M}(\dot B^{-s}_{p',r'}(\mathbb{R}^n))$.*

Proof: Let us first consider the case $s\in(0,1)$ and p,r finite (the limit cases being left to the reader). Using the characterization of homogeneous Besov semi-norms by means of finite differences (see e.g. [**5, 55**]), one may write up to an irrelevant constant:

$$\|u\circ Z\|^r_{\dot B^s_{p,r}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\biggl(\int_{\mathbb{R}^n}\frac{|u(Z(y))-u(Z(x))|^p}{|y-x|^{n+sp}}\,dy\biggr)^{\frac{r}{p}}dx.$$

So performing the change of variable $x'=Z(x)$ and $y'=Z(y)$, we see that

$$\|u\circ Z\|^r_{\dot B^s_{p,r}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\biggl(\int_{\mathbb{R}^n}\frac{|u(y')-u(x')|^p}{|Z^{-1}(y')-Z^{-1}(x')|^{n+sp}}J_{Z^{-1}}(y')\,dy'\biggr)^{\frac{r}{p}}J_{Z^{-1}}(x)\,dx',$$

whence

$$\|u\circ Z\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\leq\|J_{Z^{-1}}\|^{\frac1p+\frac1r}_{L_\infty(\mathbb{R}^n)}\|DZ\|^{s+\frac{n}{p}}_{L_\infty(\mathbb{R}^n)}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}.$$

Let us emphasize that the condition that $s<n/p$ ensures in addition that u belongs to some Lebesgue space $L_{p^*}(\mathbb{R}^n)$ with $p^*<\infty$. Hence $u\circ Z\in L_{p^*}(\mathbb{R}^n)$, too, and one may thus conclude that $u\circ Z\in\dot B^s_{p,r}(\mathbb{R}^n)$.

The result for $s\in(-1,0)$ may be achieved by duality: we have

$$\|u\circ Z\|_{\dot B^s_{p,r}(\mathbb{R}^n)}=\sup_{\|v\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)}\leq1}\int_{\mathbb{R}^n}v(z)u(Z(z))\,dz.$$

Now, setting $x=Z(z)$ yields

$$\begin{aligned}\int_{\mathbb{R}^n}v(z)u(Z(z))\,dz&=\int_{\mathbb{R}^n}u(x)v(Z^{-1}(x))J_{Z^{-1}(x)}\,dx\\&\leq\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\|J_{Z^{-1}}\,v\circ Z^{-1}\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)}\\&\leq C\|J_Z\|^{\frac{1}{p'}+\frac{1}{r'}}_{L_\infty(\mathbb{R}^n)}\|DZ^{-1}\|^{-s+\frac{n}{p'}}_{L_\infty(\mathbb{R}^n)}\|J_{Z^{-1}}\|_{\mathcal{M}(\dot B^{-s}_{p',r'}(\mathbb{R}^n))}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\|v\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)}\end{aligned}$$

the last inequality being a consequence of the first part of the proof and of the definition of multiplier spaces.

In order to show that $u\circ Z\in\mathcal{S}'_h$, one may use the fact that

$$\|\dot S_j(u\circ Z)\|_{L_\infty(\mathbb{R}^n)}=\sup_{\|v\|_{L_1(\mathbb{R}^n)}=1}\int_{\mathbb{R}^n}u\circ Z\,\dot S_jv\,dx$$

and follow the above computations. We still get

$$\int_{\mathbb{R}^n}u\circ Z\,\dot S_jv\,dx\leq\|J_Z\|^{\frac{1}{p'}+\frac{1}{r'}}_{L_\infty(\mathbb{R}^n)}\|DZ^{-1}\|^{-s+\frac{n}{p'}}_{L_\infty(\mathbb{R}^n)}\|J_{Z^{-1}}\|_{\mathcal{M}(\dot B^{-s}_{p',r'}(\mathbb{R}^n))}\|u\|_{\dot B^s_{p,r}(\mathbb{R}^n)}\|\dot S_jv\|_{\dot B^{-s}_{p',r'}(\mathbb{R}^n)}.$$

By using Bernstein inequality and the fact that v is in $L_1(\mathbb{R}^n)$, it is not difficult to conclude that $\|\dot S_j(u\circ Z)\|_{L_\infty(\mathbb{R}^n)}\to0$ when j goes to $-\infty$. This completes the proof. ■

REMARK 2.1.1. *The above lemma naturally extends to $s = 0$ by interpolation. It also may be generalized to higher order regularities if making stronger assumptions on Z. For instance, if assuming that $1 < s < 2$ then the map $u \to u \circ Z$ is continuous on $\dot{B}^s_{p,r}(\mathbb{R}^n)$ whenever $J_{Z^{-1}}$ and DZ are bounded, and*

$$DZ \in \mathcal{M}(\dot{B}^{s-1}_{p,r}(\mathbb{R}^n)).$$

Likewise, if $-2 < s < -1$ then $u \to u \circ Z$ is continuous on $\dot{B}^s_{p,r}(\mathbb{R}^n)$ whenever J_Z and DZ^{-1} are bounded, and

$$J_{Z^{-1}} \in \mathcal{M}(\dot{B}^{-s}_{p',r'}(\mathbb{R}^n)) \quad \text{and} \quad DZ^{-1} \in \mathcal{M}(\dot{B}^{-s-1}_{p,r}(\mathbb{R}^n)).$$

Proof: If $1 < s < 2$ then we look for a bound of $D(u \circ Z)$ in $\dot{B}^{s-1}_{p,r}$. Using the chain rule $D(u \circ Z) = (Du \circ Z) \cdot DZ$, the definition of multiplier spaces and the previous lemma, we may write

$$\begin{aligned} \|D(u \circ Z)\|_{\dot{B}^{s-1}_{p,r}(\mathbb{R}^n)} &\leq C\|DZ\|_{\mathcal{M}(\dot{B}^{s-1}_{p,r}(\mathbb{R}^n))}\|Du \circ Z\|_{B^{s-1}_{p,r}(\mathbb{R}^n)}, \\ &\leq C\|DZ\|_{\mathcal{M}(\dot{B}^{s-1}_{p,r}(\mathbb{R}^n))}\|J_{Z^{-1}}\|^{\frac{1}{p}+\frac{1}{r}}_{L_\infty(\mathbb{R}^n)}\|DZ\|^{s-1+\frac{n}{p}}_{L_\infty(\mathbb{R}^n)}\|Du\|_{B^{s-1}_{p,r}(\mathbb{R}^n)}. \end{aligned}$$

As for the case $-2 < s < -1$, we argue by duality:

$$\begin{aligned} \int_{\mathbb{R}^n} v(z)u(Z(z))\,dz &= \int_{\mathbb{R}^n} u(x)v(Z^{-1}(x))J_{Z^{-1}}(x)\,dx, \\ &\leq \|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)}\|v \circ Z^{-1}\|_{\dot{B}^{-s}_{p',r'}(\mathbb{R}^n)}\|J_{Z^{-1}}\|_{\mathcal{M}(\dot{B}^{-s}_{p',r'}(\mathbb{R}^n))}. \end{aligned}$$

As $1 < -s < 2$, applying the result for positive indices of regularity to $v \circ Z^{-1}$ enables us to conclude. ∎

2.2. Besov spaces on domains

We aim at extending the definition of homogeneous Besov spaces to general domains. We proceed by restriction as follows[3]:

DEFINITION 2.2.1. *For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,q}(\Omega)$ over Ω to be the restriction (in the distributional sense) of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ to Ω, that is*

$$\phi \in \dot{B}^s_{p,q}(\Omega) \iff \phi = \psi_{|\Omega} \quad \text{for some} \quad \psi \in \dot{B}^s_{p,q}(\mathbb{R}^n).$$

We then set

$$\|\phi\|_{\dot{B}^s_{p,q}(\Omega)} := \inf_{\psi_{|\Omega}=\phi} \|\psi\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}.$$

As in the $\mathbb{R}^n$ case, the Besov spaces defined above are Banach spaces with the Fatou property whenever Condition (2.5) is satisfied. Moreover, interpolation and embedding properties may be deduced from those that have been stated in Proposition 2.1.1.

Owing to the definition by restriction, we expect the product estimates to be the same as in the whole space setting. For example, for $s > 0$,

$$\|uv\|_{\dot{B}^s_{p,q}(\Omega)} \lesssim \|u\|_{L_\infty(\Omega)}\|v\|_{\dot{B}^s_{p,q}(\Omega)} + \|v\|_{L_\infty(\Omega)}\|u\|_{\dot{B}^s_{p,q}(\Omega)}. \tag{2.10}$$

So we consider some extensions $\bar{u}$ and $\bar{v}$ in $\mathbb{R}^n$ of u and v with the same regularity. Obviously, $\bar{u}\bar{v}$ is an extension of uv over $\mathbb{R}^n$. However, in general it is not clear that the restriction to Ω of $X(\mathbb{R}^n) \cap Y(\mathbb{R}^n)$ coincides with $X(\Omega) \cap Y(\Omega)$. By using extensions by zero, we will be able to justify (2.10) for $0 < s < 1/p$ (see Corollary 2.2.1 below). For larger values of s and if the domain is sufficiently smooth then there exists a bounded extension operator $E : B^s_{p,r}(\Omega) \to B^s_{p,r}(\mathbb{R}^n)$ [**1**],[**55**].

In most situations, the following result will be sufficient for our purposes:

[3] Nonhomogeneous Besov spaces on domains may be defined by the same token.

Proposition 2.2.1. *Let $b^s_{p,r}(\Omega)$ denote $\dot B^s_{p,r}(\Omega)$ or $B^s_{p,r}(\Omega)$, and Ω be a domain of $\mathbb{R}^n$. Then for any $p\in[1,\infty]$, s such that $-n/p'<s<n/p$ (or $-n/p'<s\le n/p$ if $r=1$, or $-n/p'\le s<n/p$ if $r=\infty$), the following inequality holds true:*

$$\|uv\|_{b^s_{p,r}(\Omega)}\le C\|u\|_{b^{n/q}_{q,1}(\Omega)}\|v\|_{b^s_{p,r}(\Omega)}\quad \textit{with}\ \ q=\min(p,p').$$

Proof. Let us assume for instance that $p\le 2$ (so that $p'\ge p$). Let us consider some extensions $\widetilde u$ and $\widetilde v$ of u and v in $b^{n/p}_{p,1}(\mathbb{R}^n)$ and $b^s_{p,r}(\mathbb{R}^n)$, respectively. Then applying the last item of Proposition 2.1.2 to $\widetilde u$ and $\widetilde v$ with $t=n/p-s$, and noticing that our assumption on p guarantees that

$$b^{n/p}_{p,1}(\mathbb{R}^n)\hookrightarrow b^{n/q}_{q,\infty}(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)\quad\text{for}\ \ q=p,p',$$

we get $uv=(\widetilde u\widetilde v)|_\Omega$ and

$$\|\widetilde u\,\widetilde v\|_{b^s_{p,r}(\mathbb{R}^n)}\le C\|\widetilde u\|_{b^{n/p}_{p,1}(\mathbb{R}^n)}\|\widetilde v\|_{b^s_{p,r}(\mathbb{R}^n)}.$$

As this inequality holds true (with the same constant) for *any* extensions of u and v, we get the result. □

We shall often use the following compact embedding (see [**53**]).

Proposition 2.2.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Then for any $s\in\mathbb{R}$, $(p,q)\in[1,+\infty]^2$ and $\varepsilon>0$, the space $B^s_{p,q}(\Omega)$ is compactly embedded in $B^{s-\varepsilon}_{p,q}(\Omega)$.*

In addition, any bounded sequence $(u_n)_{n\in\mathbb{N}}$ of $B^s_{p,q}(\Omega)$ converges weakly star (up to an omitted extraction) to some u in $B^s_{p,q}(\Omega)$ and we have

$$\|u\|_{B^s_{p,q}(\Omega)}\le C\liminf\|u_n\|_{B^s_{p,q}(\Omega)}\quad\textit{and}\quad u_n\longrightarrow u\ \ \textit{in any}\ \ B^{s-\varepsilon}_{p,q}(\Omega).$$

As already pointed out in the previous section, interpolation properties are a very useful feature of Besov spaces. We refer to [**6, 55**] for the proof of the following statement.

Proposition 2.2.3. *Let $b^s_{p,q}$ denote $B^s_{p,q}(\Omega)$ or $\dot B^s_{p,q}(\Omega)$; $s\in\mathbb{R}$, $p\in(1,\infty)$ and $q\in[1,\infty]$. The real interpolation of Besov spaces gives if $s_1\ne s_2$:*

$$\big(b^{s_1}_{p,q_1}(\Omega),b^{s_2}_{p,q_2}(\Omega)\big)_{\theta,q}=b^s_{p,q}(\Omega)\quad\textit{with}\ \ s:=\theta s_2+(1-\theta)s_1\ \ \textit{and}\ \ \frac1p:=\frac{\theta}{p_2}+\frac{1-\theta}{p_1}\cdot$$

Moreover, if $s_1\ne s_2$, $t_1\ne t_2$ and if $T:b^{s_1}_{p_1,q_1}(\Omega)+b^{s_2}_{p_2,q_2}(\Omega)\to b^{t_1}_{k_1,l_1}(\Omega)+b^{t_2}_{k_2,l_2}(\Omega)$ is a linear map, bounded from $b^{s_1}_{p_1,q_1}(\Omega)$ to $b^{t_1}_{k_1,l_1}(\Omega)$ and from $b^{s_2}_{p_2,q_2}(\Omega)$ to $b^{t_2}_{k_2,l_2}(\Omega)$ then for any $\theta\in(0,1)$, the map T is also bounded from $b^s_{p,q}(\Omega)$ to $b^t_{k,q}(\Omega)$ with

$$s=\theta s_2+(1-\theta)s_1,\quad t=\theta t_2+(1-\theta)t_1,\quad \frac1p=\frac{\theta}{p_2}+\frac{1-\theta}{p_1},\quad \frac1k=\frac{\theta}{k_2}+\frac{1-\theta}{k_1}\cdot$$

An important question is whether one is allowed to extend functions in domains, by 0, without changing their regularity. In the flat case, the following statement (see [**15**]) gives the answer.

Lemma 2.2.1. *For $\varepsilon>0$, denote $\Phi_\varepsilon(u):x\mapsto\eta_\varepsilon(x_n)u(x)$ with*

$$\eta_\varepsilon(t):=\begin{cases}0 & \textit{for}\quad t<\varepsilon,\\ \frac1\varepsilon t-1 & \textit{for}\quad \varepsilon\le t\le 2\varepsilon,\\ 1 & \textit{for}\quad t>2\varepsilon.\end{cases}$$

Then for all $1\le p<\infty$, $1\le q\le\infty$ and $-1+1/p<s<1/p$ the operator Φ_ε maps $\dot B^s_{p,q}(\mathbb{R}^n)$ in $\dot B^s_{p,q}(\mathbb{R}^n)$ uniformly with respect to ε. Moreover, if q is finite then for all $u\in\dot B^s_{p,q}(\mathbb{R}^n)$, we have

$$\Phi_\varepsilon(u)\longrightarrow_{\varepsilon\to0}\Phi_0(u):=1_{\mathbb{R}^n_+}u\quad\textit{in}\quad \dot B^s_{p,q}(\mathbb{R}^n).$$

As a corollary, we readily get that $1_{\mathbb{R}^n_+}$ is in $\mathcal{M}(\dot{B}^s_{p,q}(\mathbb{R}^n))$ whenever (s,p,q) are as above. More generally, as already pointed out, Lemma 2.2.1 implies that 1_A is in $\mathcal{M}(\dot{B}^s_{p,q}(\mathbb{R}^n))$ if A is any uniformly C^1 domain of $\mathbb{R}^n$. Indeed, the C^1 regularity allows to transform locally the boundary to that of the half-space case (see Lemma A.7 in [**17**] for more details).

Now, if we consider some uniformly C^1 domain Ω and $u \in \dot{B}^s_{p,q}(\Omega)$ and some arbitrary extension $\widetilde{u} \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ of u, then we deduce that $\widetilde{u}1_\Omega$ is still in $\dot{B}^s_{p,q}(\mathbb{R}^n)$ (with the expected control of the norm). Therefore we proved the following result[4]:

Corollary 2.2.1. *For any uniformly C^1 domain Ω, $(p,q) \in [1,\infty]^2$ and $s \in (-1+1/p, 1/p)$, the extension by 0 operator is continuous from $\dot{B}^s_{p,q}(\Omega)$ to $\dot{B}^s_{p,q}(\mathbb{R}^n)$.*

Remark 2.2.1. *In the case where the domain Ω is bounded, both homogeneous and nonhomogeneous Besov spaces admit trivial extension by zero onto the whole space (see* [**55**]*) for the nonhomogeneous case). Combining with Proposition 2.1.3, we deduce that*

$$B^s_{p,q}(\Omega) = \dot{B}^s_{p,q}(\Omega) \quad \text{if} \quad -1+1/p < s < 1/p \quad \text{and} \quad \Omega \quad \text{is bounded.}$$

The Besov spaces can be naturally defined on sub-manifolds using their atlas. Indeed, as already pointed out in the proof of Lemma 2.1.1 (in the $\mathbb{R}^n$ case), for positive exponents, the Besov semi-norms may be expressed in terms of finite differences. This naturally leads to the following definition of Besov spaces on manifolds:

Definition 2.2.2. *Let S be a C^1 m-dimensional submanifold and $s \in (0,1)$. Then the nonhomogeneous Besov space $B^s_{p,p}(S)$ is the set of $L_p(S)$ functions so that*

$$\|u\|_{B^s_{p,p}(S)} := \|u\|_{L_p(S)} + \|u\|_{\dot{B}^s_{p,p}(S)} < \infty \tag{2.11}$$

where $\|\cdot\|_{\dot{B}^s_{p,q}(S)}$ stands for the following homogeneous semi-norm:

$$\|u\|_{\dot{B}^s_{p,p(semi)}(S)} := \left(\int_S\int_S \frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}}\,dx\,dy\right)^{1/p}. \tag{2.12}$$

The above double integral may be used to define the *homogeneous* Besov space $\dot{B}^s_{p,p}(S)$ on S if in addition $s < m/p$ (see the remark below). The spaces $\dot{B}^s_{p,p}(S)$ with $\max(-1,-m/p') < s < 0$ may be defined by duality: we set

$$\dot{B}^s_{p,p}(S) := (\dot{B}^{-s}_{p',p'}(S))^*.$$

The remaining spaces $\dot{B}^s_{p,q}(S)$ for $1 < p < \infty$, $1 \le q \le \infty$ and $\max(-1,-m/p') < s < \min(1,m/p)$ may be defined by interpolation according to the following relation:

$$\left(\dot{B}^{s_1}_{p,q_1}(S), \dot{B}^{s_2}_{p,q_2}(S)\right)_{\theta,q} = \dot{B}^s_{p,q}(S). \tag{2.13}$$

We just have to fix some $\max(-1,-m/p') < s_1 < s_2 < \min(1,m/p)$ and take $\theta \in (0,1)$ such that $s = \theta s_2 + (1-\theta)s_1$. Note that the space $\dot{B}^s_{p,q}(S)$ with $0 < s < 1$ may be equivalently defined by finite differences as in the proof of Lemma 2.1.1.

Remark 2.2.2. *For $S = \mathbb{R}^n$, Definitions 2.2.1 and 2.2.2 give the same functional space in the case $0 < s < \min(1,n/p)$. We start with the Sobolev embedding of $\dot{B}^s_{p,p}(\mathbb{R}^n)$ in $L_m(\mathbb{R}^n)$ for some finite m. Thanks to that the behavior at the infinity is controlled (see* (2.3)*), and $\overline{C^\infty_c(\mathbb{R}^n)}^{\dot{B}^s_{p,p}(\mathbb{R}^n)}$ defines a Banach space which coincides with $\dot{B}^s_{p,p}(\mathbb{R}^n)$. For an arbitrary domain*

[4]The similar result for nonhomogeneous spaces is classical, see [**55**].

Ω one may thus define the homogeneous Besov space $\dot B^s_{p,p}(\Omega)$ (being a Banach space) by means of the following norm:

$$\|u\|_{\dot B^s_{p,p}(\Omega)} = \|u\|_{\dot B^s_{p,p(semi)}(\Omega)} + \|u\|_{L_m(\Omega)} \quad with \quad \frac{1}{p} - \frac{1}{m} = \frac{s}{n}\cdot \tag{2.14}$$

More generally, if $0 < s < n/p$ and $k = [s]$ then one may define $\dot B^s_{p,p}$ as the subset of $L_m(\Omega)$ functions u (with m as above) with $\|\nabla^k u\|_{\dot B^{s-k}_{p,p}} < \infty$.

General spaces $\dot B^s_{p,q}(\Omega)$ with $0 < s < n/p$ and $1 \le q \le \infty$ may be defined by interpolation.

From Lemma 2.2.1 and localization property, i.e.

$$\text{if} \quad \eta \in \mathcal{C}^\infty_c(\overline{\Omega}) \text{ and } u \in B^s_{p,q}(\Omega) \text{ (or } \dot B^s_{p,q}(\Omega)), \text{ then } \eta u \in B^s_{p,q}(\Omega) \text{ (or } \dot B^s_{p,q}(\Omega)), \tag{2.15}$$

one can get the following important corollary (more details may be found in [**56**], page 210).

Corollary 2.2.2. *Let Ω be a uniformly C^1 domain of $\mathbb{R}^n$. For any $(p,q) \in [1,\infty)^2$ and $s \in (-1+1/p, 1/p)$, we have $B^s_{p,q}(\Omega) = \overline{\{f \in B^s_{p,q}(\mathbb{R}^n) : \operatorname{Supp} f \subset \Omega\}}^{\|\cdot\|_{B^s_{p,q}(\Omega)}}$ and $\dot B^s_{p,q}(\Omega) = \overline{\{f \in \dot B^s_{p,q}(\mathbb{R}^n) : \operatorname{Supp} f \subset \Omega\}}^{\|\cdot\|_{\dot B^s_{p,q}(\Omega)}}$ where the nonhomogeneous and homogeneous norms are defined in (2.11) and (2.12), respectively.*

In the case $q = \infty$, density holds true for the weak $*$ topology only.

Proof. The nonhomogeneous case in standard (see e.g. [**55**]). The homogeneous case is a consequence of Corollary 2.2.1 and of the above remark. □

We shall use repeatedly the following trace theorem (see e.g. [**55**]).

Proposition 2.2.4. *Let Ω be a sufficiently smooth simply connected domain. Suppose that $1 < p < \infty$ and $s > 1/p$. The trace map from Ω to $\partial\Omega$ extends to a continuous operator from $B^s_{p,q}(\Omega)$ onto $B^{s-1/p}_{p,q}(\partial\Omega)$.*

We shall often use the following lemma (proved in the appendix of [**15**]) concerning the harmonic extension from the hyperplane $\partial\mathbb{R}^n_+$ to the half-space $\mathbb{R}^n_+$:

Lemma 2.2.2. *Let $s > 0$, $1 < p < \infty$ and $1 \le q \le \infty$. Then there exists a constant C such that for all $h \in \dot B^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+)$, we have*

$$\big\|\mathcal{F}^{-1}_{x'}[e^{-|\xi'|x_n}\mathcal{F}_{x'}[h]]\big\|_{\dot B^s_{p,q}(\mathbb{R}^n_+)} \le C\big\|h\big\|_{\dot B^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+)}, \tag{2.16}$$

where $\mathcal{F}_{x'}$ stands for the Fourier transform with respect to $x' := (x_1, \cdots, x_{n-1})$ and ξ' denotes the corresponding Fourier variable.

As a consequence, we get the following extension lemma in the nonflat situation.

Lemma 2.2.3. *Let Ω be a smooth domain with compact boundary. Then for $s \in (0, \frac{n}{p})$, $p \in (1,\infty)$ and $q \in [1,\infty]$ there is a continuous extension operator from $B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$ to $B^s_{p,q}(\Omega)$.*

Proof: Let ψ be in $B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$. Observe that the condition over s implies the space $B^{s-1/p}_{p,q}(\partial\Omega)$ to be stable under multiplication by compactly supported functions.

By compactness of $\partial\Omega$, one may find two coverings $(s_i)_{1\le i\le N}$ and $(S_i)_{1\le i\le N}$ of $\partial\Omega$ by open sets of $\mathbb{R}^n$ and such that, in addition, $s_i \subset S_i$. Then, we fix N maps $Z_i : s_i \to \mathbb{R}^n$ such that $Z_i : s_i \cap \Omega \to \mathbb{R}^n_+$ and $Z_i : s_i \cap \partial\Omega \to \partial\mathbb{R}^n_+$ (see the beginning of Section 2.4).

Let $(\eta_i)_{1\le i\le N}$ be a partition of unity associated to the covering $s_i \cap \partial\Omega$, with $\operatorname{Supp}\eta_i \subset s_i$. According to Lemma 2.1.1, we have

$$Z^*_i(\eta_i\psi) \in \dot B^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+). \tag{2.17}$$

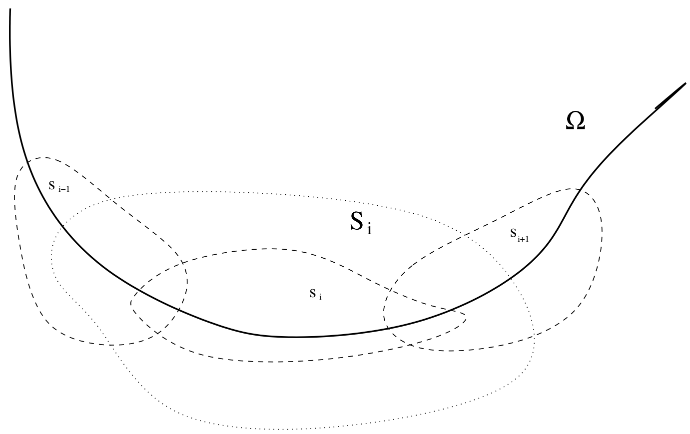

FIGURE 2.1. Covering of $\partial\Omega$

Then, thanks to Lemma 2.2.2, we may find some $\Psi_i \in \dot{B}^s_{p,q}(\mathbb{R}^n_+)$ such that

$$\|\Psi_i\|_{\dot{B}^s_{p,q}(\mathbb{R}^n_+)} \leq c\|Z_i^*(\eta_i\psi)\|_{\dot{B}^{s-1/p}_{p,q}(\partial\mathbb{R}^n_+)} \tag{2.18}$$

and $\bar{\eta}_i(Z_i^{-1})^*(\Psi_i) \in \dot{B}^s_{p,q}(\Omega)$ and $\bar{\eta}_i(Z_i^{-1})^*(\Psi_i)|_{\partial\Omega} = \bar{\eta}_i\eta_i\psi$ for $\bar{\eta}_i$ smooth such that $\bar{\eta}_i|_{s_i} \equiv 1$ and $\operatorname{Supp}\bar{\eta}_i \subset S_i$.

So taking $\Psi = \sum_i \bar{\eta}_i(Z_i^{-1})^*(\Psi_i)$ provides us with an extension such that

$$\|\Psi\|_{B^s_{p,q}(\Omega)} \leq c\|\psi\|_{B^{s-1/p}_{p,q}(\partial\Omega)} \text{ and } \Psi|_{\partial\Omega} = \psi. \tag{2.19}$$

This completes the proof of the lemma. ∎

LEMMA 2.2.4. *Let Ω be a C^1 domain with compact boundary. Denote by $\vec{n}$ the unit outer normal vector at $\partial\Omega$. Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $s \in (-1+1/p, 1/p)$. For any vector field F with coefficients in $B^\sigma_{p,q}(\Omega)$ and $\operatorname{div} F = 0$ in $\mathcal{D}'(\Omega)$, we have $(F\cdot\vec{n})|_{\partial\Omega} \in B^{s-1/p}_{p,q}(\partial\Omega)$. In addition, there exists a constant C depending only on n and such that*

$$\|(F\cdot\vec{n})|_{\partial\Omega}\|_{B^{s-1/p}_{p,q}(\partial\Omega)} \leq C\|F\|_{B^s_{p,q}(\Omega)}.$$

Proof: Using the properties of duality of Besov spaces, one may write

$$\|F\cdot\vec{n}|_{\partial\Omega}\|_{B^{s-1/p}_{p,q}(\partial\Omega)} \leq C\sup\left\{\int_{\partial\Omega} F\cdot\vec{n}\,\phi\,d\varsigma : \phi \in B^{-s+1/p}_{p',q'}(\partial\Omega) \text{ and } \|\phi\|_{B^{-s+1/p}_{p',q'}(\partial\Omega)} \leq 1\right\}.$$

Because $\operatorname{div} F = 0$, we have

$$\int_{\partial\Omega} F\cdot\vec{n}\,\phi\,d\varsigma = \int_\Omega F\cdot\nabla(E\phi)\,dx,$$

where $E\phi$ is the extension of ϕ in $B^{-s+1}_{p',q'}(\Omega)$ given by Lemma 2.2.3 – the assumptions guarantee that $-s+1 > 0$. Next, $\nabla(E\phi) \in B^{-s}_{p',q'}(\Omega)$ with $-s \in (-1+1/p', 1/p')$. Then, thanks to Corollary 2.2.1, both functions $\nabla(E\phi)$ and F can be extended by zero outside Ω. We thus get (by using the duality properties for Besov spaces on $\mathbb{R}^n$):

$$\int_{\partial\Omega} F\cdot\vec{n}\,\phi\,d\varsigma \leq C\|\nabla E\phi\|_{B^{-s}_{p',q'}(\Omega)}\|F\|_{B^s_{p,q}(\Omega)} \leq C\|\phi\|_{B^{-s+1/p}_{p',q'}(\partial\Omega)}\|F\|_{B^s_{p,q}(\Omega)}.$$

This completes the proof of the lemma. ∎

2.3. The divergence equation

Our analysis requires a more subtle description of the spaces with negative indices, an issue that depends on the problem we want to consider. As an example, let us look at the Poisson equation $\Delta u = \operatorname{div} k$ with given vector-field k with low regularity (typically $L_p(\Omega)$). The class of k for which this issue makes sense is larger if prescribing Neumann boundary conditions. To specify the definition of the divergence in lower regularity we adopt the following definition that has been proposed in our recent work [**16**]:

Definition 2.3.1. *Let Ω be a domain of $\mathbb{R}^n$ with a compact Lipschitz boundary. If k is a distribution over Ω and ζ a distribution over $\partial\Omega$ then we designate by $\mathcal{DIV}\,[k;\zeta]$ the linear functional defined on the set $\mathcal{C}_c^\infty(\overline{\Omega})$ of smooth functions with compact support in $\overline{\Omega}$, by*

$$\mathcal{DIV}\,[k;\zeta](\varphi) := -\int_\Omega k\cdot\nabla\varphi\,dx + \int_{\partial\Omega}\zeta\,\varphi\,d\sigma.$$

For $1<p<\infty$, $-1+1/p<s<1/p$, $1\le q\le\infty$, the notation $\mathcal{B}^{s-1}_{p,q}(\Omega)$ designates the set of all functionals $\mathcal{DIV}\,[k;\zeta]$ such that[5]

$$k\in B^s_{p,q}(\Omega) \quad \text{and} \quad \zeta\in B^{s-\frac1p}_{p,q}(\partial\Omega) \quad \text{with} \quad \int_{\partial\Omega}\zeta\,d\sigma = 0. \tag{2.20}$$

The space $\mathcal{B}^{s-1}_{p,q}(\Omega)$ is endowed with the following norm:

$$\|\,\mathcal{DIV}\,[k;\zeta]\|_{\mathcal{B}^{s-1}_{p,q}(\Omega)} = \inf\Big(\|\widetilde{k}\|_{B^s_{p,q}(\Omega)} + \|\widetilde{\zeta}\|_{B^{s-\frac1p}_{p,q}(\partial\Omega)}\Big), \tag{2.21}$$

where the infimum is taken over all the couples $(\widetilde{k},\widetilde{\zeta})$ satisfying (2.20) *and such that $\mathcal{DIV}\,[\widetilde{k};\widetilde{\zeta}] = \mathcal{DIV}\,[k;\zeta]$.*

Analogously, for the same range of exponents, we define the homogeneous space $\dot{\mathcal{B}}^{s-1}_{p,q}(\Omega)$ for $k\in\dot{B}^s_{p,q}(\Omega)$ and $\zeta\in\dot{B}^{s-\frac1p}_{p,q}(\partial\Omega)$ endowed with the norm

$$\|\,\mathcal{DIV}\,[k;\zeta]\|_{\dot{\mathcal{B}}^{s-1}_{p,q}(\Omega)} = \inf\Big(\|\widetilde{k}\|_{\dot{B}^s_{p,q}(\Omega)} + \|\widetilde{\zeta}\|_{\dot{B}^{s-\frac1p}_{p,q}(\partial\Omega)}\Big), \tag{2.22}$$

where the infimum is taken over all the couples $(\widetilde{k},\widetilde{\zeta})$ satisfying (2.20) *and such that $\mathcal{DIV}\,[\widetilde{k};\widetilde{\zeta}] = \mathcal{DIV}\,[k;\zeta]$.*

Let us recall that for k and ζ as above, if the vector-field v satisfies

$$\mathcal{DIV}\,[v;0] = \mathcal{DIV}\,[k;\zeta], \tag{2.23}$$

then it is a solution to the following system[6]:

$$\begin{cases} \operatorname{div} v = \operatorname{div} k & \text{in} \quad \Omega,\\ (k-v)\cdot\vec{n} = \zeta & \text{on} \quad \partial\Omega. \end{cases}$$

However, rewriting the system in terms of the functional $\mathcal{DIV}\,[k;\zeta]$ enables us to get a broader view since the boundary condition may be incorporated either in the interior part or in the boundary part of the data (see [**16, 18, 20**] for more detailed explanations).

In the present paper, the following result will be used a number of times:

[5] We make the convention that $\int_{\partial\Omega}\zeta\,d\sigma$ designates the distribution bracket $\langle\zeta,1\rangle$. That 1 is a test function comes from the fact that $\partial\Omega$ is compact.

[6] That the boundary condition makes sense stems from Lemma 2.2.4.

THEOREM 2.3.1. *Let Ω be a bounded or exterior C^2 domain. There exists a linear operator $\mathcal{B}_\Omega$ which is bounded from $\mathcal{B}^{s-1}_{p,q}(\Omega)$ to $B^s_{p,q}(\Omega;\mathbb{R}^n)$ whenever $1<p<\infty$, $1\leq q\leq\infty$ and $-1+1/p<s<1/p$, and such that for any $F=\mathcal{DIV}\,[k;\zeta]$ in $\mathcal{B}^{s-1}_{p,q}(\Omega)$, the vector-field $v:=\mathcal{B}_\Omega(F)$ fulfills* (2.23).

Furthermore if $k\in B^{m+s}_{p,q}(\Omega)$ with $m=1,2$ and $\zeta=(k\cdot\vec{n})|_{\partial\Omega}$ then v belongs to $B^{m+s}_{p,q}(\Omega)$, vanishes at the boundary and satisfies

$$\|v\|_{B^{m+s}_{p,q}(\Omega)}\leq C\|\operatorname{div}k\|_{B^{m-1+s}_{p,q}(\Omega)}\ \text{for } m=1,2. \tag{2.24}$$

Finally, if k is time dependent with k_t and $\operatorname{div}k$ in $L_1(0,T;B^s_{p,q}(\Omega))$, and $((k\cdot\vec{n})|_{\partial\Omega})_t\in L_1(0,T;B^{s-1/p}_{p,q}(\partial\Omega))$ then we have in addition the inequality

$$\|v_t\|_{L_1(0,T;B^s_{p,q}(\Omega))}\leq C\Big(\|k_t\|_{L_1(0,T;B^s_{p,q}(\Omega)}+\|((k\cdot\vec{n})|_{\partial\Omega})_t\|_{L_1(0,T;B^{s-1/p}_{p,q}(\partial\Omega))}\Big).$$

PROOF. We just sketch the proof in the case where Ω is bounded and star-shaped with respect to some ball, the reader being referred to [**16, 18, 20**] for more details. Then the following Bogovskiĭ formula provides an example of operator $\mathcal{B}_\Omega$ fulfilling (2.24):

$$\mathcal{B}_\Omega(F)(x):=\int_\Omega f(y)\frac{x-y}{|x-y|^n}\int_0^\infty\omega\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}\,dr\,dy, \tag{2.25}$$

where ω stands for a smooth function with average 1 and support in a ball with respect to which Ω is star-shaped.

In [**16**], in order to achieve distributions F with lower regularity (e.g. $F=\mathcal{DIV}\,[k;\zeta]$), we performed a formal integration by parts in (2.25) so as to decompose the outer integral into an interior integral and a boundary integral. More precisely, we introduced the following operators[7] I_Ω and J_Ω:

$$\begin{aligned} I_\Omega(k)(x)\ &:=\ -\int_\Omega k(y)\cdot\nabla_y\left[\frac{x-y}{|x-y|^n}\int_0^\infty\omega\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}dr\right]\,dy,\\ J_\Omega(\zeta)(x)\ &:=\ \int_{\partial\Omega}\zeta(y)\frac{x-y}{|x-y|^n}\int_0^\infty\omega\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{n-1}\,dr\,d\sigma_y.\end{aligned} \tag{2.26}$$

Those two operators enabled us to extend Bogovskiĭ formula to the rough case. In effect, in the smooth case where $F=\operatorname{div}k$, it is obvious that

$$v=\mathcal{B}_\Omega(\operatorname{div}k)=I_\Omega(k)+J_\Omega(\zeta)\ \ \text{with}\ \ \zeta:=(k\cdot\vec{n})|_{\partial\Omega}. \tag{2.27}$$

Under the assumptions of the theorem, it has been established in [**16**] that $v:=I_\Omega(k)+J_\Omega(\zeta)$ is indeed a solution to (2.23), and that v belongs to $B^s_{p,q}(\Omega;\mathbb{R}^n)$. More precisely, it has been shown that $I_\Omega:B^s_{p,q}(\Omega;\mathbb{R}^n)\to B^s_{p,q}(\Omega;\mathbb{R}^n)$ and $J_\Omega:B^{s-1/p}_{p,q}(\partial\Omega;\mathbb{R})\to B^s_{p,q}(\Omega;\mathbb{R}^n)$.

Finally, in the case where k is time-dependent, differentiating the above relation with respect to time yields

$$v_t=I_\Omega(k_t)+J_\Omega(\zeta_t).$$

Therefore, taking advantage of the continuity results for I_Ω and J_Ω, and integrating with respect to time gives the end of the statement. □

[7]These singular integrals have to be understood in the *principal value* meaning.

2.4. Change of coordinates

Investigating the Laplace and Stokes equations in general domains will rely on a localization of our problem. Obtaining local estimates at the interior and at the boundary of the domain will be the two key steps. The analysis at the interior is amenable to equations in the whole space. On the other hand, the analysis at the boundary will be reduced to model equations in the half-space after a suitable change of coordinates so as to 'straighten' the boundary. This section is devoted to introducing changes of coordinates so as to transform problems at the (nonflat) boundary of some C^r open set Ω ($r \geq 2$) to a problem at the boundary of $\mathbb{R}^n_+$.

Let us first present the general setting. By definition, having $\partial\Omega$ of class C^r means that for any point x_0 of $\partial\Omega$ there exists some small enough $\lambda > 0$ and a one-to-one C^r mapping

$$Z : \begin{cases} B(x_0, \lambda) & \longrightarrow & \mathbb{R}^n \\ x & \longmapsto & z \end{cases}$$

such that

i) Z is a C^r diffeomorphism from $B(x_0, \lambda)$ to $Z(B(x_0, \lambda))$;
ii) $Z(x_0) = 0$ and $D_x Z(x_0) = \mathrm{Id}$;
iii) $Z(\Omega \cap B(x_0, \lambda)) \subset \mathbb{R}^n_+$;
iv) $Z(\partial\Omega \cap B(x_0, \lambda)) = \partial\mathbb{R}^n_+ \cap Z(B(x_0, \lambda))$.

If we denote $D_x Z = \mathrm{Id} + A$ and assume that $\partial\Omega$ is uniformly C^r then there exist constants C_ℓ depending only on Ω and on $\ell \in \{1, \cdots, r-1\}$ such that

$$\|D^\ell A\|_{L_\infty(B(x_0,\lambda))} \leq C_\ell, \tag{2.28}$$

a property which implies (by the mean value formula) that

$$\|A\|_{L_\infty(B(x_0,\varepsilon))} \leq C_1\varepsilon \quad \text{if} \quad 0 < \varepsilon < \lambda, \tag{2.29}$$

hence by interpolation between the spaces $L_q(B(0,\varepsilon))$ and $W^{r-1}_q(B(0,\varepsilon))$,

$$\|A\|_{B^{\frac{n}{q}}_{q,1}(B(x_0,\varepsilon))} \leq C\varepsilon \quad \text{for all} \quad 1 \leq q < \infty \quad \text{such that} \quad n/q < r-1. \tag{2.30}$$

Let us introduce some examples of maps Z. We would like to consider a neighborhood of a point $x_0 \in \partial\Omega$. After a rigid motion we may assume that $x_0 = 0$ and $T_{x_0}\partial\Omega = \partial\mathbb{R}^n_+$ and that in addition $(B(0,\epsilon) \cap \Omega) \cap \mathbb{R}^n_+ \neq \emptyset$ for any $0 < \epsilon < \lambda$.

First example: the basic change of coordinates.

The above assumptions ensure that the *interior* unit normal vector of $\partial\Omega$ at $x_0 = 0$ is $\vec{e}_n := (0, \cdots, 0, 1)$. Then one may set

$$Z(x', x_n) := (x', x_n - \phi(x')), \tag{2.31}$$

where the graph of the function ϕ coincides with the boundary $\partial\Omega$ in some neighborhood of $x_0 = 0$, hence satisfies $\phi(0) = 0$ and $D_{x'}\phi(0) = 0$. As $\partial\Omega \in C^r$, so do ϕ and Z. In addition, (2.28) and (2.29) are satisfied.

Second example: a normal preserving change of coordinates.

We would like the value of the normal derivative at the boundary to be invariant under the change of coordinates. Hence we define Z so that for small enough t and x', we have (with ϕ as above)

$$Z\big((x', \phi(x')) + t\vec{n}\big) = (x', t), \tag{2.32}$$

where $\vec{n}$ stands for the *interior* unit normal vector at the boundary.

Differentiating the above equality with respect to t, we see that $\partial_{\vec{n}} Z$ coincides with $\vec{e}_n$. Hence in particular, for any differentiable function q,

$$\partial_{\vec{n}} q(x) = \partial_{z_n} q(z)|_{z=Z(x)} \quad \text{for} \quad x \quad \text{in a neighborhood of } x_0. \tag{2.33}$$

Third example: a measure preserving change of coordinates.

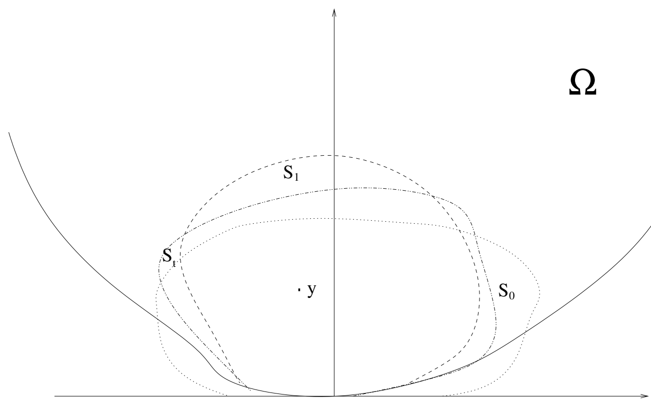

FIGURE 2.2. Construction of S_0

This last example, borrowed from e.g. [**47**], [**50**] and [**58**], is more involved. We start with some bounded open set $S_1 \subset \mathbb{R}^n_+$, star-shaped with respect to some point y inside Ω and such that ∂S_1 is a neighborhood of the point $x_0 = 0$ in $\partial\Omega$ (see the figure). We also fix another bounded open set $S_0 \subset \mathbb{R}^n_+$ such that

(1) $S_0 \cap \partial\mathbb{R}^n_+$ is a neighborhood of 0 in $\partial\mathbb{R}^n_+$,
(2) S_0 is star-shaped with respect to y,
(3) $|S_0| = |S_1|$.

We aim at constructing a *measure preserving* change of coordinates Z satisfying the requirements enumerated at the beginning of Section 2.4, and so that $Z(S_1) = S_0$.

To achieve it, we first construct intermediate sets S^*_t between S_0 and S_1 in terms of y and $t \in (0,1)$ as follows:

$$S^*_t := \{x \in \mathbb{R}^n : x = y + s\omega,\, \omega \in \mathbb{S}^{n-1},\, s \in [0, \bar{s}_t(y,\omega)]\},$$

where $\bar{s}_t(y,\omega) = (1-t)\bar{s}_0(y,\omega) + t\bar{s}_1(y,\omega)$ and $\bar{s}_i$ are given by the relation

$$y + \bar{s}_i(y,\omega)\omega \in \partial S_i \qquad \text{for } i = 0, 1.$$

In general, S^*_t need not have the same measure as S_1. Hence we define S_t to be the image of S^*_t by some suitable dilation centered at point 0. Having constructed such a family S_t, we notice that V_t, the normal speed of deformation of ∂S_t at time t, satisfies the compatibility condition

$$\int_{\partial S_t} V_t \, d\sigma = 0.$$

To show this relation it is enough to note that since the area of S_t is preserved,

$$0 = \frac{d}{dt}\int_{S_t} dx = \int_{\partial S_t} V_t \, d\sigma.$$

Hence one may solve the following system:

$$\begin{array}{rcl} \Delta P_t = 0 & \text{in} & S_t, \\ \frac{\partial P_t}{\partial \vec{n}} = V_t & \text{at} & \partial S_t. \end{array} \tag{2.34}$$

In order to define the map Z we solve the differential equation

$$\frac{dz_x}{dt}(t) = \nabla P(z_x(t),t) \quad \text{with} \quad z_x(0)=x'.$$

Then for $x=(x',t)$, we set

$$Z(x) := x + \int_0^t \nabla P(z_x(s),s)\,ds.$$

The construction guarantees that $Z(0)=0$ and Z is measure preserving since $\operatorname{div}\nabla P=0$ (Liouville theorem). In addition we are able to control the regularity of P_t (see e.g. [**27**], Th. 15): if $\partial S_t \in B^{1+s-\frac1p}_{p,q}$, then $V_t \in B^{s-\frac1p}_{p,q}$, so the solvability of (2.34) gives $\nabla P \in B^s_{p,q}$, hence eventually $Z\in B^s_{p,q}$.

To complete this section, let us explicit the effect of the above changes of coordinates on the differential operators that we shall consider throughout this paper.

We consider a general C^r-diffeomorphism $Z:\Omega\to\bar\Omega$. Let $H:\Omega\to\mathbb{R}^n$ denote some vector-field defined on Ω. Then we define the vector field $\bar H:\bar\Omega\to\mathbb{R}^n$ by $\bar H := Z^*(H) := H\circ Z^{-1}$. Similarly, for any function $f:\Omega\to\mathbb{R}$, we define $\bar f:\bar\Omega\to\mathbb{R}$ by $\bar f := Z^*(f) := f\circ Z^{-1}$. We thus have

$$\bar H(z) = H(x) \quad \text{and} \quad \bar f(z) = f(x) \quad \text{with} \quad z = Z(x).$$

From the chain rule, we get[8]

$$\operatorname{div}_x H(x) = D_z\bar H(z) : D_x Z(x) = \nabla_x Z(x) : \nabla_z \bar H(z) \quad \text{and} \quad \nabla_x f(x) = \nabla_x Z(x)\cdot\nabla_z \bar f(z). \tag{2.35}$$

Therefore,

$$\Delta_x f(x) = \operatorname{div}_x(\nabla_x f)(x) = D_z\big({}^T\bar B(z)\cdot\nabla_z\bar f(z)\big) : \bar B(z) \quad \text{with} \quad \bar B(z) = B(x) = D_x Z(x).$$

We thus deduce, with the summation convention over repeated indices, that

$$\begin{aligned}\overline{\Delta_x f} &= \partial_{z_i}(\bar B_{i,j}\bar B_{k,j}\partial_{z_k}\bar f) - (\partial_{z_i}\bar B_{i,j})\,\bar B_{k,j}\partial_{z_k}\bar f,\\ &= \partial_{z_i}(\bar B_{i,j}\bar B_{k,j}\partial_{z_k}\bar f) - \partial_{z_k}(\partial_{z_i}\bar B_{i,j}\,\bar B_{k,j}\bar f) + \bar f\partial_{z_k}(\bar B_{k,j}\partial_{z_i}\bar B_{i,j}).\end{aligned}$$

Setting $B = \operatorname{Id} + A$, that formula also reads

$$\overline{\Delta_x f} = \Delta_z\bar f + \operatorname{div}_z\Big((\bar B^T\bar B)\nabla_z\bar f - \bar f\,\bar B\operatorname{div}_z\bar A\Big) + \bar f\operatorname{div}_z\big(\bar B\operatorname{div}{}^T\!\bar A\big). \tag{2.36}$$

with the convention that $(\operatorname{div}\bar A)^j := \sum_i \partial_i\bar A_{ij}$.

