Compressible perturbation of Poiseuille type flow

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Abstract

The paper examines the issue of stability of Poiseuille type flows in regime of compressible Navier–Stokes equations in a three dimensional finite pipe-like domain. We prove the existence of stationary solutions with inhomogeneous Navier slip boundary conditions admitting nontrivial inflow condition in the vicinity of constructed generic flows. Our techniques are based on an application of a modification of the Lagrangian coordinates. Thanks to such an approach we are able to overcome difficulties coming from hyperbolicity of the continuity equation, constructing a maximal regularity estimate for a linearized system and applying the Banach fixed point theorem.

Résumé

Dans cet article on étudie la stabilité des mouvements de fluides de type Poiseuille dans le cadre des équations de Navier–Stokes compressibles dans un domaine cylindrique borné en trois dimensions. On montre l’existence de solutions stationnaires avec conditions au bord de type Navier non homogènes admettant la traversée du bord par le fluide, au voisinage d’une classe de solutions laminaires du type Poiseuille. Notre technique est fondée sur l’application de coordonnées de Lagrange modifiées. La méthode permet de prendre en compte les difficultés résultant de l’hyperbolicité de l’équation de continuité en dérivant une estimation de régularité maximale pour un système linéarisé et en appliquant un théorème de point fixe de Banach.

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1. Introduction

The mathematical description of compressible flows is important from the point of view of applications, domains such as aerodynamics and geophysics are the most natural to be mentioned here. On the other hand, complexity of the equations describing the flow delivers very interesting mathematical challenges. In spite of active research in the field, we are still far from the complete mathematical understanding of compressible flows. The only general existence results are available for weak solutions with homogeneous boundary conditions [6,11]. As far as regular solutions are

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concerned, we have so far only partial results assuming either some smallness of the data, or its special structure. The problems have been investigated mainly with homogeneous boundary conditions [4,16]. For the overview of the state of art in the theory one can consult the monograph [17].

From the point of view of the aforementioned applications it seems very important to investigate the problems with large velocity vectors, which lead in a natural way to inhomogeneous boundary conditions. Due to the hyperbolic character of the continuity equation the density must be then prescribed on the inflow part of the boundary. Existence issues for such inflow problems are investigated in [8,9,18–21,25]. The mentioned group of problems can be regarded as questions of stability of particular constant flows.

In the present article we would like to examine the issue of stability of Poiseuille type flow in pipe-like domain in compressible regime. The Poiseuille flow is a special symmetric solution to the incompressible Navier–Stokes equations in cylindrical domains. Here it is viewed as a solution to the compressible Navier–Stokes system with constant density and constant external force, parallel to axis of the cylinder, given by the pressure (gravitation-like term). Hence the pressure, unknown in the incompressible model, is recognized as a given force. Such change of ‘observer’ looks acceptable from the mechanical point of view. Thanks to that interpretation we obtain a natural physically reasonable flow in compressible regime. The mathematical objective of this article is to establish stability of such flow under some structural assumptions limiting the magnitude of admissible perturbations.

The system. Let us define the system. We consider steady flow of a viscous, barotropic fluid in a bounded, cylindrical domain in $\mathbb{R}^3$, described by the Navier–Stokes system supplied with inhomogeneous Navier slip boundary conditions. The complete system reads

$$
\begin{align*}
\rho v \cdot \nabla v - \mu \Delta v - (\mu + \nu) \nabla \cdot \nabla \pi(\rho) &= \rho F \quad \text{in } \Omega, \\
\nabla(\rho v) &= 0 \quad \text{in } \Omega, \\
\nu \cdot T(v, \pi) \cdot \tau_k + f v \cdot \tau_k &= b_k, \quad k = 1, 2 \quad \text{on } \Gamma, \\
\nu \cdot v &= d \quad \text{on } \Gamma, \\
\rho &= \rho_{in} \quad \text{on } \Gamma_{in},
\end{align*}
$$

where $\Omega = [0, L] \times \Omega_0$ with $\Omega_0 \subset \mathbb{R}^2$ of class $C^2$, $\Gamma$ denotes the boundary of $\Omega$ (see Fig. 1), $v$ is the velocity field of the fluid, $\rho$ is its density, $\mu$ and $\nu$ are viscosity constants satisfying $\mu > 0$ and $(\nu + 2\mu) > 0$, $f \geq 0$ is the friction coefficient which may be different on different components of the boundary $\Gamma$, $\pi = \pi(\rho)$ is the pressure given as a function, at least $C^1$, of the density and $F$ is an external force. $T$ denotes the Cauchy stress tensor of the form

$$
T(v, \pi) = 2\mu D(v) + \nu \nabla v \cdot I_d - \pi I_d,
$$

where $D(v) = \frac{1}{2}(\nabla v + \nabla v^T)$ is the symmetric gradient. Next, $n$ and $\tau_k$ are outer normal and tangent vectors to $\partial \Omega$. Boundary data $\rho_{in}$, $b$, $d$ will be discussed later. The boundary $\Gamma$ is naturally split into three parts:

$$
\begin{align*}
\Gamma_0 &= \{ x \in \partial \Omega : v(x) \cdot n(x) = 0 \}, \\
\Gamma_{in} &= \{ x \in \partial \Omega : v(x) \cdot n(x) < 0 \}, \\
\Gamma_{out} &= \{ x \in \partial \Omega : v(x) \cdot n(x) > 0 \}.
\end{align*}
$$

Thanks to the chosen geometry of the domain, the above decomposition is easily illustrated by Fig. 1.

We shall say few words about the physical interpretation of the system (1.1), in particular about the choice of boundary conditions (1.1)_{3,4}. We would like to model a flow through a pipe. We assume that the fluid obeys Navier slip conditions on the walls of the pipe ($\Gamma_0$ component of the boundary), hence natural conditions on $\Gamma_0$ are $d \equiv 0$ and $b_k \equiv 0$. However, the mathematical requirements impose a need to prescribe the boundary conditions on $\Gamma_{in}$ and $\Gamma_{out}$. From the physical viewpoint these parts are artificial, this is the area where the parameters of the velocity and density are measured. This gives us a freedom of choice of the type of boundary conditions on the inflow and outflow part, which can be fit to the mathematical approach, hence we choose inhomogeneous slip condition. Note that as the friction coefficient goes to infinity, then the relations (1.1)_{3,4}, at least formally, become the standard Dirichlet conditions describing the whole velocity vector at the boundary. Since the velocity does not vanish on the boundary, the hyperbolicity of the continuity equation imposes a need to prescribe the density on the inflow part, which leads to
the condition (1.1). The velocity field determines the characteristics of the continuity equation and in particular the total mass \( \int_{\Omega} \rho \, dx \) is determined implicitly by (1.1).

**The perturbed flow.** Our goal here is to analyze a perturbation of the Poiseuille type flow

\[
\bar{V} = \left[ V^P(x_2, x_3), 0, 0 \right],
\]

where \( x_1 \) points the direction of the axis of the cylinder. It is one of the classical examples of laminar flows satisfying the incompressible Navier–Stokes equations in cylindrical domains. In the classical literature the flow is considered with homogeneous Dirichlet conditions on the boundary. \( V^P \) is then found as a solution to the corresponding elliptic problem with Dirichlet boundary conditions on each \( x_1 \)-cut of \( \Omega \). Some explicit formulas on \( V^P \) in certain domains are well-known [7,10].