In the case where Z is measure preserving then formula (2.35) for the divergence operator may be alternately written

$$\operatorname{div}_x H = \operatorname{div}_z\big(\bar B\,\bar H\big). \tag{2.37}$$

This is the consequence of the following series of computation which holds true for any test function ϕ and uses the fact that $\det B\equiv 1$:

$$\begin{aligned}\int \phi\operatorname{div}_x H\,dx &= -\int D_x\phi\cdot H\,dx,\\ &= -\int D_z\bar\phi(z)\cdot\bar B(z)\cdot\bar H(z)\,dz,\\ &= \int\bar\phi(z)\operatorname{div}_z(\bar B(z)\cdot\bar H(z))\,dz,\\ &= \int\phi(x)(\operatorname{div}_z(\bar B\cdot\bar H))(Z(x))\,dx.\end{aligned}$$

[8]In all the paper, we agree that $D_x Z$ stands for the $n\times n$ matrix with entries $\partial_{x_j}Z^i$ and that $\nabla_x Z$ stands for the matrix with entries $\partial_{x_i}Z^j$. Furthermore, for M and N two $n\times n$ matrices, we set $M:N = \operatorname{Tr} MN$.

Hence we have

$$\Delta_x f = \operatorname{div}_z\big(\bar{B}^T\bar{B}\nabla_z\bar{f}\big). \tag{2.38}$$

For general diffeomorphism Z, Equality (2.37) extends as follows:

$$\operatorname{div}_x H(x) = \bar{J}_Z \operatorname{div}_z\big(\bar{J}_Z\bar{B}\bar{H}\big),$$

with $\bar{J}_Z$ being the Jacobian of the change of coordinates.

This allows to write $\Delta_x\bar{f}$ in another way :

$$\Delta_x f = \bar{J}_Z \operatorname{div}_z\big(\bar{J}_Z\bar{B}^T\bar{B}\nabla_z\bar{f}\big).$$

CHAPTER 3

The Laplace equation

In this chapter, we prove several auxiliary results for the Laplace equation with Dirichlet or Neumann boundary conditions. First we consider the equation in bounded domains, next in the whole space, then in the half-space and finally in exterior domains. Even though most of those results belong to the mathematical folklore, our approach based on a new definition of very weak solutions for the Neumann problem (see Section 3.3) sheds some new light on this classical issue.

3.1. The homogeneous Neumann problem in bounded domains

Although we are planning to investigate the system in exterior (unbounded) domains, proving first results for the bounded domain case is needed. An important issue will be the following Lemma for the Neumann problem in a smooth bounded domain.

PROPOSITION 3.1.1. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Let D be a bounded $C^{2,1/p}$ domain*[1] *of $\mathbb{R}^n$ with $n \geq 2$. Let $f \in B^{\sigma}_{p,q}(D)$ with $-1+\frac{1}{p} < \sigma < \frac{1}{p}\cdot$ Assume that $\displaystyle\int_D f\,dx = 0$. Then problem*

$$\begin{aligned} &\Delta u = f \quad in \quad D, \\ &\partial_{\vec{n}} u = 0 \quad on \quad \partial D, \end{aligned} \qquad \int_D u\,dx = 0 \tag{3.1}$$

admits a unique solution u in $B^{2+\sigma}_{p,q}(D)$ and the following estimate is valid:

$$\|u\|_{B^{2+\sigma}_{p,q}(D)} \leq C\|f\|_{B^{\sigma}_{p,q}(D)}. \tag{3.2}$$

Proof: At least in the smooth case, this result is classical (e.g. this is a consequence of Theorem 13 in [**27**], dedicated to general elliptic problems). Here we write out the details for the particular case of the Laplace equation with homogeneous Neumann boundary conditions, both for the reader convenience and because it sheds light on the key points of our approach *which is not based on any explicit representation of the solution.*

Arguing by interpolation, we see that it suffices to consider the case $q = p$. So we focus on this case only. Let us first fix some $u \in B^{\sigma+2}_{p,p}(D)$ and show that

$$\|u\|_{B^{2+\sigma}_{p,p}(D)} \leq C(\|f\|_{B^{\sigma}_{p,p}(D)} + \|u\|_{B^{1+\sigma}_{p,p}(D)}) \quad \text{with} \quad f := \Delta u. \tag{3.3}$$

For that we fix some $u \in B^{\sigma+2}_{p,p}(D)$ and introduce some partition of unity $\{\eta^0, \eta^1, \cdots, \eta^k\}$ of D such that

- η^0 is compactly supported in the interior D;
- the support of each η^l with $1 \leq l \leq k$ has diameter of order λ and has nonempty intersection with ∂D. In addition, it is required that

$$\|\partial_\alpha \eta^l\|_{L_\infty(\mathbb{R}^n)} \leq C_\alpha \lambda^{-|\alpha|} \quad \text{for all} \quad \alpha \in \mathbb{N}^n \quad \text{and} \quad 1 \leq l \leq k. \tag{3.4}$$

[1] We do not claim our regularity assumption to be optimal.

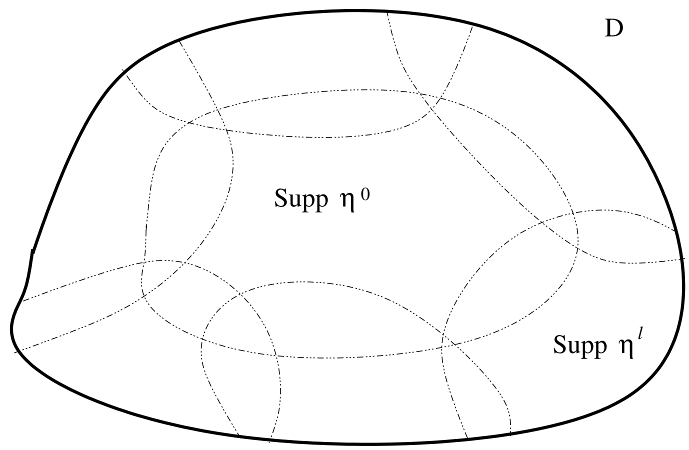

FIGURE 3.1. Partition of unity $(\eta^l)_{0\leq l\leq k}$ of D

Let $U^l := \eta^l u$ and $f^l := \eta^l f$. Note that as the functions η^l are smooth and compactly supported, Corollary 2.1.1 and our definition of Besov spaces by extension guarantee that the functions U^l (resp. f^l) are in $B^{\sigma+2}_{p,p}(D)$ (resp. $B^{\sigma}_{p,p}(D)$). The equation for U^l reads

$$\Delta U^l = 2\,\mathrm{div}\,(u\nabla\eta^l) - u\Delta\eta^l + f^l \quad \text{in } D. \tag{3.5}$$

For $l=0$, the above equation also holds in $\mathbb{R}^n$ so that estimating U^l may be done according to our results in [**15**]. We get

$$\|U^0\|_{\dot B^{\sigma+2}_{p,p}(\mathbb{R}^n)} \lesssim \|u\nabla\eta^0\|_{\dot B^{\sigma+1}_{p,p}(\mathbb{R}^n)} + \|u\Delta\eta^0\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n)} + \|f^0\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n)}.$$

Note that as all the functions involved in the above inequality are compactly supported, one may replace homogeneous norms by nonhomogeneous ones (see Proposition 2.1.3). Then taking advantage of Proposition 2.1.2, we easily get

$$\|U^0\|_{B^{\sigma+2}_{p,p}(\mathbb{R}^n)} \lesssim \|u\|_{B^{\sigma+1}_{p,p}(D)} + \|f\|_{B^{\sigma}_{p,p}(D)}.$$

In order to treat the boundary terms $U^1,\cdots,U^k$, one may introduce local coordinates so as to transform the equation into a problem over the half-space. We choose a change of coordinates Z^l which preserves the normal vector at the boundary in order to keep the homogeneity of the boundary conditions in (3.6) (see Subsection 2.4, second example). With the notation $\overline{g} = Z_l^*(g) = g\circ Z_l^{-1}$, we get

$$\begin{aligned} &\Delta_z\overline{U^l} = (\Delta_z-\Delta_x)\overline{U^l} + 2\,\overline{\mathrm{div}_x\,(u\nabla_x\eta^l)} - \overline{u\Delta_x\eta^l} + \overline{f^l} \quad &&\text{in} \quad \mathbb{R}^n_+,\\ &\partial_{z_n}\overline{U^l}|_{z_n=0} = 0 &&\text{on} \quad \partial\mathbb{R}^n_+. \end{aligned} \tag{3.6}$$

Hence, using (2.36), the above system rewrites

$$\begin{aligned} &\Delta_z\overline{U^l} = F^l \qquad \text{in} \quad \mathbb{R}^n_+,\\ &\partial_{z_n}\overline{U^l}|_{z_n=0} = 0 \quad \text{on} \quad \partial\mathbb{R}^n_+, \end{aligned} \qquad \overline{U^l}\to 0 \ \text{ at } \ \infty, \tag{3.7}$$

with, denoting $A^l := DZ_l\circ Z_l^{-1} - \mathrm{Id}\,$,

$$\begin{aligned} F^l = -\,\mathrm{div}_z\Big((A^l + {}^TA^l + A^{lT}A^l)\nabla_z\bar U^l - \overline{U^l}(\mathrm{Id}+A^l)\cdot\mathrm{div}_z\,A^l\Big)&\\ -\,\mathrm{div}_z\big((\mathrm{Id}+A^l)\cdot\mathrm{div}_z\,A^l\big)\overline{U^l} - \overline{u\Delta_x\eta^l} + 2\,\overline{\mathrm{div}_x\,(u\nabla_x\eta^l)} + \overline{f^l}.& \end{aligned}$$

As we assumed that $1/p > \sigma > -1+1/p$ the space $\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)$ admits a symmetric extension onto the whole space (see Remark 2.2.1), and we may use the results of [**15**] pertaining to the Neumann problem in the half-space, to get

$$\|\overline{U^l}\|_{\dot B^{2+\sigma}_{p,p}(\mathbb{R}^n_+)} \le C\|F^l\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)}. \tag{3.8}$$

We claim that

$$\|F^l\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} \lesssim \lambda\|\overline{U^l}\|_{\dot B^{2+\sigma}_{p,p}(\mathbb{R}^n_+)} + \|\overline{U^l}\|_{\dot B^{1+\sigma}_{p,p}(\mathbb{R}^n_+)} + \|\overline{f^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} + \lambda^{-1}\|\overline{u}\|_{\dot B^{1+\sigma}_{p,p}(B(0,\lambda))}. \tag{3.9}$$

Indeed, we first notice that, denoting $B^l = A^l + {}^TA^l + A^{l\,T}A^l$,

$$\|\operatorname{div}_z(B^l\nabla_z\overline{U^l})\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} \le C\lambda\|\overline{U^l}\|_{\dot B^{2+\sigma}_{p,p}(\mathbb{R}^n_+)}. \tag{3.10}$$

For proving that, one may use the bounds that have been stated in (2.28). Then, owing to Leibniz' rule, we have for $f \in W^3_{p;0}(B(0,\lambda))^2$,

$$\|\operatorname{div}_z(B^l\nabla_z f)\|_{\dot W^1_p(B(0,\lambda))} \le C\lambda\|f\|_{\dot W^3_p(B(0,\lambda))} + C\|f\|_{\dot W^2_p(B(0,\lambda))}, \tag{3.11}$$

but the compactness of the support allows us to take advantage of the Poincaré inequality, so (3.11) reduces to

$$\|\operatorname{div}_z(B^l\nabla_z f)\|_{\dot W^1_p(B(0,\lambda))} \lesssim \lambda\|f\|_{\dot W^3_p(B(0,\lambda))}. \tag{3.12}$$

Similarly, we find that for $f \in W^1_{p;0}(B(0,\lambda))$, we have

$$\|\operatorname{div}_z(B^l\nabla_z f)\|_{\dot W^{-1}_p(B(0,\lambda))} \lesssim \|B^l\nabla_z f\|_{L_p(B(0,\lambda))} \lesssim \lambda\|\nabla_z f\|_{L_p(B(0,\lambda))}. \tag{3.13}$$

Interpolating between (3.12) and (3.13) yields (3.10).

Bounding the other terms in F^l goes along the same lines. For instance, one has

$$\|\operatorname{div}_z((\operatorname{div}A^l)\,\overline{U^l})\|_{\dot W^1_p(\mathbb{R}^n_+)} \le C\|\overline{U^l}\|_{\dot W^2_p(\mathbb{R}^n_+)} \quad\text{and}\quad \|\operatorname{div}_z((\operatorname{div}A^l)\,\overline{U^l})\|_{\dot W^{-1}_p(\mathbb{R}^n_+)} \le C\|\overline{U^l}\|_{L_p(\mathbb{R}^n_+)},$$

hence

$$\|\operatorname{div}_z((\operatorname{div}A^l)\,\overline{U^l})\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+))} \le C\|\overline{U^l}\|_{\dot B^{1+\sigma}_{p,p}(\mathbb{R}^n_+)},$$

and

$$\|\overline{\operatorname{div}_x(u\nabla_x\eta^l)}\|_{\dot W^1_p(\mathbb{R}^n_+)} \lesssim \lambda^{-1}\|u\|_{\dot W^2_p(B(0,\lambda))} \quad\text{and}\quad \|\overline{\operatorname{div}_x(u\nabla_x\eta^l)}\|_{\dot W^{-1}_p(\mathbb{R}^n_+)} \lesssim \lambda^{-1}\|\overline{u}\|_{L_p(B(0,\lambda))},$$

whence

$$\|\overline{\operatorname{div}_x(u\nabla_x\eta^l)}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} \lesssim \lambda^{-1}\|u\|_{\dot B^{\sigma+1}_{p,p}(\mathbb{R}^n_+)}.$$

Assuming that λ is small enough, one may absorb the first term in the r.h.s. of (3.9) by the l.h.s. of (3.8). Hence, using that, by virtue of the composition lemma 2.1.1 and of remark 2.1.1, we may write

$$\|u\|_{B^{2+\sigma}_{p,p}(D)} \le \sum_l \|\eta^l u\|_{B^{2+\sigma}_{p,p}(D)} \lesssim \sum_l \|\overline{U^l}\|_{B^{2+\sigma}_{p,p}(\mathbb{R}^n_+)} \lesssim \sum_l \|F^l\|_{B^{\sigma}_{p,p}(\mathbb{R}^n_+)},$$

we get the desired estimate (3.3).

Next, we claim that

$$\|u\|_{B^{1+\sigma}_{p,p}(D)} \le C\|\Delta u\|_{B^{\sigma}_{p,p}(D)} \tag{3.14}$$

for all $u \in B^{2+\sigma}_{p,p}(D)$ such that $\displaystyle\int_D u(x)\,dx = 0$ and $\partial_{\vec n}u|_{\partial D} = 0$.

[2] $W^3_{p;0}$ stands for W^3_p with zero trace at the boundary

The proof is based on compactness arguments : if (3.14) were not true, then there would exist a sequence $\{u^k\}$ of $B^{2+\sigma}_{p,p}(D)$ functions such that

$$1 = \|u^k\|_{B^{1+\sigma}_{p,p}(D)} > k\|\Delta u^k\|_{B^{\sigma}_{p,p}(D)}. \tag{3.15}$$

According to (3.3) we have for all k:

$$\|u^k\|_{B^{2+\sigma}_{p,p}(D)} \le C(\|u^k\|_{B^{1+\sigma}_{p,p}(D)} + \|\Delta u^k\|_{B^{\sigma}_{p,p}(D)}) \le C. \tag{3.16}$$

So the compactness properties of Besov spaces (see Proposition 2.2.2) imply that there exists a subsequence $\{u^{k_n}\}$ such that

$$u^{k_n} \to u^* \qquad \text{in } B^{1+\sigma}_{p,p}(D) \tag{3.17}$$

and

$$u^{k_n} \rightharpoonup u^* \qquad \text{in } B^{2+\sigma}_{p,p}(D). \tag{3.18}$$

Relations (3.15) and (3.17) imply that $\|u^*\|_{B^{1+\sigma}_{p,p}(D)} = 1$ and $\|\Delta u^*\|_{B^{\sigma}_{p,p}(D)} = 0$. In other words, u^* fulfills the following system:

$$\begin{aligned} \Delta u^* &= 0 \quad \text{in} \quad D,\\ \partial_{\vec n} u^* &= 0 \quad \text{on} \quad \partial D, \end{aligned} \qquad \int_D u^*\,dx = 0. \tag{3.19}$$

The strong maximum principle (see e.g. [**33**]) thus implies that $u^* \equiv 0$. This stands in contradiction with the fact that $\|u^*\|_{B^{1+\sigma}_{p,p}(D)} = 1$. Hence estimate (3.3) holds true.

As an immediate consequence of (3.3) and (3.14) we get

$$\|u\|_{B^{2+\sigma}_{p,p}(D)} \le C\|f\|_{B^{\sigma}_{p,p}(D)}. \tag{3.20}$$

To end the proof of Proposition 3.1.1 we ought to prove the existence of solutions. The simplest technique proceeds by approximation. Let us write f as the limit in $B^{\sigma}_{p,p}(D)$ of a sequence of smooth functions $f_j \in \mathcal{C}^\infty(\overline{D})$ with $\displaystyle\int_D f_j\,dx = 0$. Then we seek for solutions to the following problem

$$\begin{aligned} \Delta u_j &= f_j \quad \text{in} \quad D,\\ \partial_{\vec n} u_j &= 0 \quad \text{on} \quad \partial D, \end{aligned} \qquad \int_D u_j\,dx = 0. \tag{3.21}$$

Using the L_2 approach as the Lax-Milgram theorem we obtain existence of a weak solution in $H^1(D)$, then we are able to improve their regularity to arbitrary $W^m_2(D)$, getting the classical solvability. For that we need D to be C^m to have the solvability in the class of regular solutions (see [**33**]). An alternative method is to use that, according to [**40**], solutions are in $C^{2,\frac{1}{p}}$ whenever the domain is $C^{2,\frac{1}{p}}$. Hence those solutions are also in $B^{\sigma+2}_{p,p}$ for $\sigma < 1/p$ as the domain is bounded. Furthermore, thanks to (3.2), one may write

$$\|u_j - u_k\|_{B^{2+\sigma}_{p,p}(D)} \le C\|f_j - f_k\|_{B^{\sigma}_{p,p}(D)} \quad \text{for all} \quad (j,k) \in \mathbb{N}^2.$$

Hence $(u_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $B^{2+\sigma}_{p,p}(D)$.

Eventually, as $f_j \to f \in B^{\sigma}_{p,p}(D)$, one may conclude that there exists $u \in B^{2+\sigma}_{p,p}(D)$ satisfying (3.1) and (3.20). Proposition 3.1.1 is proved. ∎

Remark 3.1.1. *One may extend Proposition 3.1.1 to more general σ. For higher regularity this requires extra compatibility conditions on f. We omit the general result (see e.g. [**27**]) since it is very technical and not needed for the analysis of the Stokes system we want to perform here. The case of very negative σ will be treated below in Lemma 3.3.3.*

3.2. The whole space case

Let us first state an important corollary of Proposition 3.1.1.

COROLLARY 3.2.1. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Consider a compactly supported function f in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ for some real number σ. For f to belong to $\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)$, it suffices that*

- *either $\sigma > -1/p'$ and*

$$\int_{\mathbb{R}^n} f\,dx = 0; \tag{3.22}$$

- *or $\sigma > 1 - n/p'$.*

Furthermore, there exists a constant C such that if the support of f is included in a ball of radius λ then

$$\|f\|_{\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)} \leq C\lambda \|f\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Proof: If $\sigma > 1 - n/p'$ then the result is an easy corollary of Proposition 2.1.3. Indeed, scaling arguments reduce the proof to the case $\lambda = 1$, and as f is compactly supported, one may write

$$\|f\|_{\dot{B}^{\sigma-1}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{B^{\sigma-1}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{B^{\sigma}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Let us now assume that $-1/p' < \sigma < \min(-1 + n/p, 0)$ and that condition (3.22) is satisfied. Then according to Proposition 2.1.1, one may write

$$\|f\|_{\dot{B}^{\sigma-1}_{p,p}(\mathbb{R}^n)} = \sup_{\|\phi\|_{\dot{B}^{1-\sigma}_{p',p'}(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} f\,\phi\,dx. \tag{3.23}$$

Because f is supported in $\overline{B(0,\lambda)}$, arguing by density we see that it suffices to consider functions ϕ in $\mathcal{C}^{\infty}_c(\mathbb{R}^n)$. Furthermore, as (3.22) is satisfied, Proposition 3.1.1 (here we need $\sigma > -1/p'$) ensures that there exists some function c in $B^{\sigma+2}_{p,p}(B(0,\lambda))$ so that

$$\begin{aligned} \Delta c = f \quad &\text{in} \quad B(0,\lambda), \\ \partial_{\vec{n}} c = 0 \quad &\text{on} \quad \partial B(0,\lambda), \end{aligned} \qquad \int_{B(0,\lambda)} c\,dx = 0, \tag{3.24}$$

and, in addition,

$$\|c\|_{B^{\sigma+2}_{p,p}(B(0,\lambda))} \leq C\|f\|_{B^{\sigma}_{p,p}(B(0,\lambda))}.$$

Next, we define

$$\widetilde{\nabla c}(x) = \begin{cases} \nabla c(x) & \text{for} \quad x \in B(0,\lambda), \\ 0 & \text{for} \quad x \in \mathbb{R}^n \setminus B(0,\lambda). \end{cases} \tag{3.25}$$

Remark that by construction, we have, owing to the homogeneous Neumann boundary condition over c and the support properties of f,

$$-\int_{\mathbb{R}^n} \widetilde{\nabla c} \cdot \nabla \phi\,dx = \int_{\mathbb{R}^n} f\phi\,dx \quad \text{for all} \quad \phi \in \mathcal{C}^{\infty}_c(\mathbb{R}^n). \tag{3.26}$$

Therefore,

$$\left| \int_{\mathbb{R}^n} f\phi\,dx \right| \leq \|\widetilde{\nabla c}\|_{L_p(\mathbb{R}^n)} \|\nabla \phi\|_{L_{p'}(\mathbb{R}^n)}. \tag{3.27}$$

To bound the right-hand side, it suffices to use the embedding result stated in Proposition 2.1.1[3] and its dual version. We get

$$\|\nabla c\|_{L_{p^*}(B(0,\lambda))} \lesssim \|\nabla c\|_{B^{\sigma+1}_{p,p}(B(0,\lambda))} \quad \text{for } \frac{n}{\sigma+1}\Big(\frac{1}{p}-\frac{1}{p^*}\Big)=1 \quad \text{because } 0<\sigma+1<n/p,$$
$$\|\nabla\phi\|_{L_{p'^*}(\mathbb{R}^n)} \lesssim \|\nabla\phi\|_{B^{-\sigma}_{p',p'}(\mathbb{R}^n)} \quad \text{for } \frac{n}{-\sigma}\Big(\frac{1}{p'}-\frac{1}{p'^*}\Big)=1 \quad \text{because } 0<-\sigma<n/p'.$$

Since $\operatorname{Supp} f\subset B(0,\lambda)$, in (3.23), is suffices to consider functions ϕ supported in a neighborhood of $B(0,\lambda)$. Now, for such functions (3.27) and the above inequalities imply that

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} f\phi\,dx\Big| &\leq C|B(0,\lambda)|^{\frac{1}{p}-\frac{1}{p^*}}\|\nabla c\|_{B^{1+\sigma}_{p,p}(B(0,\lambda))}|B(0,\lambda)|^{\frac{1}{p'}-\frac{1}{p'^*}}\|\nabla\phi\|_{B^{-\sigma}_{p',p'}(\mathbb{R}^n)},\\ &\leq C\lambda\|f\|_{B^{\sigma}_{p,p}(\mathbb{R}^n)}\|\phi\|_{B^{1-\sigma}_{p',p'}(\mathbb{R}^n)}.\end{aligned}$$

So (3.23) yields the desired inequality in the case $-1/p'<\sigma<\min(-1+n/p,0)$ and $p=q$. The remaining cases follow by interpolation. ■

One can now establish an important estimate for the Laplace equation in the whole space.

Lemma 3.2.1. *Let $p\in(1,\infty)$. Let $f\in\mathcal{D}'(\mathbb{R}^n)$ with* $\operatorname{Supp} f\subset\overline{B(0,\lambda)}$ *and* $\int_{\mathbb{R}^n} f(x)\,dx=0$. *Consider the equation*

$$\Delta b=f \quad \textit{in} \quad \mathbb{R}^n, \qquad b\to 0 \quad \textit{as} \quad |x|\to\infty. \tag{3.28}$$

If in addition $f\in\dot B^{\sigma}_{p,q}(\mathbb{R}^n)$ for some σ such that $-1+1/p<\sigma<1/p$ and $q\in[1,\infty]$ then Equation (3.28) has a unique solution b with $\nabla b\in\dot B^{\sigma}_{p,q}(\mathbb{R}^n)$ and we have

$$\|\nabla b\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n)}\leq C\lambda\|f\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Proof: In [**15**], we proved the existence of a solution operator which is continuous from $\dot B^{\sigma-1}_{p,q}(\mathbb{R}^n)$ to $\dot B^{\sigma+1}_{p,q}(\mathbb{R}^n)$. Given the assumptions on f and σ, the result is thus a straightforward consequence of Corollary 3.2.1. ■

3.3. The half-space case

Let us first concentrate on the Dirichlet problem:

$$\begin{aligned}\Delta u&=h && \text{in} \quad \mathbb{R}^n_+,\\ u|_{x_n=0}&=0 && \text{on} \quad \partial\mathbb{R}^n_+,\end{aligned} \qquad u\to 0 \quad \text{as} \quad |x|\to\infty. \tag{3.29}$$

Lemma 3.3.1. *Let $p\in[1,\infty)$, $q\in[1,\infty]$ and $\sigma\in(-1+1/p,1/p)$.*

(1) *If h is in $\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$ then (3.29) has a unique solution $u\in\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n_+)$ and we have*

$$\|u\|_{\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n_+)}\leq C\|h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

(2) *If $\nabla h\in\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$ and u satisfies (3.29) then*
   (a) *if $h\equiv 0$ at the boundary then ∇u is in $\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n_+)$ and we have*

$$\|\nabla u\|_{\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n_+)}\leq C\|\nabla h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

   (b) *with no assumption over h, we still have*

$$\|\nabla^3 u\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}\leq C\|\nabla h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

[3] Note that our definition of Besov spaces on domains ensures that embedding also holds true in this framework.

(3) *If* $\nabla^2 h \in \dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$ *and* $h \equiv 0$ *at the boundary then* $\nabla^2 u$ *is in* $\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n_+)$ *and we have*

$$\|\nabla^2 u\|_{\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n_+)} \leq C\|\nabla^2 h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

(4) *If* $h = \operatorname{div} k$ *for some* $k \in \dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$, *then* (3.29) *has a unique solution* u *with* $\nabla u \in \dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$ *and we have*

$$\|\nabla u\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)} \leq C\|k\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Proof: Let us prove the first item. Let $\widetilde{h}$ be the antisymmetric extension of h. As $\sigma \in (-1 + 1/p, 1/p)$, Remark 2.2.1 ensures that the distribution $\widetilde{h}$ is in $\dot B^{\sigma}_{p,q}(\mathbb{R}^n)$ and satisfies

$$\|\widetilde{h}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Now, according to Prop. 2 in [**15**] the Poisson equation

$$\Delta \widetilde{u} = \widetilde{h} \quad \text{in} \quad \mathbb{R}^n$$

has a unique solution $\widetilde{u}$ in $\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n)$ with

$$\|\widetilde{u}\|_{\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n)} \leq C\|\widetilde{h}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n)}. \tag{3.30}$$

Note that as the equation in invariant by reflection with respect to the hyperplane $x_n = 0$, the function $\widetilde{u}$ is antisymmetric hence has to vanish on $x_n = 0$. So taking for u the restriction of $\widetilde{u}$ to the half-space $\mathbb{R}^n_+$, the boundary conditions of equation (3.29) are satisfied, and we are done in this case.

In order to obtain the desired estimate (3.30), we just have to use the explicit formula of the solution in Fourier variables (see [**15**]), and the density of $\mathcal{S}_0(\mathbb{R}^n)$ in homogeneous Besov spaces.

Let us now go to the proof of the second item in the case where h vanishes at the boundary. Then $\nabla_{x'}\widetilde{h}$ (resp. $\partial_{x_n}\widetilde{h}$) coincides with the antisymmetric (resp. symmetric) extension of $\nabla_{x'}h$ (resp. $\partial_{x_n}h$), whence

$$\|\nabla\widetilde{h}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|\nabla h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Therefore, as

$$\Delta\nabla\widetilde{u} = \nabla\widetilde{h} \quad \text{in} \quad \mathbb{R}^n,$$

Prop. 2 in [**15**] ensures that $\nabla\widetilde{u} \in \dot B^{\sigma+2}_{p,q}(\mathbb{R}^n)$ with

$$\|\nabla\widetilde{u}\|_{\dot B^{\sigma+2}_{p,q}(\mathbb{R}^n)} \leq C\|\nabla\widetilde{h}\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|\nabla h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}$$

and we are done.

If h does not vanish at the boundary, then we differentiate (3.29) with respect to tangential directions $x' := (x_1, ..., x_{n-1})$, and get

$$\begin{aligned} \Delta\nabla_{x'}u = \nabla_{x'}h \quad &\text{in} \quad \mathbb{R}^n_+, \\ \nabla_{x'}u|_{x_n=0} = 0 \quad &\text{on} \quad \partial\mathbb{R}^n_+. \end{aligned}$$

Since $\nabla_{x'}h \in \dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$, the first item ensures that

$$\|\nabla^2\nabla_{x'}u\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)} \leq C\|\nabla_{x'}h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Let us notice that only the term $\partial^3_{x_n}u$ of $\nabla^3 u$ has not been estimated yet. However it is is given from the equation of u:

$$\partial^3_{x_n}u = \partial_{x_n}h - \Delta_{x'}\partial_{x_n}u \quad \text{in} \quad \mathbb{R}^n_+.$$

This completes the proof of the second item.

The proof of the third item is similar. We just have to notice that, owing to the trace condition over h and to the fact that $\nabla\widetilde{h}$ has no jump on $x_n=0$, one can assert that $\partial_{x_i}\partial_{x_j}\widetilde{h}$ coincides with the symmetric (resp. antisymmetric) extension of $\partial_{x_i}\partial_{x_j}h$ if $1\leq i\leq n-1$ and $j=n$ (resp. $1\leq i,j\leq n-1$ or $i=j=n$). Finally, the last item is proved in [**15**], Lemma 1. ∎

Let us recall an easy consequence of the definition of the normal trace at the boundary in the case of divergence free functions.

Lemma 3.3.2. *Let $k\in\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$ with $-1+1/p<\sigma<1/p$, $1<p<\infty$ and $1\leq q\leq\infty$. Consider the unique solution u of* (3.29) *with right-hand side* $\operatorname{div} k$. *Let $\vec e_n$ denote the unit vertical vector. Then $(\nabla u-k)\cdot\vec e_n|_{x_n=0}$ is defined as an element of $\dot B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)$ and*

$$\|(\nabla u-k)\cdot\vec e_n|_{x_n=0}\|_{\dot B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}\leq C\|k\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Proof: This is a consequence of the definition of $(\nabla u-k)\cdot\vec e_n|_{x_n=0}$ as a functional over $\dot B^{1-\sigma-1/p'}_{p',q'}(\partial\mathbb{R}^n_+)$. Indeed, we have

$$\|(\nabla u-k)\cdot\vec e_n|_{x_n=0}\|_{\dot B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)}=\sup_{\varphi}\int_{\partial\mathbb{R}^n_+}\varphi\,(\nabla u-k)\cdot\vec e_n\,dx',$$

where the supremum is taken over those functions of φ of $\dot B^{\frac1p-\sigma}_{p',q'}(\partial\mathbb{R}^n_+)$ with norm 1. Since $1/p-\sigma=1-\sigma-1/p'>0$, the function φ admits an extension $\widetilde\varphi\in\dot B^{1-\sigma}_{p',q'}(\mathbb{R}^n_+)$ (see Lemma 2.2.3). Hence, as by definition $\operatorname{div}(\nabla u-k)=0$, one may write

$$-\int_{\partial\mathbb{R}^n_+}\varphi(\nabla u-k)\cdot\vec e_n\,dx'=\int_{\mathbb{R}^n_+}\nabla\widetilde\varphi\cdot(\nabla u-k)\,dx.$$

Of course the functions $\nabla\widetilde\varphi$ and $\nabla u-k$ may be extended by 0 on $\mathbb{R}^n$, and the extension belongs to the same Besov space. Therefore, taking advantage of the duality between $\dot B^{-\sigma}_{p',q'}(\mathbb{R}^n)$ and $\dot B^{\sigma}_{p,q}(\mathbb{R}^n)$, and of the last item of Lemma 3.3.1 gives the result. ∎

Corollary 3.3.1. *Let $h\in\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$ be such that $\operatorname{Supp}h\subset\overline{B(0,\lambda)\cap\mathbb{R}^n_+}$. Then the Dirichlet problem* (3.29) *has a unique solution u with $\nabla u\in\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$ and*

$$\|\nabla u\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}\leq C\lambda\|h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Proof: Resorting to scaling arguments, we see that it suffices to consider the case $\lambda=1$. As $-1+1/p<\sigma<1/p$ the function h may be extended antisymmetrically on the whole $\mathbb{R}^n$. We name $\widetilde h$ this extension. Now, as it is compactly supported, Proposition 2.1.3 ensures that $\widetilde h\in B^{\sigma}_{p,q}(\mathbb{R}^n)$ and that

$$\|\widetilde h\|_{B^{\sigma}_{p,q}(\mathbb{R}^n)}\leq C\|h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

As the compatibility condition $\int_{B(0,1)}\widetilde h\,dx=0$ is satisfied (a consequence of antisymmetric extension), we know from [**45**] that there exists some function $k\in B^{1+s}_{p,q}(B(0,1))$ such that

$$\operatorname{div}k=\widetilde h\ \text{in}\ \ B(0,1),\quad k|_{\partial B(0,1)}=0\quad\text{and}\quad\|k\|_{B^{1+\sigma}_{p,q}(B(0,1))}\leq C\|\widetilde h\|_{B^{\sigma}_{p,q}(B(0,1))}\leq C\|h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Note also that if we extend k by 0 on $\mathbb{R}^n$ then owing to the boundary condition, we have $k\in B^{1+\sigma}_{p,q}(\mathbb{R}^n)$ and thus

$$\|k\|_{B^{1+\sigma}_{p,q}(\mathbb{R}^n)}\leq C\|h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

As k is compactly supported, applying once again Proposition 2.1.3 leads to the following series of inequalities:

$$\|k\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}\leq\|k\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n)}\leq C\|k\|_{B^{\sigma}_{p,q}(\mathbb{R}^n)}\leq C\|k\|_{B^{1+\sigma}_{p,q}(\mathbb{R}^n)}\leq C\|h\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Finally, applying the last part of Lemma 3.3.1, we get some solution u to

$$\Delta u = h = \operatorname{div} k \quad \text{in} \quad \mathbb{R}^n_+,$$
$$u|_{x_n=0} = 0 \quad \text{on} \quad \partial\mathbb{R}^n_+$$

with

$$\|\nabla u\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)} \leq C\|k\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

This completes the proof of the lemma. ■

We now turn to the study of the *Neumann problem* for the Poisson equation in the half-space, namely

$$\begin{aligned} \Delta u &= h && \text{in} \quad \mathbb{R}^n_+, \\ \partial_{x_n} u|_{x_n=0} &= 0 && \text{on} \quad \partial\mathbb{R}^n_+, \end{aligned} \qquad u \to 0 \ \text{ as } \ |x| \to \infty. \tag{3.31}$$

We are interested in the case where the source term h in (3.31) has very low regularity, so that the meaning of $\partial_{x_n} u$ at the boundary cannot be understood in the classical way. The relevant framework will be taken from Definition 2.3.1: we want to solve (3.31) with source term $h = \mathcal{DIV}\,[k;\zeta]$ in $\dot{\mathcal{B}}^{\sigma-1}_{p,q}(\mathbb{R}^n_+)$, that is to find some distribution u so that

$$\mathcal{DIV}\,[\nabla u; 0] = \mathcal{DIV}\,[k;\zeta],$$

that is to say,

$$-\int_{\mathbb{R}^n_+} \nabla u \cdot \nabla\varphi\, dx = -\int_{\mathbb{R}^n_+} k \cdot \nabla\varphi\, dx + \int_{\partial\mathbb{R}^n_+} \zeta\varphi\, d\varsigma \quad \text{for all} \ \ \varphi \in \mathcal{C}^\infty_c(\overline{\mathbb{R}^n_+}). \tag{3.32}$$

Note that as $\partial\mathbb{R}^n_+$ is noncompact, the meaning of the compatibility condition over ζ in Definition 2.3.1 is unclear. In fact, given that $\mathcal{S}_0(\partial\mathbb{R}^n_+)$ is *dense* in the space $\dot B^{\sigma-1/p}_{p,q}(\partial\mathbb{R}^n_+)$ (in the sense of the weak $*$ topology only if $q = \infty$), the compatibility condition for ζ is 'hidden' in the definition of the Besov space.

Lemma 3.3.3. *Let $h = \mathcal{DIV}\,[k;\zeta] \in \dot{\mathcal{B}}^{\sigma-1}_{p,q}(\mathbb{R}^n_+)$ with $-1 + 1/p < \sigma < 1/p$. Then equation* (3.31) *admits a unique solution u in the meaning of* (3.32) *with $\nabla u \in \dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)$. Moreover*

$$\|\nabla u\|_{\dot B^{\sigma}_{p,q}(\mathbb{R}^n_+)} \leq C\|\,\mathcal{DIV}\,[k;\zeta]\|_{\dot{\mathcal{B}}^{\sigma-1}_{p,q}(\mathbb{R}^n_+)}. \tag{3.33}$$

Proof: Uniqueness is obvious as any harmonic function over $\mathbb{R}^n_+$ with null normal derivative on $\partial\mathbb{R}^n_+$ may be extended by symmetry into an harmonic function over $\mathbb{R}^n$. Hence the condition $u \to 0$ at infinity implies that $u \equiv 0$ on $\mathbb{R}^n_+$.

In order to prove Inequality (3.33), we shall first construct some function H going to 0 at infinity, so that

$$-\int_{\mathbb{R}^n_+} \nabla H \cdot \nabla\varphi\, dx = \int_{\partial\mathbb{R}^n_+} \zeta\varphi\, d\varsigma \quad \text{for all} \ \ \varphi \in \mathcal{C}^\infty_c(\overline{\mathbb{R}^n_+}).$$

In other word, we want to solve

$$\begin{aligned} \Delta H &= 0 && \text{in} \quad \mathbb{R}^n_+, \\ -\partial_{x_n} H|_{x_n=0} &= \zeta && \text{on} \quad \partial\mathbb{R}^n_+, \end{aligned} \ , \qquad H \to 0 \ \text{ at } \ \infty. \tag{3.34}$$

This may be done by using the Fourier transform with respect to tangential variables x': we get

$$-|\xi'|^2 \mathcal{F}_{x'} H + \partial^2_{x_n x_n} \mathcal{F}_{x'} H = 0, \qquad \mathcal{F}_{x'} H \to 0 \ \text{ for } \ x_n \to +\infty,$$

the solution of which is given by the explicit formula

$$H = \mathcal{F}^{-1}_{x'}\Big[\frac{1}{|\xi'|} e^{-|\xi'|x_n} \mathcal{F}_{x'}\zeta\Big].$$

Note that without loss of generality it suffices to consider $\zeta \in \mathcal{S}_0(\partial\mathbb{R}^n_+)$, as one may argue by density. Now, Lemma 2.2.2 ensures that

$$\nabla H \in \dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+) \quad\text{and}\quad \|\nabla H\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)} \le C\|\zeta\|_{\dot{B}^{\sigma-\frac{1}{p}}_{p,q}(\partial\mathbb{R}^n_+)}.$$

Next, let us construct some distribution w satisfying

$$\int_{\mathbb{R}^n_+} \nabla w\cdot\nabla\varphi\,dx = \int_{\mathbb{R}^n_+} k\cdot\nabla\varphi\,dx \quad\text{for all}\quad \varphi\in\mathcal{C}^\infty_c(\overline{\mathbb{R}^n_+}), \tag{3.35}$$

that is to say $\mathcal{DIV}\,[\nabla w;0] = \mathcal{DIV}\,[k;0]$. For that, we consider the symmetric/antisymmetric extension $k_{\rm div}$ of k over $\mathbb{R}^n$, namely the function $k_{\rm div}$ defined by

$$k_{\rm div} = k \quad\text{on}\quad \mathbb{R}^n_+ \quad\text{and}\quad (k'_{\rm div}, k^n_{\rm div})(x',x^n) := (k',-k^n)(x',-x^n) \quad\text{for}\quad x^n<0,$$

and solve, according to Prop. 2 in [**15**],

$$\Delta w_{\rm sym} = \operatorname{div} k_{\rm div} \quad\text{in}\quad \mathbb{R}^n. \tag{3.36}$$

We get some distribution $w_{\rm sym}$ such that

$$\|\nabla w_{\rm sym}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \le C\|k_{\rm div}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}. \tag{3.37}$$

Note that (see Remark 2.2.1)

$$\|k_{\rm div}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \le C\|k\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}$$

so that the restriction w of $w_{\rm sym}$ to $\mathbb{R}^n_+$ satisfies

$$\|\nabla w\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)} \le C\|k\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

In addition, owing to the symmetry property of $k_{\rm div}$, the function $w_{\rm sym}$ has to be symmetric with respect to the hyperplane $\{x_n=0\}$. Therefore (3.36) implies (3.35), and setting $u := w+H$ completes the proof of the lemma. ∎

LEMMA 3.3.4. *Let $p\in(1,\infty)$, $q\in[1,\infty]$ and $\sigma\in(-1+1/p,1/p)$. Let h be in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ with*

$$\int_{\mathbb{R}^n_+} h(x)\,dx = 0 \quad\textit{and}\quad \operatorname{Supp} h \subset \overline{B(0,\lambda)\cap\mathbb{R}^n_+}.$$

Then the Neumann problem (3.31) *has a solution u with $\nabla u\in\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$, and we have*

$$\|\nabla u\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)} \le C\lambda\|h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}.$$

Proof: As h is in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)$ with $-1+1/p<\sigma<1/p$, the symmetric extension $\widetilde{h}$ of h belongs to $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$ and satisfies

$$\|\widetilde{h}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \le C\|h\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n_+)}, \qquad \int_{\mathbb{R}^n}\widetilde{h}(x)\,dx = 0 \quad\text{and}\quad \operatorname{Supp}\widetilde{h}\subset\overline{B(0,\lambda)}.$$

Therefore, according to Lemma 3.2.1, the problem

$$\Delta\widetilde{u} = \widetilde{h} \quad\text{in}\quad \mathbb{R}^n, \qquad \widetilde{u}\to 0 \quad\text{as}\quad x\to\infty$$

has a unique solution $\widetilde{u}$ with

$$\|\nabla\widetilde{u}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \le C\lambda\|\widetilde{h}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$

Note that here the constructed solution $\widetilde{u}$ also belongs to $\dot{B}^{\sigma+2}_{p,q}(\mathbb{R}^n_+)$ so that $\nabla\widetilde{u}$ does have a trace at $\partial\mathbb{R}^n_+$. Owing to the symmetry of $\widetilde{h}$ with respect to the hyperplane $x_n=0$, the solution $\widetilde{u}$ is symmetric, too. Hence the homogeneous Neumann boundary condition on $\{x_n=0\}$ is satisfied.

So setting $u := \widetilde{u}|_{\mathbb{R}^n_+}$ provides the desired solution for (3.31). Indeed, as $\nabla\widetilde{u}$ is an extension of ∇u, one may write

$$\|\nabla u\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)} \leq \|\nabla\widetilde{u}\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n)} \leq C\lambda\|\widetilde{h}\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n)} \leq C\lambda\|h\|_{\dot{B}^\sigma_{p,q}(\mathbb{R}^n_+)}.$$

This completes the proof of the corollary. ■

3.4. The nonhomogeneous Neumann problem in bounded or exterior domains

This section is devoted to solving the nonhomogeneous Neumann problem

$$(3.38)\qquad \begin{aligned} \Delta P &= 0 \quad \text{in} \quad \Omega, \\ \partial_{\vec{n}} P &= b \quad \text{on} \quad \partial\Omega \end{aligned}$$

in an exterior domain (that is in the complement of some simply connected compact subset of $\mathbb{R}^n$) or in a bounded domain of $\mathbb{R}^n$.

In the exterior domain case, we supplement the equation with

$$(3.39)\qquad P \to 0 \quad \text{as} \quad |x| \to \infty,$$

while, if Ω is bounded, we ask for

$$(3.40)\qquad \int_\Omega P\,dx = 0.$$

We focus on the case of rough data at the boundary. If Ω is a bounded domain then the issue is somewhat standard. If it is an exterior domain then a special case has been proved in [**42**] - Lemma 2.3 by techniques of potential theory.

The main result of this section reads:

THEOREM 3.4.1. *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ with $n \geq 3$, or a simply connected bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Let b be in $B^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)$ for some $p \in (1,\infty)$, $q \in [1,\infty]$ and $-1+1/p < \sigma < 1/p$, satisfying the compatibility condition (in the distributional meaning)*

$$\int_{\partial\Omega} b\,d\sigma = 0.$$

Then equation (3.38) *supplemented with* (3.39) *or* (3.40) *has a unique solution P such that:*

- *Bounded domain case: $P \in B^{\sigma+1}_{p,q}(\Omega)$ and*

$$(3.41)\qquad \|P\|_{B^{1+\sigma}_{p,q}(\Omega)} \leq C\|b\|_{B^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)}.$$

- *Exterior domain case: $\nabla P \in \dot{B}^\sigma_{p,q}(\Omega)$ and $P \in B^\sigma_{p,q}(K)$ for any compact subset K of $\mathbb{R}^n$ such that* $\mathrm{dist}(\partial\Omega, \Omega\setminus K) > 0$ *(see Fig. 3.4). In addition, we have*

$$(3.42)\qquad \|\nabla P\|_{\dot{B}^\sigma_{p,q}(\Omega)} \leq C\|b\|_{B^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)},$$

$$(3.43)\qquad \|P\|_{B^\sigma_{p,q}(K)} \leq C_K\|b\|_{B^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)}.$$

Proof: We shall only consider the case $q = p$ so as to benefit from localization properties of Besov spaces $\dot{B}^s_{p,p}$ (see relation (3.56) below). The general case follows by interpolation. We shall focus on the exterior domain case and will just indicate what has to be changed in the easier bounded case.