In the case of slip boundary conditions that are subject of our analysis in this paper the flow (1.3) can be also found on each cut of the cylinder as a solution to elliptic problem with corresponding boundary conditions (see Lemma 1 below). In certain domains it can also be expressed with explicit formulas (see [13] and the example below). Since we are interested in a general cylindrical domain, we will not have such a formula but we show that the solution of the form (1.3) exists provided that \( \Omega_0 \) is sufficiently regular and, under the slip boundary conditions (1.1)\(_{3,4} \), \( V^P \) does not vanish on the boundary.

**Lemma 1.** Let \( \Omega_{\infty} = \mathbb{R} \times \Omega_0 \), where \( \Omega_0 \subset \mathbb{R}^2 \) with \( \partial \Omega_0 \in C^2 \), \( \mu > 0 \) and \( f \geq 0 \). Then there exists a solution \((\bar{V}, \bar{\Pi})\), such that

\[
\bar{V} = (V^P(x_2, x_3), 0, 0) \quad \text{and} \quad \bar{\Pi} = \omega x_1
\]

(1.4)

to the incompressible Navier–Stokes system with slip boundary conditions:

\[
\begin{align*}
\bar{V} \cdot \nabla \bar{V} - \mu \Delta \bar{V} + \nabla \bar{\Pi} &= 0 & \text{in } \Omega_{\infty}, \\
\text{div } \bar{V} &= 0 & \text{in } \Omega_{\infty}, \\
n \cdot (\bar{V}, \bar{\Pi}) \cdot \tau_k + f \bar{V} \cdot \tau_k &= 0, & k = 1, 2 \quad \text{on } \mathbb{R} \times \partial \Omega_0, \\
n \cdot \bar{V} &= 0 & \text{on } \mathbb{R} \times \partial \Omega_0.
\end{align*}
\]

Moreover, there exist \( \theta = \theta(f, \mu) \) and a continuous function \( \omega(f) \) such that

\[
\bar{V}^{(1)} = V^P \geq \theta TF > 0 \quad \text{in } \bar{\Omega}_{\infty},
\]

(1.6)

and

\[
\| \nabla V^P \|_{L^{\infty}} \leq \frac{C \omega(f)}{\mu} TF,
\]

(1.7)

where

\[
TF = \int_{\Omega_0} V^P \, dx_2 \, dx_3
\]

(1.8)

is the total flux of the flow \( \bar{V} \). In addition if \((\bar{V}, \bar{\Pi})\) is the Poiseuille solution, then \((\lambda \bar{V}, \lambda \bar{\Pi})\), too.
As we already said, the pressure in the Poiseuille flow can be regarded as an external force parallel to the axis of the cylinder. Like in the classical Poiseuille flow, we assume the pressure to be a linear function of $x_1$. Hence it is natural to assume the form (1.4) for the pressure where $\mu$ is a negative constant (the sign describes the direction of the flow). The form $\vec{V} = (V^P, 0, 0)$ implies that the nonlinear term $\vec{V} \cdot \nabla \vec{V}$ vanishes. Then we see that $V^P$ is a solution to the elliptic problem

\[
\mu \Delta V^P = \Delta_{x_1} = \omega(f) < 0 \quad \text{in } \Omega_0,
\]
\[
\mu \frac{\partial V^P}{\partial n} + f V^P = 0 \quad \text{on } \partial \Omega_0. \tag{1.9}
\]

Now to prove Lemma 1 it is enough to apply the maximum principle to the system (1.9). We show the proof in Appendix A. In particular Lemma 1 gives the whole family of solutions $(\lambda \vec{V}, \lambda \vec{H})$: $\lambda > 0$ parametrized by the total flux (1.8). For the stability analysis we want to focus on a given solution to (1.9) obtained for given $\mu$ and $f|\Gamma_0$. Hence we normalize the Poiseuille profile setting given total flux $TF$.

Then we have a functional dependence $\omega = \omega(f)$. Namely, $\omega$ is determined by the compatibility condition for the system (1.9), that reads $\omega|\Omega_0 = -f \int_{\partial \Omega_0} \bar{V}^P \, d\sigma$. To justify that indeed $\omega(f)$ is a function provided that $TF$ is fixed assume that $V^f_1$ satisfies (1.9) with $\omega_1$ and $f$, while $V^f_2$ satisfies (1.9) with $\omega_2$ and the same $f$, and $TF(V^f_1) = TF(V^f_2)$. Then the linear structure of (1.9) implies that $\frac{\omega_2}{\omega_1} V^f_1$ satisfies (1.9) with $\omega_2$ and $f$, and the uniqueness for (1.9) implies $\frac{\omega_2}{\omega_1} V^f_1 = V^f_2$. We get a contradiction $TF = \frac{\omega_2}{\omega_1} TF$.

The dependence $\omega(f)$ determines the assumptions we will have to make on the viscosity, more precisely, on admissible relation of $\mu$ and $f|\Gamma_0$.

Note that for $\omega = 0$ the only solution is $V^P = 0$, provided $f \neq 0$. For $f = 0$ (perfect slip) we should obtain a constant flow, hence we put $\omega = 0$ for $f = 0$. The linear structure of (1.9) makes $\omega(f)$ continuous. Thus, we conclude $\omega(f) \to 0$ for $f \to 0$, and hence by (1.7)

\[
\|\nabla \vec{V}\|_{L^\infty} \to 0 \quad \text{for } f \to 0. \tag{1.10}
\]

This is an important conclusion since in order to show the energy estimate we have to control $\nabla \vec{V}$ with the viscosity, and so we allow the viscosity to be low provided that the friction on $\Gamma_0$ is small, what is a realistic assumption (see also the remarks after the formulation of Theorem 1). Let us illustrate the dependence $\omega = \omega(f)$ with the following example.

**Example.** Take $\Omega_0 = B(0, 1) \subset \mathbb{R}^2$ and $\mu = 1$. Then, it is natural to look for $v^f = v^f(r)$ where $r = \sqrt{x_2^2 + x_3^2}$.

The boundary condition (1.9)$_2$ then reads $v^f_1 + f v^f_1|_{r=1} = 0$. We require that $\Delta v = v_{rr} + \frac{1}{r} v_r$ depend only on $f$. Moreover, we expect to obtain a constant flow for $f = 0$ (perfect slip) and classical Poiseuille profile for $f = \infty$. The above considerations lead to the family of solutions

\[
v^f(r) = TF \left[ \frac{f + 2}{f + k_f} - \frac{f}{f + k_f} r^2 \right],
\]

where $k_f = \frac{(\pi - 2)f + 4\pi}{2}$ and $TF$ is the flux of the flow $v^f$ through $\Omega_0$ defined in (1.8).

For a perfect slip case $f = 0$ we obtain a constant flow $v^0 = \frac{2TF}{k_f}$ and for a no-slip case $f = \infty$ we get a classical Poiseuille profile $v^\infty = TF(1 - r^2)$. On the boundary we have $\theta(f) = v^f(1) = \frac{2}{f + k_f}$ what is a strictly positive constant. Finally,

\[
\omega(f) = \Delta v^f = -\frac{4f}{f + k_f} TF \quad \text{and} \quad |\nabla v^f| \sim \frac{f}{f + k_f} TF.
\]

In particular $\omega(f) < 0$ for $f > 0$ and $\omega(0) = 0$.

**The main result.** Before we formulate our main result, we need one observation concerning the boundary conditions. Note that, since $V^P$ is found on every $x_1$-cut of $\Omega$, we can impose the boundary conditions (1.5)$_{3,4}$ only on $\Gamma_0$. On the other hand, in order to define small perturbations as a solution to (1.1) we have to measure the distance
(in appropriate norms) between the solution to (1.1) and the Poiseuille flow $\bar{V}$. Hence we need to consider the traces of the quantities from the boundary conditions of (1.1) with the function $\bar{V}$ instead of $v$. Since our analysis acts on a finite cylinder we define these traces at the bottoms $\Gamma_{in} \cup \Gamma_{out}$:

$$
\tilde{b}_k |_{\Gamma_{in} \cup \Gamma_{out}} = \text{tr}_{\Gamma_{in} \cup \Gamma_{out}} \left[ n \cdot 2 \mu \mathbf{D}(\bar{V}) \cdot \tau_k + f \bar{V} \cdot \tau_k \right], \quad \tilde{d} |_{\Gamma_{in}} = -\text{tr}_{\Gamma_{in}} V^p, \quad \tilde{d} |_{\Gamma_{out}} = \text{tr}_{\Gamma_{out}} V^p,
$$

where tr denotes the trace operator. By (1.5),

$$
n \cdot 2 \mu \mathbf{D}(\bar{V}) \cdot \tau_k + f \bar{V} \cdot \tau_k = 0 \quad \text{and} \quad n \cdot \bar{V} = 0 \quad \text{at } \Gamma_0.
$$

Hence the construction of the Poiseuille flow determines $(k = 1, 2)$:

$$
\tilde{b}_k = \tilde{d} = 0 \quad \text{on } \Gamma_0. \tag{1.11}
$$

The Poiseuille flow $(\bar{V}, \Pi)$ has constant density $\bar{\rho}$, we set $\bar{\rho} = 1$. For our needs we have to reformulate it in terms of the velocity and density only. Hence in the chosen setting the couple $(\bar{V}, \bar{\rho} \equiv 1)$ fulfills the following system:

$$
\bar{\rho} \bar{V} \cdot \nabla \bar{V} - \mu \Delta \bar{V} - (\mu + v) \nabla \text{div} \bar{V} + \nabla \pi(\bar{\rho}) = -\bar{\rho} \omega(f) \hat{e}_1 \quad \text{in } \Omega,
$$

$$
\text{div}(\bar{\rho} \bar{V}) = 0 \quad \text{in } \Omega,
$$

$$
n \cdot 2 \mu \mathbf{D}(\bar{V}) \cdot \tau_k + f \bar{V} \cdot \tau_k = \tilde{b}_k, \quad k = 1, 2 \quad \text{on } \Gamma,
$$

$$
n \cdot \bar{V} = \tilde{d} \quad \text{on } \Gamma. \tag{1.12}
$$

Notice that $\nabla \pi(\bar{\rho}) = 0$, however we write this term on the l.h.s. of (1.12) as we want to look at the Poiseuille profile as a solution to the system (1.1). We keep in mind that $\nabla \Pi = \omega(f) \hat{e}_1$ and (1.11). This way we obtain compatibility of (1.12) and (1.9).

We are now in a position to formulate our main result. To this end it is convenient to define the quantity which measures the distance of the data from the Poiseuille flow:

$$
D_0 = \| F + \omega(f) \hat{e}_1 \|_{L^p(\Omega)} + \| d - \bar{d} \|_{W^{2-1/p}_p(\Gamma)} + \sum_{k=1}^2 \| b_k - \tilde{b}_k \|_{W^{1-1/p}_{p}(\Gamma)} + \| \rho_{in} - 1 \|_{W^1_p(\Gamma_{in})}. \tag{1.13}
$$

We emphasize that on $\Gamma_0$ we have $d = \tilde{d} = 0$ and $b_k = \tilde{b}_k = 0$, $k = 1, 2$. The main result of the paper reads:

**Theorem 1.** Let $(\bar{V}, \Pi)$ be the Poiseuille flow given by Lemma 1, with given total flux $TF$, such that

$$
\mu \text{ is large enough compared to } f |_{\Gamma_0}. \tag{1.14}
$$

Let $p > 3$, $F \in L^p(\Omega), d \in W^{2-1/p}_{p}(\Gamma), b_1, b_2 \in W^{1-1/p}_{p}(\Gamma), \rho_{in} \in W^1_p(\Gamma_{in})$ and assume that $D_0$ driven by (1.13) is sufficiently small. Furthermore assume that the friction $f |_{\Gamma_{in}}$ is large enough.

Then there exists a solution $(v, \rho) \in W^2_p(\Omega) \times W^1_p(\Omega)$ to the system (1.1) such that

$$
\| v - \bar{V} \|_{W^2_p(\Omega)} + \| \rho - \bar{\rho} \|_{W^1_p(\Omega)} \leq C(D_0). \tag{1.15}
$$

This solution is unique in a class of solutions satisfying (1.15).

Let us make some remarks concerning our main result. The condition on the viscosity (1.14) seems to be a serious constraint, but we need it to control the gradient of the Poiseuille flow, what yields this assumption natural (see also the remark in the proof of Lemma 6). In other words, condition (1.14) can be understood as a definition of ‘laminarity’ of the perturbed flow. As it is well known that even in the incompressible case we can expect only laminar flows to be stable, thus (1.14) becomes a natural constraint.

We recall that $\nabla \bar{V}$ depends on the friction $f$ on $\Gamma_0$, and in particular (1.10) holds. It follows that for small values of friction on $\Gamma_0$ it is enough to assume that the viscosity is large enough, but only compared to the friction. This assumption is reflected in the condition (1.14). Theorem 1 admits the case of perfect slip $f |_{\Gamma_0} = 0$, and in such a case (1.14) reduces to $\mu > 0$, so no lower bound on the viscosity is required. In this case there is no bound on the size of $\bar{V}$. However in this case $\bar{V}$ would be a constant flow. We shall recall that the friction at $\Gamma_{in}$ is chosen independently to $f$ at $\Gamma_0$. From the point of view of modeling, the data at $\Gamma_{in}$ is given, hence it is important to focus the attention at $\Gamma_0$. 


The assumption $p > 3$ is required for the imbedding $W^1_p \subset L_\infty$ [1], it is required to control the pointwise boundedness of $\nabla v$ and the density.

The full regularity $W^2_\infty$ of the velocity in the cylindrical domain $\Omega$ is possible to obtain due to the properties of geometry of $\Omega$ and slip boundary conditions (Lemma 11 in Appendix A and comments concerning (2.9)).