Let us first prove a priori estimates for smooth solutions. We consider some solution P to (3.38) such that $P \in B^{\sigma}_{p,p}(K)$ and $\nabla P \in \dot{B}^{\sigma}_{p,p}(\Omega)$. We shall first prove that

$$\|\nabla P\|_{\dot{B}^{\sigma}_{p,p}(\Omega)} \leq C(\|b\|_{B^{\sigma-\frac{1}{p}}_{p,p}(\partial\Omega)} + \|P\|_{B^{\sigma}_{p,p}(K)}) \tag{3.44}$$

and next that

$$\|P\|_{B^{\sigma}_{p,p}(K)} \leq C\|b\|_{B^{\sigma-\frac{1}{p}}_{p,p}(\partial\Omega)}. \tag{3.45}$$

To show (3.44), we localize the system by means of some partition of unity $\{\eta^l\}_{0\leq l\leq L}$ of Ω: we take a finite number of smooth functions $\eta^l : \mathbb{R}^n \to [0,1]$ $(0 \leq l \leq L)$ such that

(1) $\eta^0 \equiv 0$ on a neighborhood of $\mathbb{R}^n \setminus \Omega$ and $\eta^0 \equiv 1$ on $\Omega \setminus K$;
(2) η^l $(1 \leq l \leq L)$ is supported in some open set Ω^l of size λ that intersects $\partial\Omega$ and such that $\{\Omega^l\}_{1\leq l\leq L}$ is a covering of $\partial\Omega$;
(3) $\|\nabla\eta^l\|_{L_\infty} \leq C\lambda^{-1}$ if $l \geq 1$;
(4) $\sum_{l=0}^{L} \eta^l \equiv 1$ on Ω.

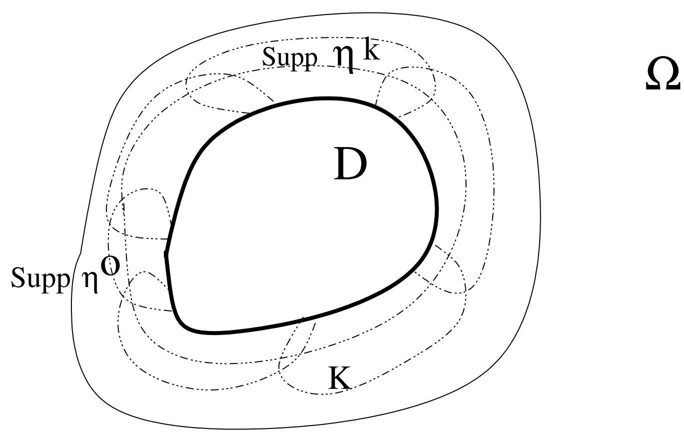

FIGURE 3.2. The subset K and the partition of unity $(\eta^k)_{0\leq k\leq L}$ of Ω.

Of course, if Ω is bounded then (1) has to be replaced with:

(1′) η^0 is supported in some compact subset K of Ω that does not intersect $\partial\Omega$.

Let $P^l := \eta^l P$ and $b^l := \eta^l b$. Corollary 2.1.1 ensures that P^l is in $B^{\sigma}_{p,p}(K)$ and that ∇P^l is in $\dot{B}^{\sigma}_{p,p}(\Omega)$ (as indeed one may write that $\nabla P^l = \eta^l \nabla P + P\nabla\eta^l$ and use the fact that $\nabla\eta^l$ is supported in K). In addition, P^l fulfills

$$\begin{aligned} &\Delta P^l = 2\,\mathrm{div}\,(P\nabla\eta^l) - P\Delta\eta^l && \text{in } \ \Omega,\\ &\partial_{\vec{n}} P^l = b^l && \text{on } \ \partial\Omega. \end{aligned} \tag{3.46}$$

In the case $l = 0$ we get the problem in the whole space since $\mathrm{Supp}\,\eta^0 \cap \partial\Omega = \emptyset$. The compatibility condition

$$\int_{\mathbb{R}^n} \left(2\,\mathrm{div}\,(P\nabla\eta^0) - P\Delta\eta^0\right) dx = 0$$

holds since P is harmonic and goes to zero at infinity.

Therefore combining Lemma 3.2.1 and Prop. 2 in [**15**] implies that

$$\|\nabla P^0\|_{\dot{B}^{\sigma}_{p,p}(\Omega)} \lesssim \|P\nabla\eta^0\|_{\dot{B}^{\sigma}_{p,p}(\Omega)} + \|P\Delta\eta^0\|_{\dot{B}^{\sigma}_{p,p}(\Omega)}.$$

Additionally we note that, owing to the definition of Besov spaces by restriction, to the fact that $\operatorname{Supp}\nabla\eta^0 \subset K$ and that $\dot{B}^\sigma_{p,p}(\mathbb{R}^n)$ is stable by multiplication by $\mathcal{C}^\infty_c(\mathbb{R}^n)$ functions, we have

$$\|P\nabla\eta^0\|_{\dot{B}^\sigma_{p,p}(\Omega)} \lesssim \|\widetilde{P}\|_{\dot{B}^\sigma_{p,p}(\mathbb{R}^n)} \quad \text{for any extension } \ \widetilde{P} \ \text{ of } \ P|_K \ \text{ on } \ \mathbb{R}^n.$$

Therefore

$$\|P\nabla\eta^0\|_{\dot{B}^\sigma_{p,p}(\Omega)} \leq C\|P\|_{B^\sigma_{p,p}(K)}.$$

Similar arguments lead to

$$\|P\Delta\eta^0\|_{\dot{B}^\sigma_{p,p}(\Omega)} \leq C\|P\|_{B^\sigma_{p,p}(K)}.$$

Putting these two inequalities together, one may thus conclude that

$$\|\nabla P^0\|_{\dot{B}^\sigma_{p,p}(\Omega)} \leq C\|P\|_{B^\sigma_{p,p}(K)}. \tag{3.47}$$

Let $l \in \{1, \cdots, L\}$. Then $\operatorname{Supp}\eta^l \cap \partial\Omega \neq \emptyset$ and we have to perform a change of variables in order to transform (3.38) into a Neumann problem in the half-space. To track the information at the boundary, we use the *normal preserving* change of coordinates $z = Z^l(x)$ (see Chapter 2 and [**46**] for details), and get:

$$\begin{aligned} &\Delta_z \overline{P^l} = 2\,\overline{\operatorname{div}_x(P\nabla_x\eta^l)} - \overline{P\Delta_x\eta^l} + (\Delta_z - \Delta_x)\overline{P^l} \quad \text{in} \quad \mathbb{R}^n_+,\\ &\partial_{z_n}\overline{P^l}|_{z_n=0} = \overline{b^l} \qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \partial\mathbb{R}^n_+. \end{aligned} \tag{3.48}$$

Setting $B^l := DZ_l \circ Z_l^{-1}$ and $A^l := B^l - \mathrm{Id}$, and taking into account that $\operatorname{div}_x = {}^TB^l : \nabla_z$ the above system recasts in

$$\begin{aligned} &\Delta_z\overline{P^l} = \operatorname{div}_z k^l + g^l \quad \text{in} \quad \mathbb{R}^n_+,\\ &\partial_{z_n}\overline{P^l}|_{z_n=0} = \overline{b^l} \qquad \text{on} \quad \partial\mathbb{R}^n_+ \end{aligned} \tag{3.49}$$

with

$$\begin{aligned} k^l &:= -B^{lT}B^l\nabla_z\overline{P^l} + \overline{P^l}B^l\operatorname{div}A^l + 2{}^TB^l\overline{P\nabla_x\eta^l},\\ g^l &:= -\overline{P^l}B^l\operatorname{div}{}^TA^l - \overline{P\Delta_x\eta^l}. \end{aligned}$$

Note that by construction of $\overline{P^l}$, we have[4]

$$-\int_{\partial\mathbb{R}^n_+} \overline{b}^l\,dx' = \int_{\mathbb{R}^n_+} g^l\,dx' - \int_{\partial\mathbb{R}^n_+} k^l_n\,dx'. \tag{3.50}$$

This compatibility condition will be important in the sequel.

We now plan to bound $\|\nabla\overline{P}^l\|_{\dot{B}^\sigma_{p,p}(\mathbb{R}^n_+)}$ according to Lemma 3.3.3. This requires our writing g^l and the boundary condition $\overline{b}^l$ in terms of the generalized divergence operator $\mathcal{DIV}$. To achieve it, let us consider the following problem

$$\begin{aligned} &\operatorname{div}L^l = E_{ant}g^l \quad \text{in} \quad B(0,\lambda),\\ &L^l = 0 \qquad\qquad\quad \text{on} \quad \partial B(0,\lambda), \end{aligned} \tag{3.51}$$

where E_{ant} denotes the antisymmetric extension operator.

[4]The minus sign is due to the downward orientation of the exterior normal on $\partial\mathbb{R}^n_+$.

The fact that $-1+1/p<\sigma<1/p$ ensures that $E_{ant}g^l\in\dot B^{\sigma}_{p,p}(\mathbb{R}^n)$ and that

$$\|E_{ant}g^l\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n)}\leq C\|g^l\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)}.$$

In addition, by construction, we have

$$\operatorname{Supp} E_{ant}g^l\subset\overline{B(0,\lambda)}\quad\text{and}\quad\int_{B(0,\lambda)}E_{ant}g^l\,dx=0.$$

Therefore Theorem 2.3.1 ensures that (3.51) has a solution L^l such that

$$\|L^l\|_{B^{1+\sigma}_{p,p}(B(0,\lambda))}\lesssim\|E_{ant}g^l\|_{B^{\sigma}_{p,p}(B(0,\lambda))}\lesssim\|g^l\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)}.$$

As, by construction,

$$L^l\in B^{1+\sigma}_{p,p}(B(0,\lambda))\quad\text{and}\quad L^l=0\ \text{ on }\ \partial B(0,\lambda),$$

the function

$$\widetilde{L}^l:=\begin{cases}L^l & \text{in}\quad B(0,\lambda),\\ 0 & \text{elsewhere}\end{cases}$$

belongs to $B^{\sigma+1}_{p,p}(\mathbb{R}^n)$ and satisfies

$$\|\widetilde{L}^l\|_{B^{1+\sigma}_{p,p}(\mathbb{R}^n)}\lesssim\|g^l\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)}.\tag{3.52}$$

So, setting $H^l:=k^l+\widetilde{L}^l$, we have

$$\operatorname{div}H^l=\operatorname{div}k^l+g^l\quad\text{in}\quad\mathbb{R}^n_+$$

and we are thus led to the problem of estimating the solution $\nabla\overline{P^l}$ to

$$\mathcal{DIV}\,[\nabla\overline{P^l};0]=\mathcal{DIV}\,[H^l;-H^l_n+\overline{b^l}].$$

In order to apply Lemma 3.3.3, it suffices to establish that

$$H^l\in\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)\quad\text{and}\quad(H^l_n-\overline{b^l})\in\dot B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+).$$

According to Proposition 2.1.3 and Corollary 2.2.1, because H^l is compactly supported, in order to establish the first condition, it is enough to show that $H^l\in B^{\sigma}_{p,p}(\mathbb{R}^n_+)$. Note also that, according to (3.50), we have

$$\int_{\partial\mathbb{R}^n_+}(\overline{b^l}-H^l_n)\,dx'=0.\tag{3.53}$$

Hence, given (3.53) and $\sigma-1/p>-1-(n-1)/p'$ (which is equivalent to $\sigma>-n/p'$), for justifying the second condition, it suffices to show that $(H^l_n-\overline{b^l})\in B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)$. This will come up as a consequence of the trace theorem (as regards H^l_n) and of the stability of nonhomogeneous spaces by right composition (see Lemma 2.1.1) and localization, as regards $\overline{b^l}$. As a final consequence, we will get that

$$\|\nabla\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)}\lesssim\|H^l\|_{B^{\sigma}_{p,p}(\mathbb{R}^n_+)}+\|(k^l_n,L^l_n,\overline{b^l})|_{z_n=0}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}.\tag{3.54}$$

Concerning $k^l_n|_{z_n=0}$ the important and nice fact is that the change of coordinates $x\to z$ preserves the normal vector. Hence the highest order term, namely $B^{lT}B^l\nabla_z\overline{P^l}\cdot\vec{e}_n|_{\partial\mathbb{R}^n_+}$, is just $\overline{b^l}$. Therefore

$$\|k^l_n|_{z_n=0}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}\lesssim\|B^l(\operatorname{div}A^l)\overline{P^l}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}+\|{}^TB^l\overline{P\nabla_x\eta^l}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}+\|\overline{b^l}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}.$$

Of course, any space $B^{\varepsilon}_{p,p}(\partial\mathbb{R}^n_+)$ with $\varepsilon>0$ embeds in $B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)$, so that the trace theorem implies that

$$\|k^l_n|_{z_n=0}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}\lesssim\|B^l(\operatorname{div}A^l)\overline{P^l}\|_{B^{\varepsilon+1/p}_{p,p}(\mathbb{R}^n_+)}+\|{}^TB^l\overline{P\nabla_x\eta^l}\|_{B^{\varepsilon+1/p}_{p,p}(\mathbb{R}^n_+)}+\|\overline{b^l}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}.$$

As nonhomogeneous Besov spaces are stable by multiplication by compactly supported smooth functions, we thus deduce that

$$\|k_n^l|_{z_n=0}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)} \lesssim \|\overline{P^l}\|_{B^{\varepsilon+1/p}_{p,p}(\mathbb{R}^n_+)} + \lambda^{-1}\|P\|_{B^{\varepsilon+1/p}_{p,p}(K)} + \|\overline{b^l}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}.$$

Note that one may take some $\varepsilon > 0$ such that $\sigma < \varepsilon + 1/p < \sigma + 1$. So arguing by interpolation, we conclude that for any small enough α, we have

$$\begin{aligned}\|k_n^l|_{z_n=0}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)} \leq \alpha\|\nabla\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} + C_\alpha\|\overline{P^l}\|_{B^{\sigma}_{p,p}(\mathbb{R}^n_+)} \\ + C\lambda^{-1}\|P\|_{B^{\epsilon+1/p}_{p,p}(K)} + \|\overline{b^l}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)}.\end{aligned} \tag{3.55}$$

Next, we see that the trace theorem, (3.52) and the definition of g^l imply that

$$\|(L_n^l)|_{z_n=0}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)} \lesssim \|\overline{P^l}B^l \operatorname{div}{}^T A^l\|_{B^{\sigma}_{p,p}(\mathbb{R}^n_+)} + \|(\operatorname{div}{}^T A^l)\overline{P^l\nabla_x\eta^l}\|_{B^{\sigma}_{p,p}(\mathbb{R}^n_+)} + \|\overline{P\Delta_x\eta^l}\|_{B^{\sigma}_{p,p}(\mathbb{R}^n_+)}.$$

So we get

$$\|(L_n^l)|_{z_n=0}\|_{B^{\sigma-1/p}_{p,p}(\partial\mathbb{R}^n_+)} \lesssim \lambda^{-1}\|\overline{P^l}\|_{B^{\sigma}_{p,p}(\mathbb{R}^n_+)} + \lambda^{-2}\|P\|_{B^{\sigma}_{p,p}(K)}.$$

Let us now turn to bounds for H^l. Most of the terms entering in its definition have already been bounded above. The only definitely new term is $A\nabla_z\overline{P^l}$. Now, from product estimates, we get (see Proposition 2.1.2)

$$\|B^{lT}B^l\nabla_z\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} \lesssim \|A^l\|_{\dot B^{n/p'}_{p',1}(\mathbb{R}^n_+)}\|\nabla_z\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)}.$$

According to (2.30), we have

$$\|A^l\|_{\dot B^{n/p'}_{p',1}(\mathbb{R}^n_+)} \leq C\lambda.$$

Hence one may conclude that

$$\|H^l\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} \lesssim \lambda\|\nabla_z\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} + \lambda^{-2}\|P\|_{B^{\sigma}_{p,p}(K)} + \lambda^{-1}\|\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)}.$$

Plugging all the previous estimates in (3.54) (we take $\alpha = \lambda$ in (3.55)), we end up with

$$\|\nabla_z\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} \lesssim \lambda\|\nabla_z\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)} + \lambda^{-2}\|P\|_{B^{\sigma}_{p,p}(K)} + \|\overline{b^l}\|_{B^{\sigma-\frac{1}{p}}_{p,p}(\partial\mathbb{R}^n_+)} + C_\lambda\|\overline{P^l}\|_{\dot B^{\sigma}_{p,p}(B_+(0,\lambda))}.$$

Of course, the first term of the r.h.s may be absorbed by the l.h.s if taking λ small enough. To conclude, we use the fact that Besov spaces $\dot B^{\sigma}_{p,p}$ have the following localization property (see e.g. [**9**], [**53**], page 69 or [**57**], Section 2.4.7):

$$\|\nabla P\|^p_{\dot B^{\sigma}_{p,p}(\Omega)} \approx \sum_l \|\nabla P^l\|^p_{\dot B^{\sigma}_{p,p}(\Omega)}. \tag{3.56}$$

Hence one may write (see Lemma 2.1.1)

$$\|\nabla P\|_{\dot B^{\sigma}_{p,p}(\Omega)} \lesssim \Big(\sum_l \|\nabla P^l\|^p_{\dot B^{\sigma}_{p,p}(\Omega)}\Big)^{\frac{1}{p}} \lesssim \Big(\sum_l \|\nabla\overline{P^l}\|^p_{\dot B^{\sigma}_{p,p}(\mathbb{R}^n_+)}\Big)^{\frac{1}{p}}.$$

So from the previous inequalities and the localization property (3.56), we get

$$\|\nabla P\|_{\dot B^{\sigma}_{p,p}(\Omega)} \leq C_\lambda\Big(\|P\|_{B^{\sigma}_{p,p}(K)} + \|b\|_{B^{\sigma-\frac{1}{p}}_{p,p}(\partial\Omega)}\Big). \tag{3.57}$$

Note that C_λ blows up as $\lambda \to 0$, but remains finite for all $\lambda > 0$, since the sum is finite.

Next we want to show the estimate

$$\|P\|_{L_p(K)} \leq C\|\partial_{\vec{n}} P\|_{B^{\sigma-1/p}_{p,p}(\partial\Omega)} \quad \text{for} \quad -1+1/p<\sigma<1/p. \tag{3.58}$$

Note that, as may be easily shown by decomposing any extension of P to $\mathbb{R}^n$ into low and high frequencies, we have for $\sigma \geq 0$ and $\epsilon > 0$:

$$\|P\|_{B^{\sigma}_{p,p}(K)} \leq \epsilon\|\nabla P\|_{B^{\sigma}_{p,p}(K)} + C(\epsilon)\|P\|_{L_p(K)}. \tag{3.59}$$

Furthermore, if $\sigma < 0$ then $B^{-\sigma}_{p',p'}(K)$ is embedded in $L_m(K)$ for some $m > p'$ hence also in $L_{p'}(K)$ as K is compact. Therefore, by duality, we get $L_p(K) \hookrightarrow B^{\sigma}_{p,p}(K)$, whence

$$\|P\|_{B^{\sigma}_{p,p}(K)} \leq C\|P\|_{L_p(K)} \quad \text{for} \quad \sigma < 0.$$

Let us first consider the easier case where Ω is bounded. Then the above inequalities hold with $K = \overline{\Omega}$ under Condition (3.40). Therefore if we assume (by contradiction) that (3.58) fails then there exists a sequence $\{P_k\}$ of functions with average 0, gradient in $B^{\sigma}_{p,p}(\Omega)$ and such that $\Delta P_k = 0$ in Ω and that

$$1 = \|P_k\|_{L_p(\Omega)} > k\|\partial_{\vec{n}} P_k\|_{B^{\sigma-\frac{1}{p}}_{p,p}(\partial\Omega)}. \tag{3.60}$$

By (3.44), this implies that $\{\nabla P_k\}$ is bounded in $B^{\sigma}_{p,p}(\Omega)$, too. Hence $(P_k)_{k\in\mathbb{N}}$ is bounded in $B^{1+\sigma}_{p,p}(\Omega)$. Since this latter space is compactly embedded in $L_p(\Omega)$, we deduce that there exists a function $P^* \in B^{1+\sigma}_{p,p}(\Omega)$, and some subsequence $(P_{k_n})_{n\in\mathbb{N}}$ so that

$$P_{k_n} \to P^* \text{ in } L_p(\Omega). \tag{3.61}$$

Note that we also have

$$\partial_{\vec{n}} P_{k_n}|_{\partial\Omega} \to 0 \text{ in } B^{\sigma-\frac{1}{p}}_{p,p}(\partial\Omega).$$

So finally, P^* must fulfill the system

$$\begin{aligned} \Delta P^* &= 0 \quad \text{in} \quad \Omega, \\ \partial_{\vec{n}} P^* &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \qquad \int_\Omega P^*\,dx = 0,$$

the only solution of which is $P^* \equiv 0$, a consequence of the strong maximum principle (as already pointed out in (3.19)). Now, the strong convergence given by (3.61) implies that $\|P^*\|_{L_p(\Omega)} = 1$, a contradiction. Hence (3.41) has been proved.

Let us now assume that Ω is an exterior domain. As before, we suppose that there exists a sequence $\{P_k\}$ of functions going to 0 at infinity, with gradient in $\dot{B}^{\sigma}_{p,p}(\Omega)$ and such that $\Delta P_k = 0$ in Ω and that

$$1 = \|P_k\|_{L_p(K)} > k\|\partial_{\vec{n}} P_k\|_{B^{\sigma-\frac{1}{p}}_{p,p}(\partial\Omega)}.$$

By (3.44), this implies that $\{\nabla P_k\}$ is bounded in $\dot{B}^{\sigma}_{p,p}(\Omega)$, too. So finally, because of (3.59),

1. $(\nabla P_k)_{k\in\mathbb{N}}$ is bounded in $\dot{B}^{\sigma}_{p,p}(\Omega)$,
2. $(P_k)_{k\in\mathbb{N}}$ is bounded in $B^{\sigma+1}_{p,p}(K)$.

Since the embedding of $B^{\sigma+1}_{p,p}$ in L_p is locally compact and since the Besov spaces have the Fatou property, we thus conclude that there exists a function P^* over Ω with $P^* \in B^{\sigma+1}_{p,p}(K)$ and $\nabla P^* \in \dot{B}^{\sigma}_{p,p}(\Omega)$, and some subsequence $(P_{k_n})_{n\in\mathbb{N}}$ so that

$$P_{k_n} \to P^* \ \text{ in } \ L_p(K) \quad \text{and} \quad \nabla P_{k_n} \rightharpoonup \nabla P^* \ \text{ in } \ \dot{B}^{\sigma}_{p,p}(\Omega). \tag{3.62}$$

Note that we also have

$$\partial_{\vec{n}} P_{k_n}|_{\partial\Omega} \to 0 \text{ in } B^{\sigma-\frac{1}{p}}_{p,p}(\partial\Omega).$$

So finally, P^* must fulfill the system (recall that $\Delta P_k = 0$)

$$\begin{aligned} \Delta P^* &= 0 \quad \text{in} \quad \Omega, \\ \partial_{\vec{n}} P^* &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.63}$$

If in addition,

$$P^* \to 0 \quad \text{as} \quad |x| \to \infty \tag{3.64}$$

then the strong maximum principle [**33**] implies that the only solution is $P^* \equiv 0$ (the regularity of P^* follows from the smoothness of the boundary, and we use the fact that $n \geq 3$). But the strong convergence given by (3.62) implies that $\|P^*\|_{L_p(K)} = 1$, a contradiction. Hence (3.45) has been proved.

In order to establish that (3.64) is fulfilled, let us first consider the case where p is so small as to satisfy $1 + \sigma < n/p$. Then, thanks to Sobolev embedding, $P_k \in L_m(\Omega)$ uniformly for some $m < \infty$, and thus $P^* \in L_m(\Omega)$, too.

In the large p case, we may find $1 < \widetilde{p} < p$ such that $1 + \sigma < n/\widetilde{p}$. Since $\partial\Omega$ is compact $B^{\sigma-1/p}_{p,1}(\partial\Omega) \subset B^{\sigma-1/\widetilde{p}}_{\widetilde{p},1}(\partial\Omega)$, and we may conclude as in the previous case that P^* satisfies (3.63) and is in $L_m(\Omega)$ for some finite m. Thus (3.64) is satisfied.

Let us finally say a few words on the proof of existence. If the boundary is smooth then we may use the L_2 approach. Take a sequence of smooth functions $b_k \in C^\infty(\partial\Omega)$ such that $b_k \to b$ in $B^{\sigma-\frac{1}{p}}_{p,1}(\partial\Omega)$. For each b_k we are able to construct a smooth solution such that $\nabla P_k \in L_2(\Omega)$ (via the Lax-Milgram theorem). In particular, if $n \geq 3$ then, owing to Sobolev embedding, $P_k \in L_{\frac{2n}{n-2}}(\Omega)$ so that $P_k \to 0$ at infinity. Furthermore, P_k satisfies (3.57). Then, passing to the limit we get the existence of our solution in the desired class of regularity, and (3.42) and (3.43) are fulfilled. ■

Remark 3.4.1. *The case of an exterior domain of* $\mathbb{R}^2$ *requires a modification, since we are not able to guarantee the condition* $P^* \to 0$ *as* $|x| \to \infty$. *We have just the knowledge that the limit is a constant. We prefer not to concentrate on this case, since it is beyond our interests in this monograph (as a matter of fact, the restriction* $n \geq 3$ *will appear for other reasons when investigating the evolutionary Stokes system in exterior domains, see the next chapter).*

3.5. Helmholtz projection

The mathematical theory of incompressible flows requires handy function spaces with the divergence-free property. Those spaces may be obtained as the image of a suitable continuous projection operator on a space X of *vector valued* functions over the domain Ω. Such an operator $\mathcal{P} : X \to X$ is often called *Helmholtz or Leray projector* and has the property that for any $f \in X$ one has

$$\operatorname{div} \mathcal{P}f = 0 \ \text{ in } \ \Omega, \quad \text{and} \quad \mathcal{P}f \cdot \vec{n} = 0 \text{ at } \ \partial\Omega.$$

At the formal level, $\mathcal{P}$ may be defined by $\mathcal{P}f := f - \nabla P$ where ∇P is a solution to

$$\begin{aligned} \Delta P &= \operatorname{div} f \qquad \text{in} \quad \Omega, \\ (\nabla P - f) \cdot \vec{n} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.65}$$

Let us first focus on solving (3.65).

Proposition 3.5.1. *Let* Ω *be an exterior domain of* $\mathbb{R}^n$ *(with* $n \geq 3$*), or a bounded domain, the whole space or the half-space (with* $n \geq 2$*). Assume that* f *has coefficients in* $\dot{B}^\sigma_{p,q}(\Omega)$ *for some* $p \in (1, \infty)$, $q \in [1, \infty]$ *and* $\sigma \in (-1 + 1/p, 1/p)$. *Then the above equation has a unique solution* ∇P *in* $\dot{B}^\sigma_{p,q}(\Omega)$ *with*

$$\|\nabla P\|_{\dot{B}^\sigma_{p,q}(\Omega)} \leq C \|f\|_{\dot{B}^\sigma_{p,q}(\Omega)}. \tag{3.66}$$

Proof: Fix some $\varepsilon > 0$ and consider an extension $\widetilde{f}$ of f on the whole space $\mathbb{R}^n$ such that
$$\|\widetilde{f}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \leq \|f\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} + \varepsilon.$$
According to [**15**] one may find $\widetilde{P}$ in $\dot{B}^{1+\sigma}_{p,q}(\mathbb{R}^n)$ satisfying
$$\Delta\widetilde{P} = \operatorname{div}\widetilde{f} \quad \text{in} \quad \mathbb{R}^n,$$
and
$$\|\nabla\widetilde{P}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)} \leq C\|\widetilde{f}\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)}.$$
At this point, setting $P := \widetilde{P}$ completes the proof in the case $\Omega = \mathbb{R}^n$ (as we have $\widetilde{f} = f$).

If $\Omega \neq \mathbb{R}^n$ then we use that, by construction, $\operatorname{div}(\widetilde{f} - \nabla\widetilde{P}) = 0$ in $\mathbb{R}^n$ and $\widetilde{f} - \nabla\widetilde{P}$ is in $\dot{B}^{\sigma}_{p,q}(\mathbb{R}^n)$. Hence the normal component $(\widetilde{f} - \nabla\widetilde{P}) \cdot \vec{n}$ has a trace at $\partial\Omega$ (this is Lemma 2.2.4) and
$$\|(\widetilde{f} - \nabla\widetilde{P}) \cdot \vec{n}\|_{\dot{B}^{\sigma-\frac{1}{p}}_{p,q}(\partial\Omega)} \leq C\|\widetilde{f} - \nabla\widetilde{P}\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} \leq C(\|f\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} + \varepsilon).$$
Therefore, setting $P := \widetilde{P} + P_{new}$, we see that P_{new} has to satisfy (3.38) with boundary data $b := (f - \nabla\widetilde{P}) \cdot \vec{n}$. Note that because $\operatorname{div}(f - \nabla\widetilde{P}) = 0$ in Ω, the compatibility condition for b holds. So applying Theorem 3.4.1, we find that $\nabla P_{new} \in \dot{B}^{\sigma}_{p,q}(\Omega)$ and that
$$\|\nabla P_{new}\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} \leq C\big(\|f\|_{\dot{B}^{\sigma}_{p,q}(\Omega)} + \varepsilon\big).$$
Finally, the half-space case may be easily deduced from Lemma 3.3.3. ■

COROLLARY 3.5.1. *Let Ω be an exterior domain of $\mathbb{R}^n$ (with $n \geq 3$) or a bounded domain, the whole space or the half-space (with $n \geq 2$). Then the Helmholtz projection $\mathcal{P}$ is a continuous self-map on $\dot{B}^{s}_{p,q}(\Omega; \mathbb{R}^n)$.*

Proof: Let $\mathcal{P}f := f - \nabla P$ with P given by Proposition 3.5.1. Then we have $\operatorname{div}\mathcal{P}f = 0$ and $\mathcal{P}f \cdot \vec{n} = 0$ at the boundary, hence
$$\|\nabla P\|_{\dot{B}^{s}_{p,q}(\Omega;\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{s}_{p,q}(\Omega;\mathbb{R}^n)}.$$
This completes the proof of the corollary. ■

CHAPTER 4

The evolutionary Stokes system

This section is devoted to proving the main result concerning the linear theory of the Stokes system. First we concentrate on the whole and half-space cases, then we consider the problem in exterior or bounded domains. Finally, we establish a low order bound for the velocity on a compact set, which will enable us to obtain the final *time independent* estimate, allowing to consider the whole time half-line.

4.1. The whole space case

We investigate the system

$$\begin{aligned} &u_t - \nu\Delta u + \nabla P = f && \text{in} \quad (0,T)\times\mathbb{R}^n,\\ &\operatorname{div} u = g && \text{in} \quad (0,T)\times\mathbb{R}^n,\\ &u|_{t=0} = u_0 && \text{on} \quad \mathbb{R}^n. \end{aligned} \tag{4.1}$$

The main result of this part reads

THEOREM 4.1.1. *Let $p \in (1,\infty)$ and $-1+1/p < s < 1/p$. Let $f \in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$, $g \in \mathcal{C}([0,T];\dot B^{s-1}_{p,1}(\mathbb{R}^n))$ with $\nabla g \in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$ and $u_0 \in \dot B^s_{p,1}(\mathbb{R}^n)$. Assume in addition that*

$$g_t = \operatorname{div} B + A, \quad \textit{with} \ \operatorname{Supp} A(t,\cdot) \subset \overline{B(0,\lambda)} \ \textit{and} \ \int_{\mathbb{R}^n} A(t,x)\,dx = 0 \tag{4.2}$$

for some $\lambda > 0$ and $A, B \in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$ and that the compatibility condition $\operatorname{div} u_0 = g|_{t=0}$ on $\mathbb{R}^n$ (in the distributional meaning) is satisfied.

Then System (4.1) has a unique solution $(u,\nabla P)$ with

$$u \in \mathcal{C}([0,T);\dot B^s_{p,1}(\mathbb{R}^n)) \quad \textit{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$$

and the following estimate is valid:

$$\begin{aligned} &\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} + \|u_t, \nu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\ &\qquad \le C(\|f,\nu\nabla g, B\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} + \lambda\|A\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} + \|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}), \end{aligned} \tag{4.3}$$

where C is an absolute constant with no dependence on ν, T and λ.

Proof: Applying the divergence operator to the equation, we see that the pressure has to satisfy

$$\Delta P = \operatorname{div} f + \nu\Delta g - \operatorname{div} B - A \qquad \text{in} \quad \mathbb{R}^n,$$

where time is treated here as a parameter and we have used the constraint $g_t = \operatorname{div} B + A$. In order to construct ∇P, we thus set $\nabla P := \nabla P_0 + \nabla P_1$ with

$$\Delta P_1 = \operatorname{div}(f + \nu\nabla g - B) \qquad \text{in} \quad \mathbb{R}^n, \tag{4.4}$$

and

$$\Delta P_0 = -A \qquad \text{in} \quad \mathbb{R}^n.$$

For solving (4.4), one may set (treating t as a parameter and denoting by $\mathcal{F}$ the Fourier transform over $\mathbb{R}^n$)

$$\nabla P_1 := \mathcal{F}^{-1}\left[\frac{\xi}{|\xi|^2}\big(\xi\cdot\mathcal{F}[f-\nu\nabla g-B]\big)\right].$$

As the above formula involves an homogeneous multiplier of degree 0, we get

$$\|\nabla P_1\|_{\dot B^s_{p,1}(\mathbb{R}^n)} \leq C\|f,\nu\nabla g, B\|_{\dot B^s_{p,1}(\mathbb{R}^n)}. \tag{4.5}$$

Constructing P_0 stems from Lemma 3.2.1 which yields

$$\|\nabla P_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)} \leq C\lambda\|A\|_{\dot B^s_{p,1}(\mathbb{R}^n)} \quad \text{whenever} \quad -1+1/p<s<1/p. \tag{4.6}$$

Hence (4.5) and (4.6) give

$$\|\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} \leq C\big(\|f,\nu\nabla g,B\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\lambda\|A\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\big). \tag{4.7}$$

Now that ∇P has been constructed, we rewrite (4.1) as

$$\begin{aligned} u_t-\nu\Delta u &= f-\nabla P \quad &&\text{in} \quad (0,T)\times\mathbb{R}^n,\\ u|_{t=0} &= u_0 &&\text{on} \quad \mathbb{R}^n. \end{aligned}$$

Standard results for the heat equation (see e.g. [**5**], Chap. 2) ensure the existence of u in the desired functional space, together with

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|u_t,\nu\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\lesssim\|f,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}. \tag{4.8}$$

This inequality is an easy consequence of the fact that there exist two constants c and C such that for all $j\in\mathbb{Z}$ and $\alpha\in\mathbb{R}^+$ one has (see e.g. [**5**])

$$\|e^{\alpha\dot\Delta}\dot\Delta_j h\|_{L_p(\mathbb{R}^n)} \leq Ce^{-c\alpha 2^{2j}}\|\dot\Delta_j h\|_{L_p(\mathbb{R}^n)}.$$

Now, as u satisfies

$$\dot\Delta_j u(t) = e^{\nu t\Delta}\dot\Delta_j u_0+\int_0^t e^{\nu(t-\tau)\Delta}\dot\Delta_j(f-\nabla P)\,d\tau,$$

we readily get

$$\|\dot\Delta_j u(t)\|_{L_p(\mathbb{R}^n)} \leq C\bigg(e^{-c\nu t2^{2j}}\|\dot\Delta_j u_0\|_{L_p(\mathbb{R}^n)}+\int_0^t e^{-c\nu(t-\tau)2^{2j}}\|\dot\Delta_j(f-\nabla P)\|_{L_p(\mathbb{R}^n)}\,d\tau\bigg),$$

whence

$$\|\dot\Delta_j u\|_{L_\infty(0,T;L_p(\mathbb{R}^n))}+\nu 2^{2j}\|\dot\Delta_j u\|_{L_1(0,T;L_p(\mathbb{R}^n))}\lesssim\|\dot\Delta_j u_0\|_{L_p(\mathbb{R}^n)}+\|\dot\Delta_j(f-\nabla P)\|_{L_1(0,T;L_p(\mathbb{R}^n))}.$$

Multiplying the inequality by 2^{js} and summing up over j yields (4.8). Remembering (4.7) implies the sought inequality (4.3).

To complete the proof of the theorem, one has to check that $\operatorname{div} u\equiv g$ on $[0,T)\times\mathbb{R}^n$. Now, applying the divergence operator to the equation for u and using the definition of ∇P and the assumption on $g|_{t=0}$, we see that

$$\partial_t(\operatorname{div} u-g)-\nu\Delta(\operatorname{div} u-g)=0, \quad (\operatorname{div} u-g)|_{t=0}=0.$$

As uniqueness holds true in $\mathcal{C}([0,T);\mathcal{S}'(\mathbb{R}^n))$, we thus have $\operatorname{div} u-g\equiv 0$ on $[0,T)\times\mathbb{R}^n$ and one may conclude that $(u,\nabla P)$ satisfies System (4.1). ■

REMARK 4.1.1. *Note that for any $q\in[1,\infty]$, the same proof would give that if $u_0\in\dot B^{s+2-\frac{2}{q}}_{p,q}(\mathbb{R}^n)$, f and ∇g are in $L_q(0,T;\dot B^s_{p,q}(\mathbb{R}^n))$, with in addition (4.2) for some A and B in $L_q(0,T;\dot B^s_{p,q}(\mathbb{R}^n))$ then $u\in L_\infty(0,T;\dot B^{s+2-\frac{2}{q}}_{p,q}(\mathbb{R}^n))$ and $(\partial_t u,\nabla^2u,\nabla P)\in L_q(0,T;\dot B^s_{p,q}(\mathbb{R}^n))$ with an estimate similar to (4.3).*

4.2. The Stokes system in the half-space

The purpose of this part is to extend Theorem 4.1.1 to the half-space case $\mathbb{R}^n_+$. We thus consider

$$(4.9)\qquad \begin{aligned} &u_t - \nu\Delta u + \nabla P = f && \text{in} \quad (0,T)\times\mathbb{R}^n_+,\\ &\operatorname{div} u = g && \text{in} \quad (0,T)\times\mathbb{R}^n_+,\\ &u|_{x_n=0} = 0 && \text{on} \quad (0,T)\times\partial\mathbb{R}^n_+,\\ &u|_{t=0} = u_0 && \text{on} \quad \mathbb{R}^n_+. \end{aligned}$$

The main result of this part reads:

THEOREM 4.2.1. *Let $p\in(1,\infty)$ and $s\in(-1+1/p,1/p)$. Assume that $f\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$, $g\in\mathcal{C}(0,T;\dot B^{s-1}_{p,1}(\mathbb{R}^n_+))$ with $\nabla g\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$, and $u_0\in\dot B^s_{p,1}(\mathbb{R}^n_+)$ with $\operatorname{div} u_0=0$ in $\mathbb{R}^n_+$ (and thus $g|_{t=0}\equiv 0$), and $u_0\cdot\vec e_n|_{\partial\mathbb{R}^n_+}\equiv 0$. Additionally we require that there exist some $A,B\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$ and $b\in L_1(0,T;\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$ such that*

$$g_t = A + \mathcal{DIV}\,[B,b] \text{ with } \operatorname{Supp} A(t,\cdot)\subset\overline{B(0,\lambda)\cap\mathbb{R}^n_+} \text{ for some } \lambda>0.$$

Then System (4.9) has a unique solution $(u,\nabla P)$ with

$$u\in\mathcal{C}([0,T];\dot B^s_{p,1}(\mathbb{R}^n_+)),\quad u_t,\nabla^2 u,\nabla P\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$$

and the following estimate is valid:

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|u_t,\nu\nabla^2 u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} \le C\Big(\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\\ +\|f,\nu\nabla g,B\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \lambda\|A\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|b\|_{L_1(0,T;\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}\Big),\end{aligned}$$

where C is an absolute constant with no dependence on ν, T and λ.

REMARK 4.2.1. *To simplify the presentation, we here restrict ourselves to the case where the sought solution u has to vanish on $\partial\mathbb{R}^n_+$. The nonhomogeneous case where u equals some given h at the boundary reduces to the homogeneous case, if assuming that h admits some extension $\widetilde h$ over $(0,T)\times\mathbb{R}^n_+$ so that $\widetilde h_t - \nu\Delta\widetilde h$ (resp. $\operatorname{div}\widetilde h$) satisfies the same assumptions as f (resp. g).*

The proof of Theorem 4.2.1 is based essentially on the results from [**15**] concerning the case $g\equiv 0$ and on our recent work in [**16**] so as to discard the nonhomogeneous condition over $\operatorname{div} u$. Recall the statement pertaining to the case $g\equiv 0$:

THEOREM 4.2.2. *Under the above assumptions with $g\equiv 0$, System (4.9) has a unique solution $(u,\nabla P)$ satisfying*

$$u\in\mathcal{C}([0,T];\dot B^s_{p,1}(\mathbb{R}^n_+)),\quad u_t,\nabla^2 u,\nabla P\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$$

and the following estimate is valid:

$$(4.10)\qquad \begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|u_t,\nu\nabla^2 u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ \le C(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}),\end{aligned}$$

where C is an absolute constant with no dependence on ν and T.

Proof of Theorem 4.2.1 : Without loss of generality, one may assume that f and g are defined on $\mathbb{R}_+\times\mathbb{R}^n$ (just extend them by 0 beyond T). We want to reduce our study to the case $u_0\equiv 0$ and $f\equiv 0$. For that, one may first consider System (4.9) with $g\equiv 0$. Then Theorem 4.2.2 provides us with a solution $(u_1,\nabla P_1)$ satisfying (4.10). Now, setting

$$(4.11)\qquad u = u_{new} + u_1 \quad\text{and}\quad \nabla P = \nabla P_{new} + \nabla P_1,$$

we see that $(u_{new}, \nabla P_{new})$ satisfies System (4.9) with $f \equiv 0$ and $u_0 \equiv 0$ and the same g (since $\operatorname{div} u_1 = 0$). Additionally having $g|_{t=0} = 0$ one may extend the system on the whole time line (it is just a matter of setting $u_{new} = \nabla P_{new} = 0$ for $t < 0$), preserving the regularity of the data. So dropping the index *new* for simplicity, System (4.9) translates into

$$\begin{aligned} &u_t - \nu\Delta u + \nabla P = 0 && \text{in} \quad \mathbb{R}\times\mathbb{R}^n_+, \\ &\operatorname{div} u = g && \text{in} \quad \mathbb{R}\times\mathbb{R}^n_+, \\ &u|_{x_n=0} = 0 && \text{on} \quad \mathbb{R}\times\partial\mathbb{R}^n_+. \end{aligned} \tag{4.12}$$

LEMMA 4.2.1. *Assume that* $\nabla g \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ *with* p, s *as above and* $g_t = A + \mathcal{DIV}\,[B, b]$ *with* $A, B \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$, $b \in L_1(\mathbb{R}; \dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$ *and, in addition,*

$$\operatorname{Supp} A(\cdot, t) \subset \overline{B(0,\lambda)\cap\mathbb{R}^n_+}.$$

Then System (4.12) *has a unique solution* $(u, \nabla P)$ *satisfying the following bound:*

$$\begin{aligned} &\|u\|_{L_\infty(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|u_t, \nu\nabla^2 u, \nabla P\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\ &\qquad \le C\big(\|b\|_{L_1(\mathbb{R};\dot{B}^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))} + \|\nu\nabla g, B\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \lambda\|A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\big). \end{aligned}$$

Proof: Setting

$$u_{new}(t,x) = \nu u_{old}(\nu^{-1}t, x), \quad P_{new}(t,x) = P_{old}(\nu^{-1}t, x), \quad g_{new}(t,x) = \nu g_{old}(\nu^{-1}t, x), \tag{4.13}$$

we see that (u_{new}, P_{new}) satisfies System (4.12) with $\nu = 1$. Hence one may assume with no loss of generality that $\nu = 1$. Next, we want to discard the source term g. For that, we define w to be the solution to the following equation (treating the time variable as a parameter):

$$\begin{aligned} &\Delta w = g && \text{in} \quad \mathbb{R}^n_+, \\ &w|_{x_n=0} = 0 && \text{on} \quad \partial\mathbb{R}^n_+. \end{aligned} \tag{4.14}$$

According to Lemma 3.3.1, we have

$$\|\nabla^3 w\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \lesssim \|\nabla g\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}. \tag{4.15}$$

Now, differentiating (4.14) with respect to the time variable and using the assumption on g_t, we discover that

$$\begin{aligned} &\Delta w_t = g_t = A + \operatorname{div} B && \text{in} \quad \mathbb{R}^n_+, \\ &w_t|_{x_n=0} = 0 && \text{on} \quad \partial\mathbb{R}^n_+. \end{aligned} \tag{4.16}$$

Hence according to Lemma 3.3.1 and Corollary 3.3.1, we must have

$$\|\nabla w_t\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \le C\big(\|B\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \lambda\|A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\big). \tag{4.17}$$

Then we look for a solution $(u, \nabla P)$ to (4.12) with $\nu = 1$ in the following form:

$$u = u_{new} + \nabla w, \qquad \nabla P = \nabla P_{new} - \nabla w_t + \Delta\nabla w.$$

Dropping the index *new*, we thus get the following system[1]:

$$\begin{aligned} &u_t - \Delta u + \nabla P = 0 && \text{in} \quad \mathbb{R}\times\mathbb{R}^n_+, \\ &\operatorname{div} u = 0 && \text{in} \quad \mathbb{R}\times\mathbb{R}^n_+, \\ &u|_{x_n=0} = -\nabla w|_{x_n=0} && \text{on} \quad \mathbb{R}\times\partial\mathbb{R}^n_+. \end{aligned} \tag{4.18}$$

Let (ξ_0, ξ') denote the Fourier variables for the Fourier transform $\mathcal{F}_{t,x'}$ with respect to t and x'. We claim that the pressure P obeys the formula

$$\widehat{P}(\xi_0, \xi', x_n) := \mathcal{F}_{t,x'}P(\xi_0, \xi', x_n) = \widehat{P}_b(\xi_0, \xi')e^{-|\xi'|x_n}, \tag{4.19}$$

[1] Note that only $\partial_{x_n} w$ may be nonzero at the boundary.

where

$$\widehat{P}_b(\xi_0,\xi') := -\left(\frac{i\xi_0}{|\xi'|}+r+|\xi'|\right)\widehat{\partial_{x_n} w}|_{x_n=0} \quad\text{and}\quad r^2 := i\xi_0+|\xi'|^2. \tag{4.20}$$

Indeed, it is only a matter of looking at (4.18) as the following heat equation:

$$u_t-\Delta u = -\nabla P, \qquad u|_{x_n=0} = -\nabla w|_{x_n=0}.$$

Then taking the Fourier transform with respect to time and to the tangential directions, and considering only the solutions which decay at $x_n\to+\infty$, we obtain from the standard theory of linear ordinary differential equations,

$$\widehat{u}(\xi_0,\xi',x_n) = \widehat{u}(\xi_0,\xi',0)e^{-rx_n}+\frac{1}{2r}\int_0^\infty [e^{-r|x_n-s_n|}-e^{-r(x_n+s_n)}]\begin{pmatrix} -i\xi' \\ |\xi'| \end{pmatrix}\widehat{P}_b(\xi_0,\xi')e^{-|\xi'|s_n}\,ds_n.$$

So differentiating the n-th component with respect to x_n and letting x_n go to 0 gives (see [**15**] for more details):

$$\partial_{x_n}\widehat{u}_n(\xi_0,\xi',x_n)|_{x_n=0} = r\widehat{\partial_{x_n} w}|_{x_n=0}+\left(\int_0^\infty |\xi'|e^{-(r+|\xi'|)s_n}\,ds_n\right)\widehat{P}_b(\xi_0,\xi').$$

We know that the tangential parts of the boundary data is zero and that $\operatorname{div} u = 0$, thus $\partial_{x_n} u_n|_{x_n=0} = 0$. Therefore, the above equality implies that

$$\frac{|\xi'|}{r+|\xi'|}\widehat{P}_b(\xi_0,\xi') = -r\widehat{\partial_{x_n} w}|_{x_n=0}.$$

This yields formula (4.20).