Let us compare Theorem 1 with the results of [19]. The crucial point is that in the present paper we investigate stability of nonconstant flows. What is important, the shape of the Poiseuille profile (1.4) depends on the friction on the wall of the cylinder, while in [19] the perturbed flow was constant and hence independent of $f$. In this sense the flow (1.4) can be regarded as a natural solution to the system (1.1) with external force given by the pressure.

Another, more technical point which should be noticed is that after linearizing Eqs. (2.8) around the flow (1.4) we end up with a system with variable coefficients (3.1). For this system we develop an accurate treatment adapting the concept of Lagrangian coordinates to the steady case, treating the direction of the axis of the cylinder as a ‘time’ variable. This approach, although exploiting some concepts from [19], is to our knowledge new in the theory of strong solutions to the compressible Navier–Stokes equations.

Let us explain the main idea of the proof. We will follow an idea of Lagrangian-type coordinates [5,12,15,23]. The continuity equation is of hyperbolic type and contains a term $u \cdot \nabla w$ (where $u$ and $w$ are perturbations to the velocity and density introduced in the next section), which makes serious troubles for the issues of existence in case of inhomogeneous boundary conditions, see [8,20,17–19]. Here we overcome this obstacle by changing the system of coordinates in such a way that this term disappears (2.11). We obtain a more complex system but with structure suitable for an application of the Banach fixed point theorem. On the other hand our solutions are regular enough, thus we are able to go back to the original system keeping the well posedness of the original model. Our approach works since we are equipped with the maximal regularity estimate for a linearization of the original equations – Theorem 2. This tool gives a complete control of the regularity of solutions.

Let us make a remark concerning the solvability of the continuity equation. In general it is not solvable for arbitrary boundary data $d$ since the condition $\int_{\Gamma} \rho d = 0$ must be satisfied. This is however not our case as we consider small perturbations of an admissible flow. Namely, as a consequence of the fact that $d$ is close to $\vec{V} \cdot n$ on the boundary, the velocity does not vanish hence the continuity equation can be solved with the method of characteristics.

The rest of the paper is organized as follows. In Section 2 we introduce the perturbations as unknown variables obtaining the system (2.8). Next we introduce the Lagrangian-type coordinates that lead to the system (2.23) and we derive the necessary estimates for the Lagrangian transformation. In Section 3 we deal with the linearization of (2.23). For the linear system we show the estimate in $W^2_p(\Omega) \times W^1_p(\Omega)$. It is given by Theorem 2. The first step is the energy estimate (3.2). Then we consider the vorticity of the velocity and the Helmholtz decomposition to reduce the continuity equation to a sort of transport equation (3.26) that enables us to find the bound on $\|w\|_{W^2_1(\Omega)}$. This result together with the properties of the Lamé system lets us conclude Theorem 2. In the second part of this section we apply the estimates to solve the linear system and hence show that $T$ given by (3.32) is well defined. In Section 4 we show the contraction principle for $T$. To this end we consider the system for the difference of two solutions and write it in a form (4.1) which has a structure of (3.1). The contraction results from the estimate (3.34) and bounds on the norms on of the r.h.s. of the system for the difference. At the end of Section 4 we apply the Banach fixed point theorem to solve the system (2.23) and conclude the proof of Theorem 1.

Let us finish this introductory part with some remarks concerning notation. By $C$ we shall denote a constant that is controlled, but not necessarily small. $E$ shall denote a constant that can be arbitrarily small provided the data is small enough. Sometimes we will write $E(\cdot)$ to underline that we need the smallness of certain quantity. The functional spaces on $\Omega$ will be denoted without the symbol of the set, for example we will write $W^1_0$ instead of $W^1_0(\Omega)$ for standard Sobolev spaces of functions integrable with the $p$-th power with derivatives up to order $k$, $W^{1-1/p}_p(\partial \Omega)$ denotes the Slobodeckij spaces, defining regularity of traces from $W^1_p(\Omega)$ [1]. Please note that to define trace spaces it is enough to consider domains with Lipschitz boundaries, hence in particular for our boundary $\Gamma$ such spaces are well defined. Finally, we will need to consider the density in the space $L_\infty(0, L; L_2(\Omega_0))$. For simplicity we denote it as $L_\infty(L_2)$. We do not use different notation for scalar and vector valued functions, while matrix valued functions are written in bolded font. The coordinates of a vector are denoted by $(\cdot)$, i.e. $u = (u^{(1)}, u^{(2)}, u^{(3)})$. 

2. Preliminaries

In this section we introduce perturbations of the Poiseuille flow \((\bar{V}, \bar{\rho})\) as unknown variables, what leads to the system (2.8). Then we introduce a change of variables that straightens the characteristics of the continuity equation. We obtain the system (2.23). The simplified form of the continuity equation in this Lagrangian framework makes it possible to apply the Banach fixed point theorem to the system (2.23).

2.1. Reformulation of the problem

We come back to the main system (1.1). Since we are interested in solutions that are small perturbations of \((\bar{V}, \bar{\rho})\), it is convenient to consider the perturbations as unknown functions. For technical reasons it is better to have \(u \cdot n = 0\) on the boundary, where \(u\) is the perturbation of the velocity. Hence we start introducing \(u_0 \in W^2_p(\Omega)\) such that \(u_0 \cdot n|_\Gamma = d - \bar{d}\) (recall that \(\bar{d} = \bar{V} \cdot n\)). It can be found as

\[
\phi = \nabla \phi, \quad (2.1)
\]

where \(\phi\) solves a Neumann problem:

\[
\Delta \phi = \begin{cases} \frac{1}{|\Omega|} \int_{\Gamma} d - \bar{d} = \text{const}, & \frac{\partial \phi}{\partial n} \bigg|_{\Gamma} = d - \bar{d}. 
\end{cases} \quad (2.2)
\]

The elliptic regularity estimate in our cylindrical domain for the above problem can be obtained using symmetry arguments similar to the proof of Lemma 11. It yields

\[
\|u_0\|_{W^2_p(\Omega)} \leq E(D_0), \quad (2.3)
\]

where \(D_0\) is defined in (1.13). We assume that \(\|d - \bar{d}\|_{W^{1/p-1/p}_p(\Gamma)}\) is small enough for

\[
V^P + u_0^{(1)}|_{\Omega} \geq \theta_1 \quad (2.4)
\]

to hold for some \(\theta_1 > 0\). In fact this is not really a restriction as we consider small perturbations of \((\bar{V}, \bar{\rho})\) and in Lemma 1 we have shown that \(V^P\) is separated from zero. Now we take

\[
u = v - \bar{V} - u_0. \quad (2.5)
\]

In particular we want our perturbed flow \(v\) to have the first component also separated from zero. This is quite natural constraint if we consider small perturbations of \(\bar{V}\). With the above definition of \(\bar{V}\) this constraint reads

\[
V^P + u^{(1)} + u_0^{(1)} \geq \theta_2 > 0. \quad (2.6)
\]

Notice that (2.6) involves not only the assumption on smallness of the data but also a restriction on the size of the solution. The latter however depends on the size of the data, hence (2.6) will be satisfied for \(D_0\) sufficiently small.