In order to construct the pressure, we proceed as follows:

(1) we construct an extension G_1 of $\mathcal{F}^{-1}_{t,x'}[|\xi'|\widehat{\partial_{x_n} w}|_{x_n=0}]$ with $\nabla G_1 \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$;
(2) we construct an extension G_2 of $\mathcal{F}^{-1}_{t,x'}[r\widehat{\partial_{x_n} w}|_{x_n=0}]$ with $\nabla G_2 \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$;
(3) we construct an extension V of $\mathcal{F}^{-1}_{t,x'}[i\xi_0/|\xi'|\widehat{\partial_{x_n} w}|_{x_n=0}]$ with $\nabla V \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$;
(4) keeping in mind that P has to be harmonic, we write $P = P_{new}-G_1-G_2-V$ where P_{new} is a solution to

$$\Delta P_{new} = \Delta(G_1+G_2+V) \quad\text{in}\ \ \mathbb{R}^n_+, \qquad P_{new}|_{x_n=0} = 0 \quad\text{on}\ \ \partial\mathbb{R}^n_+,$$

and establish that $\nabla P_{new} \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ with a suitable estimate.

First step: construction of G_1. We just have to set $G_1 := |D'|\partial_{x_n} w$ where the pseudo-differential operator $|D'|$ is defined by

$$|D'|z := \mathcal{F}^{-1}_{x'}(|\xi'|\widehat{z}).$$

Indeed, we have $\nabla^3 w \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$, hence

$$\nabla|D'|\partial_{x_n} w \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+)).$$

Second step: construction of G_2. It suffices to set $G_2 := -\partial_{x_n} y$ with y the solution to

$$\begin{aligned} &y_t-\Delta y = 0 && \text{in} \quad \mathbb{R}\times\mathbb{R}^n_+, \\ &y|_{x_n=0} = (\partial_{x_n} w)|_{x_n=0} && \text{on} \quad \mathbb{R}\times\partial\mathbb{R}^n_+, \end{aligned} \qquad y\to 0 \ \text{ at } \ \infty. \tag{4.21}$$

Indeed, we observe that $\mathcal{F}_{t,x'} y = e^{-rx_n}\widehat{\partial_{x_n} w}|_{x_n=0}$, whence

$$\partial_{x_n} y = -\mathcal{F}^{-1}_{t,x'}[re^{-rx_n}\widehat{\partial_{x_n} w}|_{x_n=0}],$$

and $\partial_{x_n} y$ is thus an extension of $-\mathcal{F}^{-1}_{t,x'}(r\widehat{\partial_{x_n} w}|_{x_n=0})$ to $\mathbb{R}\times\mathbb{R}^n_+$.

In order to solve (4.21), we decompose y into $y = z + \partial_{x_n} w$ with z the solution to

$$\begin{cases} z_t - \Delta z = \Delta(\partial_{x_n} w) - (\partial_{x_n} w)_t & \text{in} \quad \mathbb{R}\times\mathbb{R}^n_+, \\ z|_{x_n=0} = 0 & \text{on} \quad \mathbb{R}\times\partial\mathbb{R}^n_+, \end{cases} \qquad z \to 0 \ \text{ at } \ \infty.$$

Note that the right-hand side is in $L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))$ by construction of w and satisfies, owing to (4.15) and (4.17),

$$\|\Delta(\partial_{x_n} w) - (\partial_{x_n} w)_t\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))} \leq C\|\nabla g, B, \lambda A\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}. \tag{4.22}$$

Thus, as a consequence of Prop. 6 in [**15**] or Th. 6.2. in [**19**], we get that $z \in L_1(\mathbb{R};\dot B^{2+s}_{p,1}(\mathbb{R}^n_+))$ and that z_t and $\nabla^2 z$ are bounded by the right-hand side of (4.22).

Third step: construction of V. We now want to extend the term coming from

$$\mathcal{F}^{-1}_{t,x'}\left(\frac{i\xi_0}{|\xi'|}\widehat{\partial_{x_n} w}|_{x_n=0}\right) = |D'|^{-1}\partial_{x_n} w_t|_{x_n=0} \quad \text{with} \quad w_t \ \text{ fulfilling (4.16).}$$

It is natural to set

$$V = \mathcal{F}^{-1}_{x'}\big(e^{-|\xi'|x_n}|\xi'|^{-1}\mathcal{F}_{x'}(\partial_{x_n} w_t|_{x_n=0})\big)$$

and to use Lemma 2.2.2 to bound ∇V, after observing that

$$|D'|V = -\partial_{x_n} V = \mathcal{F}^{-1}_{x'}\big(e^{-|\xi'|x_n}\mathcal{F}_{x'}(\partial_{x_n} w_t|_{x_n=0})\big).$$

However, we first have to justify that $\partial_{x_n} w_t|_{x_n=0}$ is in $L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$. This will based on the last part of Lemma 3.3.1 (treating the time as a parameter), once we will have found a suitable vector-field H such that

$$\operatorname{div} H = \operatorname{div} B + A.$$

In other words, we have to express A as the divergence of some vector field in $\dot B^s_{p,1}(\mathbb{R}^n_+)$. Considering the antisymmetric extension of A on the whole space we are guaranteed that

$$\int_{\mathbb{R}^n} E_{anti} A\, dx = 0.$$

Hence Proposition 3.1.1 enables us to solve the following problem

$$\begin{cases} \Delta a = E_{anti} A & \text{in} \quad B(0,\lambda), \\ \partial_{\vec n} a = 0 & \text{on} \quad \partial B(0,\lambda), \end{cases} \qquad \int_{B(0,\lambda)} a\, dx = 0 \tag{4.23}$$

and Proposition 3.1.1 provides us with the following bound in nonhomogeneous Besov space:

$$\|a\|_{B^{s+2}_{p,1}(B(0,\lambda))} \leq C\|A\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}. \tag{4.24}$$

The above inequality combined with the scaling argument of Corollary 3.2.1 yields

$$\|\nabla a\|_{\dot B^s_{p,1}(B(0,\lambda))} \leq C\lambda\|A\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}, \tag{4.25}$$

where C is independent of λ.

Next we consider the extension $\widetilde{\nabla a}$ of ∇a by 0, on $\mathbb{R}^n$. In light of Corollary 2.2.1, $\widetilde{\nabla a}$ is in $L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n))$ and satisfies (4.25). Additionally, owing to $\nabla a\cdot\vec n|_{\partial B(0,\lambda)} = 0$, we have

$$-\int_{B(0,\lambda)} \nabla a\cdot\nabla\varphi\, dx = \int_{B(0,\lambda)} E_{anti} A\,\varphi\, dx \quad \text{for all} \quad \varphi \in \mathcal{C}^\infty(\overline{B(0,\lambda)}).$$

Hence, we gather that

$$\operatorname{div}\widetilde{\nabla a} = E_{anti} A \quad \text{in the space} \quad L_1(\mathbb{R};\dot B^{s-1}_{p,1}(\mathbb{R}^n)).$$

In conclusion, we may write $A + \operatorname{div} B = \operatorname{div} H$ with $H := B + \widetilde{\nabla a}|_{\mathbb{R}^n_+}$ satisfying

$$\|H\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))} \leq C\big(\lambda\|A\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|B\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\big). \tag{4.26}$$

The above computations allow us to write $A+\mathcal{DIV}\,[B;b]=\mathcal{DIV}\,[H;h]$ with $H=B+\nabla a$ satisfying (4.26), and $h=b-\partial_{x_n}a$ (the minus sign is due to the orientation of the exterior normal unit vector at $\partial\mathbb{R}^n_+$). Thanks to (4.23) the trace of $\partial_{x_n}a$ on $\partial\mathbb{R}^n_+\cap B(0,\lambda)$ is well defined in $L_1(\mathbb{R};B^{s+1-1/p}_{p,1}(\partial\mathbb{R}^n_+\cap B(0,\lambda)))$ hence h makes sense in $L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$, and we have

$$\|h\|_{L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}\leq C\big(\lambda\|A\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|b\|_{L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}\big).\tag{4.27}$$

Now, differentiation of (4.14) by t gives

$$\operatorname{div}(\nabla w_t-H)=0\quad\text{in}\quad\mathbb{R}^n_+.$$

Therefore, according to Lemma 2.2.4, $(\partial_{x_n}w_t-H^n)|_{x_n=0}$ in $L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$ and satisfies

$$\|(\partial_{x_n}w_t-H^n)|_{x_n=0}\|_{L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}\leq C\|\nabla w_t-H\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))},$$

whence, owing to (4.17) and (4.26),

$$\|(\partial_{x_n}w_t-H^n)|_{x_n=0}\|_{L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}\leq C\big(\|B\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}+\lambda\|A\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\big).$$

Because $H^n|_{x_n=0}=-h$ with h satisfying (4.27), one may deduce that $\partial_{x_n}w_t|_{x_n=0}$ is defined as an element of $L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))$. Therefore V is our sought extension: we have $\nabla V\in L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))$ and

$$\|\nabla V\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|B\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}+\lambda\|A\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|b\|_{L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}.\tag{4.28}$$

Remark 4.2.2. *To justify the construction of V we need to clarify one point. We have to be able to find a family (ψ_ϵ) of smooth approximations of $\mathcal{DIV}\,[H,h]$, that is*

$$(\nabla\psi^\epsilon,\partial_{x_n}\psi^\epsilon)\to(H,h)\ \text{in}\ \dot B^s_{p,1}(\mathbb{R}^n_+)\times\dot B^{s-\frac1p}_{p,1}(\partial\mathbb{R}^n_+).\tag{4.29}$$

For $\varepsilon>0$, we solve the following system

$$\begin{aligned}\Delta\psi^\epsilon&=\operatorname{div}H^\epsilon\quad &&\text{in}\quad\mathbb{R}^n_+,\\ \partial_{x_n}\psi^\epsilon&=h^\epsilon &&\text{at}\quad\partial\mathbb{R}^n_+,\end{aligned}$$

where H^ϵ,h^ϵ are mollifications of H and h.

Next, we split the system into two parts:

$$\begin{aligned}\Delta\psi^\epsilon_0&=0, &\qquad \Delta\psi^\epsilon_1&=\operatorname{div}H^\epsilon\quad &&\text{in}\quad\mathbb{R}^n_+,\\ \partial_{x_n}\psi^\epsilon_0&=h^\epsilon &\qquad \partial_{x_n}\psi^\epsilon_1&=0 &&\text{at}\quad\partial\mathbb{R}^n_+.\end{aligned}$$

Thanks to Lemma 3.3.3 the solvability is ensured, and the estimates and linearity imply (4.29).

Last step: construction of the pressure and velocity. Recall that the pressure defined in (4.19) has to fulfill the system

$$\begin{aligned}\Delta P&=0 &&\text{in}\quad\mathbb{R}^n_+,\\ P|_{x_n=0}&=P_b &&\text{on}\quad\partial\mathbb{R}^n_+.\end{aligned}$$

Now, setting $EP_b:=-G_1-G_2-V$, the previous steps ensure that $\nabla EP_b\in L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))$ and $EP_b|_{x_n=0}=P_b$. In addition, (4.22) and (4.28) yield

$$\|\nabla EP_b\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|\nabla g,B,\lambda A\|_{L_1(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|b\|_{L_1(\mathbb{R};\dot B^{s-1/p}_{p,1}(\partial\mathbb{R}^n_+))}.$$

Hence decomposing P into $P=P_{new}+EP_b$ and dropping the index *new* as usual, we see that it suffices to consider the system

$$\begin{aligned}\Delta P&=-\operatorname{div}\nabla EP_b &&\text{in}\quad\mathbb{R}^n_+,\\ P|_{x_n=0}&=0 &&\text{on}\quad\partial\mathbb{R}^n_+.\end{aligned}$$

According to Lemma 3.3.1, this system has a unique solution P going to 0 at infinity, with $\nabla P \in L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))$. Collecting all the steps and changes of unknown functions, one may conclude that the pressure – solution to (4.12) obeys the estimate

$$\|\nabla P\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leq C\|\nabla g, B, \lambda A\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))}.$$

Once the pressure term has been estimated, it is easy to define the velocity u as the solution to the following heat equation:

$$\begin{aligned} u_t - \Delta u &= -\nabla P \qquad \text{in} \qquad \mathbb{R}^n_+ \times \mathbb{R}, \\ u|_{x_n=0} &= 0 \qquad\qquad \text{on} \qquad \partial\mathbb{R}^n_+. \end{aligned}$$

Solving this equation in our functional framework has been done in [**15**], Prop. 6. We get a velocity u with the required property. Lemma 4.2.1 is proved. Subsequently, the proof of Theorem 4.2.1 is complete, too. ∎

REMARK 4.2.3. *A direct approach, based on the explicit solution formula as in* [**21**] *is possible if one assumes that* $g = \operatorname{div} R$ *with* $\partial_t(\mathcal{H}(R|_{x_n=0})) \in L_1(\mathbb{R}_+;\dot{B}^s_{p,1})$, *where* $\mathcal{H}(R|_{x_n=0})$ *stands for the harmonic extension of* $R|_{x_n=0}$, *as in* (3.34).

REMARK 4.2.4. *If we assume that the function g (and thus also B) is* compactly *supported then one may provide a* shorter *proof based on our work in* [**16**]. *Indeed, in this case, it is possible to remove directly the divergence part of u by means of the (generalized) Bogovskiĭ formula, resorting to the $\mathcal{DIV}$ functional introduced in Chapter 2. Here, we treated the general case where g is not compactly supported because it shows how the approach of* [**15**] *has to be adapted so as to handle nonzero divergence condition. Besides, it will be useful to investigate systems for incompressible fluids in Lagrangian coordinates.*

4.3. The exterior domain case

This section is devoted to solving the time-dependent Stokes system

$$\begin{aligned} u_t - \nu\Delta u + \nabla P = f &\quad \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div} u = g &\quad \text{in} \quad (0,T)\times\Omega, \\ u = 0 &\quad \text{on} \quad (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0 &\quad \text{on} \quad \Omega, \end{aligned} \tag{4.30}$$

in the case of an *exterior* or *bounded* domain Ω.

Extending the results of the previous section to this new situation is the main aim of this part. We shall focus on the unbounded case which is more tricky and just indicate at the end of this section what has to be changed in the bounded domain case.

4.3.1. Proof of time-dependent estimates. As a preliminary step, we shall establish the following time-dependent estimates for (4.30):

THEOREM 4.3.1. *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ with $n \geq 3$. Let $1 < p < \infty$, $-1+1/p < s < 1/p$. Let $u_0 \in \dot{B}^s_{p,1}(\Omega)$, $f \in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$ and $g \in \mathcal{C}([0,T);\dot{B}^{s-1}_{p,1}(\Omega))$ with $g(0) = \operatorname{div} u_0$. Assume in addition that $\nabla g \in L_1(0,T;\dot{B}^s_{p,1}(\Omega))$ and that $g = \operatorname{div} R$ for some vector field R with the following properties*[2]*:*

(1) *R is in $L_1(0,T;L_m(\Omega))$ for some $m \in (1,\infty)$ and $R|_K \in L_1(0,T;B^{1+s}_{p,1}(K))$ where K stands for some bounded subset of Ω such that* $\operatorname{dist}(\partial\Omega,\Omega\setminus K) > 0$ *(see Figure 3.4);*

[2]Note that the last two properties imply that $g_t = \mathcal{DIV}\,[R_t;\varrho]$. That $\frac{d}{dt}((R\cdot\vec{n})|_{\partial\Omega})$ is defined in the sense of distributions is due to the fact that $R|_K$ is in $L_1(0,T;B^{1+s}_{p,1}(K))$, hence its trace on ∂K (and thus on $\partial\Omega$) is well defined (see Proposition 2.2.4).

(2) $\displaystyle\int_{\partial\Omega} R\cdot\vec{n}\,dx=0$;

(3) $R_t\in L_1(0,T;\dot B^s_{p,1}(\Omega))$;

(4) $\varrho:=\frac{d}{dt}((R\cdot\vec n)|_{\partial\Omega})\in L_1(0,T;B^{s-1/p}_{p,1}(\partial\Omega))$.

Then System (4.30) *has a unique solution* $(u,\nabla P)$ *such that*

$$u\in\mathcal{C}([0,T];\dot B^s_{p,1}(\Omega)),\qquad \partial_t u,\nabla^2u,\nabla P\in L_1(0,T;\dot B^s_{p,1}(\Omega)),\tag{4.31}$$

and the following estimate is valid:

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|u_t,\nu\nabla^2u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}&\leq Ce^{CT\nu}\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)}\\&+\|f,\nu\nabla g,R_t\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|\varrho\|_{L_1(0,T;B^{s-1/p}_{p,1}(\partial\Omega))}\\&+\nu\|R\|_{L_1(0,T;L_m(\Omega))}+\nu\|R|_K\|_{L_1(0,T;B^{1+s}_{p,1}(K))}\big),\end{aligned}\tag{4.32}$$

where the constant C *depends only on* K, Ω, s, m *and* p.

Additionally, there holds

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|u_t,\nu\nabla^2u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}&\leq C\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)}\\+\|f,\nu\nabla g,R_t\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\nu\|R\|_{L_1(0,T;L_m(\Omega))}&+\|\varrho\|_{L_1(0,T;B^{s-1/p}_{p,1}(\partial\Omega))}\\+\nu\|R|_K\|_{L_1(0,T;B^{1+s}_{p,1}(K))}&+\nu\|u|_K\|_{L_1(0,T;B^s_{p,1}(K))}\big),\end{aligned}\tag{4.33}$$

where C *depends only on* K, Ω, s, m *and* p.

Proof: Let us first say a few words about the existence. According to the first step of our proof below, we see that one may restrict to the case where $g\equiv0$. Even in this case, the task is more complex than for parabolic systems. To overcome this difficulty we have to control the influence of the pressure. Here one may use a suitable approximation in L_2-type spaces or the results of Giga-Sohr [**31**] and Maremonti-Solonikov [**42**], to obtain the solvability for smooth data together with (4.32). This latter inequality enables us to pass to the limit so as to get a solution satisfying (4.31), as it ensures that the sequence corresponding to smooth data is a *Cauchy sequence* in the space defined in (4.31).

The rest of the proof is devoted to establishing estimates (4.32) and (4.33). We may suppose that we are given a smooth enough solution. Note also that performing the change of variables (4.13) reduces the study to the case $\nu=1$. So we shall make this assumption in all that follows.

First step: removing g. In order to remove the inhomogeneity from the r.h.s. of $(4.30)_2$, we may construct a vector field $v\in L_1(0,T;\dot B^{2+s}_{p,1}(\Omega))$ satisfying $v_t\in L_1(0,T;\dot B^s_{p,1}(\Omega))$ and

$$\operatorname{div}v=g\ \text{ in }\ \Omega,\qquad\qquad v=0\ \text{ on }\ \partial\Omega\tag{4.34}$$

and also, owing to our assumption on g_t,

$$\mathcal{DIV}\,[v_t,0]=\mathcal{DIV}\,[R_t,\varrho].\tag{4.35}$$

In view of our results in [**16**], if g *has compact support* then the most natural approach is to construct v by means of the (generalized) Bogovskiĭ formula associated to the domain Ω: we set

$$v=\mathcal{B}_\Omega(g)=I_\Omega(R)+J_\Omega\big((R\cdot\vec n)|_{\partial\Omega}\big),$$

where the operators I_Ω and J_Ω have been defined in the proof of Theorem 2.3.1. Then, owing to the properties of these two operators (see [**16**]), we may write that

$$-\int_\Omega v\cdot\nabla\varphi\,dx=-\int_\Omega R\cdot\nabla\varphi+\int_{\partial\Omega}(R\cdot\vec n)\varphi\,d\sigma=\int_\Omega\varphi\,g\,dx\quad\text{for all }\ \varphi\in\mathcal{C}^\infty_c(\overline{\Omega}).$$

Differentiating with respect to time we thus get

$$-\int_\Omega v_t\cdot\nabla\varphi\,dx=-\int_\Omega R_t\cdot\nabla\varphi+\int_{\partial\Omega}\varrho\varphi\,d\sigma\quad\text{for all}\quad\varphi\in\mathcal{C}_c^\infty(\overline{\Omega}),$$

which is exactly what we wanted.

That approach works whenever g has compact support for it suffices to solve (4.34) in a bounded subdomain of Ω. The result is given by Theorem 2.3.1, and as nonhomogeneous and homogeneous Besov norms coincide for compactly supported functions, we are done.

Let us now establish that one may construct such a vector field v *even if g is not compactly supported.*

LEMMA 4.3.1. *There exists a vector field v fulfilling* (4.34) *and* (4.35), *supported in $\overline{\Omega}$ and such that*

$$\begin{aligned}\|v\|_{L_1(0,T;\dot B^{s+2}_{p,1}(\Omega))}&\leq C\big(\|\operatorname{div}R\|_{L_1(0,T;\dot B^{s+1}_{p,1}(\Omega))}+\|R\|_{L_1(0,T;L_m(\Omega))}+\|R|_K\|_{L_1(0,T;B^{1+s}_{p,1}(K))}\big),\\ \|v_t\|_{L_1(0,T;\dot B^{s}_{p,1}(\Omega))}&\leq C\big(\|R_t\|_{L_1(0,T;\dot B^{s}_{p,1}(\Omega))}+\|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}\big).\end{aligned}$$

Proof: We aim at reducing the study to the compact support case. As the time variable does not play any role here, we just omit it in the following computations.

Let $\eta^0:\mathbb{R}^n\to[0,1]$ be a smooth cut-off function such that $\eta^0\equiv0$ on a neighborhood of $\mathbb{R}^n\setminus\Omega$ and $\eta^0\equiv1$ on a neighborhood of $\Omega\setminus K$ (see Figure 3.4). Let us consider the following problem:

$$\Delta G=\operatorname{div}(\eta^0R)\quad\text{in}\quad\mathbb{R}^n.\tag{4.36}$$

The solution to (4.36) with gradient going to 0 at infinity satisfies

$$\nabla G:=-(-\Delta)^{-1}\nabla\operatorname{div}(\eta^0R),$$

so that we also have

$$\nabla G_t=-(-\Delta)^{-1}\nabla\operatorname{div}(\eta^0R_t).$$

Because $(-\Delta)^{-1}\nabla\operatorname{div}$ is an homogeneous multiplier of degree 0, we gather from [**5**] and standard results on singular integrals that

$$\|\nabla^2G\|_{\dot B^{1+s}_{p,1}(\mathbb{R}^n)}\lesssim\|\operatorname{div}(\eta^0R)\|_{\dot B^{1+s}_{p,1}(\mathbb{R}^n)},$$
$$\|\nabla G\|_{L_m(\mathbb{R}^n)}\lesssim\|\eta^0R\|_{L_m(\mathbb{R}^n)}\quad\text{and}\quad\|\nabla G_t\|_{\dot B^{s}_{p,1}(\mathbb{R}^n)}\lesssim\|\eta^0R_t\|_{\dot B^{s}_{p,1}(\mathbb{R}^n)}.$$

Therefore, using the decomposition

$$\operatorname{div}(\eta^0R)=\eta^0g+R\nabla\eta^0\quad\text{on}\quad\Omega,$$

we get

$$\begin{aligned}&\|\nabla^2G\|_{\dot B^{1+s}_{p,1}(\mathbb{R}^n)}\lesssim\|g\|_{\dot B^{1+s}_{p,1}(\Omega)}+\|R|_K\|_{B^{1+s}_{p,1}(K)},\\ &\|\nabla G\|_{L_m(\mathbb{R}^n)}\lesssim\|R\|_{L_m(\Omega)},\qquad\|\nabla G_t\|_{\dot B^{s}_{p,1}(\mathbb{R}^n)}\lesssim\|R_t\|_{\dot B^{s}_{p,1}(\Omega)}.\end{aligned}\tag{4.37}$$

Then we look for v in the following form

$$v=w_0+\nabla G.\tag{4.38}$$

The new function w_0 thus has to fulfill

$$\begin{aligned}\operatorname{div}w_0&=g-\operatorname{div}(\eta^0R)&&\text{in}\quad\Omega,\\ w_0|_{\partial\Omega}&=-\nabla G|_{\partial\Omega}&&\text{on}\quad\partial\Omega.\end{aligned}$$

We further decomposer w_0 into

$$w_0=(\eta^0-1)\nabla G+w_1\tag{4.39}$$

with

$$\operatorname{div} w_1 = \operatorname{div}\big((1-\eta^0)R\big) + \operatorname{div}\big((1-\eta^0)\nabla G\big) \ \text{ in } \ \Omega \qquad \text{and} \qquad w_1|_{\partial\Omega} = 0. \tag{4.40}$$

Decomposing $(\eta^0 - 1)\nabla G$ into[3]

$$(\eta^0-1)\nabla G = (\eta^0-1)\dot S_0 \nabla G + (\eta^0-1)(\mathrm{Id} - \dot S_0)\nabla G,$$

and using product estimates in $\mathbb{R}^n$ (recall that $\eta^0 - 1$ is compactly supported), and Bernstein inequality, we get

$$\|(\eta^0-1)\nabla G\|_{\dot B^{s+2}_{p,1}(\mathbb{R}^n)} \lesssim \|\nabla G\|_{L_m(\mathbb{R}^n)} + \|\nabla^2 G\|_{\dot B^{1+s}_{p,1}(\mathbb{R}^n)}.$$

Hence, combining with (4.37),

$$\|(\eta^0-1)\nabla G\|_{\dot B^{s+2}_{p,1}(\mathbb{R}^n)} \lesssim \|g\|_{\dot B^{1+s}_{p,1}(\Omega)} + \|R|_K\|_{B^{1+s}_{p,1}(K)} + \|R\|_{L_m(\Omega)}. \tag{4.41}$$

In order to reduce (4.40) to solving some divergence equation on the bounded set K we have to prove that the average over K of the r.h.s. of (4.40) vanishes. As a matter of fact, owing to the support properties of $1-\eta^0$ and to Condition (2) in Theorem 4.3.1, we can write

$$\begin{aligned}\int_K \operatorname{div}\big((1-\eta^0)(R+\nabla G)\big)dx &= \int_{\partial\Omega} \big(\vec n \cdot (R+\nabla G)\big)d\sigma \\ &= \int_{\partial\Omega} \vec n\cdot\nabla G\, d\sigma = -\int_{\mathbb{R}^n\setminus\Omega} \operatorname{div}(\eta^0 R)\,dx = 0.\end{aligned}$$

Hence we may solve (4.40) via the Bogovskiĭ formula in the set K according to Theorem 2.3.1: setting

$$\begin{aligned} w_1 &= \mathcal{B}_K\big[\operatorname{div}\big((1-\eta^0)(R+\nabla G)\big)\big] \\ &= \mathcal{DIV}_K\big[(1-\eta^0)(R+\nabla G), (1-\eta^0)(\varrho + (\vec n\cdot\nabla G)|_{\partial K})\big],\end{aligned} \tag{4.42}$$

we readily get, by virtue of continuity results for $\mathcal{B}_K$,

$$\begin{aligned}\|w_1\|_{B^{s+2}_{p,1}(K)} &\lesssim \|(1-\eta^0)(\operatorname{div} R + \Delta G)\|_{B^{s+1}_{p,1}(K)} + \|\nabla\eta^0\cdot(R+\nabla G)\|_{B^{1+s}_{p,1}(K)} \\ &\lesssim \|g\|_{B^{s+1}_{p,1}(K)} + \|R|_K\|_{B^{1+s}_{p,1}(K)}.\end{aligned} \tag{4.43}$$

Therefore by (4.37), (4.38), (4.39), (4.41) and (4.43), we conclude that

$$\|v\|_{\dot B^{s+2}_{p,1}(\Omega)} \lesssim \|g\|_{\dot B^{s+1}_{p,1}(\Omega)} + \|R\|_{L_m(\Omega)} + \|R|_K\|_{B^{s+1}_{p,1}(K)}. \tag{4.44}$$

Let us now concentrate on the proof of estimates for v_t. We have

$$v_t = \nabla G_t + (\eta^0-1)\nabla G_t + w_{1,t}.$$

Differentiating (4.42) yields

$$w_{1,t} = I_K[(1-\eta^0)(R_t + \nabla G_t)] + J_K[(1-\eta^0)(\varrho + \partial_{\vec n} G_t|_{\partial K})].$$

That $\partial_{\vec n} G_t$ has a trace at the boundary ∂K is a consequence of Lemma 2.2.4 as, by construction, $\operatorname{div}(\nabla G_t) = 0$ in $\mathbb{R}^n\setminus\Omega$. Furthermore, there holds for a.e. $t>0$,

$$\|\partial_{\vec n} G_t|_{\partial\Omega}\|_{\dot B^{s-1/p}_{p,1}(\partial\Omega)} \lesssim \|\nabla G_t\|_{\dot B^s_{p,1}(\mathbb{R}^n\setminus\Omega)} \lesssim \|R_t\|_{\dot B^s_{p,1}(\Omega)}.$$

Thus, we obtain

$$\|w_{1,t}\|_{\dot B^s_{p,1}(\Omega)} \lesssim \|R_t\|_{\dot B^s_{p,1}(\Omega)} + \|\varrho\|_{B^{s-1/p}_{p,1}(\partial\Omega)}. \tag{4.45}$$

Putting (4.37) and (4.45) together, one may conclude that

$$\|v_t\|_{\dot B^s_{p,1}(\Omega)} \lesssim \|R_t\|_{\dot B^s_{p,1}(\Omega)} + \|\varrho\|_{B^{s-1/p}_{p,1}(\partial\Omega)}. \tag{4.46}$$

[3]Recall that $\dot S_0$ has been defined in (2.2).

Integrating (4.44) with respect to time completes the proof of the lemma. ■

The construction of v given by Lemma 4.3.1 reduces the proof of Theorem 4.3.1 to the case $g \equiv 0$. Indeed, if we set

$$u_{new} = u_{old} - v \quad \text{and} \quad f_{new} = f_{old} - v_t + \Delta v \tag{4.47}$$

then $u = u_{new}$ has to satisfy

$$\begin{aligned} &u_t - \Delta u + \nabla P = f_{new} && \text{in} \quad (0,T) \times \Omega, \\ &\operatorname{div} u = 0 && \text{in} \quad (0,T) \times \Omega, \\ &u = 0 && \text{on} \quad (0,T) \times \partial\Omega, \\ &u|_{t=0} = u_0 && \text{on} \quad \Omega. \end{aligned} \tag{4.48}$$

According to (4.44) and (4.46), the regularity of the new function f is preserved and

$$\begin{aligned} \|f_{new}\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \lesssim{}& \|f_{old}, \nabla g, R_t\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \\ &+ \|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))} + \|R\|_{L_1(0,T;L_m(\Omega))} + \|R\|_{L_1(0,T;B^{1+s}_{p,1}(K))}. \end{aligned} \tag{4.49}$$

Remark 4.3.1. *As already pointed out, in the case of a bounded domain* Ω, *one may directly apply Theorem 2.3.1 so as to remove* g. *Then making the change of unknown* (4.47), *we eventually get* (4.48) *with* f_{new} *satisfying*

$$\|f_{new}\|_{L_1(0,T;B^s_{p,1}(\Omega))} \lesssim \|f_{old}, \nabla g, R_t\|_{L_1(0,T;B^s_{p,1}(\Omega))} + \|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}.$$

Remark 4.3.2. *Note that in this first step, the time variable is just treated as a parameter. Hence reducing the study to the divergence free case may be done in any Besov space* $\dot B^s_{p,r}(\Omega)$ *with* $1 \le r \le \infty$, $1 < p < \infty$ *and* $-1 + 1/p < s < 1/p$.

Second step: an estimate for the pressure. At this point, one may remove the potential part of f and its normal component at the boundary. Indeed, Proposition 3.5.1 enables us to solve the following problem:

$$\begin{aligned} &\Delta Q = \operatorname{div} f && \text{in} \quad \Omega, \\ &(\nabla Q - f) \cdot \vec n = 0 && \text{on} \quad \partial\Omega. \end{aligned}$$

Then one may change f to $f - \nabla Q$, putting ∇Q in the pressure.

So we may assume from now on that

$$\operatorname{div} f = 0 \quad \text{in} \ \Omega \quad \text{and} \quad f \cdot \vec n = 0 \ \text{on} \ \partial\Omega. \tag{4.50}$$

Since the Stokes system is not quite of parabolic type, an extra information about the pressure is needed so as to adapt the purely parabolic techniques of e.g. [**37, 39**]. One of the difficulties is that the basic energy estimate does not supply any reasonable bound for the pressure. The estimates that we shall obtain below will enable us to control lower order terms which will appear as a consequence of the localization procedure, and to 'close the estimates' for small enough times.

In order to get this extra information over the pressure, we take the divergence of $(4.30)_1$. Under assumption (4.50), we obtain:

$$\begin{aligned} &\Delta P = 0 && \text{in} \quad \Omega, \\ &\partial_{\vec n} P = \Delta u \cdot \vec n && \text{on} \quad \partial\Omega, \end{aligned} \qquad P \to 0 \ \text{at} \ \infty. \tag{4.51}$$

The boundary condition is just taken directly from the equation $(4.30)_1$. Note that, as $\operatorname{div} f = 0$ we have $\operatorname{div} \Delta u = 0$ with $\Delta u \in L_1(0,T;\dot B^s_{p,1}(\Omega))$, hence the boundary condition makes sense according to Lemma 2.2.4.

LEMMA 4.3.2. *Let $-1+\frac{1}{p} < s < \frac{1}{p}$ and u be a divergence free vector field over Ω with $u\cdot\vec{n}=0$ at $\partial\Omega$ and $\Delta u \in L_1(0,T;\dot{B}^s_{p,1}(\Omega)) \cap L_\infty(0,T;\dot{B}^{-2+s}_{p,1}(\Omega))$. Then there exists a unique distributional solution P to (4.51) such that*

$$\begin{aligned} &\|P\|_{L_1(0,T;B^{s-2a}_{p,1}(K))} + \|\nabla P\|_{L_1(0,T;\dot{B}^{s-2a}_{p,1}(\Omega))} \\ &\qquad \leq C\|\Delta u\|^{1-a}_{L_1(0,T;B^s_{p,1}(K))}\|\Delta u\|^a_{L_1(0,T;B^{-2+s}_{p,1}(K))}, \end{aligned} \tag{4.52}$$

$$\begin{aligned} &\|P\|_{L_{1+\kappa}(0,T;B^{s-2a}_{p,1}(K))} + \|\nabla P\|_{L_{1+\kappa}(0,T;\dot{B}^{s-2a}_{p,1}(\Omega))} \\ &\qquad \leq C\|\Delta u\|^{1-a}_{L_1(0,T;B^s_{p,1}(K))}\|\Delta u\|^a_{L_\infty(0,T;B^{-2+s}_{p,1}(K))}, \end{aligned} \tag{4.53}$$

where the constant C is independent of T and $-1+\frac{1}{p} < s-2a < \frac{1}{p}$ with $\frac{1}{1+\kappa} = \frac{1-a}{1} + \frac{a}{\infty}$.

Proof: Of course we have

$$\Delta u \in L_1(0,T;\dot{B}^s_{p,1}(K)) \cap L_\infty(0,T;\dot{B}^{-2+s}_{p,1}(K)),$$

and this implies, by interpolation that $\Delta u \in L_{1+\kappa}(0,T;\dot{B}^{s-2a}_{p,1}(K))$ with κ defined as in the statement. If $s-2a > -1+1/p$ then, owing to the compactness of K, we have $\Delta u \in L_{1+\kappa}(0,T;B^{s-2a}_{p,1}(K))$, and by the trace Lemma (see Lemma 2.2.4) combined with the fact that $\operatorname{div}\Delta u = 0$ in K, we are guaranteed that $\Delta u\cdot\vec{n}|_{\partial\Omega}$ is defined in $L_1(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))$ and that

$$\|\Delta u\cdot\vec{n}\|_{L_1(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))} \leq C\|\Delta u\|_{L_1(0,T;B^{s-2a}_{p,1}(K))}.$$

Combining with the interpolation inequality, we thus get

$$\|\Delta u\cdot\vec{n}\|_{L_1(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))} \leq C\|\Delta u\|^{1-a}_{L_1(0,T;\dot{B}^s_{p,1}(K))}\|\Delta u\|^a_{L_1(0,T;\dot{B}^{-2+s}_{p,1}(K))}$$

where C is independent of T. Now, applying Theorem 3.4.1 with $b := \Delta u\cdot\vec{n}$ gives (4.52).

Let us now turn to the proof of (4.53). Starting from the fact that

$$\|\Delta u\cdot\vec{n}\|_{L_{1+\kappa}(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))} \leq C\|\Delta u\|_{L_{1+\kappa}(0,T;B^{s-2a}_{p,1}(K))},$$

and using once again the equivalence between homogeneous and nonhomogeneous norms in our context, we get by means of an elementary interpolation argument:

$$\|\Delta u\cdot\vec{n}\|_{L_{1+\kappa}(0,T;B^{s-2a-1/p}_{p,1}(\partial\Omega))} \leq C\|\Delta u\|^{1-a}_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))}\|\Delta u\|^a_{L_\infty(0,T;\dot{B}^{-2+s}_{p,1}(\Omega))},$$

where C is independent of T.

Therefore, applying Theorem 3.4.1 with $b := \Delta u\cdot\vec{n}$ gives (4.53). ■

REMARK 4.3.3. *For Ω a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ and $K = \bar{\Omega}$, the proof works the same. Under the above conditions, we get*

$$\|P\|_{L_1(0,T;B^{s+1-2a}_{p,1}(\Omega))} \leq C\|\Delta u\|^{1-a}_{L_1(0,T;B^s_{p,1}(\Omega))}\|\Delta u\|^a_{L_1(0,T;B^{-2+s}_{p,1}(\Omega))},$$

$$\|P\|_{L_{1+\kappa}(0,T;\dot{B}^{s+1-2a}_{p,1}(\Omega))} \leq C\|\Delta u\|^{1-a}_{L_1(0,T;B^s_{p,1}(\Omega))}\|\Delta u\|^a_{L_\infty(0,T;B^{-2+s}_{p,1}(\Omega))}.$$

Now we can tackle the proof of estimates in Theorem 4.3.1 under the assumption $g \equiv 0$, $\operatorname{div} f \equiv 0$ and $f\cdot\vec{n}|\partial\Omega \equiv 0$. Throughout we fix some covering $(\Omega^l)_{0\leq l\leq L}$ of Ω such that $\operatorname{Supp}\eta^0 \subset \Omega^0$ with $\Omega^0 \cap \partial\Omega = \emptyset$ (here η^0 is the function introduced in the first step of the proof just after (4.37)), and $(\Omega^l)_{1\leq l\leq L}$ constitutes a covering of $\partial\Omega$ with $\Omega^l \subset \Omega$, $\Omega^l \cap \partial\Omega \neq \emptyset$ and $\Omega \cap \Omega^l$ star-shaped with respect to some ball, and $\operatorname{diam}(\Omega^l) \approx \lambda$ (see Figure 3.4). Then we consider a subordinate partition of unity $(\eta^l)_{1\leq l\leq L}$ such that:

(1) $\sum_{0\leq l\leq L} \eta^l = 1 \quad \text{on} \quad \Omega;$
(2) $\|\nabla^k \eta^l\|_{L_\infty(\mathbb{R}^n)} \leq C_k \lambda^{-k}$ for $k \in \mathbb{N}$ and $1 \leq l \leq L$;
(3) $\operatorname{Supp} \eta^l \subset \Omega^l$.

We also introduce a smooth function $\widetilde{\eta}^0$ supported in K and with value 1 on $\operatorname{Supp} \nabla\eta^0$ and smooth functions $\widetilde{\eta}^1, \cdots, \widetilde{\eta}^L$ with compact support in Ω^l and such that $\widetilde{\eta}^l \equiv 1$ on $\operatorname{Supp} \eta^l$. Obviously those functions can be defined on the whole space.

Note that, for $l \in \{1, \cdots, L\}$, the bounds for the derivatives of η^l together with the fact that $\big|\operatorname{Supp} \nabla\eta^l\big| \approx \lambda^n$ and interpolation implies that for $k = 1, 2$ and any $q \in [1, \infty]$, we have

$$\|\nabla^k \eta^l\|_{\dot{B}^{n/q}_{q,1}(\mathbb{R}^n)} \lesssim \lambda^{-k}. \tag{4.54}$$

The same holds for the functions $\widetilde{\eta}^l$.

Throughout, we set $U^l := u\eta^l$ and $P^l := P\eta^l$. We first prove *an interior estimate* (that is an estimate for (U^0, P^0)) then *boundary estimates*, which will eventually lead to the desired estimates (4.32) and (4.33).

Third step: the interior estimate. The couple (U^0, P^0) satisfies:

$$\begin{aligned} U^0_t - \Delta U^0 + \nabla P^0 &= f^0 + \eta^0 f && \text{in} \quad (0, T) \times \mathbb{R}^n, \\ \operatorname{div} U^0 &= g^0 && \text{in} \quad (0, T) \times \mathbb{R}^n, \\ U^0|_{t=0} &= u_0\eta^0 && \text{on} \quad \mathbb{R}^n, \end{aligned} \tag{4.55}$$

with

$$f^0 := -2\nabla\eta^0 \cdot \nabla u + u\Delta\eta^0 + P\nabla\eta^0 \quad \text{and} \quad g^0 := u \cdot \nabla\eta^0.$$

The localization procedure destroys the divergence-free assumption. Hence we have to check whether the r.h.s. of $(4.55)_2$ is of the form that we considered in Theorem 4.1.1. Let us observe that

$$g^0_t = u_t \cdot \nabla\eta^0 = (u_t - f) \cdot \nabla\eta^0 + f \cdot \nabla\eta^0 = (\Delta u - \nabla P) \cdot \nabla\eta^0 + f \cdot \nabla\eta^0.$$

Hence, denoting $\mathbb{D}u := {}^t\nabla u + \nabla u$, one may write

$$g^0_t = \operatorname{div} [\mathbb{D}u \cdot \nabla\eta^0 - P\nabla\eta^0] - \mathbb{D}u : \nabla^2\eta^0 + P\Delta\eta^0 + f \cdot \nabla\eta^0.$$

To match the assumptions of Theorem 4.1.1, we thus set

$$B^0 := \mathbb{D}u \cdot \nabla\eta^0 - P\nabla\eta^0 \ \text{ and } \ A^0 := -\mathbb{D}u : \nabla^2\eta^0 + P\Delta\eta^0 + f \cdot \nabla\eta^0.$$

Next we notice that by virtue of the Stokes formula,

$$\int_{\mathbb{R}^n} A^0\,dx = \int_\Omega \operatorname{div}(\partial_t U^0 - B^0)\,dx = \int_{\partial\Omega} (\partial_t U^0 - B^0) \cdot \vec{n}\,d\sigma = 0,$$

hence Theorem 4.1.1 yields:

$$\begin{aligned} &\|U^0\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|U^0_t, \nabla^2 U^0, \nabla P^0\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} \\ &\qquad \leq C(\|\eta^0 f, f^0, \nabla g^0, B^0\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|A^0\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|\eta^0 u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)}). \end{aligned}$$

Let us emphasize that as A^0, B^0, f^0 and g^0 are compactly supported, we may replace the homogeneous norms by nonhomogeneous ones. As a consequence, because the function $\nabla\eta^0$ is in $\mathcal{C}^\infty_c(\mathbb{R}^n)$ and $\widetilde{\eta}^0 \equiv 1$ on $\operatorname{Supp} \nabla\eta^0$, Corollary 2.1.1 ensures that

$$\|\nabla g^0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} + \|A^0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} + \|B^0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} \lesssim \|\widetilde{\eta}^0 u\|_{B^{s+1}_{p,1}(\mathbb{R}^n)} + \|\widetilde{\eta}^0 P\|_{B^s_{p,1}(\mathbb{R}^n)} + \|\widetilde{\eta}^0 f\|_{B^s_{p,1}(\mathbb{R}^n)}.$$

Therefore,

$$\|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^0_t,\nabla^2U^0,\nabla P^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\lesssim\|\widetilde\eta^0 f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|\widetilde\eta^0 u\|_{L_1(0,T;B^{s+1}_{p,1}(\mathbb{R}^n))}+\|\widetilde\eta^0 P\|_{L_1(0,T;B^s_{p,1}(\mathbb{R}^n))}. \tag{4.56}$$

By interpolation, u belongs to $L_2(0,T;\dot B^{1+s}_{p,1}(\Omega))$ and

$$\|u\|_{L_2(0,T;\dot B^{1+s}_{p,1}(\Omega))}\le C\|u\|^{\frac12}_{L_1(0,T;\dot B^{2+s}_{p,1}(\Omega))}\|u\|^{\frac12}_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}.$$

Additionally, as it is obvious that (use the definition of Besov spaces by restriction)

$$\|\widetilde\eta^0 P\|_{B^s_{p,1}(\mathbb{R}^n)}\le C\|P\|_{B^s_{p,1}(K)},$$

combining Inequality (4.53) and Hölder inequality gives

$$\|\widetilde\eta^0 P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\le CT^a\|u\|^{1-a}_{L_1(0,T;B^{2+s}_{p,1}(K))}\|u\|^a_{L_\infty(0,T;B^s_{p,1}(K))}$$

whenever $-1+1/p<s-2a<1/p$.

In this way, we may conclude that

$$\|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^0_t,\nabla^2U^0,\nabla P^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\lesssim\|u_0\|_{\dot B^s_{p,1}(\Omega)}+\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+(T^{1/2}+T^a)\|u\|_{L_1(0,T;\dot B^{2+s}_{p,1}(K))\cap L_\infty(0,T;\dot B^s_{p,1}(K))}. \tag{4.57}$$

On the other hand, if we want to prove (4.33) then we estimate the terms from (4.56) as follows:

$$\|f^0,\nabla g^0,B^0,A^0\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\le C\Big(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|u\|^{1/2}_{L_1(0,T;B^{2+s}_{p,1}(K))}\|u\|^{1/2}_{L_1(0,T;B^s_{p,1}(K))}+\|P\|_{L_1(0,T;\dot B^s_{p,1}(K))}+\|u_0\|_{\dot B^s_{p,1}(\Omega)}\Big) \tag{4.58}$$

and we use the following estimate for the pressure (a consequence of (4.52) and of an interpolation inequality involving the two terms in the l.h.s. of (4.52)):

$$\|P\|_{L_1(0,T;\dot B^s_{p,1}(K))}\le C\|u\|^{1-a}_{L_1(0,T;B^{2+s}_{p,1}(K))}\|u\|^a_{L_1(0,T;B^s_{p,1}(K))}.$$

Hence, using Young's inequality, we find the following interior inequality for all $\varepsilon>0$:

$$\begin{aligned}&\|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^0_t,\nabla^2U^0,\nabla P^0\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\\&\quad\lesssim\|u_0\|_{\dot B^s_{p,1}(\Omega)}+\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\varepsilon\|u\|_{L_1(0,T;B^{2+s}_{p,1}(K))}+c(\varepsilon)\|u\|_{L_1(0,T;B^s_{p,1}(K))}.\end{aligned} \tag{4.59}$$

Fourth step: the boundary estimate. We now consider an index $l\in\{1,\cdots,L\}$ so that $\operatorname{Supp}\eta^l\cap\partial\Omega\neq\emptyset$. The localization leads to the following problem:

$$\begin{aligned}&U^l_t-\Delta U^l+\nabla P^l=f^l &&\text{in}\quad(0,T)\times\Omega,\\&\operatorname{div}U^l=g^l &&\text{in}\quad(0,T)\times\Omega,\\&U^l=0 &&\text{on}\quad(0,T)\times\partial\Omega,\\&U^l_t|_{t=0}=u_0\eta^l &&\text{on}\quad\Omega,\end{aligned} \tag{4.60}$$

with

$$f^l:=-2\nabla\eta^l\cdot\nabla u+u\Delta\eta^l+P\nabla\eta^l+\eta^l f\quad\text{and}\quad g^l:=u\cdot\nabla\eta^l.$$

As a first step for proving the boundary estimate, we want to reduce the problem to the case $\operatorname{div}U^l\equiv0$. For that, we shall resort once again to the (generalized) Bogovskiĭ formula.