Next we introduce the perturbation of the density (recall that \(\bar{\rho} \equiv 1\)):

\[
\rho = \rho - 1. \quad (2.7)
\]

Substituting (2.5) and (2.7) to (1.1) and (1.12) we arrive at

\[
\begin{align*}
\nu \cdot \nabla \bar{V} + V^P \partial_{x_1} u - \mu \Delta u - (\mu + v) \nabla \text{div} u + \gamma \nabla w + \omega(f) \hat{e}_1 w = F(u, w), \\
V^P \partial_{x_1} w + \nabla \text{div} u + (u + u_0) \cdot \nabla w = G(u, w), \\
n \cdot 2 \mu D(u) \cdot \tau_k + f(u \cdot \tau_k)|_{\Gamma} = B_k, \\
n \cdot u|_{\Gamma} = 0, \quad w|_{\Gamma_{in}} = w_{in},
\end{align*} \quad (2.8)
\]

where \(\gamma = \pi'(1)\) and

\[
F(u, w) = (u + u_0) \cdot \nabla (u + u_0) - u_0 \cdot \nabla \bar{V} - V^P \partial_{x_1} u_0 - \left[\pi'(w + 1) - \pi'(1)\right] \nabla w + (w + 1)(F + \omega(f) \hat{e}_1) - w(u + u_0 + \bar{V}) \cdot \nabla (u + u_0 + \bar{V}).
\]
Lemma 2. Let the following lemma.

\[ G(u, w) = -w \, \text{div} \, u - (w + 1) \, \text{div} u, \]
\[ B_k = b_k - n \cdot 2\mu \mathbf{D}(\bar{V} + u_0) \cdot \tau_k - f(\bar{V} + u_0) \cdot \tau_k. \]

We no longer have \( B_k|_{\Gamma_0} \equiv 0 \). However, the geometry of \( \Omega \), the condition \( u_0 \cdot n|_{\Gamma_0} = 0 \) and assumption \( d - \bar{a} \in W_p^{2-1/p}(\Gamma) \) imply that \( n \cdot \mathbf{D}(u_0) \cdot \tau_1 = 0 \) on \( (\bar{\Gamma}_\text{in} \cap \bar{\Gamma}_0) \cup (\bar{\Gamma}_\text{out} \cap \bar{\Gamma}_0) - n \) and \( \tau_1 \) are normal vectors to these curves. Hence

\[ B_1 = 0 \quad \text{on} \quad (\bar{\Gamma}_\text{in} \cap \bar{\Gamma}_0) \cup (\bar{\Gamma}_\text{out} \cap \bar{\Gamma}_0). \]  

(2.9)

This observation will be important in the proof of the regularity for the Lamé system in Lemma 11. From now on we focus on the system (2.8).

Notice that \( F \) and \( G \) contain only the terms which are quadratic or cubic w.r.t. \( u \) and \( w \), linear terms multiplied by small quantities and terms independent on \( u \) and \( w \) which depend only on the data. In particular \( \bar{V} \cdot \nabla \bar{V} = 0 \), hence the term \(-w(u + u_0 + \bar{V}) \cdot \nabla(u + u_0 + \bar{V}) \) is a higher order term and so the form of \( F \) and \( G \) implies immediately the following lemma.

Lemma 2. Let \( F(u, w) \) and \( G(u, w) \) be given as above, then

\[
\| F(u, w) \|_{L_p} + \| G(u, w) \|_{W_p^1} \\
\lesssim C \left[ \left( \| u \|_{W_p^2} + \| w \|_{W_p^1} \right)^3 + \left( \| u \|_{W_p^2} + \| w \|_{W_p^1} \right)^2 + E \left( \| u \|_{W_p^2} + \| w \|_{W_p^1} \right) + D_0 \right],
\]

(2.10)

where \( E \) denotes a small, compared to \( \mu \), positive constant.

2.2. Change of variables

With our smallness assumptions it is quite natural to solve (2.8) with a fixed point argument. However, a direct application of this method fails because of the nonlinear term \( u \cdot \nabla w \) in the hyperbolic continuity equation. The idea to overcome this problem is to introduce a change of variables such that this awkward term vanishes. We look for the appropriate transformation as \( x = \psi_{\bar{u} + u_0}(z) \) satisfying the identity

\[
\partial_z x_1 = \partial_x x_1 + \frac{(u + u_0)^{(2)}}{V_P + (u + u_0)^{(1)}} \partial_x x_2 + \frac{(u + u_0)^{(3)}}{V_P + (u + u_0)^{(1)}} \partial_x x_3.
\]

(2.11)

In the following lemma we construct the mapping \( \psi_{\bar{u}} \) for arbitrary function \( \bar{u} \) small in \( W_p^2 \) with vanishing normal component on the boundary \( \Gamma_0 \).

Lemma 3. Let \( \| \bar{u} \|_{W_p^2} \) be small enough and \( \bar{u} \cdot n|_{\Gamma_0} = 0 \). Then there exists a diffeomorphism \( x = \psi_{\bar{u}}(z) \) defined on \( \Omega \) such that \( \Omega = \psi_{\bar{u}}(\Omega) \) and (2.11) holds with \( u + u_0 = \bar{u} \).

Proof. A key point in the proof is the fact that by (1.6) we have \( V_P \geq c > 0 \) for some given constant \( c \) (recall that we consider a normalized profile with \( TF = 1 \)). Hence for \( \| \bar{u} \|_{W_p^2} \) small enough by the imbedding \( W_p^1 \in L_\infty \) we have

\[
V_P + \bar{u}^{(1)} \geq \theta_3 > 0.
\]

(2.12)

In particular we are able to divide (2.11) by \( V_P + \bar{u}^{(1)} \) obtaining

\[
\partial_{\bar{z}_1} = \partial_{x_1} + \bar{u}^{(2)} \partial_{x_2} + \bar{u}^{(3)} \partial_{x_3},
\]

where \( \bar{u} = \frac{\bar{u}}{V_P + \bar{u}^{(1)}}. \) Since \( \bar{u}, V_P \in W_p^2 \) we have \( \bar{u} \in W_p^2 \) and, by (2.12) we can assume that

\[
\| \bar{u} \|_{W_p^2} \ll 1.
\]

(2.13)

Now we can follow the proof from [19, Lemma 7] and look for \( \psi(z_1, z_2, z_3) = \psi_{z_2, z_3}(z_1) \), where for each \((z_2, z_3) \in \Gamma_0\) the function \( \psi_{z_2, z_3} \) is a solution to
due to (2.13) we solve (2.14) for \((z_2, z_3) \in \Gamma_0\) following [19] and show that there exists a set \(\Omega_{\tilde{u}}\) such that \(\psi(\Omega_{\tilde{u}}) \rightarrow \Omega\) is a diffeomorphism. It remains to show that \(\Omega_{\tilde{u}} = \Omega\). To this end we examine the derivatives of \(\psi\). We have \(D\psi = \text{Id} + E\), where

\[
E = \begin{bmatrix}
0 & \tilde{u}^{(2)}(\psi(z)) & \tilde{u}^{(3)}(\psi(z)) \\
\tilde{u}^{(2)}(\psi(z)) & \int_0^{z_1} \tilde{u}^{(2)}(\psi(s, \tilde{z}))ds & \int_0^{z_1} \tilde{u}^{(3)}(\psi(s, \tilde{z}))ds \\
\tilde{u}^{(3)}(\psi(z)) & \int_0^{z_1} \tilde{u}^{(3)}(\psi(s, \tilde{z}))ds & \int_0^{z_1} \tilde{u}^{(3)}(\psi(s, \tilde{z}))ds
\end{bmatrix}
\]

(2.15)

and \(\tilde{z} := (z_2, z_3)\). Hence

\[
D\psi([1, 0, 0]) = \left[1, \tilde{u}^{(2)}(\psi(z)), \tilde{u}^{(3)}(\psi(z))\right].
\]

Now take \(x_n \in \Omega, x_n \rightarrow x_0 \in \Gamma_0\) and \(z_n = \phi(x_n)\) where \(\phi = \psi^{-1}\). Then by the continuity of \(D\psi\) we have

\[
\lim_{n \rightarrow \infty} D\psi(z_n)([1, 0, 0]) = [1, \tilde{u}^{(2)}(x_0), \tilde{u}^{(3)}(x_0)]
\]

The latter is parallel to \(\Gamma_0\) since \(\tilde{u} \cdot n|_{\Gamma_0} = 0\) and \(n^{(1)}|_{\Gamma_0} = 0\), hence we have \(\phi(\Gamma_0) = \Gamma_0\) (precisely we say it in a sense of tangent spaces since \(\phi\) is defined only on \(\Omega\).

To examine the behavior of tangent vectors on \(\Gamma_{out}\) notice that

\[
D\psi(z)([0, \tau_1, \tau_2]) = \left([0, \tilde{\tau}_1(\psi(z)), \tilde{\tau}_2(\psi(z))\right],
\]

where \(\tilde{\tau}_i\) are given by appropriate entries of \(D\psi\), the important fact is that the first coordinate vanishes. Hence if we take \(x_n \in \Omega, x_n \rightarrow x_0\) and \(z_n = \phi(x_n)\), this time with \(x_0 \in \Gamma_{out}\), then

\[
\lim_{n \rightarrow \infty} D\psi(z_n)([0, \tau_1, \tau_2]) = [0, \tilde{\tau}_1(x_0), \tilde{\tau}_2(x_0)]
\]

what is parallel to \(\Gamma_{out}\). The same argument shows that \(\psi(\Omega_{x_1})\) is parallel to \(\Omega_{x_1}\) for and \(x_1 \in (0, L)\), where \(\Omega_{x_1}\) is an \(x_1\)-cut of \(\Omega\), i.e. a set \(\{x_1\} \times \Gamma_{int}\). But on the other hand \(\phi(\Gamma_0)\) is parallel to \(\Gamma_0\) and, by the definition of \(\phi\) we have \(\phi(\Gamma_{in}) = \Gamma_{in}\). We conclude that \(\phi\) conserves any \(x_1\)-cut of \(\Omega\) and \(\phi(\Gamma_{out}) = \Gamma_{out}\). Taking all above into account we see that \(\Omega_{\tilde{u}} = \Omega\), which completes the proof. \(\square\)

The next lemma gives \(L_p\) estimates on the derivatives of \(\psi\) and \(\phi = \psi^{-1}\).

**Lemma 4.** We have

\[
\sum_i \left\| \frac{\partial \psi^{(i)}}{\partial z_i} - 1 \right\|_{L_p} + \sum_{i \neq j} \left\| \frac{\partial \psi^{(i)}}{\partial z_j} \right\|_{L_p} \leq E,
\]

(2.16)

\[
\| D^2 \psi \|_{L_p} \leq E,
\]

(2.17)

\[
\sum_i \left\| \frac{\partial \phi^{(i)}}{\partial x_i} - 1 \right\|_{L_p} + \sum_{i \neq j} \left\| \frac{\partial \phi^{(i)}}{\partial x_j} \right\|_{L_p} \leq E,
\]

(2.18)

\[
\| D^2 \phi \|_{L_p} \leq E.
\]

(2.19)

where \(E = E(\|\tilde{u}\|_{W^2_p})\) can be arbitrarily small for \(\|\tilde{u}\|_{W^2_p}\) small enough.

**Proof.** The core of the proof is in the imbedding \(W^1_p \subset L_{\infty}\). We start with (2.16). We estimate \(L_p\) norm of the entries of \(E\) (2.15). This result quite directly from the form of \(E\), but needs certain attention as \(E\) depends on \(\psi\) implicitly. The entries of \(E\) without integrals will be small provided that \(\psi\) is bounded what obviously holds true. For the entries involving integrals we change the order of integration and derivative obtaining
\[
\partial_{z_i} \psi^{(j)} = \partial_{z_i} \int_0^{z_1} \tilde{u}^{(j)}(\psi(s, \tilde{z})) \, ds = \int_0^{z_1} \left[ \nabla_x \tilde{u}^{(j)}(\psi(s, \tilde{z})) \cdot \partial_{z_i} \psi(s, \tilde{z}) \right] \, ds, \quad i, j = 2, 3.
\]

By Jensen’s inequality we have
\[
\left| \partial_{z_i} \psi^{(j)} \right|^p = \left| z_1 \right|^p \left| \int_0^{z_1} \nabla_x \tilde{u}^{(j)}(\psi(s, \tilde{z})) \cdot \partial_{z_i} \psi(s, \tilde{z}) \, ds \right|^p \leq \left| z_1 \right|^{p-1} \int_0^{z_1} \left| \nabla_x \tilde{u}^{(j)}(\psi(s, \tilde{z})) \cdot \partial_{z_i} \psi(s, \tilde{z}) \right|^p \, ds \leq \left| z_1 \right|^{p-1} \left\| \nabla_x \tilde{u} \right\|_{L^\infty}^p \int_0^{z_1} \left| \partial_{z_i} \psi^{(k)}(s, \tilde{z}) \right|^p \, ds.
\]

Integrating the last inequality over \( \Omega \) we get
\[
\| E_{ij} \|_{L^p} \leq C \left( 1 + \sum_{k,l} \| E_{kl} \|_{L^p} \right) \left\| \nabla_x \tilde{u} \right\|_{L^\infty}.
\] (2.20)

The smallness of \( \tilde{u} \) in \( W^2_p \) and the imbedding \( W^1_p \subset L^\infty \) gives (2.16).