Since $g^l=\nabla\eta^l\cdot u$, it is compactly supported (in Ω^l for instance). In addition, we notice that

$$\int_{\Omega^l}g^l\,dx=\int_{\partial\Omega}\eta^l u\cdot\vec n\,d\sigma-\int_\Omega\eta^l\operatorname{div}u\,dx=0\quad\text{and}\quad g^l=0\ \text{ on }\ \partial\Omega^l. \tag{4.61}$$

Therefore, setting

$$v^l := \mathcal{B}_{\Omega^l}(g^l), \tag{4.62}$$

where $\mathcal{B}_{\Omega^l}$ stands for the Bogovskiĭ operator defined in (2.25), we get a vector field $v^l \in L_1(0,T;B^{2+s}_{p,1}(\mathbb{R}^n))$ such that

$$\|v^l\|_{L_1(0,T;B^{2+s}_{p,1}(\mathbb{R}^n))} \lesssim \|g^l\|_{L_1(0,T;B^{1+s}_{p,1}(\Omega))}, \quad \operatorname{div} v^l = g^l \ \text{ in } \ \Omega \ \text{ and } \ \operatorname{Supp} v^l(t,\cdot) \subset \overline{\Omega^l}.$$

Then, using the stability of Besov spaces by multiplication by smooth compactly supported functions, we conclude that

$$\|v^l\|_{L_1(0,T;B^{2+s}_{p,1}(\mathbb{R}^n))} \leq C_\lambda \|\widetilde{\eta}^l u\|_{L_1(0,T;B^{1+s}_{p,1}(\Omega))}, \tag{4.63}$$

where the constant C_l depends only on (s,p), on Ω_l and on Ω.

Next, differentiating (4.62) with respect to time yields

$$v^l_t = \mathcal{B}_{\Omega^l}(A^l) + I_{\Omega^l}(B^l) + J_{\Omega^l}(B^l_n) \tag{4.64}$$

with I_{Ω^l} and J_{Ω^l} defined in (2.26),

$$A^l := -\mathbb{D}u : D^2\eta^l + P\Delta\eta^l + f\cdot\nabla\eta^l \ \text{ and } \ B^l := \mathbb{D}u\cdot\nabla\eta^l - P\nabla\eta^l.$$

Using again Corollary 2.1.1, we easily find that

$$\|A^l\|_{B^s_{p,1}(\Omega)} \leq C_\lambda\Big(\|\widetilde{\eta}^l f\|_{B^s_{p,1}(\Omega)} + \|\widetilde{\eta}^l P\|_{B^s_{p,1}(\Omega)} + \|\widetilde{\eta}^l u\|_{B^{s+1}_{p,1}(\Omega)}\Big) \tag{4.65}$$

and that

$$\|B^l\|_{B^s_{p,1}(\Omega)} \leq C_\lambda\Big(\|\widetilde{\eta}^l P\|_{B^s_{p,1}(\Omega)} + \|\widetilde{\eta}^l u\|_{B^{s+1}_{p,1}(\Omega)}\Big). \tag{4.66}$$

Besides, $u \in L_1(0,T;\dot{B}^{s+2}_{p,1}(\Omega))$ and $P \in L_1(0,T;\dot{B}^{s+1}_{p,1}(K))$. Hence the product laws in Besov spaces (Corollary 2.1.1) ensure that B^l belongs to $L_1(0,T;B^{s+1}_{p,1}(\Omega))$. Therefore $B^l_n := B^l\cdot\vec{n}$ has a trace at the boundary and Relation (4.64) is thus valid.

Now, differentiating (4.61) with respect to time implies that

$$\int_{\Omega^l} (A^l + \operatorname{div} B^l)\,dx = 0.$$

As A^l and B^l are compactly supported in $\bar{\Omega}^l$, we deduce that the compatibility condition for A^l and B^l is satisfied on Ω^l. By virtue of (4.65), we thus have

$$\|\mathcal{B}_{\Omega^l}(A^l)\|_{L_1(0,T;B^{s+1}_{p,1}(\Omega))} \leq C_l\Big(\|(\widetilde{\eta}^l f, \widetilde{\eta}^l P)\|_{L_1(0,T;B^s_{p,1}(\Omega))} + \|\widetilde{\eta}^l u\|_{L_1(0,T;B^{s+1}_{p,1}(\Omega))}\Big). \tag{4.67}$$

For bounding $I_{\Omega^l}(B^l)$, it is only a matter of using the results of [**16**] and (4.66). We get

$$\|I_{\Omega^l}(B^l)\|_{B^s_{p,1}(\Omega)} \leq C_l\Big(\|\widetilde{\eta}^l P\|_{B^s_{p,1}(\Omega)} + \|\widetilde{\eta}^l u\|_{B^{s+1}_{p,1}(\Omega)}\Big). \tag{4.68}$$

As usual, owing to the compact support of $\mathcal{B}_{\Omega^l}(A^l)$ and of $I_{\Omega^l}(B^l)$, the nonhomogeneous norms may be replaced with homogeneous ones in the left-hand side of (4.67) and (4.68).

For bounding B^l_n in $B^{s-\frac{1}{p}}_{p,1}(\partial\Omega^l)$, we take advantage of $B^l \in L_1(0,T;B^{s+1}_{p,1}(\Omega^l))$. Hence applying Proposition 2.2.4 yields for any ε in $(0, s+1-1/p]$,

$$\|B^l_n\|_{B^\varepsilon_{p,1}(\partial\Omega^l)} \leq C_\lambda\|B^l\|_{B^{\varepsilon+1/p}_{p,1}(\Omega^l)}.$$

If in addition $\varepsilon + 1/p < n/p$, then, owing to Proposition 2.2.1 and Inequality (4.54),

$$\|B^l\|_{B^{\varepsilon+1/p}_{p,1}(\Omega^l)} \leq C_\lambda\big(\|\widetilde{\eta}^l P\|_{B^{\varepsilon+1/p}_{p,1}(\Omega)} + \|\widetilde{\eta}^l u\|_{B^{\varepsilon+1/p+1}_{p,1}(\Omega)}\big),$$

whence

$$\|B_n^l\|_{B^{s-1/p}_{p,1}(\partial\Omega^l)} \leq C\|B_n^l\|_{B^{\varepsilon}_{p,1}(\partial\Omega^l)} \leq C_\lambda\bigl(\|\widetilde{\eta}^l P\|_{B^{\varepsilon+1/p}_{p,1}(\Omega)} + \|\widetilde{\eta}^l u\|_{B^{\varepsilon+1/p+1}_{p,1}(\Omega)}\bigr).$$

Then using the results from [**16**], one may conclude that

$$\|J_{\Omega^l}(B_n^l)\|_{B^s_{p,1}(\Omega^l)} \leq C_\lambda\bigl(\|\widetilde{\eta}^l P\|_{B^{\varepsilon+1/p}_{p,1}(\Omega)} + \|\widetilde{\eta}^l u\|_{B^{\varepsilon+1/p+1}_{p,1}(\Omega)}\bigr).$$

So finally, putting together the above inequalities and bearing in mind (4.63), we get

$$\|v_t^l, \nabla^2 v^l\|_{L_1(0,T;B^s_{p,1}(\Omega))} \leq C_\lambda\Bigl(\|\widetilde{\eta}^l f\|_{L_1(0,T;B^s_{p,1}(\Omega))} + \|\widetilde{\eta}^l P\|_{L_1(0,T;B^{\varepsilon+1/p}_{p,1}(\Omega))} + \|\widetilde{\eta}^l u\|_{L_1(0,T;B^{\varepsilon+1/p+1}_{p,1}(\Omega))}\Bigr). \tag{4.69}$$

Next, making use of v^l we modify System (4.60) into

$$\begin{array}{lll} U_t^l - \Delta U^l + \nabla P^l = f^l_{new} & \text{in} & (0,T)\times\Omega, \\ \operatorname{div} U^l = 0 & \text{in} & (0,T)\times\Omega, \\ U^l = 0 & \text{on} & (0,T)\times\partial\Omega, \\ U^l|_{t=0} = u_0\eta^l - \mathcal{B}_{\Omega^l}\operatorname{div}(u_0\eta^l) & \text{on} & \Omega, \end{array}$$

with

$$U^l_{new} := U^l_{old} - v^l \quad \text{and} \quad f^l_{new} := f^l_{old} - v^l_t + \Delta v^l.$$

Note that (4.69) implies that

$$\begin{aligned} \|f^l_{new}\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} &\leq \|\widetilde{\eta}^l f^l_{old}\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \\ &\quad + C_\lambda\bigl(\|\widetilde{\eta}^l P\|_{L_1(0,T;B^{\varepsilon+1/p}_{p,1}(\Omega))} + \|\widetilde{\eta}^l u\|_{L_1(0,T;B^{\varepsilon+1/p+1}_{p,1}(\Omega))}\bigr) \end{aligned} \tag{4.70}$$

and it is also clear that

$$\|U^l|_{t=0}\|_{\dot B^s_{p,1}(\Omega)} \leq C_\lambda \|u_0\|_{\dot B^s_{p,1}(\Omega)}. \tag{4.71}$$

Let us now recast (4.30) on $\mathbb{R}^n_+$ according to the *volume preserving* change of coordinates introduced in Chapter 2. Let $V^l := Z_l^* U^l := U^l \circ Z_l^{-1}$ and $Q^l := Z_l^* P^l$. The system satisfied by (V^l, Q^l) reads

$$\begin{array}{lll} V_t^l - \Delta_z V^l + \nabla_z Q^l = F^l & \text{in} & (0,T)\times\mathbb{R}^n_+, \\ \operatorname{div}_z V^l = G^l & \text{in} & (0,T)\times\mathbb{R}^n_+, \\ V^l|_{z_n=0} = 0 & \text{on} & (0,T)\times\partial\mathbb{R}^n_+, \\ V^l|_{t=0} = Z_l^*(U^l|_{t=0}) & \text{on} & \partial\mathbb{R}^n_+, \end{array}$$

with

$$F^l := Z_l^* f^l + (\Delta_x - \Delta_z)V^l - (\nabla_x - \nabla_z)Q^l \quad \text{and} \quad G^l := (\operatorname{div}_z - \operatorname{div}_x)V^l.$$

Let us stress that, according to Chapter 2, we have

$$G^l = -{}^T\!A^l : \nabla_z V^l = -\operatorname{div}_z(A^l V^l) \quad \text{with} \quad A^l(z) := D_x Z_l(x) - \operatorname{Id}.$$

Hence $G^l|_{t=0} = 0$ and

$$G^l_t = -\operatorname{div}_z(A^l V^l_t).$$

According to Theorem 4.2.1, we thus get

$$\begin{aligned} &\|V^l\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|V^l_t, \nabla^2 V^l, \nabla Q^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} \\ &\quad\lesssim \|Z_l^* f^l, (\Delta_x - \Delta_z)V^l, (\nabla_z - \nabla_x)Q^l, \nabla G^l, A^l V^l_t\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|Z_l^*(U^l|_{t=0})\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}. \end{aligned}$$

The first and last terms in the right-hand side may be dealt with thanks to Lemma 2.1.1: we have

$$\|Z_l^* f^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim \|f^l\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\quad\text{and}\quad \|Z_l^*(U^l|_{t=0})\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\lesssim \|U^l|_{t=0}\|_{\dot B^s_{p,1}(\Omega)}.$$

So the definitely new terms are $(\nabla_x-\nabla_z)Q^l$, $A^lV^l_t$, $(\Delta_x-\Delta_z)V^l$ and ∇G_l. First, we notice that

$$(\nabla_x-\nabla_z)Q^l={}^TB^l\nabla_zQ^l={}^T\widetilde B^l\nabla_zQ^l\quad\text{with}\quad \widetilde B^l:=B^l\,Z_l^*\widetilde\eta^l.$$

Hence Proposition 2.2.1 ensures that

$$\|(\nabla_x-\nabla_z)Q^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\leq C\|{}^TB^l\,Z_l^*\widetilde\eta^l\|_{\dot B^{\frac np}_{p,1}(\mathbb{R}^n_+)\cap\dot B^{\frac n{p'}}_{p',1}(\mathbb{R}^n_+)}\|\nabla_zQ^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.$$

Together with Lemma 2.1.1 and Inequality (2.30), this implies that

$$\|(\nabla_x-\nabla_z)Q^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\leq C\lambda\|\nabla Q^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.$$

From similar arguments, we get

$$\|A^lV^l_t\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\leq C\lambda\|V^l_t\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.$$

Bounding $(\Delta_x-\Delta_z)V^l$ is more involved. It relies on the formula

$$(\Delta_x-\Delta_z)V^l=\operatorname{div}_z(\widetilde A^l\cdot(\mathrm{Id}+{}^TA^l)\cdot\nabla_zV^l)+\operatorname{div}_z({}^T\widetilde A^l\cdot\nabla_zV^l).\tag{4.72}$$

Using the fact that

$$\operatorname{div}_z({}^T\widetilde A^l\cdot\nabla_zV^l)=(\nabla{}^T\widetilde A^l)\cdot\nabla_zV^l+{}^T\widetilde A^l\cdot\nabla_z\operatorname{div}_zV^l,$$

we may write, by virtue of Proposition 2.2.1,

$$\|\operatorname{div}_z({}^T\widetilde A^l\cdot\nabla_zV^l)\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\lesssim\|\nabla_z\widetilde A^l\|_{\dot B^{\frac np}_{p,1}(\mathbb{R}^n_+)}\|\nabla_zV^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}+\|\widetilde A^l\|_{\dot B^{\frac np}_{p,1}(\mathbb{R}^n_+)}\|\nabla_z^2V^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.$$

The first term in the right-hand side of (4.72) obeys a similar inequality. Hence, using Inequality (2.30), one may easily conclude that

$$\|(\Delta_x-\Delta_z)V^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\lesssim\lambda\|\nabla_z^2V^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}+\|\nabla_zV^l\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.$$

The last term, ∇_zG^l, may be treated in the same way. Hence, putting together the above inequalities, we finally get

$$\begin{aligned}\|V^l\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^l_t,\nabla^2V^l,\nabla Q^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&\lesssim\|U^l|_{t=0}\|_{\dot B^s_{p,1}(\Omega)}\\&+\|f^l\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\lambda\|V^l_t,\nabla^2V^l,\nabla Q^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla V^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.\end{aligned}$$

By interpolation, we have

$$\|\nabla V^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\leq\|\nabla^2V^l\|^{1/2}_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\|V^l\|^{1/2}_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.$$

Now, using Young's inequality to handle the term with ∇B^l, taking λ so small as the term

$$\lambda\|V^l_t,\nabla^2V^l,\nabla Q^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}$$

to be absorbed by the l.h.s., and using (4.70) and (4.71), we end up with

$$\begin{aligned}\|V^l\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^l_t,\nabla^2V^l,\nabla Q^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&\lesssim\|u^l_0\|_{\dot B^s_{p,1}(\Omega)}+\|\widetilde\eta^lf\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\&+\lambda^{-1}\big(\|V^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\widetilde\eta^lP\|_{L_1(0,T;B^{\varepsilon+1/p}_{p,1}(\Omega))}+\|\widetilde\eta^lu\|_{L_1(0,T;B^{\varepsilon+1/p+1}_{p,1}(\Omega))}\big).\end{aligned}$$

In order to handle the last two terms, there are two ways of proceeding depending on whether we want a time dependent constant or not. Throughout, we fix some $a\in(0,1/2)$ given by Lemma 4.3.2 and choose ε so that $s+1-2a=\varepsilon+1/p$. The first possibility is to write that, by interpolation and Hölder's inequality,

$$\|\widetilde{\eta}^l u\|_{L_1(0,T;\dot B^{s+2-2a}_{p,1}(\Omega))} \leq T^a\|\widetilde{\eta}^l u\|_{L_1(0,T;\dot B^{s+2}_{p,1}(\Omega))\cap L_\infty(0,T;\dot B^s_{p,1}(\Omega))},$$

and that, according to (4.53), we have

$$\|\widetilde{\eta}^l P\|_{L_1(0,T;\dot B^{\varepsilon+1/p}_{p,1}(\Omega)\cap\dot B^s_{p,1}(\Omega))} \lesssim T^a\|u\|_{L_1(0,T;\dot B^{s+2}_{p,1}(K))\cap L_\infty(0,T;\dot B^s_{p,1}(K))}.$$

This yields

$$\begin{aligned}\|V^l\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^l_t,\nabla^2V^l,\nabla Q^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|u^l_0\|_{\dot B^s_{p,1}(\Omega)}\\ +\|\widetilde{\eta}^l f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\lambda^{-1}T^a\|u\|_{L_1(0,T;\dot B^{s+2}_{p,1}(K))\cap L_\infty(0,T;\dot B^s_{p,1}(K))}\\ +\lambda^{-1}T\|V^l\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.\end{aligned}\tag{4.73}$$

The second possibility is to write that

$$\|\widetilde{\eta}^l u\|_{L_1(0,T;\dot B^{s+2-2a}_{p,1}(\Omega))} \leq \|\widetilde{\eta}^l u\|^{1-a}_{L_1(0,T;\dot B^{s+2}_{p,1}(\Omega))}\|\widetilde{\eta}^l u\|^a_{L_1(0,T;\dot B^s_{p,1}(\Omega))},$$

and to bound the pressure term according to (4.52). We eventually get

$$\begin{aligned}\|V^l\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^l_t,\nabla^2V^l,\nabla Q^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ \lesssim\|u^l_0\|_{\dot B^s_{p,1}(\Omega)}+\|\widetilde{\eta}^l f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\lambda^{-1}\|V^l\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ +\lambda^{-1}\|u\|^{1-a}_{L_1(0,T;\dot B^{2+s}_{p,1}(K))}\|u\|^a_{L_1(0,T;\dot B^s_{p,1}(K))}.\end{aligned}\tag{4.74}$$

Fifth step: global a priori estimates. Now, in view of Lemma 2.1.1, we may write

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))} &\leq \sum_l\|U^l\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}\\ &\lesssim \|U^0\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\sum_{l\geq1}\|V^l\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},\end{aligned}$$

and similar inequalities for the other terms of the l.h.s of (4.73). Of course, Proposition 2.1.2 ensures that

$$\|u^l_0\|_{\dot B^s_{p,1}(\Omega)}\lesssim\|u_0\|_{\dot B^s_{p,1}(\Omega)}\quad\text{and}\quad\|\widetilde{\eta}^l f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\lesssim\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}.$$

So using also (4.57) and the fact that $L\approx\lambda^{-n}$, and bearing in mind (4.49), we get

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|(u_t,\nabla^2u,\nabla P)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\lesssim\lambda^{-n}\Big(\|u_0\|_{B^s_{p,1}(\mathbb{R}^n)}\\ +\|(f,\nabla g,R_t)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}+\|R\|_{L_1(0,T;L_m(\Omega)\cap L_1(0,T;B^{1+s}_{p,1}(K))}\Big)\\ +\lambda^{-n-1}(T^a+T)\|u\|_{L_1(0,T;\dot B^{s+2}_{p,1}(K))\cap L_\infty(0,T;\dot B^s_{p,1}(K))}.\end{aligned}$$

Hence

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1})}+\|u_t,\nabla^2u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1})}\leq C\Big(\|u_0\|_{B^s_{p,1}(\mathbb{R}^n)}\\ +\|(f,\nabla g,R_t)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}+\|R\|_{L_1(0,T;L_m(\Omega)\cap B^{1+s}_{p,1}(K))}\Big)\end{aligned}$$

for a very short time T depending only on λ.

Repeating the argument over the interval $[T,2T]$ and so on, we get exactly Inequality (4.32) with the constant Ce^{CT} for some suitably large constant C.

If we want to remove the time-dependency, this is just a matter of starting from (4.59) and (4.74) instead of (4.57) and (4.73). After a few computation and use of (4.49), we get for some constant C depending on λ,

$$\|u_t,\nabla^2u,\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\leq C\Big(\|u_0\|_{\dot B^s_{p,1}(\Omega)}+\|f,\nabla g,R_t\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}$$
$$+\|R\|_{L_1(0,T;L_m(\Omega)\cap L_1(0,T;B^{1+s}_{p,1}(K))}+\|u\|^{1/2}_{L_1(0,T;\dot B^{2+s}_{p,1}(K))}\|u\|^{1/2}_{L_1(0,T;\dot B^s_{p,1}(K))}$$
$$+\|u\|^{1-a}_{L_1(0,T;\dot B^{2+s}_{p,1}(K))}\|u\|^{a}_{L_1(0,T;\dot B^s_{p,1}(K))}+\|u\|_{L_1(0,T;B^s_{p,1}(K))}\Big).$$

Using Young's inequality, it is easy to absorb the second line, up to an error term which may be bounded by the last term. ■

We end this section with a few remarks concerning the case where the domain Ω is bounded. We still have to introduce some resolution of unity $(\eta^l)_{0\leq l\leq L}$ where, now, η^0 is supported in the interior of Ω hence has compact support. Step one (removing g) is directly based on our work in [**16**]. The main difference is in step 2 because Proposition 3.5.1 holds true in bounded domains for *any* $n\geq 2$. Hence, Theorem 4.3.1 holds true for $n\geq 2$, with K replaced by Ω.

REMARK 4.3.4. *Let us emphasize that the term $\|u\|_{L_1(0,T;B^s_{p,1}(K))}$ may be replaced by any lower norm taken over a compact set K. In particular s can be put to zero.*

4.3.2. A low order bound for the velocity on a compact set. The goal of this section is to establish *time-independent* bounds in terms of the data for the velocity satisfying System (4.30). We here aim at bounding $\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}$ (which appears in the right-hand side of (4.33)), in terms of the data only, *independently of* T. This is a natural generalization of estimates proved in [**43**] in the L_p-framework.

Let us emphasize that this lower order term does not appear when removing the divergence part of the velocity (first step of the proof of Theorem 4.3.1). Therefore, it suffices to consider the case of a divergence free flow, namely System (4.48). We claim that

LEMMA 4.3.3. *Assume that $n\geq 3$ and that $1<p<n/2$. There exists some $s_p>0$ (depending only on p and n) such that for all $s\in(-s_p,s_p)$ sufficiently smooth solutions to* (4.48) *fulfill*

$$\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}\leq C(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|u_0\|_{\dot B^s_{p,1}(\Omega)}),$$

where C is independent of T.

Proof: Thanks to the linearity of the system and to the uniqueness of solutions provided by (4.32), one may split the solution u into two parts, the first one u_1 being the solution of the system with zero initial data and source term f, and the second one u_2, the solution of the system with no source term and initial data u_0. In other words, $u=u_1+u_2$ with u_1 and u_2 satisfying

$$\begin{array}{llll} u_{1,t}-\Delta u_1+\nabla P_1=f, & u_{2,t}-\Delta u_2+\nabla P_2=0 & \text{in} & (0,T)\times\Omega,\\ \operatorname{div}u_1=0, & \operatorname{div}u_2=0 & \text{in} & (0,T)\times\Omega,\\ u_1=0, & u_2=0 & \text{on} & (0,T)\times\partial\Omega,\\ u_1|_{t=0}=0, & u_2|_{t=0}=u_0 & \text{on} & \Omega.\end{array}$$

Let us first focus on u_1. Arguing by duality, one may write that

$$\|u_1(t)\|_{\dot B^s_{p,1}(K)}\leq C\sup\int_K u_1(t,x)\cdot\psi(x)\,dx,$$

where the supremum is taken over $\psi \in \dot{B}^{-s}_{p',\infty}(K;\mathbb{R}^n)$ such that $\|\psi\|_{\dot{B}^{-s}_{p',\infty}(K)} = 1$. Recall that, by virtue of Corollary 2.2.1, such functions may be extended by 0 over $\mathbb{R}^n$. So we may assume that the supremum is taken over those functions ψ satisfying

$$\psi \in \dot{B}^{-s}_{p',\infty}(\mathbb{R}^n;\mathbb{R}^n) \quad \text{with norm } 1 \text{ and } \operatorname{Supp}\psi \subset K. \tag{4.75}$$

In what follows, it will be important to restrict our attention to functions ψ which are divergence free and satisfy $\psi \cdot \vec{n}|_{\partial\Omega} = 0$. Note that according to Corollary 3.5.1, the Helmholtz projector $\mathcal{P}$ is a selfmap over $\dot{B}^{-s}_{p',\infty}(\Omega)$ and that, since u_1 is divergence free, we may write

$$\int_K u_1(t,x)\cdot\psi(x)\,dx = \int_\Omega u_1(t,x)\cdot\mathcal{P}\psi(x)\,dx.$$

Set $\eta_0 := \mathcal{P}\psi$ (with ψ as above) and consider the solution η to the following problem:

$$\begin{aligned} &\eta_t - \Delta\eta + \nabla Q = 0 && \text{in} \quad (0,T)\times\Omega,\\ &\operatorname{div}\eta = 0 && \text{in} \quad (0,T)\times\Omega,\\ &\eta = 0 && \text{on} \quad (0,T)\times\partial\Omega,\\ &\eta|_{t=0} = \eta_0 && \text{on} \quad \Omega. \end{aligned} \tag{4.76}$$

Testing the equation for u_1 by $\eta(t-\cdot)$ we discover that

$$\int_\Omega u_1(t,x)\cdot\eta_0(x)\,dx = \int_0^t\int_\Omega f(\tau,x)\cdot\eta(t-\tau,x)\,dx\,d\tau. \tag{4.77}$$

Following the approach of [**43**] we have to prove a suitable $L_a - L_b$ estimate for the semigroup generated by the Stokes system (4.76). The general theory for the Stokes operator in exterior domains (see e.g. [**36, 42**]) implies the following estimates in dimension $n \geq 2$:

$$\|\eta(t)\|_{L_a(\Omega)} \leq C\|\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})} \quad \text{for} \quad 1 < b \leq a < \infty, \tag{4.78}$$

$$\|\nabla\eta(t)\|_{L_a(\Omega)} \leq C\|\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})-\frac{1}{2}} \quad \text{for} \quad 1 < b \leq a \leq n. \tag{4.79}$$

We claim that for all $1 < p \leq q < \infty$, there exists some (small) positive $s_{p,q}$ depending only on n, p, q such that

$$\|\eta(t)\|_{\dot{B}^s_{q,q}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^s_{p,p}(\Omega)} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } s \in (0, s_{p,q}). \tag{4.80}$$

Indeed, first we use the classical embedding theorem (see e.g. [**53**], page 31)[4]:

$$L_a(\mathbb{R}^n) \hookrightarrow \dot{B}^{-s_0}_{\bar{a},\bar{a}}(\mathbb{R}^n) \quad \text{and} \quad \dot{B}^{s_0}_{\bar{b},\bar{b}}(\mathbb{R}^n) \hookrightarrow L_b(\mathbb{R}^n)$$

for

$$\frac{1}{a} - \frac{1}{\bar{a}} = \frac{s_0}{n} = \frac{1}{\bar{b}} - \frac{1}{b} \tag{4.81}$$

and $1 < a < \bar{a} < \infty$, $1 < \bar{b} < b < \infty$ to write that, according to (4.78),

$$\|\eta(t)\|_{\dot{B}^{-s_0}_{\bar{a},\bar{a}}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^{s_0}_{\bar{b},\bar{b}}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{4.82}$$

By the same token, (4.79) implies that

$$\|\eta(t)\|_{\dot{B}^{1-s_1}_{\bar{a}_1,\bar{a}_1}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^{s_1}_{\bar{b}_1,\bar{b}_1}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b_1}-\frac{1}{a_1})-\frac{1}{2}}, \tag{4.83}$$

provided

$$\frac{1}{a_1} - \frac{1}{\bar{a}_1} = \frac{s_1}{n} = \frac{1}{\bar{b}_1} - \frac{1}{b_1} \tag{4.84}$$

[4] which naturally extends to general domains, owing to our definition of spaces by restriction.

under the restriction $1 < b_1 \leq a_1 \leq n$, $1 < \bar{b}_1 < b_1$ and $\bar{a}_1 > a_1$.

Now, let us recall the interpolation property:

$$(\dot{B}^{s_1}_{a,a}(\Omega), \dot{B}^{s_2}_{b,b}(\Omega))_{\theta,q} = \dot{B}^{s}_{q,q}(\Omega)$$

with

$$\theta s_2 + (1-\theta)s_1 = s, \qquad \frac{1}{q} = \frac{\theta}{b} + \frac{1-\theta}{a}\cdot$$

Let us fix some small enough positive s. Then we see that combining Inequalities (4.82) and (4.83) with the above interpolation property gives (4.80) with decay exponent σ given by

$$\sigma := \frac{n}{2}\left(\frac{1}{b} - \frac{1}{a}\right)\theta + \frac{n}{2}\left(\frac{1}{b_1} - \frac{1}{a_1}\right)(1-\theta) + \frac{1}{2}(1-\theta)$$

provided one may find some $\theta \in (0,1)$, a, $\bar{a}$, b, $\bar{b}$, a_1, $\bar{a}_1$, b_1, $\bar{b}_1$, s_0 and s_1 so that (4.81) and (4.84) are satisfied together with,

$$\begin{aligned}&\theta(-s_0) + (1-\theta)(1-s_1) = \theta s_0 + (1-\theta)s_1 = s,\\ &\frac{1}{q} = \frac{\theta}{\bar{a}} + \frac{1-\theta}{\bar{a}_1}, \qquad \frac{1}{p} = \frac{\theta}{\bar{b}} + \frac{1-\theta}{\bar{b}_1}\cdot\end{aligned}$$

Thus we see that θ must satisfy

$$s = \frac{1}{2}(1-\theta).$$

Finally, we get

$$\begin{aligned}\sigma &= \frac{n}{2}\left(\frac{1}{\bar{b}} - \frac{1}{\bar{a}} - \frac{2s_0}{n}\right)\theta + \frac{n}{2}\left(\frac{1}{\bar{b}_1} - \frac{1}{\bar{a}_1} - \frac{2s_1}{n}\right)(1-\theta) + \frac{1}{2}(1-\theta)\\ &= \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - s + \frac{1}{2}(1-\theta) = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right),\end{aligned}$$

and we are done.

Let us emphasize that if s is close to zero then θ is close to 1. Hence $a, b, \bar{a}, \bar{b}$ may be chosen very close to p, q. Therefore Inequality (4.80) is valid *for all* $1 < p \leq q < \infty$.

In order to extend (4.80) to negative indices s and $1 < q < \infty$, we consider the dual problem

$$\begin{array}{lll}\zeta_t - \Delta\zeta + \nabla Q' = 0 & \text{in} & (0,T)\times\Omega,\\ \operatorname{div}\zeta = 0 & \text{in} & (0,T)\times\Omega,\\ \zeta = 0 & \text{on} & (0,T)\times\partial\Omega,\\ \zeta|_{t=0} = \zeta_0 & \text{on} & \Omega\end{array} \tag{4.85}$$

where $\zeta_0 \in B^{-s}_{b',q'}(\Omega)$, is divergence free and satisfies $\zeta_0 \cdot \vec{n} = 0$ at the boundary.

Testing (4.85) by $\eta(t-\cdot)$ yields

$$\int_\Omega \eta(t,x)\cdot\zeta_0(x)\,dx = \int_\Omega \eta_0(x)\cdot\zeta(t,x)\,dx. \tag{4.86}$$

Let us observe that

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \leq C\sup_{\zeta_0}\int_\Omega \eta(t,x)\cdot\zeta_0(x)\,dx,$$

where the supremum is taken over all $\zeta_0 \in \dot{B}^{-s}_{a',q'}(\Omega)$ such that $\operatorname{div}\zeta_0 = 0$, $\zeta\cdot\vec{n}|_{\partial\Omega} = 0$ and $\|\zeta_0\|_{\dot{B}^{-s}_{a',q'}(\Omega)} = 1$. Thus by virtue of (4.86), we get:

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \leq C\sup_{\zeta_0}\int_\Omega \eta_0(x)\cdot\zeta(t,x)\,dx \leq \|\eta_0\|_{\dot{B}^s_{b,q}(\Omega)}\sup_{\zeta_0}\|\zeta(t)\|_{\dot{B}^{-s}_{b',q'}(\Omega)}\cdot$$

Because $-s$ is positive we can apply (4.80) (if s is close enough to 0) and get

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \le C\|\eta_0\|_{\dot{B}^s_{b,q}(\Omega)} t^{-\frac{n}{2}(\frac{1}{a'}-\frac{1}{b'})} \sup_{\zeta_0} \|\zeta_0\|_{\dot{B}^{-s}_{a',q'}(\Omega)}.$$

Since $\frac{1}{a'}-\frac{1}{b'}=\frac{1}{b}-\frac{1}{a}$, we conclude that

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \le C\|\eta_0\|_{\dot{B}^s_{b,q}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{4.87}$$

In order to get the remaining case $s=0$, one may argue by interpolation between (4.80) and (4.87). One can thus conclude that for all $1<b\le a<\infty$, $q\in[1,\infty]$ and s close enough to 0, we have

$$\|\eta(t)\|_{\dot{B}^s_{a,q}(\Omega)} \le C\|\eta_0\|_{\dot{B}^s_{b,q}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{4.88}$$

Now we return to the initial problem of bounding u_1. Starting from (4.77) and using duality, one may write

$$\left|\int_\Omega u_1(t,x)\cdot\eta_0(x)\,dx\right| \lesssim \int_0^t \|f(\tau)\|_{\dot{B}^s_{p,1}(\Omega)}\|\eta(t-\tau)\|_{\dot{B}^{-s}_{p',\infty}(\Omega)}\,d\tau.$$

Hence splitting the interval $(0,t)$ into $(0,\max(0,t-1))$ and $(\max(0,t-1),t)$ and applying (4.88) yields for small enough ε,

$$\begin{aligned}\left|\int_\Omega u_1(t,x)\cdot\eta_0(x)\,dx\right| \lesssim &\int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot{B}^s_{p,1}(\Omega)}\|\eta_0\|_{\dot{B}^{-s}_{p',\infty}(\Omega)}\,d\tau\\ &+\int_0^{\max(0,t-1)} \|f(\tau)\|_{\dot{B}^s_{p,1}(\Omega)}\|\eta_0\|_{\dot{B}^{-s}_{\frac{1}{1-\varepsilon},\infty}(\Omega)}(t-\tau)^{-\frac{n}{2}(\frac{1}{p}-\varepsilon)}\,d\tau.\end{aligned}$$

Recall that $\eta_0=\mathcal{P}\psi$ with ψ satisfying (4.75). Using the properties of continuity of $\mathcal{P}$, we can thus write for $1<a\le p'$,

$$\|\eta_0\|_{\dot{B}^{-s}_{a,\infty}(\Omega)} \le C\|\psi\|_{\dot{B}^{-s}_{a,\infty}(\Omega)}.$$

Now, as ψ is supported in K, one has

$$\|\psi\|_{\dot{B}^{-s}_{a,\infty}(\Omega)} \le C|K|^{\frac{1}{p}+\frac{1}{a}-1}\|\psi\|_{\dot{B}^{-s}_{p',\infty}(\Omega)}.$$

This may be proved by introducing a suitable smooth cut-off function with value 1 over K, taking advantage of Proposition 2.1.2. A scaling argument yields the dependency of the norm of the embedding with respect to $|K|$. Hence we have for some constant C depending on K,

$$\|\eta_0\|_{\dot{B}^{-s}_{\frac{1}{1-\varepsilon},\infty}} \le C\|\psi\|_{\dot{B}^{-s}_{p',\infty}(\Omega)}.$$

So, keeping in mind (4.77) and the fact that the supremum is taken over all the functions ψ satisfying (4.75), we deduce that

$$\|u_1(t)\|_{\dot{B}^s_{p,1}(K)} \le C\bigg(\int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot{B}^s_{p,1}(\Omega)}\,ds + \int_0^{\max(0,t-1)} (t-\tau)^{-\frac{n}{2}(\frac{1}{p}-\epsilon)}\|f(\tau)\|_{\dot{B}^s_{p,1}(\Omega)}\,ds\bigg).$$

Therefore,

$$\int_1^T \|u_1\|_{\dot{B}^s_{p,1}(K)}dt \le C\bigg(1+\int_1^T s^{-\frac{n}{2}(\frac{1}{p}-\epsilon)}\,ds\bigg)\int_0^T \|f(t)\|_{\dot{B}^s_{p,1}(\Omega)}\,dt. \tag{4.89}$$

For the time interval $[0,1]$, we merely have

$$\int_0^1 \|u_1(t)\|_{\dot{B}^s_{p,1}(K)}\,dt \le C\int_0^1 \|f(t)\|_{\dot{B}^s_{p,1}(\Omega)}\,dt. \tag{4.90}$$

Now, provided that one may find some $\varepsilon > 0$ such that

$$\frac{n}{2}(\frac{1}{p} - \epsilon) > 1,$$

a condition which is equivalent to $p < n/2$, the constant in (4.89) may be made independent of T, and we conclude that for some time independent constant C,

$$\int_0^T \|u_1\|_{\dot{B}^s_{p,1}(K)}\,dt \leq C\int_0^T \|f\|_{\dot{B}^s_{p,1}(\Omega)}\,dt.$$

Let us now bound u_2. We first write that

$$\|u_2(t)\|_{\dot{B}^s_{p,1}(K)} \leq C\|u_0\|_{\dot{B}^s_{p,1}(\Omega)}$$

and

$$\|u_2(t)\|_{\dot{B}^s_{p,1}(K)} \leq C|K|^{\frac{1}{p}-\epsilon}\|u_2(t)\|_{\dot{B}^s_{\frac{1}{\epsilon},1}(K)} \leq C|K|^{\frac{1}{p}-\epsilon}\|u_0\|_{\dot{B}^s_{p,1}(\Omega)}t^{-\frac{n}{2}(\frac{1}{p}-\epsilon)}.$$

Then decomposing the integral on $[0, \min(1,T)]$ into an integral on $[0,1]$ and on $[1, \min(1,T)]$, we easily get

$$\int_0^T \|u_2(t)\|_{\dot{B}^s_{p,1}(K)}\,dt \leq C\biggl(1 + \int_{\min(1,T)}^T t^{-\frac{n}{2}(\frac{1}{p}-\epsilon)}\,dt\biggr)\|u_0\|_{\dot{B}^s_{p,1}(\Omega)},$$

$$\int_0^T \|u_2(t)\|_{\dot{B}^s_{p,1}(K)}\,dt \leq C\|u_0\|_{\dot{B}^s_{p,1}(\Omega)}.$$

Putting this together with (4.89) and (4.90) completes the proof of the lemma. ■

We end this section with a few remarks concerning the case where Ω is a bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Then it is standard (a consequence of e.g. [**30**]) that the solution η to (4.76) satisfies for some $c > 0$,

$$\|\eta(t)\|_{L_p(\Omega)} \leq Ce^{-ct}\|\eta_0\|_{L_p(\Omega)},$$

and it is also true that, denoting by A the Stokes operator,

$$\|A\eta(t)\|_{L_p(\Omega)} \leq Ce^{-ct}\|A\eta_0\|_{L_p(\Omega)}.$$

By interpolation, we thus have for any $1 < p < \infty$ and $-1 + 1/p < s < 1/p$,

$$\|\eta(t)\|_{\dot{B}^s_{p,1}(\Omega)} \leq Ce^{-ct}\|\eta_0\|_{\dot{B}^s_{p,1}(\Omega)}.$$

Therefore we may write

$$\left|\int_\Omega u_1(t,x)\cdot\eta_0(x)\,dx\right| \leq C\|\eta_0\|_{\dot{B}^{-s}_{p',\infty}(\Omega)}\int_0^t \|f(\tau)\|_{\dot{B}^s_{p,1}(\Omega)}e^{-c(t-\tau)}\,d\tau,$$

thus giving

$$\|u_1\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} \leq C\|f\|_{L^1(0,T;\dot{B}^s_{p,1}(\Omega))}.$$

Similarly, we have

$$\|u_2\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} \leq C\|u_0\|_{\dot{B}^s_{p,1}(\Omega)}.$$

So one may finally conclude to the following statement:

LEMMA 4.3.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Then for all $1 < p < \infty$ and $-1 + 1/p < s < 1/p$, there exists a constant C such that for all $T > 0$, sufficiently smooth solutions to* (4.48) *fulfill*

$$\|u\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} \leq C(\|f\|_{L_1(0,T;\dot{B}^s_{p,1}(\Omega))} + \|u_0\|_{\dot{B}^s_{p,1}(\Omega)}).$$

4.3.3. The final result. Putting together Theorem 4.3.1 with $K=\Omega$, and Lemma 4.3.4, we easily get the following statement:

THEOREM 4.3.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with $n\geq 2$. Let $1<p<\infty$ and $-1+1/p<s<1/p$. Assume that $u_0\in\dot B^s_{p,1}(\Omega)$, $f\in L_1(0,T;\dot B^s_{p,1}(\Omega))$, $g\in\mathcal{C}([0,T);\dot B^{s-1}_{p,1}(\Omega))$ with $g(0)=\operatorname{div}u_0$, $\nabla g\in L_1(0,T;\dot B^s_{p,1}(\Omega))$ and $g=\operatorname{div}R$ with R satisfying all the conditions of Theorem 4.3.1.*

Then there exists a unique solution $(u,\nabla P)$ to System (4.30) such that

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|(u_t,\nu\nabla^2u,\nabla P)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\leq C\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)}+\|(f,\nu\nabla g,R_t)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}(\partial\Omega))}\big),$$

where C is independent of T.

Let us now give the statement in the more involved case where Ω is an exterior domain of $\mathbb{R}^n$ with $n\geq 3$.

THEOREM 4.3.3. *Let $1<q\leq p<\infty$ with $q<n/2$. Let $-1+1/p<s<1/p$ and s' close enough to 0. Assume that*

$$u_0\in\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega),\qquad f\in L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)),$$

$$g\in\mathcal{C}([0,T);\dot B^{s-1}_{p,1}\cap\dot B^{s'-1}_{q,1}(\Omega))\quad\textit{with}\quad g(0)\equiv\operatorname{div}u_0,\quad\nabla g\in L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)),$$

and $g=\operatorname{div}R$ with R satisfying the conditions of Theorem 4.3.1 with respect to (s,p) and (s',q).

Then there exists a unique solution $(u,\nabla P)$ to System (4.30) such that

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}&+\|(u_t,\nu\nabla^2u,\nabla P)\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}+\nu\|u|_K\|_{L_1(0,T;B^{s'}_{q,1}(K))}\\&\leq C\big(\|u_0\|_{\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)}+\|(f,\nu\nabla g,R_t)\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}\\&\qquad+\|\varrho\|_{L_1(0,T;B^{s-\frac1p}_{p,1}\cap B^{s'-\frac1q}_{q,1}(\partial\Omega))}+\nu\|R\|_{L_1(0,T;L_m(\Omega)\cap B^{1+s'}_{q,1}(K)\cap B^{1+s}_{p,1}(K))}\big),\end{aligned}\tag{4.91}$$

where C in (4.91) is independent of T.

Proof: Granted with Theorem 4.3.1 and Inequality (4.33), it is enough to show that

$$\|u|_K\|_{L_1(0,T;B^s_{p,1}(K)\cap B^{s'}_{q,1}(K))}$$

may be bounded by the right-hand side of (4.91). In addition, as $q\leq p$ and K is bounded, it suffices to bound $\|u\|_{L_1(0,T;B^s_{p,1}(K))}$ (because one may assume that $s'\leq s$ with no loss of generality). Note also that following the first step of the proof of Theorem 4.3.1 reduces the study to the case $g\equiv 0$. Hence, if $p<n/2$ then inequality (4.91) stems from Lemma 4.3.3.

In the case where $p\geq n/2$, then we may write

$$\dot B^{s'+2}_{q,1}(\Omega)\subset\dot B^{s'}_{q^*,1}(\Omega)\text{ with }\frac{1}{q^*}=\frac1q-\frac2n\cdot$$

Therefore, if $p\leq q^*$ and, say $s'=s$, then one may combine interpolation and Lemma 4.3.3 so as to cancel out the term $\|u\|_{L_1(0,T;B^s_{p,1}(K))}$ by the left-hand side of (4.91), changing the constant C if necessary.

Finally, if $p>q*$ then one may bound $\|u\|_{L_1(0,T;B^s_{p,1}(K))}$ by means of $\|u\|_{L_1(0,T;B^s_{q^*,1}(K))}$ and $\|u\|_{L_1(0,T;B^{s+2}_{p,1}(K))}$, by interpolation and embedding. As the former bound may be bounded in terms of the data (by the argument just above with $p=q^*$), it is not difficult to absorb the term pertaining to $\|u\|_{L_1(0,T;B^{s+2}_{p,1}(K))}$, by the left-hand side of (4.91). Theorem 4.3.3 is proved. ∎

CHAPTER 5

Inhomogeneous Navier-Stokes equations in exterior domains

This chapter presents a first important application of the results that we established so far for the Stokes system. We analyze here the Navier-Stokes equations modeling flows of incompressible *and* inhomogeneous fluids. In this context, the density is constant along the stream lines. We shall see that the L_1-integrability in time property for the velocity field that has been proved in the previous chapter enables us to recast the whole system of equations in the Lagrangian coordinates. This will allow us to construct unique strong solutions for quite general initial data : as regards the initial density : piecewise constant initial configurations may be considered for instance. Let us emphasize that according to several recent works [**20, 35, 52**] it is even possible to build strong unique solutions assuming that the initial density is only bounded and bounded away from zero. However, the velocity has to be smooth enough therein. Here, following our recent work in [**17**] devoted the case where the fluid domain is the whole space $\mathbb{R}^n$, we focus on the case where the initial velocity has *critical regularity*, which requires to slightly enhance the regularity assumptions on the density. Nevertheless we shall see that initial densities as in (1.4) are admissible data.

This chapter unfolds as follows. In the first section, we present the inhomogeneous Navier-Stokes equations in Eulerian and Lagrangian coordinates. The next section is devoted to the study of a suitable linearization of those equations. This will enable us to prove the local (resp. global) existence of strong solutions for large (resp. small) data in the rest of the chapter.

5.1. Lagrangian stream lines setting

In the Eulerian coordinates, the inhomogeneous incompressible Navier-Stokes equations read

$$\left\{\begin{array}{lll} \rho_t + u\cdot\nabla\rho = 0, & \text{in} & (0,T)\times\Omega,\\ \rho(u_t+u\cdot\nabla u)-\mu\Delta u+\nabla P=0 & \text{in} & (0,T)\times\Omega,\\ \operatorname{div} u = 0 & \text{in} & (0,T)\times\Omega,\\ u|_{\partial\Omega}=0 & \text{on} & (0,T)\times\partial\Omega,\\ u|_{t=0}=u_0 & \text{in} & \Omega. \end{array}\right. \tag{5.1}$$

Above, $\rho=\rho(t,x)\in\mathbb{R}_+$, $u=u(t,x)\in\mathbb{R}^n$ and $P=P(t,x)\in\mathbb{R}$ stand for the density, velocity field and pressure of the fluid, respectively. The viscosity coefficient μ is positive and constant. For simplicity, we assume that there is no external force. We aim at constructing solutions $(\rho,u,\nabla P)$ so that ∇u is in $L_1(\mathbb{R}_+;L_\infty(\Omega))$. This will imply that the velocity field u has a unique measure preserving flow X, defined on $\mathbb{R}_+\times\Omega$. It will be thus possible to recast System (5.1) in Lagrangian coordinates, and to prove uniqueness under rather mild assumptions on the density (in particular small jumps are admitted). Using so-called *critical spaces*, that is, in our context, functional spaces with norm invariant for all $\ell>0$ by the following transform

$$(\rho,u,\nabla P)(t,x)\longmapsto(\rho,\ell u,\ell^3\nabla P)(\ell^2 t,\ell x) \tag{5.2}$$

has become a classical approach nowadays, in the case where $\Omega=\mathbb{R}^n$ (see [**13**]). For more general domains, the above rescaling is no longer relevant as it changes the domain. However it still gives some hint on the minimal regularity that has to be assumed for the data, so as to prove the well-posedness of the equations. In the Besov spaces scale for instance, this suggests our taking u_0 in $\dot{B}^{n/p-1}_{p,1}(\Omega)$, which is in fact the only possibility ensuring the constructed velocity u to be in $L_1(\mathbb{R}_+;L_\infty(\Omega))$. Supplementary 'out of scaling' conditions are needed to control the decay of u, if Ω is an exterior domain.