To show (2.17) we differentiate the entries of \( \tilde{E} \), let us focus on entries with integrals. We have (we omit the sum over \( k \)):
\[
\partial_{z_i} \partial_{z_l} \psi^{(j)} = \partial_{z_l} \int_0^{z_1} \left[ \partial_{z_k} \tilde{u}^{(j)}(\psi(s, \tilde{z})) \partial_{z_l} \psi^{(k)}(s, \tilde{z}) \right] \, ds
\]
\[
= \int_0^{z_1} \partial_{z_l} \left[ \partial_{z_k} \tilde{u}^{(j)}(\psi(s, \tilde{z})) \right] \partial_{z_l} \psi^{(k)}(s, \tilde{z}) \, ds + \int_0^{z_1} \partial_{z_k} \tilde{u}^{(j)}(\psi(s, \tilde{z})) \partial_{z_l} \partial_{z_k} \psi^{(k)}(s, \tilde{z}) \, ds =: I_1 + I_2.
\]

Again by Jensen’s inequality,
\[
|I_1|^p = \left| \int_0^{z_1} \partial_{z_m} \partial_{z_k} \tilde{u}^{(j)}(\psi(s, \tilde{z})) \partial_{z_l} \psi^{(m)}(s, \tilde{z}) \partial_{z_l} \psi^{(k)}(s, \tilde{z}) \, ds \right|^p \leq \left\| \nabla_{\tilde{z}} \psi \right\|_{L^\infty}^2 |z_1|^{p-1} \int_0^{z_1} \left| \partial_{z_m} \partial_{z_k} \tilde{u}^{(j)}(s, \tilde{z}) \right|^p \, ds \] (2.21)

and
\[
|I_2|^p \leq \left\| \nabla_x \tilde{u} \right\|_{L^\infty} \int_0^{z_1} \left| \partial_{z_l} \partial_{z_k} \psi^{(k)}(s, \tilde{z}) \right|^p \, ds.
\]

Integrating the above inequalities over \( \Omega \) we arrive at
\[
\left\| \nabla^2 \psi \right\|_{L^p} \leq C(\Omega) \left[ \left\| \nabla_x \tilde{u} \right\|_{L^\infty} \left\| \nabla^2 \psi \right\|_{L^p} + \left\| \nabla_x^2 \tilde{u} \right\|_{L^p} \right] \] (2.22)

where the term \( \left\| \nabla_{\tilde{z}} \psi \right\|_{L^\infty}^2 \) from (2.21) has been put into the constant. Like in the previous estimate, the imbedding \( W^1_p \subset L^\infty \) and the smallness of \( \tilde{u} \) in \( W^2_p \) yield (2.17).

To show (2.18) note that the smallness of \( \tilde{E} \) given by (2.16) combined with the imbedding \( W^1_p \subset L^\infty \) implies that \( \det D\psi \geq c > 0 \), hence \( D\psi \) is invertible and we have
\[
D\varphi = D\psi^{-1} = \mathrm{Id} + \tilde{E},
\]
where the elements of \( \tilde{E} \) can be explicitly computed in terms of \( E \). The smallness of \( \tilde{u} \) together with the fact that \( W^1_p \) is an algebra implies smallness in \( L^p \) of the entries of \( \tilde{E} \), which gives (2.18). Finally (2.19) is obtained by taking derivatives if \( D\varphi \) similarly to (2.17). \( \Box \)
Now we proceed with transformation of the system (2.8). As a vector field \( \tilde{u} \) satisfying the assumptions of Lemma 3 we take \( u + u_0 \) where \( u \) is the solution of (2.8). So far we do not know if this solution exists, our goal is to show its existence. Hence our approach can be regarded as working in a kind of Lagrangian coordinates [15]; assuming that the solution exists we rewrite the system in the new variables \( \psi_{u+u_0} \) induced by \( u + u_0 \) through (2.11). Then in the new coordinates we hope to be able to apply a fixed point method to show the existence of a solution. Since \((u, w)\) are perturbations that are assumed to be small, we can assume that \( \| u \|_{W^2_p} \) is small enough that the assumptions of Lemma 3 are satisfied. Hence the solution gives a well defined transformation \( x = \psi_{u+u_0}(z) \), that we denote for simplicity by \( \psi \). If we show in addition its uniqueness in a class of small perturbations, then denoting \( \phi = \psi^{-1} \) we have \( \psi(\Omega_u) = \Omega \) and we come back to the original coordinates where our solution solves (2.8). Rewriting the system (2.8) in coordinates \( z \) yields

\[
\begin{align*}
\tilde{F}(u, w) &= F(u, w) - u \cdot R(\tilde{V}, \nabla) - V^P R(u, \partial_{x_i}) + \mu R(u, \Delta) + (\mu + v)R(u, \nabla \text{div}) - \gamma R(w, \nabla) \\
\tilde{G}(u, w) &= G(u, w) - R(u, \text{div}).
\end{align*}
\]

(2.24) and (2.25)

Here the first variable in the commutator \( R(\cdot, \cdot) \) denotes a function and the second is a differential operator. For example, \( R(w, \text{div}) := \nabla_x w - \nabla_z w \) and its \( i \)-th coordinate reads

\[
R(i)(w, \nabla)(z) = (\partial_{x_i} w - \partial_{z_i} w)(\phi(x)) = \left[ \partial_{z_i} w(\phi^{(i)} - 1) + \sum_{j \neq i} w_{z_j} \phi^{(j)} \right](x).
\]

We shall not give here precise formulas for the other commutators. Instead, we are now ready to give some heuristic arguments that will show what regularity we expect from the change of variables \( \phi \). To this end note that the commutators of the operators of order \( k \) depend on the derivatives of \( \phi \) up to order \( k \). More precisely, commutators of order one contain only components of the form

\[
\nabla_z u \cdot \nabla_x \phi,
\]

while the second-order commutators contain the terms \( \nabla^2_z u \cdot (\nabla_x \phi)^2 \), \( \nabla_z u \cdot \nabla^2_x \phi \) and the terms of lower order. Hence in order to find the estimates on \( \| \tilde{F}(u, w) \|_{L_p} \) and \( \| \tilde{G}(u, w) \|_{W^1_p} \) we need

\[
\| \nabla_x \phi \|_{L_{\infty}}, \quad \| \nabla^2_x \phi \|_{L_p},
\]

what will be satisfied provided that \( \phi \in W^2_p \) due to the imbedding \( W^1_p \subset L_{\infty} \) (recall that \( p > 3 \)). We should also note that, for simplicity of notation, the functions \( F(u, w) \) and \( G(u, w) \) in (2.24) and (2.25) denote exactly the same quantities as before. Hence we should keep in mind that they contain differential operators and so now they also contain some commutators that we will have to control to repeat the estimate (2.10). To this end we will apply (2.16)–(2.19).