Let us give more details on the 'Lagrangian approach'. The change from the Eulerian coordinates (t,x) to the Lagrangian coordinates (t,y) is defined by setting $x=X(t,y)$ with X the solution to the following (integrated) Ordinary Differential Equation:

$$X(t,y)=y+\int_0^t u(\tau,X(\tau,y))\,d\tau.$$

Because $u=0$ at the boundary, this transform preserves the domain of the fluid : we have $X(t,\Omega)=\Omega$. Then we set

$$\bar\rho(t,y):=\rho(t,X(t,y)),\quad \bar P(t,y):=P(t,X(t,y))\ \text{ and }\ \bar u(t,y):=u(t,X(t,y)).$$

Given the definition of X and according to the chain rule, it is obvious that

$$\partial_t\bar\rho(t,y)=(\partial_t\rho+u\cdot\nabla_x\rho)(t,X(t,y))\quad\text{and}\quad \partial_t\bar u(t,y)=(\partial_t u+u\cdot\nabla_x u)(t,X(t,y)).$$

Besides, denoting by $Y(t,\cdot)$ the inverse diffeomorphism of $X(t,\cdot)$:

$$\nabla_x P(t,x)={}^T\bar B(t,y)\cdot\nabla_y\bar P(t,y)\quad\text{with}\ \ x:=X(t,y)\ \text{ and }\ \bar B(t,y):=D_xY(t,x).\tag{5.3}$$

Furthermore, the fact that X is measure preserving implies that for any smooth enough vector field H one has (see (2.37) and (2.38))

$$\operatorname{div}_x H={}^T\bar B:\nabla_y\bar u=\operatorname{div}_y(\bar B\bar H),\tag{5.4}$$

$$\Delta_x H^i=\operatorname{div}_x\nabla_x H^i=\operatorname{div}_y(\bar B^T\bar B\nabla_y\bar H^i).\tag{5.5}$$

So finally, we see that, at least formally, $(\rho,u,\nabla_x P)$ satisfies (5.1) if and only if $\bar\rho\equiv\rho_0$ and $(\bar u,\nabla_y\bar P)$ satisfies

$$\begin{cases}\rho_0\bar u_t-\mu\operatorname{div}_y(\bar B^T\bar B\nabla_y u)+{}^T\bar B\nabla_y\bar P=0,\\ \operatorname{div}_y(\bar B\bar u)=0\end{cases}\tag{5.6}$$

with

$$\bar B(t,y)=D_xY(t,x)=(D_yX(t,y))^{-1}\quad\text{and}\quad X(t,y)=y+\int_0^t\bar u(\tau,y)\,d\tau.\tag{5.7}$$

By adapting the arguments that have been used in [**17**] to the domain setting, one may show that systems (5.1) and (5.6) are equivalent in the functional framework that we shall use. Hence we shall focus on solving (5.6) rather than (5.1), in the rest of this chapter.

5.2. The linearized equations

This section concerns the analysis of the following linearization of System (5.6), namely

$$\begin{array}{lll}\rho_t+v\cdot\nabla\rho=0 & \text{in} & (0,T)\times\Omega,\\ q(u_t+v\cdot\nabla u)-\mu\Delta u+\nabla P=0 & \text{in} & (0,T)\times\Omega,\\ \operatorname{div}u=0 & \text{in} & (0,T)\times\Omega,\\ u=0 & \text{at} & (0,T)\times\partial\Omega,\\ u|_{t=0}=u_0 & \text{at} & \Omega.\end{array}\tag{5.8}$$

In the above system the vector-field v and the positive function q are given. We assume in addition that $\operatorname{div}v=0$ and that the trace of v is zero at the boundary.

Introducing Lagrangian coordinates with respect to the vector field v, that is setting $y := Y_v(t,x)$ with $Y_v(t,\cdot) := (X_v(t,\cdot))^{-1}$ and X_v defined by

$$X_v(t,y) = y + \int_0^t v(\tau, X_v(\tau,y))\,d\tau, \tag{5.9}$$

and

$$\bar{B}_v(t,y) := D_x Y_v(t,x) = (D_y X_v(t,y))^{-1}, \quad \bar{\rho}(t,y) := \rho(t, X_v(t,y)),$$
$$\bar{P}(t,y) := P(t, X_v(t,y)) \;\text{ and }\; \bar{u}(t,y) := u(t, X_v(t,y)),$$

we see that, under suitable regularity assumptions, $(\rho, u, \nabla_x P)$ satisfies (5.8) if and only if $\bar{\rho}(t,y) = \rho_0(y)$ and $(\bar{u}, \nabla_y \bar{P})$ satisfies

$$\begin{array}{lll} \rho_0 \bar{u}_t - \mu\, \mathrm{div}_y\,(\bar{B}_v{}^T \bar{B}_v \nabla_y \bar{u}) + {}^T\bar{B}_v \nabla_y \bar{P} = 0 & \text{in} & (0,T)\times\Omega,\\ \mathrm{div}_y\,(\bar{B}_v \bar{u}) = 0 & \text{in} & (0,T)\times\Omega,\\ \bar{u} = 0 & \text{at} & (0,T)\times\partial\Omega,\\ \bar{u}|_{t=0} = u_0 & \text{at} & \Omega. \end{array} \tag{5.10}$$

Here we aim at proving existence results for (5.8) in the critical functional framework in which (5.1) and (5.6) are going to be solved.

Before giving the main statement, let us introduce a few notation. First, we denote by X_T^p the set of $(u, \nabla P)$ so that $u|_{\partial\Omega} = 0$,

$$u \in \mathcal{C}([0,T]; \dot{B}^{n/p-1}_{p,1}(\Omega)) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_1(0,T; \dot{B}^{n/p-1}_{p,1}(\Omega)),$$

and set

$$\|(u,\nabla P)\|_{X_T^p} := \|u\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))} + \|u_t, \mu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}. \tag{5.11}$$

PROPOSITION 5.2.1. *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ ($n \geq 3$) or smooth bounded domain ($n \geq 2$). Let $\bar{T} > 0$ and $p \in (n-1, 2n)$. Assume that $v \in \mathcal{C}([0,\bar{T}]; \dot{B}^{n/p-1}_{p,1}(\Omega))$ with $\mathrm{div}\, v = 0$, $\nabla v \in L_1(0,\bar{T}; \dot{B}^{n/p}_{p,1}(\Omega))$ and $v|_{\partial\Omega} = 0$. There exists a constant $c = c(n,p,\Omega)$ so that if*

$$\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))} + \|\nabla \bar{v}\|_{L_1(0,\bar{T};\dot{B}^{n/p}_{p,1}(\Omega))} \leq c, \tag{5.12}$$

then for any divergence-free data $u_0 \in \dot{B}^{n/p-1}_{p,1}(\Omega)$ with $(u_0\cdot\vec{n})|_{\partial\Omega} = 0$ System (5.10) has a unique solution $(\bar{u}, \nabla\bar{P})$ on $[0,\bar{T}]$, belonging to $X^p_{\bar{T}}$ and so that for some constant $C = C(n,p,\Omega)$ we have

$$\|(\bar{u},\nabla\bar{P})\|_{X^p_{\bar{T}}} \leq Ce^{C\bar{T}}\|u_0\|_{\dot{B}^{n/p-1}_{p,1}(\Omega)}. \tag{5.13}$$

Proof: We focus on the exterior domain case. Using (4.13) enables us to restrict ourselves to the case $\mu = 1$. So we make this assumption throughout. The proof is based on Theorem 4.3.1 and on the Banach fixed point theorem after observing that System (5.10) recasts in

$$\begin{array}{lll} \bar{u}_t - \Delta_y \bar{u} + \nabla_y \bar{P} = f_v(\bar{u}, \nabla\bar{P}) & \text{in} & (0,T)\times\Omega,\\ \mathrm{div}_y\, \bar{u} = g_v(\bar{u}) & \text{in} & (0,T)\times\Omega,\\ \bar{u} = 0 & \text{at} & (0,T)\times\partial\Omega,\\ \bar{u}|_{t=0} = u_0 & \text{at} & \Omega, \end{array} \tag{5.14}$$

with[1]

$$f_v(\bar{u},\nabla\bar{P}) \;:=\; (1-\rho_0)\partial_t\bar{u} + \mathrm{div}\,((\bar{B}_v{}^T\bar{B}_v - \mathrm{Id}\,)\nabla_y\bar{u}) + (\mathrm{Id}\, - {}^T\bar{B}_v)\nabla_y\bar{P},$$
$$g_v(\bar{u}) \;:=\; (\mathrm{Id}\, - {}^T\bar{B}_v) : \nabla_y\bar{u} = \mathrm{div}\, R_v(\bar{u}) \quad \text{with} \quad R_v(\bar{u}) := (\mathrm{Id}\, - \bar{B}_v)\bar{u}.$$

[1]That $g_v(\bar{u})$ may be written in two different ways is a consequence of (5.4) because $\mathrm{div}\, v = 0$; this is of course fundamental.

Hence to show existence for (5.14), it suffices to find a fixed point for the map

$$\Phi : (\bar{w}, \nabla\bar{Q}) \to (\bar{u}, \nabla\bar{P}) \tag{5.15}$$

with $(\bar{u}, \nabla\bar{P})$ the solution to

$$\begin{array}{lll} \bar{u}_t - \Delta\bar{u} + \nabla\bar{P} = f_v(\bar{w}, \nabla\bar{Q}) & \text{in} & (0,T)\times\Omega, \\ \operatorname{div}\bar{u} = g_v(\bar{w}) & \text{in} & (0,T)\times\Omega, \\ \bar{u} = 0 & \text{at} & (0,T)\times\partial\Omega, \\ \bar{u}|_{t=0} = u_0 & \text{at} & \Omega. \end{array} \tag{5.16}$$

Let us decompose $\Phi(\bar{w}, \nabla\bar{Q})$ into

$$\Phi(\bar{w}, \nabla\bar{Q}) = (u_L, \nabla P_L) + \Psi(\bar{w}, \nabla\bar{Q})$$

where $(u_L, \nabla P_L)$ stands for the free solution to the Stokes system with initial data u_0, namely

$$\begin{array}{lll} \partial_t u_L - \mu\Delta u_L + \nabla P_L = 0 & \text{in} & (0,T)\times\Omega, \\ \operatorname{div} u_L = 0 & \text{in} & (0,T)\times\Omega, \\ u_L = 0 & \text{at} & (0,T)\times\partial\Omega, \\ u_L|_{t=0} = u_0 & \text{at} & \Omega. \end{array} \tag{5.17}$$

Theorem 4.3.1 guarantees that $(u_L, \nabla P_L) \in X_T^p$ for all $T \in \mathbb{R}_+$, and that

$$\|(u_L, \nabla P_L)\|_{X_T^p} \le Ce^{CT\mu}\|u_0\|_{\dot{B}^s_{p,1}(\Omega)}. \tag{5.18}$$

Hence in order to establish that the map Φ fulfills the conditions of Banach fixed point theorem, it is only a matter of finding a condition under which the linear map Ψ is a self-map on X_T^p, with norm smaller than, say, 1/2. Now, we notice that $R_v(\bar{w})$ vanishes at the boundary and one may thus apply Theorem 4.3.1 to bound $\Psi(\bar{w}, \nabla\bar{Q})$ in X_T^p. Taking $s = n/p - 1$ and $m = 2n$ leads to

$$\begin{aligned} \|\Psi(\bar{w}, \nabla\bar{Q})\|_{X_T^p} \le Ce^{CT}\Big(&\|(f_v(\bar{w}, \nabla\bar{Q}), \nabla g_v(\bar{w}), (R_v(\bar{w}))_t)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))} \\ &+ \|R_v(\bar{w})\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))}\Big), \end{aligned} \tag{5.19}$$

where K is any bounded subset of Ω with $d(\Omega\setminus K, \partial\Omega) > 0$.

In the following computations, we agree that $(s, r) = (n/p - 1, p)$. First, from the expression of f_v and the definition of multiplier spaces, we readily have

$$\begin{aligned} \|f_v(\bar{w}, \nabla\bar{Q})\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))} &\lesssim \|1-\rho_0\|_{\mathcal{M}(\dot{B}^s_{r,1}(\Omega))}\|\bar{w}_t\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))} \\ &+ \|\bar{B}_v{}^T\bar{B}_v - \operatorname{Id}\|_{L_\infty(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\|\nabla^2\bar{w}\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))} \\ &+ \|\nabla(\bar{B}_v{}^T\bar{B}_v)\|_{L_\infty(0,T;\dot{B}^s_{r,1}(\Omega))}\|\nabla\bar{w}\|_{L_1(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))} \\ &+ \|\operatorname{Id} - \bar{B}_v\|_{L_\infty(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\|\nabla\bar{Q}\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}. \end{aligned} \tag{5.20}$$

Next, we have

$$\begin{aligned} \|\nabla g_v(\bar{w})\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))} &\lesssim \|\operatorname{Id} - \bar{B}_v\|_{L_\infty(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\|\nabla^2\bar{w}\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))} \\ &+ \|\nabla\bar{B}_v\|_{L_\infty(0,T;\dot{B}^s_{r,1})}\|\nabla\bar{w}\|_{L_1(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}, \end{aligned} \tag{5.21}$$

$$\begin{aligned} \|(R_v(\bar{w}))_t\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))} &\lesssim \|\bar{w}\|_{L_\infty(0,T;\dot{B}^s_{r,1}(\Omega))}\|(\bar{B}_v)_t\|_{L_1(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))} \\ &+ \|\operatorname{Id} - \bar{B}_v\|_{L_\infty(0,T;\mathcal{M}(\dot{B}^s_{r,1}(\Omega)))}\|\bar{w}_t\|_{L_1(0,T;\dot{B}^s_{r,1}(\Omega))}. \end{aligned} \tag{5.22}$$

In order to go further in the computations, we have to use Lemma 5.5.1 below which implies that all the above multiplier norms are controlled by the norm in $\dot{B}^{n/p}_{p,1}$. Therefore,

$$\|f_v(\bar{w},\nabla\bar{Q})\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\nabla g_v(\bar{w})\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|(R_v(\bar{w}))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ \lesssim \big(\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1})}\big)\|(\bar{w},\nabla\bar{Q})\|_{X^p_T}. \tag{5.23}$$

We also have (use that $B^{n/p}_{p,1}(K)$ is an algebra, Proposition 2.1.3, and interpolation):

$$\begin{aligned}\|R_v(\bar{w})\|_{L_1(0,T;B^{n/p}_{p,1}(K))} &\le \|\mathrm{Id}-\bar{B}_v\|_{L_\infty(0,T;B^{n/p}_{p,1}(K))}\|\bar{w}\|_{L_1(0,T;B^{n/p}_{p,1}(K))}\\ &\lesssim T^{1/2}\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big(\|\nabla\bar{w}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}+\|\bar{w}\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big),\end{aligned}$$

and, because $\dot{B}^{n/p-1/2}_{p,1}(\Omega)$ embeds in $L_{2n}(\Omega)$,

$$\begin{aligned}\|R_v(\bar{w})\|_{L_1(0,T;L_{2n}(\Omega))} &\le \|\mathrm{Id}-\bar{B}_v\|_{L_\infty(0,T;L_\infty(\Omega))}\|\bar{w}\|_{L_1(0,T;L_{2n}(\Omega))}\\ &\lesssim \|\nabla\bar{v}\|_{L_1(0,T;L_\infty(\Omega))}\|\bar{w}\|_{L_1(0,T;\dot{B}^{n/p-1/2}_{p,1}(\Omega))}\\ &\lesssim T^{3/4}\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big(\|\nabla\bar{w}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}+\|\bar{w}\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big).\end{aligned}$$

So plugging all the previous estimates in (5.19), we conclude that

$$\|\Psi(\bar{w},\nabla\bar{Q})\|_{X^p_T}\le Ce^{CT}\big(\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big)\|(\bar{w},\nabla\bar{Q})\|_{X^p_T}.$$

Therefore, if we take $\eta>0$ so that $e^{C\eta}\le 2$ and assume that

$$8C\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}\le 1,$$

then we have

$$\|\Psi(\bar{w},\nabla\bar{Q})\|_{X^p_T}\le\frac{1}{2}\|(\bar{w},\nabla\bar{Q})\|_{X^p_T}, \tag{5.24}$$

whenever

$$8C\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1})}\le 1\quad\text{and}\quad T\le\eta. \tag{5.25}$$

In fact, if (5.25) is satisfied for $T=\bar{T}$, then one can get rid of the condition that $\bar{T}\le\eta$: it suffices to split the interval $[0,\bar{T}]$ into subintervals $[T_i,T_{i+1}]$ $(i=0,\cdots,k-1)$ of size at most η, and to use the norm (with obvious notation, see (5.11))

$$\|(\bar{w},\nabla\bar{Q})\|_{\widetilde{X}^p_{\bar{T}}}:=\sum_{i=0}^{k-1}\|(\bar{w},\nabla\bar{Q})\|_{X^p_{T_i,T_{i+1}}}.$$

Now the above argument leading to (5.24) applies on every subinterval $[T_i,T_{i+1}]$ and we end up with

$$\|\Psi(\bar{w},\nabla\bar{Q})\|_{\widetilde{X}^p_{\bar{T}}}\le\frac{1}{2}\|(\bar{w},\nabla\bar{Q})\|_{\widetilde{X}^p_{\bar{T}}}.$$

Then applying the fixed point theorem in $X^p_{\bar{T}}$ endowed with the norm $\|\cdot\|_{\widetilde{X}^p_{\bar{T}}}$ ensures the existence of a solution $(\bar{u},\nabla\bar{P})$ in $X^p_{\bar{T}}$ for (5.10). Note that by construction we have,

$$\|(\bar{u},\nabla\bar{P})\|_{\widetilde{X}^p_{\bar{T}}}\le 2\|(u_L,\nabla P_L)\|_{\widetilde{X}^p_{\bar{T}}},$$

which yields Inequality (5.13). ■

REMARK 5.2.1. *It is possible to extend the above proposition to other regularity exponents. However, owing to the properties of the multipliers spaces involved in* (5.20), (5.21) *and* (5.22), *we have to assume regularity in intersection of Besov spaces and the computations become quite cumbersome.*

The above proposition will enable us to establish the local-in-time existence for (5.1). At the same time, it is not suitable for proving a global statement as *it does not* provide any bound on the gradient of the constructed velocity field in $L^1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))$. In order to overcome this, we shall establish a second existence result for the linear system (5.8), based on Theorem 4.3.3 so as to avoid the time dependency in the estimates.

We shall work in the subspace $X^{p,q}_T$ of couples $(u,\nabla P)$ of X^p_T (see the definition in (5.11)) satisfying the additional property that

$$u\in\mathcal{C}([0,T];\dot B^0_{q,1}(\Omega))\quad\text{and}\quad \partial_t u,\nabla^2u,\nabla P\in L_1(0,T;\dot B^0_{q,1}(\Omega)),$$

and we shall set

$$\|(u,\nabla P)\|_{X^{p,q}_T}=\|(u,\nabla P)\|_{X^p_T}+\|(u,\nabla P)\|_{X^q_T}. \tag{5.26}$$

We agree that $X^{p,q}$ corresponds to the above definition with $T=+\infty$.

Proposition 5.2.2. *Let $1<q\le p<2n$ with $q<n/2$ and $p>n-1$. Let v be a divergence-free vector field in $\mathcal{C}(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}(\Omega))$ with $v|_{\partial\Omega}=0$. There exists a constant c so that if*

$$\|\nabla v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}\cap\dot B^1_{q,1}(\Omega))}\le c \tag{5.27}$$

and

$$\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\le c \tag{5.28}$$

then System (5.10) with divergence free initial velocity $u_0\in\dot B^{n/p-1}_{p,1}(\Omega)\cap\dot B^0_{q,1}(\Omega)$ satisfying $u_0\cdot\vec n|_{\partial\Omega}=0$ has a unique global solution $(\bar u,\nabla\bar P)$ in $X^{p,q}$, and we have for some constant $C=C(n,p,q)$,

$$\|(\bar u,\nabla\bar P)\|_{X^{p,q}}\le C\|u_0\|_{\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega)}.$$

Proof: The proof is similar to that of the previous proposition except that it is now based on Theorem 4.3.3 to have *time independent* estimates. We readily get for any $m\in(1,\infty)$

$$\begin{aligned}\|\Psi(\bar w,\nabla\bar Q)\|_{X^{p,q}}\le C\Big(&\|(f_v(\bar w,\nabla\bar Q),\nabla g_v(\bar w),(R_v(\bar w))_t)\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\\&+\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;L_m(\Omega)\cap B^1_{q,1}(K)\cap B^{n/p}_{p,1}(K))}\Big),\end{aligned} \tag{5.29}$$

where K stands for any bounded subset of Ω with $d(\Omega\setminus K,\partial\Omega)>0$ (see Fig. 3.4).

In order to go further in the computations, we have to use Lemma 5.5.1 which implies that all the multiplier norms in (5.20), (5.21) and (5.22) with $(s,r)=(n/p-1,p)$ of $(s,r)=(0,q)$ are controlled by the norm in $\dot B^{n/p}_{p,1}$. We get

$$\begin{aligned}&\|f_v(\bar w,\nabla\bar Q),\nabla g_v(\bar w),(R_v(\bar w))_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\\&\qquad\lesssim\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\|\bar w_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\\&+\big(\|\bar B_v-\mathrm{Id}\,\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}+\|\bar B_v{}^T\bar B_v-\mathrm{Id}\,\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big)\|\nabla^2\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\\&\qquad+\|\mathrm{Id}\,-\bar B_v\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla Q\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\\&+\big(\|\nabla\bar B_v\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}+\|\nabla(\bar B_v{}^T\bar B_v)\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\big)\|\nabla\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\\&\qquad+\|(\bar B_v)_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\bar w\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}.\end{aligned}$$

Using also Inequalities (5.52) to (5.57) below, we readily get

$$
\begin{aligned}
(5.30)\quad &\|f_v(\bar w,\nabla\bar Q),\nabla g_v(\bar w),(R_v(\bar w))_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\\
&\qquad\leq C\Big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\|\bar w_t\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\\
&\qquad+\|\nabla\bar v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|(\bar w,\nabla\bar Q)\|_{X^{p,q}}+\|\nabla^2 v\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\|\nabla\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\Big).
\end{aligned}
$$

Let us observe that, owing to $1<q<n/2$, the Besov space $\dot B^2_{q,1}(\Omega)$ embeds in the Lebesgue space $L_m(\Omega)$ with $m=\frac{qn}{n-2q}\cdot$ So we shall take this value of m in (5.29). We get

$$
\begin{aligned}
\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;L_m(\Omega))}&\leq\|\mathrm{Id}-B_v\|_{L_\infty(\mathbb{R}_+;L_\infty(\Omega))}\|\bar w\|_{L_1(\mathbb{R}_+;L_m(\Omega))}\\
&\leq\|\nabla\bar v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\bar w\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}.
\end{aligned}\tag{5.31}
$$

Let us now bound $R_v(\bar w)$ in $L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))$. We have

$$
\begin{aligned}
\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}&\lesssim\|\mathrm{Id}-B_v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\bar w\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\\
&\lesssim\|\nabla\bar v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\bar w\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.
\end{aligned}
$$

Because $q<n/2$, it is not difficult to prove (just use the corresponding inequality in $\mathbb{R}^n$ and some suitable extension operator) that

$$
\|\bar w\|_{B^{n/p}_{p,1}(K)}\lesssim\|\nabla\bar w\|_{B^{n/p}_{p,1}(K)}+\|\bar w\|_{B^2_{q,1}(K)}.\tag{5.32}
$$

Therefore

$$
\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\lesssim\|\nabla\bar v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(\Omega))}\big(\|\bar w\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}+\|\nabla\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big).\tag{5.33}
$$

Let us finally bound $R_v(\bar w)$ in $L_1(\mathbb{R}_+;B^1_{q,1}(K))$. We use the fact that, because K is bounded and $q\leq m$,

$$
\begin{aligned}
\|R_v(\bar w)\|_{B^1_{q,1}(K)}&\lesssim\|R_v(\bar w)\|_{L_q(K)}+\|\nabla(R_v(\bar w))\|_{B^0_{q,1}(K)}\\
&\lesssim\|R_v(\bar w)\|_{L_m(\Omega)}+\|\nabla(R_v(\bar w))\|_{B^0_{q,1}(K)}.
\end{aligned}\tag{5.34}
$$

The first term in the r.h.s. may be handled according to (5.31). We decompose the second one into

$$
\nabla(R_v(\bar w))=(\mathrm{Id}-B_v)\nabla\bar w-\nabla B_v\,\bar w,
$$

and use (5.51). Combining with the results of Section 5.5, we end up with

$$
\begin{aligned}
\|\nabla(R_v(\bar w))\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\lesssim\|\nabla\bar v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}&\|\nabla\bar w\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\\
&+\|\nabla^2\bar v\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\|\bar w\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.
\end{aligned}
$$

Again, one may use (5.32), and the fact that

$$
\|\nabla\bar w\|_{B^0_{q,1}(K)}\lesssim\|\bar w\|_{L_m(K)}+\|\bar w\|_{\dot B^2_{q,1}(K)}\lesssim\|\bar w\|_{\dot B^2_{q,1}(\Omega)},\tag{5.35}
$$

owing to the boundedness of K. So finally

$$
\begin{aligned}
(5.36)\quad &\|R_v(\bar w)\|_{L_1(\mathbb{R}_+;B^1_{q,1}(K))}\\
&\qquad\lesssim\|\nabla\bar v\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}\cap\dot B^{n/p}_{p,1}(\Omega))}\big(\|\bar w\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}(\Omega))}+\|\nabla\bar w\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big).
\end{aligned}
$$

Plugging Inequalities (5.30) to (5.36) in (5.29), we conclude that

$$
\|\Psi(\bar w,\nabla\bar Q)\|_{X^{p,q}}\leq C\Big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}+\|\nabla\bar v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}\cap\dot B^1_{q,1}(\Omega))}\Big)\|(\bar w,\nabla\bar Q)\|_{X^{p,q}}.
$$

Therefore assuming that c is small enough in (5.27) and (5.28), we conclude that

$$\|\Psi(\bar{w},\nabla\bar{Q})\|_{X^{p,q}}\leq\frac{1}{2}\|(\bar{w},\nabla\bar{Q})\|_{X^{p,q}}.$$

Applying the fixed point theorem in the Banach space $X^{p,q}$ completes the proof of Proposition 5.2.2. ∎

5.3. Local-in-time existence

This section is devoted to proving local-in-time existence for System (5.1) with slightly nonhomogeneous density and arbitrarily large initial velocity field. Here is the main statement:

THEOREM 5.3.1. *Let* $p\in(n-1,2n)$ *and* $u_0\in\dot{B}^{n/p-1}_{p,1}(\Omega)$ *with* $\operatorname{div}u_0=0$ *and* $u_0\cdot\vec{n}|_{\partial\Omega}=0$. *Assume that* $\rho_0\in\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))$ *and that, for a small enough constant* c,

$$\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}\leq c. \tag{5.37}$$

There exists $T>0$ *such that System* (5.6) *has a unique solution* $(\bar{u},\nabla\bar{P})$ *in the space* X^p_T *defined in* (5.11)*, with*

$$\|(\bar{u},\nabla\bar{P})\|_{X^p_T}\leq C\|u_0\|_{\dot{B}^{n/p-1}_{p,1}}.$$

Proof: We consider the map

$$\mathcal{T}:(\bar{v},\nabla\bar{Q})\mapsto(\bar{u},\nabla\bar{P}),$$

where $(\bar{u},\nabla\bar{P})$ is the solution to (5.10) with $\bar{v}$ defining $\bar{B}_v$ constructed in Proposition 5.2.1.

We claim that $\mathcal{T}$ is a contraction in some suitable closed ball of X^p_T with sufficiently small T. As we aim at considering *large* initial velocity u_0 with critical regularity however, we take a ball centered at the solution $(u_L,\nabla P_L)$ to the homogeneous Stokes system (5.17) (that satisfies (5.18)). Then we focus on the discrepancy to $(u_L,\nabla P_L)$, namely $(\widetilde{u},\nabla\widetilde{P}):=(\bar{u}-u_L,\nabla(\bar{P}-P_L))$ and $(\widetilde{v},\nabla\widetilde{Q}):=(\bar{v}-u_L,\nabla(\bar{Q}-P_L))$. The couple $(\widetilde{u},\nabla\widetilde{P})$ satisfies the following modification of (5.10) (if $\mu=1$ for simplicity):

$$\begin{array}{lll}\partial_t\widetilde{u}-\Delta\widetilde{u}+\nabla\widetilde{P}=f_v(\widetilde{u},\nabla\widetilde{P})+f_v(u_L,\nabla P_L) & \text{in} & \Omega\times(0,T),\\ \operatorname{div}\widetilde{u}=g_v(\widetilde{u})+g_v(u_L) & \text{in} & \Omega\times(0,T),\\ \widetilde{u}|_{\partial\Omega}=0 & \text{at} & \partial\Omega\times(0,T),\\ \widetilde{u}|_{t=0}=0 & \text{at} & \Omega.\end{array} \tag{5.38}$$

Thanks to Proposition 5.2.1 we are ensured that solutions to (5.38) exist at least on a short time interval $[0,T]$, so far as

$$\int_0^T\|\nabla\bar{v}\|_{\dot{B}^{n/p}_{p,1}(\Omega)}\,dt\leq c. \tag{5.39}$$

We claim that there exists $R>0$ and $T>0$ (depending only on u_L) so that the map

$$\widetilde{\mathcal{T}}:(\widetilde{v},\nabla\widetilde{Q})\longmapsto(\widetilde{u},\nabla\widetilde{P}) \tag{5.40}$$

is a contraction from $\bar{B}_{X^p_T}(0,R)$ to itself. First, applying Theorem 4.3.1 yields

$$\begin{aligned}\|(\widetilde{u},\nabla\widetilde{P})\|_{X^p_T}\leq Ce^{CT}\Big(&\|(f_v(\widetilde{u},\nabla\widetilde{P}),\nabla g_v(\widetilde{u}),(R_v(\widetilde{u}))_t)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &+\|(f_v(u_L,\nabla P_L),\nabla g_v(u_L),(R_v(u_L))_t)\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &+\|R_v(\widetilde{u})\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))}+\|R_v(u_L)\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))}\Big).\end{aligned}$$

Thus, arguing as in the proof of Proposition 5.2.1, we easily get

$$
\begin{aligned}
(5.41)\quad \|(\widetilde{u},\nabla\widetilde{P})\|_{X_T^p} \leq Ce^{CT}\Bigg(&\Big(\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\Big)\Big(\|(\widetilde{v},\nabla\widetilde{Q})\|_{X_T^p}\\
&+\|\partial_t u_L,\nabla^2 u_L,\nabla P_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\Big)\\
&+(T^{1/2}+T^{3/4})\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\Big(\|\widetilde{u},u_L\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\nabla\widetilde{u},\nabla u_L\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\Big)\\
&+\|u_L\|_{L_2(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\nabla v\|_{L_2(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\Bigg).
\end{aligned}
$$

This may be obtained by using (5.20), (5.21) and so on for $f_v(u_L,\nabla P_L)$, $g_v(u_L)$ and $R_v(u_L)$. The only difference lies in the use of (5.22) : we now write that

$$
\begin{aligned}
\|(R_v(u_L))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))} \lesssim{}& \|u_L\|_{L_2(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\nabla v\|_{L_2(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\
&+\|\partial_t u_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\|\nabla\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}.
\end{aligned}
$$

Bounding the first term in that way is important as we need to show that it is small when T goes to 0, *even if the initial velocity is large.*

Now, if we assume that $e^{CT}\leq 2$ and that c in (5.37) and (5.39) is small enough, Inequality (5.41) and interpolation imply that

$$
\begin{aligned}
\|(\widetilde{u},\nabla\widetilde{P})\|_{X_T^p} \leq{}& \frac{1}{4}\Big(\|(\widetilde{v},\nabla\widetilde{Q})\|_{X_T^p}+\|\partial_t u_L,\nabla^2 u_L,\nabla P_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\
&+T^{1/2}\big(\|u_L\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\nabla u_L\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big)\Big)\\
&+C\|u_L\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\|\nabla v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}+C\|\nabla u_L\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|v\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}.
\end{aligned}
$$

It is now clear that if one takes $R=2cC\|u_0\|_{\dot{B}^{n/p-1}_{p,1}(\Omega)}$ and T fulfilling in addition

$$
\begin{aligned}
\|\partial_t u_L,\nabla^2 u_L,\nabla P_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+T^{1/2}\big(\|u_L\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\nabla u_L\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big)&\\
+\|u_0\|_{\dot{B}^{n/p-1}_{p,1}(\Omega)}\|\nabla u_L\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\leq\eta\|u_0\|_{\dot{B}^{n/p-1}_{p,1}(\Omega)}&
\end{aligned}
$$

for a small enough $\eta>0$, then the above inequality implies that, whenever $\|(\widetilde{v},\nabla\widetilde{Q})\|_{X_T^p}\leq R$ we have $\|(\widetilde{u},\nabla\widetilde{P})\|_{X_T^p}\leq R$, too.

In order to prove that the map $\mathcal{T}$ is a contraction if T is sufficiently small, let us consider two data $(v_1,\nabla Q_1)$ and $(v_2,\nabla Q_2)$ in $\bar{B}_{X_T^p}(u_L,R)$, and set $(u_i,\nabla P_i)=\mathcal{T}(v_i,\nabla Q_i)$, $i=1,2$. We also use the notation $f_i:=f_{v_i}$, $g_i:=g_{v_i}$ and $R_i:=R_{v_i}$ for $i=1,2$. Then we may look at $(\delta u,\nabla\delta P):=(u_2-u_1,\nabla(P_2-P_1))$ as the solution to the following evolutionary Stokes system on $[0,T]\times\Omega$:

$$
\begin{cases}
\partial_t\delta u-\Delta\delta u+\nabla\delta P=f_2(\widetilde{u}_2,\nabla\widetilde{P}_2)-f_1(\widetilde{u}_1,\nabla\widetilde{P}_1)+f_2(u_L,\nabla P_L)-f_1(u_L,\nabla P_L),\\
\operatorname{div}\delta u=g_2(\widetilde{u}_2)-g_1(\widetilde{u}_1)+g_2(u_L)-g_1(u_L)=\operatorname{div}\big(R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)+R_2(u_L)-R_1(u_L)\big).
\end{cases}
$$

Therefore, applying Theorem 4.3.1 implies under the small time condition of the previous step

$$
\begin{aligned}
(5.42)\quad \|(\delta u,\nabla\delta P)\|_{X^p_T} \le C\Big(&\|f_2(\widetilde u_2,\nabla\widetilde P_2)-f_1(\widetilde u_1,\nabla\widetilde P_1)\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&+\|f_2(u_L,\nabla P_L)-f_1(u_L,\nabla P_L)\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&+\|g_2(\widetilde u_2)-g_1(\widetilde u_1)\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}+\|g_2(u_L)-g_1(u_L)\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\\
&+\|(R_2(\widetilde u_2)-R_1(\widetilde u_1))_t\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}+\|(R_2(\widetilde u_L)-R_1(\widetilde u_L))_t\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&+\|R_2(\widetilde u_2)-R_1(\widetilde u_1)\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))}+\|R_2(\widetilde u_L)-R_1(\widetilde u_L)\|_{L_1(0,T;L_{2n}(\Omega)\cap B^{n/p}_{p,1}(K))}\Big).
\end{aligned}
$$

Keeping in mind that both v_1 and v_2 satisfy (5.39), we may bound the right-hand side as follows:

- *Term* $f_2(\widetilde u_2,\nabla\widetilde P_2)-f_1(\widetilde u_1,\nabla\widetilde P_1)$. We rewrite this term as (with $B_i:=\bar B_{v_i}$ for $i=1,2$):

$$
\begin{aligned}
f_2(\widetilde u_2,\nabla\widetilde P_2)-f_1(\widetilde u_1,\nabla\widetilde P_1)&=(1-\rho_0)\partial_t\delta u+\operatorname{div}\big((B_2{}^TB_2-B_1{}^TB_1)\nabla\widetilde u_2)\big)\\
&\quad+\operatorname{div}\big((B_1{}^TB_1-\mathrm{Id}\,)\nabla\delta u\big)+{}^T(B_1-B_2)\nabla\widetilde P_2+(\mathrm{Id}\,-{}^TB_1)\nabla\delta P.
\end{aligned}
$$

Hence, using Lemma 5.5.1 and (5.54), (5.55), (5.56),

$$
\begin{aligned}
\|f_2(\widetilde u_2,\nabla\widetilde P_2)-f_1(\widetilde u_1,\nabla\widetilde P_1)\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}&\lesssim\|1-\rho_0\|_{\mathcal M(\dot B^{n/p-1}_{p,1}(\Omega))}\|\partial_t\delta u\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
+\big(\|\nabla\widetilde u_2\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}&+\|\nabla\widetilde P_2\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\\
+\|\nabla v_1\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}&\big(\|\nabla\delta u\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}+\|\nabla\delta P\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\big).
\end{aligned}
$$

Above we used Inequalities (5.59) and (5.60).

- *Term* $f_2(u_L,\nabla P_L)-f_1(u_L,\nabla P_L)$. Because

$$
f_2(u_L,\nabla P_L)-f_1(u_L,\nabla P_L)=\operatorname{div}\big((B_2{}^TB_2-B_1{}^TB_1)\nabla u_L)\big)+{}^T(B_1-B_2)\nabla P_L,
$$

we readily get

$$
\begin{aligned}
&\|f_2(u_L,\nabla P_L)-f_1(u_L,\nabla P_L)\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\
&\qquad\lesssim\big(\|\nabla u_L\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}+\|\nabla P_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}.
\end{aligned}
$$

- *Term* $\|g_2(\widetilde u_2)-g_1(\widetilde u_1)\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}$. We write

$$
g_2(\widetilde u_2)-g_1(\widetilde u_1)=(\mathrm{Id}\,-{}^T\bar B_1):\nabla\delta u+{}^T(B_1-B_2):\nabla\widetilde u_2.
$$

Now, because $\dot B^{n/p}_{p,1}(\Omega)$ is an algebra, we readily get, by virtue of (5.54),

$$
\begin{aligned}
\|g_2(\widetilde u_2)-g_1(\widetilde u_1)\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}&\lesssim\|\nabla\delta v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla\widetilde u_2\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\\
&\quad+\|\nabla v_1\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla\delta u\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}.
\end{aligned}
$$

- *Term* $\|g_2(u_L)-g_1(u_L)\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}$. Because $g_2(u_L)-g_1(u_L)={}^T(B_1-B_2):\nabla u_L$, we get

$$
\|g_2(u_L)-g_1(u_L)\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\lesssim\|\nabla\delta v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla u_L\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}.
$$

- *Term* $\|(R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}$. We use the fact that

$$(R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1))_t=-(B_1)_t\delta u+(\mathrm{Id}\,-B_1)\partial_t\delta u+(B_1-B_2)_t\widetilde{u}_2+(B_1-B_2)\partial_t\widetilde{u}_2.$$

Hence using (5.54), (5.57), (5.59) and (5.62),

$$\begin{aligned}\|(R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ \lesssim\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\big(\|\delta u\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\partial_t\delta u\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big)\\ +\big(\|\widetilde{u}_2\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\partial_t\widetilde{u}_2\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}.\end{aligned}$$

- *Term* $\|(R_2(\widetilde{u}_L)-R_1(\widetilde{u}_L))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}$. We just have to write that

$$(R_2(u_L)-R_1(u_L))_t=(B_1-B_2)_tu_L+(B_1-B_2)\partial_tu_L.$$

If we proceed as for bounding the previous term then we get the term $\|u_L\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}$ in the r.h.s. that does not need to be small for T going to 0. Hence, we proceed slightly differently: we apply (5.62) with $s=n/p-1$ in order to bound the term $(B_1-B_2)_t$. We eventually get

$$\begin{aligned}\|(R_2(u_L)-R_1(u_L))_t\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ \lesssim\|u_L\|_{L_2(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\nabla\delta v\|_{L_2(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\partial_tu_L\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1}(\Omega))}\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}.\end{aligned}$$

- *Term* $\|R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)\|_{L_1(0,T;L_{2n}(\Omega)\cap\dot{B}^{n/p}_{p,1}(K))}$. We start with the following expansion:

$$R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)=(\mathrm{Id}\,-B_1)\delta u+(B_1-B_2)\widetilde{u}_2.$$

Then applying Hölder inequality, embedding and interpolation inequality as in the proof of Proposition 5.2.1,

$$\begin{aligned}\|R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)\|_{L_1(0,T;L_{2n}(\Omega))}&\lesssim\|\nabla v_1\|_{L_1(0,T;L_\infty(\Omega))}\|\delta u\|_{L_1(0,T;L_{2n}(\Omega))}\\&\qquad+\|\nabla\delta v\|_{L_1(0,T;L_\infty(\Omega))}\|\widetilde{u}_2\|_{L_1(0,T;L_{2n}(\Omega))}\\&\lesssim T^{3/4}\Big(\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(\delta u,\nabla\delta P)\|_{X^p_T}\\&\qquad+\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(\widetilde{u}_2,\nabla\widetilde{P}_2)\|_{X^p_T}\Big).\end{aligned}$$

For bounding the norm in $L_1(0,T;\dot{B}^{n/p}_{p,1}(K))$, we just use the corresponding norm on the larger set Ω and write that

$$\begin{aligned}\|R_2(\widetilde{u}_2)-R_1(\widetilde{u}_1)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}&\lesssim\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\delta u\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\\&\qquad+\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|\widetilde{u}_2\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\\&\lesssim T^{1/2}\big(\|\nabla v_1\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(\delta u,\nabla\delta P)\|_{X^p_T}\\&\qquad+\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(\widetilde{u}_2,\nabla\widetilde{P}_2)\|_{X^p_T}\big).\end{aligned}$$

- *Term* $\|R_2(u_L)-R_1(u_L)\|_{L_1(0,T;L_{2n}(\Omega)\cap\dot{B}^{n/p}_{p,1}(\Omega))}$. We have $R_{v_2}(u_L)-R_{v_1}(u_L)=(B_1-B_2)u_L$. Hence arguing as in the previous item, we get

$$\|R_2(u_L)-R_1(u_L)\|_{L_1(0,T;L_{2n}(\Omega))}\lesssim T^{3/4}\|\nabla\delta v\|_{L_1(0,T;L_\infty(\Omega))}\|(u_L,\nabla P_L)\|_{X^p_T}$$

and

$$\|R_2(u_L)-R_1(u_L)\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\lesssim T^{1/2}\|\nabla\delta v\|_{L_1(0,T;\dot{B}^{n/p}_{p,1}(\Omega))}\|(u_L,\nabla P_L)\|_{X^p_T}.$$

Putting all the above inequalities in (5.42), using the definition of the norm of X_T^p and the fact that T is small eventually yields, up to a harmless change of C,

$$\begin{aligned}\|(\delta u,\nabla\delta P)\|_{X_T^p}\leq C\Big(\big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))}+\|\nabla v_1\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\big)\|(\delta u,\nabla\delta P)\|_{X_T^p}\\ +\big(T^{1/2}\|u_0\|_{\dot B^{n/p-1}_{p,1}(\Omega)}+\|u_L\|_{L_2(0,T;\dot B^{n/p}_{p,1}(\Omega))}+\|\nabla u_L\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}+\|\partial_t u_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}\\ +\|\nabla P_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1}(\Omega))}+\|(\widetilde u_2,\nabla\widetilde P_2)\|_{X_T^p}\big)\|(\delta v,\nabla\delta Q)\|_{X_T^p}\Big).\end{aligned}$$

Now, according to (5.37) and (5.39), the first term of the r.h.s. may be absorbed by the l.h.s. As the factor of $\|(\delta v,\nabla\delta Q)\|_{X_T^p}$ becomes less than R as T tends to 0, we conclude that the map $\mathcal{T}$ is indeed a contraction on $\bar B_{X_T^p}(u_L,R)$, if T and R have been chosen small enough. This completes the proof of existence.

Proving uniqueness or the continuity of the flow map stems from similar arguments. The details are left to the reader. Theorem 5.3.1 is proved. ∎

REMARK 5.3.1. *Let us emphasize that the smallness of* $\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1}(\Omega))}$ *is completely independent of the largeness of the velocity data. In effect, it is only needed because* $1-\rho_0$ *appears as a factor of* u_t*. As pointed out in* [**17**]*, this allows to consider discontinuous initial densities of the type* $\rho_0=c_1 1_{A_0}+c_2 1_{A_0^c}$ *with* A_0 *any uniformly* C^1 *domain, provided* $|c_2-c_1|$ *is small enough.*

REMARK 5.3.2. *Theorem 5.3.1 also holds for the original system* (5.1) *in Eulerian coordinates. In the functional framework we used, the two formulations turn out to be equivalent whenever the velocity satisfies* (5.12) *(see the Appendix of* [**17**] *for more details).*

5.4. Global in time existence

This section is devoted to proving the main result of this chapter for the inhomogeneous incompressible Navier-Stokes equations (5.1). The idea is to apply the global maximal regularity estimate for solutions to the Stokes system in exterior domains (namely Theorem 4.3.3) so as to get a global-in-time existence result for small data.

THEOREM 5.4.1. *Assume that* $1<q\leq p<2n$ *with* $q<n/2$ *and* $p>n-1$*. Let* u_0 *be in* $\dot B^{n/p-1}_{p,1}\cap\dot B^0_{q,1}(\Omega)$ *with* $\operatorname{div}u_0=0$, $u_0\cdot\vec n=0$ *at the boundary. There exists a small positive constant* c *such that if*

$$\|\rho_0-1\|_{\mathcal{M}(\dot B^{\frac np-1}_{p,1}\cap\dot B^0_{q,1}(\Omega))}\leq c\quad\text{and}\quad\|u_0\|_{\dot B^{\frac np-1}_{p,1}\cap\dot B^0_{q,1}(\Omega)}\leq c\mu,\tag{5.43}$$

then System (5.1) *has a unique global solution* $(\rho,u,\nabla P)$ *with* $\rho\in\mathcal{C}([0,T];\mathcal{M}(\dot B^{\frac np-1}_{p,1}\cap\dot B^0_{q,1}(\Omega)))$ *and* $(u,\nabla P)\in X^{p,q}$ *(see the definition in* (5.26)*). Besides, there exists some constant* C *so that*

$$\|(u,\nabla P)\|_{X^{p,q}}\leq C\|u_0\|_{\dot B^0_{q,1}\cap\dot B^{n/p-1}_{p,1}(\Omega)}.$$

Proof: The method is essentially the same as for proving local existence except that we now have to resort to Theorem 4.3.3 and Proposition 5.2.2. As usual, we restrict to $\mu=1$. We have already established, under smallness conditions (5.27) and (5.28) the existence of the solution map

$$\mathcal{T}:(\bar v,\nabla\bar Q)\to(\bar u,\nabla\bar P)\tag{5.44}$$

to System (5.10), from the subset of $X^{p,q}$ with $\bar v$ satisfying (5.27), to $X^{p,q}$. Hence, in order to complete the proof of the theorem, it is only a matter of exhibiting some positive R so small as

(5.27) to be satisfied, so that $\mathcal{T}$ maps the closed ball $\bar{B}_{X^{p,q}}(0,R)$ into itself, and is contractive. In light of Proposition 5.2.2, we have

$$\|(\bar{u},\nabla\bar{P})\|_{X^{p,q}}\leq C\|u_0\|_{\dot{B}^{n/p-1}_{p,1}\cap\dot{B}^0_{q,1}(\Omega)}.$$

Hence one may take $R=C\|u_0\|_{\dot{B}^{n/p-1}_{p,1}\cap\dot{B}^0_{q,1}(\Omega)}$ if assuming that $\|u_0\|_{\dot{B}^{n/p-1}_{p,1}\cap\dot{B}^0_{q,1}(\Omega)}$ is small enough (in order that $\|(\bar{v},\nabla\bar{Q})\|_{X^{p,q}}\leq C\|u_0\|_{\dot{B}^{n/p-1}_{p,1}\cap\dot{B}^0_{q,1}(\Omega)}$ implies (5.27)).

Let us now go to the proof of contractivity. Using the same notations as in the proof of Theorem 5.3.1, we have

$$\begin{cases}\partial_t\delta u-\Delta\delta u+\nabla\delta P=f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1),\\ \operatorname{div}\delta u=g_2(u_2)-g_1(u_1)=\operatorname{div}\big(R_2(u_2)-R_1(u_1)\big).\end{cases}$$

Applying Theorem 4.3.3 with $m:=\frac{nq}{n-2q}$, we thus get

$$\begin{aligned}(5.45)\quad \|(\delta u,\nabla\delta P)\|_{X^{p,q}}&\lesssim\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega)\cap B^1_{q,1}(K)\cap B^{n/p}_{p,1}(K))}\\ &+\|f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1),g_2(u_2)-g_1(u_1),(R_2(u_2)-R_1(u_1))_t)\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1}\cap\dot{B}^0_{q,1}(\Omega))}.\end{aligned}$$

Following the computations of the proof of Theorem 5.3.1, we readily get

$$\begin{aligned}&\|f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1),g_2(u_2)-g_1(u_1),(R_2(u_2)-R_1(u_1))_t)\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &\lesssim\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1}(\Omega))}\|\partial_t\delta u\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\big(\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}+\|\nabla P_2\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1}(\Omega))}\\ &\qquad+\|u_2\|_{L_\infty(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1}(\Omega))}+\|\partial_t u_2\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}\\ &\qquad\qquad+\|\nabla v_1,\nabla v_2\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}\big(\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}+\|\nabla\delta P\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1}(\Omega))}\big).\end{aligned}$$

Next, let us go to the proof of estimates in $L^1(\mathbb{R}_+;\dot{B}^0_{q,1}(\Omega))$ for $f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1)$. Again, we use the decomposition

$$\begin{aligned}&f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1)=(1-\rho_0)\partial_t\delta u\\ &+\operatorname{div}\big(({B_2}^TB_2-{B_1}^TB_1)\nabla u_2\big)+\operatorname{div}\big(({B_1}^TB_1-\mathrm{Id})\nabla\delta u\big)+{}^T(B_1-B_2)\nabla P_2+(\mathrm{Id}-{}^TB_1)\nabla\delta P.\end{aligned}$$

We further write that

$$\begin{aligned}\|\operatorname{div}\big(({B_2}^TB_2-{B_1}^TB_1)\nabla u_2\big)\|_{\dot{B}^0_{q,1}(\Omega)}&\leq\|({B_2}^TB_2-{B_1}^TB_1)\otimes\nabla^2u_2\|_{\dot{B}^0_{q,1}(\Omega)}\\ &+\|\nabla({B_2}^TB_2-{B_1}^TB_1)\otimes\nabla u_2\|_{\dot{B}^0_{q,1}(\Omega)}.\end{aligned}$$

Hence, using Lemma 5.5.1 and flow estimates in Section 5.5, we get

$$\begin{aligned}(5.46)\quad\|\operatorname{div}\big(({B_2}^TB_2-{B_1}^TB_1)\nabla u_2\big)\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}(\Omega))}&\lesssim\|\nabla^2u_2\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}(\Omega))}\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}\\ +\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}\big(\|\nabla^2\delta v\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}(\Omega))}&+\|\nabla^2v_1\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}(\Omega))}\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}\big).\end{aligned}$$

Similarly, using that

$$\|\operatorname{div}\big(({B_1}^TB_1-\mathrm{Id})\nabla\delta u\big)\|_{\dot{B}^0_{q,1}(\Omega)}\leq\|\nabla({B_1}^TB_1)\otimes\nabla\delta u\|_{\dot{B}^0_{q,1}(\Omega)}+\|({B_1}^TB_1-\mathrm{Id})\nabla^2\delta u\|_{\dot{B}^0_{q,1}(\Omega)},$$

we get

$$\begin{aligned}(5.47)\quad\|\operatorname{div}\big(({B_1}^TB_1-\mathrm{Id})\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}(\Omega))}&\lesssim\|\nabla^2v_1\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}(\Omega))}\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}\\ &+\|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1}(\Omega))}\|\nabla^2\delta u\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}(\Omega))}.\end{aligned}$$

Bounding the last two terms ${}^T(B_1-B_2)\nabla P_2$ and $(\mathrm{Id}-{}^TB_1)\nabla\delta P$ also follows from Lemma 5.5.1 and estimates for the flow. As it is totally similar to the above terms, we do not provide more details. We eventually get

$$\begin{aligned}\|f_2(u_2,\nabla P_2)-f_1(u_1,\nabla P_1)\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}&\lesssim\|1-\rho_0\|_{\mathcal{M}(\dot B^0_{q,1}(\Omega))}\|\partial_t\delta u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\\&+\big(\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}(\Omega))}+\|\nabla P_2\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\\&+\|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla\delta P\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}+\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\nabla^2\delta v\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}.\end{aligned}$$

Bounding $g_2(u_2)-g_1(u_1)$ is the same. As for $(R_2(u_2)-R(u_1))_t$, we write

$$(R_2(u_2)-R(u_1))_t=-(B_1)_t\delta u+(\mathrm{Id}-B_1)\partial_t\delta u+(B_1-B_2)_tu_2+(B_1-B_2)\partial_tu_2.$$

Given that the product maps $\dot B^{n/p}_{p,1}(\Omega)\times\dot B^0_{q,1}(\Omega)$ in $\dot B^0_{q,1}(\Omega)$, one may proceed exactly as in the proof of Theorem 5.3.1. Indeed, all the terms pertaining to B_1 or B_2 just have to be bounded in spaces $L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))$ or $L_\infty(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))$. We end up with

$$\begin{aligned}\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}&\lesssim\|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big(\|\delta u\|_{L_\infty(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}+\|\partial_t\delta u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\big)\\&+\big(\|u_2\|_{L_\infty(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}+\|\partial_tu_2\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}(\Omega))}.\end{aligned}$$

Let us now bound $R_2(u_2)-R_1(u_1)$ in $L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))$. Recall that

$$R_2(u_2)-R_1(u_1)=(\mathrm{Id}-B_1)\delta u+(B_1-B_2)u_2.\tag{5.48}$$

Hence, using that $B^{n/p}_{p,1}(K)$ is an algebra, one may write that

$$\begin{aligned}\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}&\lesssim\|\nabla v_1\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\delta u\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\\&+\|u_2\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\nabla\delta v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.\end{aligned}$$

Then using (5.32) enables us to get

$$\begin{aligned}\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}&\lesssim\|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big(\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}+\|\delta u\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}\big)\\&+\big(\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}+\|u_2\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}\big)\|\nabla\delta v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.\end{aligned}$$

Let us finally bound $R_2(u_2)-R_1(u_1)$ in $L_1(\mathbb{R}_+;L_m(\Omega)\cap B^1_{q,1}(K))$. We shall use again that, according to (5.34),

$$\begin{aligned}&\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega)\cap B^1_{q,1}(K))}\\&\qquad\lesssim\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega))}+\|\nabla(R_2(u_2)-R_1(u_1))\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}.\end{aligned}$$

For the first term, using the decomposition (5.48) and the bounds for $\mathrm{Id}-B_1$ and B_1-B_2 in Section 5.5, we find out that

$$\begin{aligned}\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega))}&\lesssim\|\nabla v_1\|_{L_1(\mathbb{R}_+;L_\infty(\Omega))}\|\delta u\|_{L_1(\mathbb{R}_+;L_m(\Omega))}\\&+\|\nabla\delta v\|_{L_1(\mathbb{R}_+;L_\infty(\Omega))}\|u_2\|_{L_1(\mathbb{R}_+;L_m(\Omega))},\end{aligned}$$

whence, using the embeddings $\dot B^{n/p}_{p,1}(\Omega)\hookrightarrow L_\infty(\Omega)$ and $\dot B^2_{q,1}(\Omega)\hookrightarrow L_m(\Omega)$,

$$\begin{aligned}\|R_2(u_2)-R_1(u_1)\|_{L_1(\mathbb{R}_+;L_m(\Omega))}&\lesssim\|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\delta u\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}\\&+\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|u_2\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}K))}.\end{aligned}\tag{5.49}$$

Finally, differentiating (5.48) with respect to x yields

$$\nabla(R_2(u_2)-R_1(u_1))=-\nabla(B_1)\delta u+\nabla(B_1-B_2)\,u_2+(\mathrm{Id}-B_1)\nabla\delta u+(B_1-B_2)\,\nabla u_2.$$

We have (see Section 5.5)

$$\begin{aligned}
\|\nabla(B_1)\delta u\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))} &\lesssim \|\nabla^2 v_1\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\|\delta u\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))},\\
\|\nabla(B_1-B_2)\,u_2\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))} &\lesssim \|\nabla\delta v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\nabla u_2\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))},\\
\|(\mathrm{Id}-B_1)\nabla\delta u\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))} &\lesssim \|\nabla v_1\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}\|\nabla\delta u\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))},\\
\|(B_1-B_2)\,\nabla u_2\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))} &\lesssim \|\nabla u_2\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))}\|\nabla\delta v\|_{L_1(\mathbb{R}_+;B^{n/p}_{p,1}(K))}.
\end{aligned}$$

Therefore, using (5.35),

$$\begin{aligned}
(5.50)\quad \|\nabla(R_2(u_2)-R_1(u_1))\|_{L_1(\mathbb{R}_+;B^0_{q,1}(K))} &\lesssim \|\nabla v_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\|\delta u\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}\\
&+\|\nabla^2 v_1\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}\big(\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}+\|\nabla\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big)\\
&+\big(\|\nabla^2\delta v\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}(\Omega))}+\|\nabla\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1}(\Omega))}\big)\|\nabla u_2\|_{L_1(\mathbb{R}_+;\dot B^1_{q,1}(\Omega)\cap\dot B^{n/p}_{p,1}(\Omega))}.
\end{aligned}$$

Plugging all the above estimates in (5.45), we conclude that

$$\begin{aligned}
\|(\delta u,\nabla\delta P)\|_{X^{p,q}} \lesssim \|(u_2,\nabla Q_2)\|_{X^{p,q}}\|(\delta v,\nabla\delta Q)\|_{X^{p,q}} &+ \big(\|(u_1,\nabla Q_1)\|_{X^{p,q}}+\|(u_2,\nabla Q_2)\|_{X^{p,q}}\\
&+\|(v_1,\nabla P_1)\|_{X^{p,q}}+\|1-\rho_0\|_{\mathcal{M}(\dot B^0_{q,1}\cap\dot B^{n/p}_{p,1}(\Omega))}\big)\|(\delta u,\nabla\delta P)\|_{X^{p,q}}.
\end{aligned}$$

It is now clear that if R and ρ_0 have been chosen so that

$$2\big(\|1-\rho_0\|_{\mathcal{M}(\dot B^0_{q,1}\cap\dot B^{n/p}_{p,1}(\Omega))}+CR\big)<1,$$

then the above inequality entails that

$$\|(\delta u,\nabla\delta P)\|_{X^{p,q}}\leq\kappa\|(\delta v,\nabla\delta Q)\|_{X^{p,q}}$$

for some $\kappa<1$ whenever $(v_1,\nabla Q_1)$ and $(v_2,\nabla Q_2)$ are in $\bar B_{X^{p,q}}(0,R)$. This completes the proof of the global existence. Proving uniqueness follows from arguments similar to that of contractivity. The details are left to the reader. ■

5.5. Estimates of nonlinearities

In this section we establish a few estimates for nonlinear terms in Besov spaces. First, let us give some insight on the structure of the multiplier spaces $\mathcal{M}(\dot B^s_{p,1})$.

LEMMA 5.5.1. *The following inequality holds true:*

$$\|u\|_{\mathcal{M}(\dot B^s_{p,1}(\Omega))}+\|u\|_{\mathcal{M}(\dot B^0_{q,1}(\Omega))}\lesssim\|u\|_{\dot B^{n/p}_{p,1}(\Omega)}$$

whenever $1<p<\infty$, $-\min(n/p,n/p')<s\leq n/p$ *and* $1<q<\infty$.

If in addition $\max(p,q)\leq n$ *then we also have*

$$\|u\|_{\mathcal{M}(\dot B^1_{q,1}(\Omega))}\lesssim\|u\|_{\dot B^{n/p}_{p,1}(\Omega)}.$$

PROOF. For the first item, it suffices to establish that the product maps $\dot B^s_{p,1}(\Omega)\times\dot B^{n/p}_{p,1}(\Omega)$ in $\dot B^s_{p,1}(\Omega)$, and $\dot B^0_{q,1}(\Omega)\times\dot B^{n/p}_{p,1}(\Omega)$ in $\dot B^0_{q,1}(\Omega)$. Our definition of Besov norms by restriction allows us to consider only the case $\Omega=\mathbb{R}^n$. Then the result is sort of classical. Note that the result for $\mathcal{M}(\dot B^s_{p,1}(\mathbb{R}^n))$ has already been proved in Proposition 2.2.1. As for $\mathcal{M}(\dot B^0_{q,1}(\mathbb{R}^n))$, one may use continuity results for the paraproduct, and functional embedding. Indeed, we have $uv=T_uv+R(u,v)+T_vu$, and we may use that

- T maps $\dot B^{n/p}_{p,1}(\mathbb{R}^n)\times\dot B^0_{q,1}(\mathbb{R}^n)$ in $\dot B^0_{q,1}(\mathbb{R}^n)$ (because $\dot B^{n/p}_{p,1}(\mathbb{R}^n)$ embeds in $L_\infty(\mathbb{R}^n)$),
- R maps $\dot B^{n/p}_{p,1}(\mathbb{R}^n)\times\dot B^0_{q,1}(\mathbb{R}^n)$ in $\dot B^0_{q,1}(\mathbb{R}^n)$ because $n/p+0>n\max(0,1/q-1/p')$,

- T maps $\dot B^0_{q,1}(\mathbb{R}^n)\times\dot B^{n/p}_{p,1}(\mathbb{R}^n)$ in $\dot B^0_{q,1}(\mathbb{R}^n)$ (use first that $\dot B^0_{q,1}(\mathbb{R}^n)$ is embedded in $\dot B^{n/p'-n/q}_{p',1}(\mathbb{R}^n)$ if $q\leq p'$).

In order to prove the last item, we use the fact that T and R map $L_\infty(\mathbb{R}^n)\times\dot B^1_{q,1}(\mathbb{R}^n)$ in $\dot B^1_{q,1}(\mathbb{R}^n)$, together with the embedding $\dot B^{n/p}_{p,1}(\mathbb{R}^n)$ in $L_\infty(\mathbb{R}^n)$, and also that T maps $\dot B^1_{q,1}(\mathbb{R}^n)\times\dot B^{n/p}_{p,1}(\mathbb{R}^n)$ in $\dot B^1_{q,1}(\mathbb{R}^n)$, if $\max(p,q)\leq n$. □

Let us finally prove some useful 'flow estimates'. The important fact that we shall use repeatedly is that, as a consequence of the above lemma, the space $\dot B^{n/p}_{p,1}(\Omega)$ is a (quasi)-Banach algebra (and of course so does $B^{n/p}_{p,1}(K)$). Hence if $\int_0^t D\bar v\,d\tau$ is small enough, a condition that will be ensured by the smallness of the data, then one may just write

$$\bar B_v=\sum_{k\geq0}\Bigl(-\int_0^t D\bar v\,d\tau\Bigr)^k. \tag{5.51}$$

Therefore, whenever $\|\nabla\bar v\|_{L_1(0,T;L_\infty(\Omega))}<1$, we have $\mathrm{Id}-\bar B_v\in L_\infty(0,T;L_\infty(\Omega))$ and

$$\|\mathrm{Id}-\bar B_v\|_{L_\infty(0,T;L_\infty(\Omega))}\leq\frac{\|\nabla\bar v\|_{L_1(0,T;L_\infty(\Omega))}}{1-\|\nabla\bar v\|_{L_1(0,T;L_\infty(\Omega))}}. \tag{5.52}$$

Likewise, since $\dot B^{n/p}_{p,1}(\Omega)$ is a quasi-Banach algebra, there exist two constants $c=c(n,p)$ and $C=c(n,p)$ such that if

$$\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}<c, \tag{5.53}$$

then

$$\|\mathrm{Id}-\bar B_v\|_{L_\infty(0,T;\dot B^{n/p}_{p,1}(\Omega))}\leq C\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}. \tag{5.54}$$

Let us emphasize that we have

$$D_xv(t,x)=D_y\bar v(t,y)\bar B_v(t,y)=D_y\bar v(t,y)\sum_{k\geq0}\Bigl(-\int_0^t D_y\bar B(\tau,y)\,d\tau\Bigr)^k.$$

Hence condition (5.53) for $\bar v$ or v are equivalent (up to a harmless change of c), a fact that we used freely and repeatedly throughout this chapter.

Note also that, as

$$\mathrm{Id}-\bar B_v{}^T\bar B_v=(\mathrm{Id}-\bar B_v)({}^T\bar B_v-\mathrm{Id})+(\mathrm{Id}-\bar B_v)+(\mathrm{Id}-{}^T\bar B_v),$$

we also have, under Condition (5.54),

$$\|\mathrm{Id}-\bar B_v{}^T\bar B_v\|_{L_\infty(0,T;\dot B^{n/p}_{p,1}(\Omega))}\leq C\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}. \tag{5.55}$$

Similarly, by taking the gradient of (5.51) and using $\nabla(\bar B_v\,{}^T\bar B_v)=\nabla\bar B_v\,{}^T\bar B_v+\bar B_v\nabla({}^T\bar B_v)$, we find out that for $(s,r)\in\{(n/p-1,p),(0,q)\}$, we have

$$\|\nabla\bar B_v\|_{L_\infty(0,T;\dot B^s_{r,1}(\Omega))}+\|\nabla(\bar B_v\,{}^T\bar B_v)\|_{L_\infty(0,T;\dot B^s_{r,1}(\Omega))}\leq C\|\nabla^2\bar v\|_{L_1(0,T;\dot B^s_{r,1}(\Omega))}. \tag{5.56}$$

Finally, by taking one time derivative of (5.51), we get

$$\|(\bar B_v)_t\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}\leq C\|\nabla\bar v\|_{L_1(0,T;\dot B^{n/p}_{p,1}(\Omega))}. \tag{5.57}$$

In order to prove stability, we need some estimates on $B_2 - B_1$ where $B_i := \bar{B}_{v_i}$ for $i = 1, 2$. The starting point is that, owing to (5.51), we may write if both v_1 and v_2 satisfy (5.53), the following identity[2]:

$$B_2 - B_1 = \sum_{k\geq 1}\biggl(\Bigl(-\int_0^t D\bar{v}_2\,d\tau\Bigr)^k - \Bigl(-\int_0^t D\bar{v}_1\,d\tau\Bigr)^k\biggr)$$

$$= -\Bigl(\int_0^t D\delta v\,d\tau\Bigr)\sum_{k\geq 1}\sum_{j=0}^{k-1}\Bigl(-\int_0^t D\bar{v}_2\,d\tau\Bigr)^j\Bigl(-\int_0^t D\bar{v}_1\,d\tau\Bigr)^{k-1-j}. \tag{5.58}$$

Therefore Lemma 5.5.1 guarantees that we have for all positive t,

$$\|(B_2 - B_1)(t)\|_{\dot{B}^s_{r,1}(\Omega)} \lesssim \|\nabla\delta v\|_{L_1(0,t;\dot{B}^s_{r,1}(\Omega))} \tag{5.59}$$

whenever $1 < p < \infty$ and $-\min(n/p, n/p') < s \leq n/p$ if $r = p$, or with $s = 0$ and $r \in (1, \infty)$.

Note that because

$$B_2{}^T B_2 - B_1{}^T B_1 = (B_2 - B_1)^T B_2 + B_1{}^T(B_2 - B_1),$$

we also have, for the same couples (s, r) as in (5.59),

$$\|(B_2{}^T B_2 - B_1{}^T B_1)(t)\|_{\dot{B}^s_{r,1}(\Omega)} \lesssim \|\nabla\delta v\|_{L_1(0,t;\dot{B}^s_{r,1}(\Omega))}. \tag{5.60}$$

Next, we want to estimate $D(B_2 - B_1)$. Differentiating (5.58), we get

$$D(B_2 - B_1) = -\int_0^t D^2\delta v\,d\tau \sum_{k\geq 1} k\Bigl(-\int_0^t D\bar{v}_2\,d\tau\Bigr)^{k-1}$$
$$+\sum_{k\geq 2} k\Bigl(\int_0^t D^2 v_1\,d\tau\Bigr)\Bigl(\int_0^t D\delta v\,d\tau\Bigr)\sum_{j=0}^{k-2}\Bigl(-\int_0^t D\bar{v}_2\,d\tau\Bigr)^j\Bigl(-\int_0^t D\bar{v}_1\,d\tau\Bigr)^{k-2-j}.$$

Hence, still assuming (5.53) for v_1 and v_2, and using Lemma 5.5.1,

$$\|\nabla(B_1 - B_2)(t)\|_{\dot{B}^s_{r,1}(\Omega)} \lesssim \|\nabla^2\delta v\|_{L_1(0,t;\dot{B}^s_{r,1}(\Omega))} + \|\nabla^2 v_1\|_{L_1(0,t;\dot{B}^s_{r,1}(\Omega))}\|\nabla\delta v\|_{L_1(0,t;\dot{B}^{n/p}_{p,1}(\Omega))}. \tag{5.61}$$

Finally, taking one time derivative of (5.58) yields

$$D(B_1 - B_2) = -D\delta v \sum_{k\geq 1} k\Bigl(-\int_0^t D\bar{v}_2\,d\tau\Bigr)^{k-1}$$
$$+\sum_{k\geq 2} kD\bar{v}_1\Bigl(\int_0^t D\delta v\,d\tau\Bigr)\sum_{j=0}^{k-2}\Bigl(-\int_0^t D\bar{v}_2\,d\tau\Bigr)^j\Bigl(-\int_0^t D\bar{v}_1\,d\tau\Bigr)^{k-2-j}.$$

Hence,

$$\|\partial_t(B_1 - B_2)(t)\|_{\dot{B}^s_{r,1}(\Omega)} \lesssim \|\nabla\delta v(t)\|_{\dot{B}^s_{r,1}(\Omega)}\|\nabla\bar{v}_2\|_{L_1(0,t;\dot{B}^{n/p}_{p,1}(\Omega))} + \|\nabla\bar{v}_1(t)\|_{\dot{B}^s_{r,1}(\Omega)}\|\nabla\delta v\|_{L_1(0,t;\dot{B}^{n/p}_{p,1}(\Omega))}. \tag{5.62}$$

[2]Rigorously speaking (5.58) is not quite correct for the different matrices involved need not to commute. From the point of view of a priori estimates, everything happens as if they did, though.

CHAPTER 6

The low Mach number system

The last part of the memoir is dedicated to the analysis of a limit system for the Navier-Stokes-Fourier equations that may be derived in the low Mach number asymptotics and has been studied recently in [**14**] in the whole space setting. This system is a nonlinear coupling between a Stokes-like system and a heat-like equation. As a consequence of its derivation, the divergence of the velocity is determined by the heat flux and is thus nonzero if the fluid is heat-conductive. In contrast with the previous chapter, the full system is of parabolic type even in the Eulerian coordinates framework. Hence it will not be necessary to switch to Lagrangian coordinates to solve the system by means of the Banach fixed point theorem.

We here aim at extending the results of [**14**] to the case where the fluid domain is an exterior domain of $\mathbb{R}^n$ with $n \geq 3$. To simplify the presentation, we shall concentrate on the proof of global-in-time solutions with critical regularity. In passing, we will establish a new regularity result for the heat equation with Neumann boundary condition in exterior domains, which is of independent interest.

6.1. The system

We aim at investigating the following type of systems:

$$\begin{aligned}
&\beta(\vartheta)(\partial_t\vartheta + u\cdot\nabla\vartheta) - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) = 0 && \text{in} \quad \mathbb{R}_+\times\Omega,\\
&\rho(\vartheta)(\partial_t u + u\cdot\nabla u) - \operatorname{div}\tau + \nabla P = 0 && \text{in} \quad \mathbb{R}_+\times\Omega,\\
&\operatorname{div} u = a(\vartheta)\operatorname{div}(\kappa(\vartheta)\nabla\vartheta) && \text{in} \quad \mathbb{R}_+\times\Omega,
\end{aligned}$$

with $\tau := \mu\mathbb{D}u + \lambda(\operatorname{div} u)\mathrm{Id}$, where $\mathbb{D}u$ stands for (twice) the deformation tensor of the fluid, that is $\mathbb{D}u = \nabla u + {}^T\nabla u$. We suppose that ρ (the density of the fluid) and also β, λ, μ, κ and a are given smooth functions of ϑ satisfying

$$\kappa > 0, \quad \rho > 0, \quad \beta > 0, \quad \mu > 0 \ \text{ and } \ \lambda + 2\mu > 0.$$

This type of system may be derived in the low Mach number asymptotics in the large entropy variations regime (see e.g. [**14, 41**] and the references therein). For simplicity, we here consider only perfect gases. Then we have for some reference positive constant pressure P_0,

$$\rho(\vartheta) = \frac{P_0}{R\vartheta}, \quad \beta(\vartheta) = C_p\rho(\vartheta) = \frac{\gamma}{\gamma-1}R\rho(\vartheta) \quad \text{and} \quad a(\vartheta) = \frac{\gamma-1}{\gamma P_0} \quad \text{with } \ R>0, \quad \gamma>1.$$

For simplicity, we here focus on small perturbations of some constant positive reference temperature, say 1. Setting $\theta := \vartheta - 1$, $\nabla Q := \nabla(P + \lambda\operatorname{div} u)$, and keeping the same notation for the functions β, ρ, κ and μ, expressed in terms of θ, we eventually get

$$\begin{aligned}
&\beta(\theta)(\partial_t\theta + u\cdot\nabla\theta) - \operatorname{div}(\kappa(\theta)\nabla\theta) = 0 && \text{in} \quad \mathbb{R}_+\times\Omega,\\
&\rho(\theta)(\partial_t u + u\cdot\nabla u) - \operatorname{div}(\mu(\theta)\mathbb{D}u) + \nabla Q = 0 && \text{in} \quad \mathbb{R}_+\times\Omega,\\
&\operatorname{div} u = a\ \operatorname{div}(\kappa(\theta)\nabla\theta) && \text{in} \quad \mathbb{R}_+\times\Omega.
\end{aligned} \tag{6.1}$$

We supplement System (6.1) with the boundary constraints

$$u = 0, \quad \partial_{\vec{n}}\theta = 0 \qquad \text{at} \ \ \partial\Omega \tag{6.2}$$

and the initial data

$$u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0 \qquad \text{at} \quad \Omega, \tag{6.3}$$

interrelated through the compatibility condition $\operatorname{div} u_0 = a \operatorname{div}(\kappa(\theta_0)\nabla\theta_0)$.

Scaling arguments similar to those of Chapter 5 suggest us to use a functional framework in which the temperature has one more derivative than the velocity. Besides, in order to have some control on the conductivity and viscosity coefficients (that may depend on the temperature), we need the temperature to be at least continuous. Keeping our maximal regularity results in mind, this eventually leads to consider the initial velocity u_0 in the space $\dot{B}^{n/p-1}_{p,1}$ and the initial (relative) temperature θ_0 in $\dot{B}^{n/p}_{p,1}$ with $n/p-1$ close to 0.

Looking at the structure of the linearization System (6.1), we see that we have to deal with the heat equation and the Stokes system *with some non-divergence free constraint*. Therefore, the full system is of (generalized) parabolic type and using the Eulerian coordinates will enable us to show the well-posedness by means of the Banach fixed point theorem (in contrast with Chapter 5 where we had to switch to the Lagrangian coordinates). Note that the coupling between the temperature and velocity equations is rather harmless: once θ has been determined as a solution to a transport-diffusion type equation, the velocity may be controlled almost as if solving the homogeneous incompressible Navier-Stokes equation. The dependency of $\operatorname{div} u$ with respect to θ will turn out to be compatible with the statement of Theorem 4.3.3.

Before starting our investigation of System (6.1), we have to establish a new maximal regularity result concerning the heat equation in exterior domains, in the spirit of [**19**], but for *Neumann* boundary conditions. Besides, as we plan to handle initial temperatures in $\dot{B}^{n/p}_{p,1}$ with n/p close to 1, we will have to prove regularity estimates as well.

6.2. The heat equation with Neumann boundary conditions

Let us first recall the following classical result in the whole space (see e.g. [**5**]).

THEOREM 6.2.1. *Let $p\in[1,\infty]$ and $s\in\mathbb{R}$. Let $f\in L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))$ and $u_0\in\dot{B}^s_{p,1}(\mathbb{R}^n)$. The following system*

$$\begin{aligned} &u_t - \nu\Delta u = f && \text{in} \quad (0,T)\times\mathbb{R}^n,\\ &u = u_0 && \text{on} \qquad\quad \mathbb{R}^n \end{aligned}$$

has a unique solution u in

$$\mathcal{C}([0,T);\dot{B}^s_{p,1}(\mathbb{R}^n)) \quad \text{with} \quad \partial_t u, \nabla^2 u \in L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))$$

and the following inequality holds true:

$$\|u\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|u_t,\nu\nabla^2 u\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} \le C(\|f\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)}).$$

6.2.1. The heat equation in the half-space.

The purpose of this paragraph is to extend Theorem 6.2.1 to the half-space case $\mathbb{R}^n_+$, namely

$$\begin{aligned} &u_t - \nu\Delta u = f && \text{in} \quad (0,T)\times\mathbb{R}^n_+,\\ &\partial_{x_n} u|_{x_n=0} = 0 && \text{on} \quad (0,T)\times\partial\mathbb{R}^n_+,\\ &u|_{t=0} = u_0 && \text{on} \quad \mathbb{R}^n_+. \end{aligned} \tag{6.4}$$

THEOREM 6.2.2. *Let $u_0\in\dot{B}^s_{p,1}(\mathbb{R}^n_+)$ and $f\in L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ with $p\in[1,\infty)$ and $s\in(-1+1/p,1/p)$. Then (6.4) has a unique solution u satisfying*

$$u\in\mathcal{C}([0,T);\dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad u_t,\nabla^2 u\in L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))$$

and the following estimate is valid:

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|u_t,\nu\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\leq C(\|f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}),\tag{6.5}$$

where C is an absolute constant with no dependence on ν and T.

If in addition $\nabla u_0\in\dot B^s_{p,1}(\mathbb{R}^n_+)$ and $\nabla f\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))$ then we have

$$\nabla u\in\mathcal{C}([0,T);\dot B^s_{p,1}(\mathbb{R}^n_+)),\quad \nabla u_t,\nabla^3u\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+)),\qquad \textit{and}$$

$$\begin{aligned}\|\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&+\|\nabla u_t,\nu\nabla^2\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\leq C(\|\nabla f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}).\end{aligned}\tag{6.6}$$

Proof. Let $\widetilde u_0$ and $\widetilde f$ be the symmetric extensions over $\mathbb{R}^n$ of the data u_0 and f. Because $-1+1/p<s<1/p$, we have $\widetilde u_0\in\dot B^s_{p,1}(\mathbb{R}^n)$, $\widetilde f\in L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))$ with (see Corollary 2.2.1)

$$\|\widetilde u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}\approx\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\quad\text{and}\quad\|\widetilde f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\approx\|f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.$$

Let $\widetilde u$ be the solution given by Theorem 6.2.1. As this solution is unique in the corresponding functional framework, the symmetry properties of the data ensure that $\widetilde u$ is symmetric with respect to $\{x_n=0\}$. Note that $\nabla\widetilde u\in L_1(0,T;\dot B^{s+1}_{p,1}(\mathbb{R}^n))$ and that $s+1>1/p$. Hence $\partial_{x_n}\widetilde u|_{x_n=0}$ is well defined and vanishes, owing to the antisymmetry of $\partial_{x_n}\widetilde u$. In short, the restriction u of $\widetilde u$ to the half-space satisfies (6.4), and

- $\widetilde u_t$ coincides with the symmetric extension of u_t,
- $\nabla^2_{x'}\widetilde u$ coincides with the symmetric extension of $\nabla^2_{x'}u$,
- $\nabla_{x'}\partial_{x_n}\widetilde u$ coincides with the antisymmetric extension of $\nabla_{x'}\partial_{x_n}u$,
- $\partial^2_{x_n,x_n}\widetilde u=(\Delta-\Delta_{x'})\widetilde u$ hence coincides with $\widetilde u_t-\widetilde f-\Delta_{x'}\widetilde u$.

Hence one may conclude that

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&+\|u_t,\nu\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\leq\|\widetilde u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|\widetilde u_t,\nu\nabla^2\widetilde u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))},\end{aligned}$$

which implies (6.5).

To prove higher regularity estimates, we differentiate (6.4) with respect to horizontal variables:

$$\begin{aligned}&(\nabla_{x'}u)_t-\nu\Delta\nabla_{x'}u=\nabla_{x'}f&&\text{in}\quad(0,T)\times\mathbb{R}^n_+,\\&\partial_{x_n}(\nabla_{x'}u)|_{x_n=0}=0&&\text{on}\quad(0,T)\times\partial\mathbb{R}^n_+,\\&\nabla_{x'}u|_{t=0}=\nabla_{x'}u_0&&\text{on}\quad\mathbb{R}^n_+,\end{aligned}$$

hence applying the above result implies that

$$\begin{aligned}\|\nabla_{x'}u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}&+\|\nabla_{x'}u_t,\nu\nabla^2\nabla_{x'}u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\&\leq C(\|\nabla_{x'}f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla_{x'}u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}).\end{aligned}$$

As regards the vertical derivative, we notice that

$$\begin{aligned}&(\partial_{x_n}u)_t-\nu\Delta\partial_{x_n}u=\partial_{x_n}f&&\text{in}\quad(0,T)\times\mathbb{R}^n_+,\\&\partial_{x_n}u|_{x_n=0}=0&&\text{on}\quad(0,T)\times\partial\mathbb{R}^n_+,\\&\partial_{x_n}u|_{t=0}=\partial_{x_n}u_0&&\text{on}\quad\mathbb{R}^n_+.\end{aligned}$$

Applying the results for the heat equation *with Dirichlet boundary conditions* (see [**15**]) yields

$$\|\partial_{x_n} u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_{x_n} u_t, \nu\nabla^2\partial_{x_n} u\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} \leq C(\|\partial_{x_n} f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_{x_n} u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}).$$

This completes the proof of the theorem. □

6.2.2. The exterior domain case. Here we extend Theorem 6.2.1 to the case where Ω is a smooth exterior domain.

THEOREM 6.2.3. *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ with $n \geq 3$. Let $1 < q \leq p < \infty$ with $q < n/2$. Let $-1+1/p < s < 1/p$ and $-1+1/q < s' < 1/q - 2/n$. Let*

$$u_0 \in \dot B^s_{p,1} \cap \dot B^{s'}_{q,1}(\Omega) \quad \textit{and} \quad f \in L_1(0,T;\dot B^s_{p,1} \cap \dot B^{s'}_{q,1}(\Omega)).$$

Then there exists a unique solution u to

$$\begin{array}{lll} u_t - \nu\Delta u = f & \textit{in} & (0,T)\times\Omega, \\ \partial_{\vec n} u = 0 & \textit{at} & (0,T)\times\partial\Omega, \\ u = u_0 & \textit{on} & \Omega \end{array} \tag{6.7}$$

such that

$$u \in \mathcal{C}([0,T];\dot B^s_{p,1} \cap \dot B^{s'}_{q,1}(\Omega)), \qquad u_t, \nabla^2 u \in L_1(0,T;\dot B^s_{p,1} \cap B^{s'}_{q,1}(\Omega))$$

and the following inequality is satisfied:

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))} + \|u_t, \nu\nabla^2 u\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))} \leq C\big(\|u_0\|_{\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)} + \|f\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}\big), \tag{6.8}$$

where the constant C is independent of T and ν.

If in addition $\nabla u_0 \in \dot B^s_{p,1} \cap \dot B^{s'}_{q,1}(\Omega)$ and $\nabla f \in L^1(0,T;\dot B^s_{p,1} \cap \dot B^{s'}_{q,1}(\Omega))$, then u also satisfies

$$\nabla u \in \mathcal{C}([0,T];\dot B^s_{p,1} \cap \dot B^{s'}_{q,1}(\Omega)), \qquad \nabla u_t, \nabla^3 u \in L_1(0,T;\dot B^s_{p,1} \cap B^{s'}_{q,1}(\Omega))$$

and

$$\|\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))} + \|\nabla u_t, \nu\nabla^3 u\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))} \leq C\big(\|u_0, \nabla u_0\|_{\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega)} + \|f, \nabla f\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(\Omega))}\big). \tag{6.9}$$

Proving this theorem relies on the following statement, and on lower order estimates (see Lemma 6.2.1 below) which will enable us to remove the time dependency.

THEOREM 6.2.4. *Let Ω be a smooth exterior domain of $\mathbb{R}^n$ with $n \geq 2$. Let $1 < p < \infty$, $-1+1/p < s < 1/p$, $f \in L_1(0,T;\dot B^s_{p,1}(\Omega))$, and $u_0 \in \dot B^s_{p,1}(\Omega)$. Then Equation* (6.7) *has a unique solution u such that*

$$u \in \mathcal{C}([0,T];\dot B^s_{p,1}(\Omega)), \qquad \partial_t u, \nabla^2 u \in L_1(0,T;\dot B^s_{p,1}(\Omega))$$

and the following estimates are valid:

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))} + \|u_t, \nu\nabla^2 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq Ce^{CT\nu}\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)} + \|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\big), \tag{6.10}$$

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))} + \|u_t, \nu\nabla^2 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq C_K\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)} + \|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} + \nu\|u|_K\|_{L_1(0,T;\dot B^s_{p,1}(K))}\big), \tag{6.11}$$

where K stands for any compact subset of Ω such that $\operatorname{dist}(\partial\Omega, \Omega\setminus K) > 0$.

If in addition $\nabla u_0 \in \dot B^s_{p,1}(\Omega)$ *and* $\nabla f \in L^1(0,T;\dot B^s_{p,1}(\Omega))$ *then we also have*

$$\|\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|\nabla u_t,\nu\nabla^3 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq Ce^{CT\nu}\big(\|u_0,\nabla u_0\|_{\dot B^s_{p,1}(\Omega)}+\|f,\nabla f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\big), \tag{6.12}$$

$$\|\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|\nabla u_t,\nu\nabla^3 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} \leq C_K\big(\|u_0,\nabla u_0\|_{\dot B^s_{p,1}(\Omega)}+\|f,\nabla f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\nu\|u|_K\|_{L_1(0,T;\dot B^s_{p,1}(K))}\big). \tag{6.13}$$

Proof. We suppose that we are given a smooth enough solution and focus on the proof of the estimates. We shall do it in three steps: first we prove interior estimates, next boundary estimates and finally global estimates after summation. By performing a suitable change of time variable and source term, one may reduce our study to the case $\nu=1$.

Throughout we fix some covering $(B(x^\ell,\lambda))_{1\leq\ell\leq L}$ of K by balls of radius λ and take some neighborhood $\Omega^0\subset\Omega$ of $\mathbb{R}^n\setminus K$ such that $d(\Omega^0,\partial\Omega)>0$. We assume in addition that the first M balls do not intersect K while the last $L-M$ balls are centered at some point of $\partial\Omega$.

Let $\eta^0:\mathbb{R}^n\to[0,1]$ be a smooth function supported in Ω^0 and with value 1 on a neighborhood of $\Omega\backslash K$. Then we consider a subordinate partition of unity $(\eta^\ell)_{1\leq\ell\leq L}$ such that (see e.g. [**39, 46**]):

1. $\sum_{0\leq l\leq L}\eta^\ell=1\quad\text{on}\quad\Omega$;
2. $\|\nabla^k\eta^\ell\|_{L_\infty(\mathbb{R}^n)}\leq C_k\lambda^{-k}$ for $k\in\mathbb{N}$ and $1\leq\ell\leq L$;
3. $\operatorname{Supp}\eta^\ell\subset B(x^\ell,\lambda)$,
4. $\partial_{\vec n}\eta^\ell=0$ on $\partial\Omega$.

We also introduce another smooth function $\widetilde\eta^0$ supported in K and with value 1 on $\operatorname{Supp}\nabla\eta^0$ and smooth functions $\widetilde\eta^1,\cdots,\widetilde\eta^L$ with support in $B(x^\ell,\lambda)$ and such that $\widetilde\eta^\ell\equiv1$ on $\operatorname{Supp}\eta^\ell$.

Note that for $\ell\in\{1,\cdots,L\}$, the bounds for the derivatives of η^ℓ together with the fact that $\big|\operatorname{Supp}\nabla\eta^\ell\big|\approx\lambda^n$ implies that for any $k\in\mathbb{N}$ and $q\in[1,\infty]$, we have

$$\|\nabla^k\eta^\ell\|_{\dot B^{n/q}_{q,1}(\mathbb{R}^n)}+\|\nabla^k\widetilde\eta^\ell\|_{\dot B^{n/q}_{q,1}(\mathbb{R}^n)}\lesssim\lambda^{-k}.$$

First step: the interior estimate. For $\ell\in\{0,\cdots,M\}$, the vector-field $U^\ell:=u\eta^\ell$ satisfies

$$\begin{aligned} U^\ell_t-\Delta U^\ell&=f^\ell\quad\text{in}\quad(0,T)\times\mathbb{R}^n,\\ U^\ell|_{t=0}&=u_0\eta^\ell\quad\text{on}\quad\mathbb{R}^n\end{aligned}$$

with

$$f^\ell:=\eta^\ell f-2\nabla\eta^\ell\cdot\nabla u-u\Delta\eta^\ell. \tag{6.14}$$

Applying Theorem 6.2.1 yields the estimates:

$$\|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|U^\ell_t,\nabla^2U^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\lesssim\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))},$$

$$\|\nabla U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\|\nabla U^\ell_t,\nabla^3U^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}\lesssim\|\nabla(\eta^\ell u_0)\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|\nabla f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}.$$

Because the function $\nabla\eta^\ell$ is in $\mathcal{C}^\infty_c(\mathbb{R}^n)$ and $\widetilde\eta^\ell\equiv1$ on $\operatorname{Supp}\nabla\eta^\ell$, we get according to the results of Chapter 2,

$$\|f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\leq\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+C_\lambda\|\widetilde\eta^\ell u,\widetilde\eta^\ell\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(K))}. \tag{6.15}$$

As may be proved by writing that $\nabla(\eta^\ell z)=z\nabla\eta^\ell+\eta^\ell\nabla z$, for any $z\in\dot B^s_{p,1}(\mathbb{R}^n)$ with $-n/p'<s\leq n/p$, we have

$$\|\nabla(\eta^\ell z)\|_{\dot B^s_{p,1}(\mathbb{R}^n)}\leq C\big(\|\widetilde\eta^l z\|_{\dot B^s_{p,1}(\mathbb{R}^n)}+\|\widetilde\eta^\ell\nabla z\|_{\dot B^s_{p,1}(\mathbb{R}^n)}\big). \tag{6.16}$$

Hence, we also have

$$\|\nabla f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\leq C\|\eta^\ell f,\widetilde\eta^l\nabla f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+C_\lambda\|\widetilde\eta^\ell u,\widetilde\eta^\ell\nabla u,\widetilde\eta^\ell\nabla^2u\|_{L_1(0,T;\dot B^s_{p,1}(K))}. \tag{6.17}$$

Plugging (6.15) and (6.17) in the inequalities for U^ℓ and ∇U^ℓ, we end up with

$$\begin{aligned}\|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb R^n))}&+\|U^\ell_t,\nabla^2U^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb R^n))}\\&\leq C\big(\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\mathbb R^n)}+\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb R^n))}\big)+C_\lambda\|\widetilde\eta^\ell(u,\nabla u)\|_{L_1(0,T;\dot B^s_{p,1}(K))},\end{aligned}\tag{6.18}$$

$$\begin{aligned}\|\nabla U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb R^n))}+\|\nabla U^\ell_t,\nabla^2\nabla U^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb R^n))}\leq C\big(\|\widetilde\eta^\ell(u_0,\nabla u_0)\|_{\dot B^s_{p,1}(\mathbb R^n)}\\+\|\widetilde\eta^\ell(f,\nabla f)\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb R^n))}\big)+C_\lambda\|\widetilde\eta^\ell(u,\nabla u,\nabla^2u)\|_{L_1(0,T;\dot B^s_{p,1}(K))}.\end{aligned}\tag{6.19}$$

Second step: the boundary estimate. We now consider an index $\ell\in\{L+1,\cdots,M\}$ so that $B(x^\ell,\lambda)$ is centered at a point of $\partial\Omega$. The localization leads to the following problem:

$$\begin{array}{lll}U^\ell_t-\Delta U^\ell=f^\ell & \text{in} & (0,T)\times\Omega,\\ \partial_{\vec n}U^\ell=0 & \text{on} & (0,T)\times\partial\Omega,\\ U^\ell|_{t=0}=u_0\eta^\ell & \text{on} & \Omega,\end{array}\tag{6.20}$$

with f^ℓ defined by (6.14), hence satisfying (6.15).

Let us now make a change of variables so as to recast (6.20) in the half-space. As $\partial\Omega$ is smooth and compact, if λ has been chosen small enough then for fixed ℓ we may find a map Z_ℓ so that (see Chapter 2)

i) Z_ℓ is a $\mathcal C^\infty$ diffeomorphism from $B(x^\ell,\lambda)$ to $Z_\ell(B(x^\ell,\lambda))$;
ii) $Z_\ell(x^\ell)=0$ and $D_xZ(x^\ell)=\mathrm{Id}\,$;
iii) $Z_\ell(\Omega\cap B(x^\ell,\lambda))\subset\mathbb R^n_+$;
iv) $Z_\ell(\partial\Omega\cap B(x^\ell,\lambda))=\partial\mathbb R^n_+\cap Z_\ell(B(x^\ell,\lambda))$;
v) Z_ℓ is normal preserving.

Setting $\nabla_xZ_\ell(x)=\mathrm{Id}+A_\ell(z)$ then one may assume in addition that there exist constants C_j depending only on Ω and on $j\in\mathbb N$ such that

$$\|D^jA_\ell\|_{L_\infty(B(x^\ell,\lambda))}\leq C_j,\tag{6.21}$$

a property which implies (by the mean value formula) that

$$\|A_\ell\|_{L_\infty(B(x^\ell,\lambda))}\leq C_1\lambda,\tag{6.22}$$

hence by interpolation between the spaces $L_q(B(x^\ell,\lambda))$ and $W^k_q(B(x^\ell,\lambda))$,

$$\|A_\ell\|_{B^{\frac nq}_{q,1}(B(x^\ell,\lambda))}\leq C\lambda\quad\text{for all}\quad 1\leq q<\infty.\tag{6.23}$$

Let $V^\ell:=Z^*_\ell U^\ell:=U^\ell\circ Z^{-1}_\ell$. The system satisfied by V^ℓ reads

$$\begin{array}{lll}V^\ell_t-\Delta_zV^\ell=F^\ell & \text{in} & (0,T)\times\mathbb R^n_+,\\ \partial_{z_n}V^\ell|_{z_n=0}=0 & \text{on} & (0,T)\times\partial\mathbb R^n_+,\\ V^\ell|_{t=0}=Z^*_\ell(U^\ell|_{t=0}) & \text{on} & \partial\mathbb R^n_+,\end{array}\tag{6.24}$$

with

$$F^\ell:=Z^*_\ell f^\ell+(\Delta_x-\Delta_z)V^\ell.$$

According to Theorem 6.2.2, we thus get

$$\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^\ell_t,\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}$$
$$\lesssim\|Z^*_\ell(U^\ell|_{t=0})\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}+\|Z^*_\ell f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|(\Delta_x-\Delta_z)V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.$$

The first two terms in the right-hand side may be dealt with thanks to composition estimates:

$$\|Z^*_\ell(U^\ell|_{t=0})\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\lesssim\|U^\ell|_{t=0}\|_{\dot B^s_{p,1}(\Omega)},$$
$$\|Z^*_\ell f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|f^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}.$$

Compared to the first step, the only definitely new term is $(\Delta_x-\Delta_z)V^\ell$. Routine computations show that

$$(\Delta_x-\Delta_z)V^\ell=\operatorname{div}_z(\widetilde A^\ell\cdot(\mathrm{Id}+{}^TA^\ell)\cdot\nabla_zV^\ell)+\operatorname{div}_z({}^T\widetilde A^\ell\cdot\nabla_zV^\ell)\quad\text{with}\quad\widetilde A^\ell=Z^*_\ell\widetilde\eta^\ell A^\ell.$$

Therefore

$$\|(\Delta_x-\Delta_z)V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|A_\ell\otimes\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla_zA_\ell\otimes\nabla_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.$$

Owing to product estimates and to the support properties of the terms involved in the inequalities, we have

$$\|A_\ell\otimes\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|A_\ell\|_{\dot B^{\frac nq}_{q,1}(B(x^\ell,\lambda))}\|\nabla_z^2V^\ell\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\ \text{ with }\ q=\min(p,p').$$

Therefore, thanks to (6.22) and to (6.23),

$$\|A_\ell\otimes\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\lambda\|\nabla_z^2V^\ell\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.$$

Similarly, we have

$$\|\nabla_zA_\ell\otimes\nabla_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|\nabla_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.$$

Therefore

$$\|(\Delta_x-\Delta_z)V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\lambda\|\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.$$

Putting together the above inequalities and remembering (6.15), we thus get for small enough λ,

$$\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^\ell_t,\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\le C\big(\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\Omega)}$$
$$+\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|\nabla_zV^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\big)+C_\lambda\|\widetilde\eta^\ell(u,\nabla_xu)\|_{L_1(0,T;\dot B^s_{p,1}(K))}.$$

Recall the following interpolation inequality (for any smooth domain D – see [**7**], Chap. 18):

$$\|\nabla W\|_{\dot B^s_{p,1}(D)}\lesssim\|\nabla^2W\|^{1/2}_{\dot B^s_{p,1}(D)}\|W\|^{1/2}_{\dot B^s_{p,1}(D)}+\|W\|_{\dot B^s_{p,1}(D)}.\tag{6.25}$$

Applying it to $G=\nabla_zV^\ell$ and $D=\mathbb{R}^n_+$ and using Young's inequality allows to reduce the above inequality to

$$\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|V^\ell_t,\nabla_z^2V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\le C\big(\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\Omega)}$$
$$+\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\big)+C_\lambda\|\widetilde\eta^\ell(u,\nabla_xu)\|_{L_1(0,T;\dot B^s_{p,1}(K))}.\tag{6.26}$$

In order to prove regularity estimates, we apply Inequality (6.6) to (6.24), and thus get

$$\|\nabla_zV^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|(\nabla_zV^\ell)_t,\nabla_z^3V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|\nabla_z(Z^*_\ell(U^\ell|_{t=0}))\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}$$
$$+\|\nabla_z(Z^*_\ell f^\ell)\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla_z(\Delta_x-\Delta_z)V^\ell)\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.\tag{6.27}$$

Because $\nabla_z(Z_\ell^* g) = Z_\ell^* \nabla_x g \cdot \nabla_z Z_\ell^{-1}$ for $g = U^\ell|_{t=0}$ and $g = f^\ell$, composition and product estimates together with (6.23) and (6.14) ensure that

$$\begin{aligned}\|\nabla_z(Z_\ell^*(U^\ell|_{t=0}))\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)} &\leq C\|\eta^\ell\nabla_x u_0\|_{\dot B^s_{p,1}(\Omega)},\\ \|\nabla_z(Z_\ell^* f^\ell)\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\leq C_\lambda\|\widetilde\eta^\ell(u,\nabla_x u,\nabla_x^2 u)\|_{L_1(0,T;\dot B^s_{p,1}(K)}.\end{aligned}$$

We also see that $\nabla_z(\Delta_x-\Delta_z)V^\ell$ is a linear combination of components of $\nabla_z^3 V_\ell\otimes A^\ell$, $\nabla_z^2 V_\ell\otimes\nabla_z A^\ell$ and $\nabla_z V_\ell\otimes\nabla_z^2 A^\ell$. Now, for λ small enough, Inequalities (6.21), (6.22) and (6.23) guarantee that

$$\begin{aligned}\|\nabla_z^3 V_\ell\otimes A^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\leq C\lambda\|\nabla_z^3 V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},\\ \|\nabla_z^2 V_\ell\otimes \nabla_z A^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\leq C\|\nabla_z^2 V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},\\ \|\nabla_z V_\ell\otimes \nabla_z^2 A^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} &\leq C\|\nabla_z V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}.\end{aligned}$$

Resuming to (6.27), taking λ small enough and using the interpolation inequality (6.25) so as to eliminate the term pertaining to $\nabla_z^2 V^\ell$, we conclude that

$$\begin{aligned}(6.28)\quad &\|\nabla_z V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|(\nabla_z V^\ell)_t,\nabla_z^3 V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\leq C\big(\|\widetilde\eta^\ell(u_0,\nabla_x u_0)\|_{\dot B^s_{p,1}(\Omega)}\\ &+\|\widetilde\eta^\ell(f,\nabla_x f)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|\nabla_z V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\big)+C_\lambda\|\widetilde\eta^\ell(u,\nabla_x u,\nabla_x^2 u)\|_{L_1(0,T;\dot B^s_{p,1}(K))}.\end{aligned}$$

Third step: global a priori estimates : low regularity. In order to establish (6.8) and (6.10), we start from the observation that, according to (6.18) and (6.26),

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))} &\leq \sum_\ell \|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}\\ &\lesssim \sum_{0\leq\ell\leq M}\|U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\sum_{M<\ell\leq L}\|V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\lesssim \sum_{0\leq\ell\leq L}\Big(\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\Omega)}+\|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|\widetilde\eta^\ell(u,\nabla u)\|_{L_1(0,T;\dot B^s_{p,1}(K))}\Big)\\ &\qquad\qquad+\sum_{M<\ell\leq L}\|V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},\end{aligned}$$

and similar inequalities for $\|\partial_t u,\nabla^2 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}$.

As the space $\dot B^s_{p,1}(\Omega)$ has the localization property (because $-n/p' < s \leq n/p$), we may write

$$\|\eta^\ell u_0\|_{\dot B^s_{p,1}(\Omega)}\lesssim\|u_0\|_{\dot B^s_{p,1}(\Omega)},\qquad \|\eta^\ell f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\lesssim\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))},$$
$$\|\widetilde\eta^\ell(u,\nabla u)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\lesssim\|(u,\nabla u)\|_{L_1(0,T;\dot B^s_{p,1}(K))},\qquad \|V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\lesssim\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}.$$

Therefore

$$\begin{aligned}\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|u_t,\nabla^2 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}&\lesssim\|u_0\|_{\dot B^s_{p,1}(\Omega)}\\ &+\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}+\|\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(K))}.\end{aligned}$$

Once again, using (6.25) enables us to eliminate the last term, and we thus end up with Inequality (6.11). Now, if we use the fact that

$$\|u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\leq T\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))},$$

then the last term of (6.11) may be absorbed by the left-hand side if T is small enough. Repeating the argument over the interval $[T,2T]$ and so on, eventually leads to (6.10).

Fourth step: global a priori estimates : high regularity. Owing to (6.19) and (6.28),

$$\begin{aligned}\|\nabla_x u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))} &\lesssim \sum_{0\leq\ell\leq M}\|\nabla_x U^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))}+\sum_{M<\ell\leq L}\|\nabla_z V^\ell\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\lesssim \sum_{0\leq\ell\leq L}\Big(\|\widetilde\eta^\ell(u_0,\nabla_x u_0)\|_{\dot B^s_{p,1}(\Omega)}+\|\widetilde\eta^\ell(f,\nabla_x f)\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\\ &\qquad+\|\widetilde\eta^\ell(u,\nabla_x u,\nabla_x^2 u)\|_{L_1(0,T;\dot B^s_{p,1}(K))}\Big)+\sum_{M<\ell\leq L}\|\nabla_z V^\ell\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))},\end{aligned}$$

and similar inequalities for $\|\nabla u_t,\nabla^3 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}$.

By using the fact that $\nabla_z V^\ell = Z^*_\ell\nabla_x U^\ell\cdot\nabla_z Z_\ell^{-1}$ and arguing as in the previous step, we get

$$\begin{aligned}\|\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))}+\|\nabla u_t,\nabla^3 u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))} &\lesssim \|u_0,\nabla u_0\|_{\dot B^s_{p,1}(\Omega)}\\ +\|f,\nabla f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}+\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}&+\|\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(K))}+\|\nabla^2 u\|_{L_1(0,T;\dot B^s_{p,1}(K))}.\end{aligned}$$

The last term may be handled by interpolation, and eliminated, and we get (6.13). If we use the fact that

$$\|u,\nabla u\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\leq T\|u,\nabla u\|_{L_\infty(0,T;\dot B^s_{p,1}(\Omega))},$$

and add up to Inequality (6.11), then we get (6.12) on a small time interval $[0,T]$. Then repeating the argument leads to (6.12) on $\mathbb{R}_+$.

Proving the existence is a rather standard issue (see e.g. [**39**]). We may consider smooth approximations of data f and u_0, which will generate W^k_p approximate solutions with k large. Estimates (6.10), (6.12) may thus be derived not only for those approximate solutions but also for the differences of them. We readily get that the sequence of approximate solutions is indeed a Cauchy sequence in the required space and it is then easy to pass to the limit in (6.7). □

In order to complete the proof of Theorem 6.2.3, we have to bound the last term of (6.11) and (6.13), *independently of* T. This is the goal of the next lemma. We here adapt Lemma 4.3.3 to the heat equation.

Lemma 6.2.1. *Assume that* $n\geq 3$ *and that* $1<p<n/2$. *Then for any* $s\in(-1+1/p,1/p-2/n)$ *sufficiently smooth solutions to* (6.7) *with* $\nu=1$ *fulfill*

$$\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}\leq C\big(\|u_0\|_{\dot B^s_{p,1}(\Omega)}+\|f\|_{L_1(0,T;\dot B^s_{p,1}(\Omega))}\big),$$

where C *is* independent of T.

Proof. We split the solution u to (6.7) in $u=u_1+u_2$ with

$$\begin{array}{lll} u_{1,t}-\Delta u_1=f & \text{in} & (0,T)\times\Omega,\\ \partial_{\vec n}u_1=0 & \text{on} & (0,T)\times\partial\Omega,\\ u_1|_{t=0}=0 & \text{on} & \Omega,\end{array}\qquad \begin{array}{lll} u_{2,t}-\Delta u_2=0 & \text{in} & (0,T)\times\Omega,\\ \partial_{\vec n}u_2=0 & \text{on} & (0,T)\times\partial\Omega,\\ u_2|_{t=0}=u_0 & \text{on} & \Omega.\end{array}$$

Let us first focus on u_1. From Corollary 2.2.1 and duality properties of Besov spaces, we infer that

$$\|u_1(t)\|_{\dot B^s_{p,1}(K)}=\sup\int_K u_1(t,x)\eta_0(x)\,dx,$$

where the supremum is taken over all

$$\eta_0\in\dot B^{-s}_{p',\infty}(\mathbb{R}^n)\quad\text{with}\quad\|\eta_0\|_{\dot B^{-s}_{p',\infty}(\mathbb{R}^n)}=1\quad\text{and}\quad\operatorname{Supp}\eta_0\subset K.\tag{6.29}$$

Consider the solution η to the following problem:

$$\begin{array}{lll}\eta_t-\Delta\eta=0 & \text{in} & (0,T)\times\Omega,\\ \partial_{\vec n}\eta=0 & \text{on} & (0,T)\times\partial\Omega,\\ \eta|_{t=0}=\eta_0 & \text{on} & \Omega.\end{array}\tag{6.30}$$

Testing the equation for u_1 by $\eta(t-\cdot)$ we discover that

$$\int_\Omega u_1(t,x)\eta_0(x)\,dx = \int_0^t\int_\Omega f(\tau,x)\eta(t-\tau,x)\,dx\,d\tau. \tag{6.31}$$

The general theory for the heat operator in exterior domains implies that

$$\|\eta(t)\|_{L_a(\Omega)} \leq C\|\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})} \quad \text{for} \quad 1<b\leq a<\infty. \tag{6.32}$$

This is a consequence of Gaussian estimates for the kernel pertaining to the heat equation with Neumann boundary condition. More precisely, in [**34**], Th. 2 (see also [**22**]), it has been proved that for fairly general domains there exists some function $\theta\in L_1(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)$ so that for all $t>0$ and $x\in\Omega$, we have

$$\eta(t,x) = \int_\Omega \frac{1}{t^{n/2}}\theta\Big(\frac{x-y}{\sqrt{t}}\Big)\eta_0(y)\,dy.$$

Using standard convolution estimates obviously yields (6.32).

Next, we observe that smooth solutions to (6.30) satisfy $\partial_{\vec{n}}\Delta\eta|_{\partial\Omega}=\partial_{\vec{n}}\eta_t|_{\partial\Omega}=0$. Hence Inequality (6.32) applies to $\Delta\eta$ and we eventually get

$$\|\nabla^2\eta(t)\|_{L_a(\Omega)} \leq C\|\Delta\eta(t)\|_{L_a(\Omega)} \leq C'\|\Delta\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{6.33}$$

Interpolating between (6.32) and (6.33) thus yields for $0<s<1/b$ and $1\leq r\leq\infty$,

$$\|\eta(t)\|_{\dot{B}^s_{b,r}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^s_{a,r}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})} \quad \text{if} \ \ 1<a\leq b<\infty. \tag{6.34}$$

In order to extend this inequality to negative indices s, we consider the dual problem:

$$\begin{array}{lll} \zeta_t-\Delta\zeta=0 & \text{in} & (0,T)\times\Omega, \\ \partial_{\vec{n}}\zeta=0 & \text{on} & (0,T)\times\partial\Omega, \\ \zeta|_{t=0}=\zeta_0 & \text{on} & \Omega, \end{array} \qquad \zeta_0\in B^{-s}_{b',r'}(\Omega). \tag{6.35}$$

Testing (6.35) by $\eta(t-\cdot)$ yields

$$\int_\Omega \eta(t,x)\zeta_0(x)\,dx = \int_\Omega \eta_0(x)\zeta(t,x)\,dx.$$

Thus we get:

$$\|\eta(t)\|_{\dot{B}^s_{b,r}(\Omega)} = \sup_{\zeta_0}\int_\Omega \eta_0(x)\zeta(t,x)\,dx \leq \sup_{\zeta_0}\Big(\|\eta_0\|_{\dot{B}^s_{a,r}(\Omega)}\|\zeta(t)\|_{\dot{B}^{-s}_{a',r'}(\Omega)}\Big),$$

where the supremum is taken over all $\zeta_0\in\dot{B}^{-s}_{b',r'}(\Omega)$ such that $\|\zeta_0\|_{\dot{B}^{-s}_{b',r'}(\Omega)}=1$.

Since $-s$ is positive we can apply (6.34) to bound $\|\zeta(t)\|_{\dot{B}^{-s}_{a',r'}(\Omega)}$. We conclude that

$$\|\eta(t)\|_{\dot{B}^s_{a,r}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^s_{b,r}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})} \quad \text{if} \ \ s>-1+1/a.$$

The remaining case $s=0$ follows by interpolation. So finally for all $1<b\leq a<\infty$, $q\in[1,\infty]$ and $-1+1/a<s<1/b$, we have

$$\|\eta(t)\|_{\dot{B}^s_{a,r}(\Omega)} \leq C\|\eta_0\|_{\dot{B}^s_{b,r}(\Omega)} t^{-\frac{n}{2}(\frac{1}{b}-\frac{1}{a})}. \tag{6.36}$$

Resuming to the initial problem of bounding u_1 and starting from (6.31), one may write

$$\Big|\int_\Omega u_1(t,x)\eta_0(x)\,dx\Big| \lesssim \int_0^t \|f(\tau)\|_{\dot{B}^s_{p,1}(\Omega)}\|\eta(t-\tau)\|_{\dot{B}^{-s}_{p',\infty}(\Omega)}\,d\tau.$$

Hence applying (6.36) yields for any $\varepsilon \in (\max(0,s),1)$,

$$\left|\int_\Omega u_1(t,x)\eta_0(x)\,dx\right| \lesssim \int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\|\eta_0\|_{\dot B^{-s}_{p',\infty}(\Omega)}\,d\tau \\ + \int_0^{\max(0,t-1)} \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\|\eta_0\|_{\dot B^{-s}_{\frac{1}{1-\varepsilon},\infty}(\Omega)}(t-\tau)^{-\frac n2(\frac1p-\varepsilon)}\,d\tau.$$

As η_0 is supported in K, one has for some constant C depending on K:

$$\|\eta_0\|_{\dot B^{-s}_{\frac{1}{1-\varepsilon},\infty}(\Omega)} \leq C\|\eta_0\|_{\dot B^{-s}_{p',\infty}(\Omega)}.$$

So, keeping in mind (6.31) and the fact that the supremum is taken over all the functions η_0 satisfying (6.29), we deduce that

$$\|u_1(t)\|_{\dot B^s_{p,1}(K)} \leq C\biggl(\int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\,d\tau + \int_0^{\max(0,t-1)} (t-\tau)^{-\frac n2(\frac1p-\varepsilon)}\|f(\tau)\|_{\dot B^s_{p,1}(\Omega)}\,d\tau\biggr).$$

Therefore,

$$\int_1^T \|u_1\|_{\dot B^s_{p,1}(K)}\,dt \leq C\biggl(1+\int_1^T \tau^{-\frac n2(\frac1p-\varepsilon)}\,d\tau\biggr)\int_0^T \|f\|_{\dot B^s_{p,1}(\Omega)}\,dt. \tag{6.37}$$

On $[0,1]$, we merely have

$$\int_0^1 \|u_1\|_{\dot B^s_{p,1}(K)}\,dt \leq C\int_0^1 \|f\|_{\dot B^s_{p,1}(\Omega)}\,dt.$$

Now, provided $\max(0,s) < \varepsilon < 1/p - 2/n$, a condition which is equivalent to $p < n/2$, the constant in (6.37) may be made independent of T and we conclude that

$$\int_0^T \|u_1\|_{\dot B^s_{p,1}(K)}\,dt \leq C\int_0^T \|f\|_{\dot B^s_{p,1}(\Omega)}\,dt.$$

Let us finally bound u_2. We first write that

$$\|u_2(t)\|_{\dot B^s_{p,1}(K)} \leq C\|u_0\|_{\dot B^s_{p,1}(\Omega)} \tag{6.38}$$

and, if $-1+\varepsilon < s < 1/p$,

$$\|u_2(t)\|_{\dot B^s_{p,1}(K)} \leq C_K\|u_2(t)\|_{\dot B^s_{\frac1\varepsilon,1}(K)} \leq C_K\|u_0\|_{\dot B^s_{p,1}(\Omega)}t^{-\frac n2(\frac1p-\varepsilon)}.$$

Then decomposing the integral over $[0,T]$ into an integral over $[0,\min(1,T)]$ and $[\min(1,T),T]$, we easily get

$$\int_0^T \|u_2(t)\|_{\dot B^s_{p,1}(K)}\,dt \leq C\biggl(1+\int_{\min(1,T)}^T t^{-\frac n2(\frac1p-\varepsilon)}\,dt\biggr)\|u_0\|_{\dot B^s_{p,1}(\Omega)}. \tag{6.39}$$

The integrant in the r.h.s. of (6.39) is finite for $\frac n2\bigl(\frac1p-\varepsilon\bigr) > 1$. Hence,

$$\int_0^T \|u_2(t)\|_{\dot B^s_{p,1}(K)}\,dt \leq C\|u_0\|_{\dot B^s_{p,1}(\Omega)}. \tag{6.40}$$

Putting this together with (6.37) and (6.2.2) completes the proof of the lemma. □

We are now ready to prove Theorem 6.2.3. According to (6.11), in order to prove (6.8), it suffices to show that

$$\|u\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(K))} \lesssim \|u_0\|_{\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(K)} + \|f\|_{L_1(0,T;\dot B^s_{p,1}\cap\dot B^{s'}_{q,1}(K))}.$$

Of course, $\|u\|_{L_1(0,T;\dot B^{s'}_{q,1}(K))}$ may be directly bounded from Lemma 6.2.1, and it is also the case of $\|u\|_{L_1(0,T;\dot B^s_{p,1}(K))}$ if $p < n/2$ and $s < 1/p - 2/n$.

If $p \geq n/2$, then we use the fact that

$$\dot{B}^{s'+2}_{q,1}(\Omega) \subset \dot{B}^{s}_{q^*,1}(\Omega) \quad \text{with} \quad \frac{1}{q^*} = \frac{1}{q} - \frac{2}{n} + \frac{s-s'}{n}\cdot$$

Therefore, if $q < n/2 \leq p < q^*$ then one may combine interpolation and Lemma 6.2.1 so as to absorb $\|u\|_{L_1(0,T;\dot{B}^s_{p,1}(K))}$ by the left-hand side of (6.8), changing the constant C if necessary.

If $p \geq q^*$ then one may repeat the argument again and again until the all possible values of p in $(n/2, \infty)$ are exhausted. This completes the proof of (6.8).

In order to establish the regularity estimate (6.9), we add up Inequalities (6.11) and (6.13) (pertaining to Besov spaces $\dot{B}^s_{p,1}(\Omega)$ and $\dot{B}^{s'}_{q,1}$) and use the interpolation inequality (6.25) so as to eliminate the term $\|\nabla u\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(K))}$. We eventually get

$$\begin{aligned}\|u, \nabla u\|_{L_\infty(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))} &+ \|u_t, \nabla u_t, \nabla^2 u, \nabla^3 u\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))} \\ &\lesssim \|u_0, \nabla u_0\|_{\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega)} + \|f, \nabla f\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(\Omega))} + \|u\|_{L_1(0,T;\dot{B}^s_{p,1}\cap\dot{B}^{s'}_{q,1}(K))}.\end{aligned}$$

The last term may be handled by means of Lemma 6.2.1, as in the proof of (6.8). This completes the proof of Theorem 6.2.3.

REMARK 6.2.1. *Here we decided to concentrate on the exterior domain case. Similar results hold true for the solutions to* (6.7) *supplemented with the condition that* $\int_\Omega u\,dx = 0$ *in any smooth bounded domain* Ω *of* $\mathbb{R}^n$ *with* $n \geq 2$ *(instead of* $n \geq 3$ *for exterior domains). The first part of the analysis, namely the proof of Theorem 6.2.4, works the same, and Lemma 6.2.1 may be improved given that* $L^p - L^q$ *estimates may be replaced by exponential decay.*

6.3. Solving a low Mach number system

We are now ready to tackle the well-posedness issue of System (6.1) for data having critical regularity, and Ω being an exterior domain of $\mathbb{R}^n$ with $n \geq 3$. For simplicity, we focus on the global-in-time stability for small perturbations of the trivial constant state $(\theta, u) = (0, 0)$ and consider only data (θ_0, u_0) in the critical spaces $\dot{B}^{n/p}_{p,1}(\Omega) \times \dot{B}^{n/p-1}_{p,1}(\Omega)$ with $p = n$ (see [**14**] for more general results in the case $\Omega = \mathbb{R}^n$).

Stating our main result requires our introducing a few notation. For $s \in \mathbb{R}$ and $1 \leq p, q \leq +\infty$, we denote by $X^s_{p,q}$ the set of functions $z : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with $z \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^s_{p,1} \cap \dot{B}^s_{q,1}(\Omega))$ and $\partial_t z, D^2 z \in L_1(\mathbb{R}_+; \dot{B}^s_{p,1} \cap \dot{B}^s_{q,1}(\Omega))$ endowed with the norm

$$\|z\|_{X^s_{p,q}} := \|z\|_{L_\infty(\mathbb{R}_+;\dot{B}^s_{p,1}\cap\dot{B}^s_{q,1}(\Omega))} + \|\partial_t u, \nabla^2 z\|_{L_1(\mathbb{R}_+;\dot{B}^s_{p,1}\cap\dot{B}^s_{q,1}(\Omega))}. \tag{6.41}$$

We shall keep the same notation for vector fields with components in $X^s_{p,q}$.

Next, we denote by $\widetilde{X}^s_{p,q}$ the subspace of functions $\theta \in X^{s-1}_{p,q}$ satisfying $\nabla\theta \in X^{s-1}_{p,q}$, and set

$$\|\theta\|_{\widetilde{X}^s_{p,q}} := \|\theta\|_{X^{s-1}_{p,q}} + \|\nabla\theta\|_{X^{s-1}_{p,q}}. \tag{6.42}$$

It will also be convenient to use the notation $\widetilde{B}^s_{p,1}(\Omega)$ to designate the space of those functions $\theta \in \dot{B}^{s-1}_{p,1}(\Omega)$ so that $\nabla\theta \in \dot{B}^{s-1}_{p,1}(\Omega)$, endowed with the norm

$$\|\theta\|_{\widetilde{B}^s_{p,1}(\Omega)} := \|\theta\|_{\dot{B}^{s-1}_{p,1}(\Omega)} + \|\nabla\theta\|_{\dot{B}^{s-1}_{p,1}(\Omega)}. \tag{6.43}$$

Here is our main global well-posedness result for System (6.1).

THEOREM 6.3.1. *Assume that* Ω *is an exterior domain of* $\mathbb{R}^n$ *with* $n \geq 3$. *Let* $\theta_0 \in \widetilde{B}^1_{q,1} \cap \dot{B}^1_{n,1}(\Omega)$ *and* $u_0 \in \dot{B}^0_{q,1} \cap \dot{B}^0_{n,1}(\Omega)$ *with* $1 < q < n/2$. *If the compatibility condition*

$$\operatorname{div} u_0 = a \operatorname{div}(\kappa(\theta_0)\nabla\theta_0) \tag{6.44}$$

is satisfied, and

$$\|\theta_0\|_{\widetilde{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega)} + \|u_0\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)} \leq c \tag{6.45}$$

for sufficiently small c, *then there exists a unique global-in-time solution* $(\theta, u, \nabla Q)$ *to System* (6.1) *such that*

$$(\theta, u) \in \widetilde{X}^1_{n,q} \times X^0_{n,q} \quad \textit{and} \quad \nabla Q \in L_1(\mathbb{R}_+; \dot{B}^0_{q,1} \cap \dot{B}^0_{n,1}(\Omega)). \tag{6.46}$$

Besides, for some constant $C = C(n, q, \Omega)$,

$$\|\nabla Q\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))} + \|\theta\|_{\widetilde{X}^1_{n,q}} + \|u\|_{X^0_{n,q}} \leq C\big(\|\theta_0\|_{\widetilde{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega)} + \|u_0\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\big). \tag{6.47}$$

Some comments are in order, concerning the data. First, as in [**14**], we should be able to consider large data $\theta_0 \in \widetilde{B}^1_{n,1}(\Omega)$ and $u_0 \in \dot{B}^0_{n,1}(\Omega)$ and get the local existence of unique strong solutions, provided that there is no vacuum initially (or equivalently that ϑ_0 is positive). Second, we did not assume that θ_0 is in $\dot{B}^0_{n,1}(\Omega)$ because it is guaranteed by the following embedding:

$$\dot{B}^0_{q,1} \cap \dot{B}^1_{n,1} \cap \dot{B}^1_{q,1}(\Omega) \hookrightarrow \dot{B}^0_{n,1}(\Omega). \tag{6.48}$$

Finally, we expect a similar statement to be true for more general gases where the function a smoothly depends on ϑ. However, this would require us to generalize our maximal regularity estimates to the Stokes system with a divergence constraint which reads $\operatorname{div} R + A$.

Proof: Proving the existence and uniqueness of a solution for (6.1) is based on the Banach fixed point theorem. As a preliminary step, we shall derive a priori estimates. This will help us to exhibit the right functional framework, and an appropriate smallness condition on the data so as to get a global-in-time control on the solutions. Next, we shall introduce a suitable map $\mathcal{T}$ the fixed points of which are global solutions to (6.1). Slight modifications of the estimates obtained in the preliminary step will enable us to justify that the hypotheses of the fixed point theorem are indeed fulfilled. This will complete the proof of the existence of a global solution. Proving uniqueness is almost the same as proving that $\mathcal{T}$ is contractive on a suitably small ball, and is thus omitted.

Step 1. A priori estimates. Let $\bar\rho := \rho(0)$, $\bar\mu := \mu(0)$, $\bar\kappa := \kappa(0)$ and $\bar\beta := \beta(0)$. Set $\widetilde\rho := \rho - \bar\rho$, $\widetilde\mu := \mu - \bar\mu$, $\widetilde\kappa := \kappa - \bar\kappa$, and $\widetilde\beta := \beta - \bar\beta$. Let us recast System (6.1) as follows:

$$\begin{aligned}
&\bar\beta\partial_t\theta - \operatorname{div}(\bar\kappa\nabla\theta) = \operatorname{div}(\widetilde\kappa(\theta)\nabla\theta) - \beta(\theta)u\cdot\nabla\theta - \widetilde\beta(\theta)\partial_t\theta && \text{in} \quad \mathbb{R}_+ \times \Omega,\\
&\bar\rho\partial_t u - \operatorname{div}(\bar\mu\mathbb{D}u) + \nabla Q = \operatorname{div}(\widetilde\mu(\theta)\mathbb{D}u) - \rho(\theta)u\cdot\nabla u - \widetilde\rho(\theta)\partial_t u && \text{in} \quad \mathbb{R}_+ \times \Omega,\\
&\operatorname{div} u = a \operatorname{div}(\kappa(\theta)\nabla\theta) && \text{in} \quad \mathbb{R}_+ \times \Omega,\\
&u = 0, \quad \partial_{\vec n}\theta = 0 && \text{at} \quad \mathbb{R}_+ \times \partial\Omega,\\
&u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0 && \text{at} \quad \Omega.
\end{aligned} \tag{6.49}$$

Before going further in the computations, let us point out that our results for the Stokes system in Section 4.3 hold for $\operatorname{div}\mathbb{D}u$ instead of Δu. Indeed, as a first step, we removed the compressibility (the right-hand side of $(6.49)_3$) to obtain a divergence-free vector field. Then $\operatorname{div}\mathbb{D}u = \Delta u$ for $\operatorname{div} u = 0$. One may alternately remark that for a general vector-field we have $\operatorname{div}(\mathbb{D}u) = \Delta u + \nabla \operatorname{div} u$, and incorporate the last term in the pressure. In any case, our bounds for (6.49) will follow from Theorem 4.3.3 for the Stokes system and Theorem 6.2.4 for the heat equation.

More precisely, on the one hand, applying Theorem 6.2.4 yields

$$\|\theta\|_{\widetilde{X}^1_{n,q}} \lesssim \|\theta_0\|_{\widetilde{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega)} + \|\operatorname{div}(\widetilde{\kappa}(\theta)\nabla\theta) - \beta(\theta)u\cdot\nabla\theta - \widetilde{\beta}(\theta)\partial_t\theta\|_{L_1(\mathbb{R}_+;\widetilde{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega))}. \tag{6.50}$$

On the other hand, applying Theorem 4.3.3 to the momentum equation (recall that $\kappa(\theta)\nabla\theta\cdot\vec{n}\equiv 0$ at $\partial\Omega$), we get for all $1<m<\infty$,

$$\begin{aligned}\|u\|_{X^0_{n,q}} + \|\nabla Q\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))} &\lesssim \|u_0\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\ + \|\operatorname{div}(\widetilde{\mu}(\theta)\mathbb{D}u) - \widetilde{\rho}(\theta)\partial_t u - \rho(\theta)u\cdot\nabla u\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))} &+ \|\operatorname{div}(\kappa(\theta)\nabla\theta)\|_{L_1(\mathbb{R}_+;\dot{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega))}\\ + \|(\kappa(\theta)\nabla\theta)_t\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))} &+ \|\kappa(\theta)\nabla\theta\|_{L_1(\mathbb{R}_+;L_m(\Omega))}.\end{aligned} \tag{6.51}$$

Note that no bounds for the pressure are needed to close the estimates, as it does not appear in the right-hand sides of (6.50) and (6.51). This quantity can be controlled at the end of our analysis. The immediate observation is that the left-hand side of (6.50) allows to estimate the highest order term in the temperature in the right-hand side of (6.51). To close the a priori estimates, we need to get suitable bounds for all the terms in the right-hand side of (6.50) and (6.51). As the full proof is quite repetitive, we just consider a few terms by way of example.

For instance, we have, keeping (6.48) in mind,

$$\begin{aligned}\|\operatorname{div}(\widetilde{\kappa}(\theta)\nabla\theta)\|_{\widetilde{B}^1_{q,1}\cap\widetilde{B}^1_{n,1}(\Omega)} &\lesssim \|\widetilde{\kappa}(\theta)\nabla\Delta\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)} + \|\widetilde{\kappa}'(\theta)\nabla\theta\otimes\nabla^2\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\ &+ \|\widetilde{\kappa}''(\theta)|\nabla\theta|^3\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)} + \|\widetilde{\kappa}(\theta)\Delta\theta + \widetilde{\kappa}'(\theta)|\nabla\theta|^2\|_{\dot{B}^0_{q,1}(\Omega)}.\end{aligned} \tag{6.52}$$

First let us note that

$$\|\theta\|_{\mathcal{M}(\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))} \lesssim \|\theta\|_{\dot{B}^1_{n,1}(\Omega)} \lesssim \|\theta,\nabla\theta\|_{\dot{B}^0_{n,1}(\Omega)}. \tag{6.53}$$

The first inequality stems from Lemma 5.5.1 and the second one from the corresponding inequality in $\mathbb{R}^n$ (use a standard extension operator after reducing the proof to the bounded domain case). We eventually get, after using composition estimates:

$$\|\widetilde{\kappa}(\theta)\nabla^3\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))} \lesssim \|\theta,\nabla\theta\|_{L_\infty(\mathbb{R}_+;\dot{B}^0_{n,1}(\Omega))}\|\nabla^3\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}. \tag{6.54}$$

The second term in (6.52) may be handled similarly. Using (6.53), we end up with

$$\|\widetilde{\kappa}'(\theta)\nabla\theta\otimes\nabla^2\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))} \lesssim \|\theta,\nabla\theta\|_{L_\infty(\mathbb{R}_+;\dot{B}^0_{n,1}(\Omega))}\|\nabla^2\theta,\nabla^3\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}.$$

Using the same arguments, we see that the third term obeys

$$\||\nabla\theta|^2\|_{L_1(\mathbb{R}_+;\dot{B}^1_{n,1}(\Omega))} \lesssim \|\theta,\nabla\theta\|^3_{L_\infty(\mathbb{R}_+;\dot{B}^0_{n,1}(\Omega))}\|\nabla^2\theta,\nabla^3\theta\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}.$$

The (lower order) last term in (6.52) may be handled according to similar arguments.

Next we have to bound $\beta(\theta)u\cdot\nabla\theta$ in $L_1(\mathbb{R}_+;\widetilde{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega))$. It follows from the fact that $\beta(\theta)\in L_\infty(\mathbb{R}_+;\mathcal{M}(\dot{B}^0_{n,1}))$ and $u\in L_2(\mathbb{R}_+;\mathcal{M}(\dot{B}^0_{n,1}))$. We also see that

$$\begin{aligned}\|\widetilde{\beta}(\theta)\partial_t\theta\|_{\widetilde{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega)} &\lesssim \|\widetilde{\beta}(\theta)\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)} + \|\widetilde{\beta}'(\theta)\nabla\theta\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)} + \|\widetilde{\beta}(\theta)\nabla\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\ &\lesssim \|\theta\|_{\dot{B}^1_{n,1}(\Omega)}\big(\|\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)} + \|\nabla\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\|\partial_t\theta\|_{\dot{B}^1_{n,1}(\Omega)}\\ &\qquad + \|\nabla\partial_t\theta\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\big).\end{aligned}$$

Finally, let us bound some terms in the right-hand side of (6.51). We observe for instance that

$$\|\operatorname{div}(\kappa(\theta)\nabla\theta)\|_{L_1(\mathbb{R}_+;\dot{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega))} \lesssim \|\nabla^2\theta\|_{L_1(\mathbb{R}_+;\dot{B}^1_{q,1}\cap\dot{B}^1_{n,1}(\Omega))} + \|\theta\|^2_{\widetilde{X}^1_{n,q}}$$

and

$$\begin{aligned}\|(\kappa(\theta)\nabla\theta)_t\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))} &\leq \|\kappa'(\theta)\theta_t\nabla\theta\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}+\|\kappa(\theta)\nabla\theta_t\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\ &\lesssim \|\theta_t\|_{L_1(\mathbb{R}_+;\mathcal{M}(\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\|\nabla\theta\|_{L_\infty(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\ &\qquad+\|\nabla\theta_t\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\ &\lesssim \|\theta\|_{\widetilde X^1_{n,q}}(1+\|\theta\|_{\widetilde X^1_{n,q}}).\end{aligned}$$

Note also that taking $m=qn/(n-2q)$ and using the embedding $\dot B^2_{q,1}(\Omega)\hookrightarrow L_m(\Omega)$ enables us to write that

$$\|\kappa(\theta)\nabla\theta\|_{L_1(\mathbb{R}_+;L_m(\Omega))}\leq C\|\theta\|_{L_1(\mathbb{R}_+;\dot B^2_{q,1}(\Omega))}.$$

Moreover,

$$\begin{aligned}\|\,\mathrm{div}\,(\widetilde\mu(\theta)\mathbb{D}(u))\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))} &\lesssim \|\theta\|_{L_\infty(\mathbb{R}_+;\dot B^1_{n,1}(\Omega))}\|\nabla^2u\|_{L_1(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\ &\qquad+\|\nabla\theta\|_{L_\infty(\mathbb{R}_+;\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\|\nabla u\|_{L_1(\mathbb{R}_+;\mathcal{M}(\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega))}\\ &\lesssim \|\theta\|_{\widetilde X^1_{n,q}}\|u\|_{X^0_{n,q}}.\end{aligned}$$

Putting together all the above estimates, we end up with

$$\begin{aligned}\|u\|_{X^0_{n,q}}+\|\theta\|_{\widetilde X^1_{n,q}}\leq C\Big(&\|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}+\|\theta_0\|_{\widetilde B^1_{q,1}\cap\dot B^1_{n,1}(\Omega)}\\ &+\big(\|u\|_{X^0_{n,q}}+\|\theta\|_{\widetilde X^1_{n,q}}\big)^2+\big(\|u\|_{X^0_{n,q}}+\|\theta\|_{\widetilde X^1_{n,q}}\big)^4\Big).\end{aligned}\tag{6.55}$$

Hence we deduce from an elementary bootstrap argument that (6.47) follows from (6.45) if c has been taken small enough.

Step 2. The proof of the existence for small data. The proof of the existence will be an elementary consequence of the Banach fixed point theorem. Let us introduce the map

$$\mathcal{T}:E^0_{n,q}\longrightarrow E^0_{n,q}\tag{6.56}$$

with $E^0_{n,q}:=X^0_{n,q}\times\widetilde X^1_{n,q}$ and

$$\mathcal{T}(\bar u,\bar\theta)=(u,\theta)\tag{6.57}$$

such that (u,θ) is the solution to the following linear system:

$$\begin{array}{lll}\bar\theta\partial_t\theta-\,\mathrm{div}\,(\bar\kappa\nabla\theta)=\,\mathrm{div}\,(\widetilde\kappa(\bar\theta)\nabla\bar\theta)-\beta(\bar\theta)\bar u\cdot\nabla\bar\theta-\widetilde\beta(\bar\theta)\partial_t\bar\theta & \text{in} & (0,T)\times\Omega,\\ \bar\rho\partial_t u-\,\mathrm{div}\,(\bar\mu\mathbb{D}u)+\nabla Q & & \\ \qquad=\,\mathrm{div}\,(\widetilde\mu(\bar\theta)\mathbb{D}\bar u)-\widetilde\rho(\bar\theta)\partial_t\bar u-\rho(\bar\theta)\bar u\cdot\nabla\bar u & \text{in} & (0,T)\times\Omega,\\ \mathrm{div}\,u=a\,\mathrm{div}\,(\kappa(\bar\theta)\nabla\bar\theta) & \text{in} & (0,T)\times\Omega,\\ u=0\ \text{ and }\ \partial_{\vec n}\theta=0 & \text{at} & (0,T)\times\partial\Omega,\\ u|_{t=0}=u_0,\quad \theta|_{t=0}=\theta_0 & \text{at} & \Omega.\end{array}\tag{6.58}$$

The solvability of System (6.58) in the space $E^0_{n,q}$ follows from the fact that the left-hand sides are just heat equation with Neumann boundary conditions, and the Stokes system with Dirichlet boundary conditions. Hence one may directly apply Theorems 4.3.3 and 6.2.3 and follows the computations of the previous step. More precisely, (6.55) (with $(\bar\theta,\bar u)$ in the right-hand side) implies that $\mathcal{T}$ maps the closed ball $\bar B(0,R)$ of $E^0_{n,q}$ into itself if choosing

$$R=2C\big(\|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}+\|\theta_0\|_{\dot B^1_{q,1}\cap\widetilde B^1_{n,1}(\Omega)}\big)\tag{6.59}$$

and assuming that

$$4C^2\big(\|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}+\|\theta_0\|_{\dot B^1_{q,1}\cap\widetilde B^1_{n,1}(\Omega)}\big)+16C^4\big(\|u_0\|_{\dot B^0_{q,1}\cap\dot B^0_{n,1}(\Omega)}+\|\theta_0\|_{\dot B^1_{q,1}\cap\widetilde B^1_{n,1}(\Omega)}\big)^3\leq 1.$$

To conclude the proof of existence of a global solution, it thus suffices to show that $\mathcal{T}$ is a contraction on $\bar{B}(0,R)$, namely that for any $(\bar{u}_1,\bar{\theta}_1)$ and $(\bar{u}_2,\bar{\theta}_2)$ in $\bar{B}(0,R)$, we have

$$\|\mathcal{T}(\bar{u}_1,\bar{\theta}_1)-\mathcal{T}(\bar{u}_2,\bar{\theta}_2)\|_{E^0_{n,q}}\leq\frac{1}{2}\|(\bar{u}_1-\bar{u}_2,\bar{\theta}_1-\bar{\theta}_2)\|_{E^0_{n,q}}. \tag{6.60}$$

In order to guarantee (6.58) we consider the following system being a subtraction of (6.58) for the first and second solution. Setting $\delta\theta:=\theta_1-\theta_2$, $\bar{\delta\theta}:=\bar{\theta}_1-\bar{\theta}_2$, and so on, we get

$$\begin{aligned}
&\bar{\beta}\partial_t\delta\theta-\operatorname{div}(\bar{\kappa}\nabla\delta\theta)=\operatorname{div}(\widetilde{\kappa}(\bar{\theta}_1)\nabla\bar{\theta}_1)-\beta(\bar{\theta}_1)\bar{u}_1\cdot\nabla\bar{\theta}_1-\widetilde{\beta}(\bar{\theta}_1)\partial_t\bar{\theta}_1\\
&\qquad\qquad-\operatorname{div}(\widetilde{\kappa}(\bar{\theta}_2)\nabla\bar{\theta}_2)+\beta(\bar{\theta}_2)\bar{u}_2\cdot\nabla\bar{\theta}_2+\widetilde{\beta}(\bar{\theta}_2)\partial_t\bar{\theta}_2 &&\text{in}\quad(0,T)\times\Omega,\\
&\bar{\rho}\partial_t\delta u-\operatorname{div}(\widetilde{\mu}\mathbb{D}\delta u)+\nabla\delta Q=-\widetilde{\rho}(\bar{\theta}_1)\partial_t\bar{u}_1-\rho(\bar{\theta}_1)\bar{u}_1\cdot\nabla\bar{u}_1\\
&\quad+\operatorname{div}(\widetilde{\mu}(\bar{\theta}_1)\mathbb{D}\bar{u}_1)+\widetilde{\rho}(\bar{\theta}_2)\partial_t\bar{u}_2+\rho(\bar{\theta}_2)\bar{u}_2\cdot\nabla\bar{u}_2-\operatorname{div}(\widetilde{\mu}(\bar{\theta}_2)\mathbb{D}\bar{u}_2) &&\text{in}\quad(0,T)\times\Omega,\\
&\operatorname{div}\delta u=a\operatorname{div}(\kappa(\bar{\theta}_1)\nabla\bar{\theta}_1-\kappa(\bar{\theta}_2)\nabla\bar{\theta}_2) &&\text{in}\quad(0,T)\times\Omega,\\
&\delta u=0\ \text{ and }\partial_{\vec{n}}\delta\theta=0 &&\text{at}\quad(0,T)\times\partial\Omega,\\
&\delta u|_{t=0}=0,\quad\delta\theta|_{t=0}=0 &&\text{at}\quad\Omega.
\end{aligned}\tag{6.61}$$

Applying the results for the linear systems (for the left-hand sides of (6.61)) we get, up to a change of C,

$$\|(\delta u,\delta\theta)\|_{E^0_{n,q}}\leq CR\|(\bar{\delta u},\bar{\delta\theta})\|_{E^0_{n,q}}. \tag{6.62}$$

In order to justify the above inequality we just show how to bound $\partial_t\big(\kappa(\bar{\theta}_1)\nabla\bar{\theta}_1-\kappa(\bar{\theta}_2)\nabla\bar{\theta}_2\big)$ by way of example. We write that

$$\begin{aligned}
\|\partial_t\big(\kappa(\bar{\theta}_1)\nabla\bar{\theta}_1-\kappa(\bar{\theta}_2)\nabla\bar{\theta}_2\big)\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}&\lesssim\|\partial_t(\kappa(\bar{\theta}_1)\nabla\delta\bar{\theta}),\partial_t(((\kappa(\bar{\theta}_1)-\kappa(\bar{\theta}_2))\nabla\bar{\theta}_2)\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\
&\lesssim\|\kappa(\bar{\theta}_1)\nabla\delta\bar{\theta}_t,\kappa'(\bar{\theta})\bar{\theta}_t\nabla\delta\bar{\theta},(\kappa(\bar{\theta}_1)-\kappa(\bar{\theta}_2))\nabla\bar{\theta}_{2,t}\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}\\
&\quad+\|\kappa'(\bar{\theta}_1)\delta\bar{\theta}_t\nabla\bar{\theta}_2,(\kappa'(\bar{\theta}_1)-\kappa'(\bar{\theta}_2))\bar{\theta}_{2,t}\nabla\bar{\theta}_2\|_{\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega)}.
\end{aligned}$$

Performing a time integration and using again (6.53) several times, it is easy to conclude that

$$\|\partial_t\big(\kappa(\bar{\theta}_1)\nabla\bar{\theta}_1-\kappa(\bar{\theta}_2)\nabla\bar{\theta}_2\big)\|_{L_1(\mathbb{R}_+;\dot{B}^0_{q,1}\cap\dot{B}^0_{n,1}(\Omega))}\leq CR\|\delta\bar{\theta}\|_{\widetilde{X}^1_{n,q}}.$$

Taking c small enough in (6.45), and keeping the definition of R as in (6.59), it is clear that one may ensure that $CR\leq 1/2$. The contraction mapping theorem ensures the existence of a fixed point for the map $\mathcal{T}$, which defines a unique solution to the original problem (6.1). Theorem 6.3.1 is proved. ■

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