Now we are ready to show the basic estimate on the r.h.s. of (2.23):

**Lemma 5.** Let \( \tilde{F} \) and \( \tilde{G} \) be defined by (2.24) and (2.25). Then we have

\[
\begin{align*}
\| \tilde{F}(u, w) \|_{L_p} + \| \tilde{G}(u, w) \|_{W^1_p} &\leq C \left[ (\| u \|_{W^2_p} + \| w \|_{W^1_p})^3 + (\| u \|_{W^2_p} + \| w \|_{W^1_p})^2 + D_0 \right] + E(\| u \|_{W^2_p} + \| w \|_{W^1_p}).
\end{align*}
\]

(2.26)
**Proof.** The bound on \(\|F(u, w)\|_{L^p} + \|G(u, w)\|_{W^1_p}\) results from (2.10) in Lemma 2 (there are also some commutators since \(F\) and \(G\) involve differential operators, but these can be estimated as follows). We briefly justify the bounds on the commutators in \(\tilde{F}\). To start with, the first-order commutators contain the given function \(\tilde{V}\), the functions \(u\) and \(w\) and the derivatives of \(\phi\), but only of the form
\[
\phi_x^{(i)}, \quad i \neq j \quad \text{and} \quad \phi_x^{(i)} = 1. \tag{2.27}
\]
Hence applying Lemma 4 we get
\[
\|u \cdot \nabla (\tilde{V}, \nabla)\|_{L^p} + \|\nabla P(u, \partial_x)\|_{L^p} + \|R(w, \nabla)\|_{L^p} \leq E\left(\|u\|_{W^2_p} + \|w\|_{W^1_p}\right). \tag{2.28}
\]
The second-order commutators contain the second-order derivatives of \(\phi\) and the first-order derivatives only of the form (2.27). Hence the application of Lemma 4 yields
\[
\|R(u, \Delta)\|_{L^p} + \|R(u, \nabla \nabla)\|_{L^p} \leq E\|u\|_{W^2_p} \tag{2.29}
\]
and we conclude the bound on \(\|\tilde{F}\|_{L^p}\). In order to estimate \(\|\tilde{G}(u, w)\|_{W^1_p}\) we differentiate the commutator
\[
R(u, \nabla) = \sum_i \left[u_x^{(i)}(\phi_x^{(i)} - 1) + \sum_{i \neq j} u_x^{(i)} \phi_x^{(j)} + u_x^{(i)} \phi_x^{(j)} \psi_{xj} + \sum_{i \neq j} u_x^{(i)} \phi_x^{(j)} \psi_{xj} \right].
\]
what yields
\[
\partial_{\gamma k} R(u, \nabla) = \sum_i \left[u_x^{(i)} \phi_x^{(i)} - 1 + u_x^{(i)} \phi_x^{(j)} \psi_{xj} + \sum_{i \neq j} u_x^{(i)} \phi_x^{(j)} \psi_{xj} \right].
\]
Applying again Lemma 4 we get
\[
\|R(u, \nabla)\|_{W^1_p} \leq E\|u\|_{W^2_p} \tag{2.30}
\]
and the proof is complete. \(\Box\)

From now on we focus on the system (2.23) instead of (2.8). It is of crucial importance for us that we can solve (2.23) in the fixed domain \(\Omega\), what results from our choice of the transformation \(\psi\) (2.11).

Now we are in a position to define the operator
\[
T : W^2_p(\Omega) \times W^1_p(\Omega) \to W^2_p(\Omega) \times W^1_p(\Omega), \tag{2.31}
\]
to which we want to apply the Banach fixed point theorem in order to solve the system (2.23). Namely, we set \((u, w) = T(u^*, w^*)\) if
\[
\begin{align*}
\quad u \cdot \nabla \tilde{V} + (V^p \circ \psi_{u^*+u_0}) \partial_{x_k} u - \mu \Delta u - (\mu + v) \nabla \gamma \nabla \gamma w &= \tilde{F}(u^*, w^*), \\
((V^p + (u^* + u_0)^{1}) \circ \psi_{u^*+u_0}) \partial_{x_k} w + \nabla \phi \phi \gamma w &= \tilde{G}(u^*, w^*), \\
n \cdot 2 \mu D_x (u) \cdot \tau_k + f(u \cdot \tau_k) |_{\gamma} &= B_k, \\
n \cdot u |_{\gamma} &= 0, \quad w|_{\gamma_{in}} = w_{in}. \tag{2.32}
\end{align*}
\]
The point is that the term \(V^p \partial_{x_k} w + (u^* + u_0) \cdot \nabla w\) is replaced by \((V^p + (u^* + u_0)^{1}) \circ \psi_{u^*+u_0}) \partial_{x_k} w\), and for this term we find a bound in \(W^1_p\), what is necessary to show the contraction property of \(T\).

3. **A priori bounds and solution of the linear system**

In this section we deal with the linear system:
\[
\begin{align*}
\quad u \cdot \nabla \tilde{V} + (V^p \circ \psi_{u}) \partial_{x_k} u - \mu \Delta u - (\mu + v) \nabla \gamma \nabla \gamma w &= F, \\
(V^p + \tilde{u}^{1}) \circ \psi_{u} \partial_{x_k} w + \nabla \phi \phi \gamma w &= G, \\
n \cdot 2 \mu D_x (u) \cdot \tau_k + f(u \cdot \tau_k) |_{\gamma} &= B_k, \\
n \cdot u |_{\gamma} &= 0, \quad w|_{\gamma_{in}} = w_{in}. \tag{3.1}
\end{align*}
\]
with given functions $F$, $G$, $B_k$, $\bar{u} \in V^* \times L_2 \times L_2(\Gamma) \times W^2_p$. Note that in (3.1) $\bar{u}$ replaces $u^* + u_0$ from (2.32) where $u^*$ was a given function small in $W^2_p$ and $u_0$ was the extension of the boundary data of the original problem introduced in (2.1)–(2.2). Throughout this section $\bar{u}$ should be understood as a given function which by (2.3) can be assumed small in $W^2_p$.

Note that we take the superposition of $V^P$ and $\psi \bar{u}$ to obtain $V^P \circ \psi_{\bar{u}}$ for the original system and hence $V^P$ when we pass to the original system of coordinates. The same remark concerns the other functions in (3.1), but it does not change anything in the computations due to Lemma 4. We need to solve the system (3.1) to show that $T$ is well defined by (2.32). To this end we need the appropriate estimates that we show in the first part of this section. In the second part the linear system is solved.

### 3.1. A priori bounds

In this section we show the estimate in $W^2_p \times W^1_p$ for the solution of the linear system (3.1) in the maximal regularity regime. The first step is the energy estimate. It is given by the following

**Lemma 6.** Let $(u, w)$ be a solution to the system (3.1) with given $(F, G, B_k, \bar{u}) \in V^* \times L_2 \times L_2(\Gamma) \times W^2_p$, where $\bar{u}$ is small enough for (2.12) to hold. Assume that the friction $f$ is large enough on $\Gamma_0$ and the viscosity $\mu$ and the friction on $\Gamma_0$ satisfy (1.14). Then

$$
\|u\|_{W^2_p} + \|w\|_{L^\infty(\Omega)} \leq C \left[ \|F\|_{V^*} + \|G\|_{L_2} + \|B_k\|_{L_2(\Gamma)} + \|w_0\|_{L_2(\Gamma_0)} \right],
$$

(3.2)

where

$$
V = \{ v \in W^1_p(\Omega): v \cdot n|_{\Gamma} = 0 \}
$$

(3.3)

and $V^*$ is the dual space of $V$.

**Proof.** We start with two basic observations. First, since $V^*_1 \equiv 0$, by (2.14) we have

$$
\partial_{\bar{e}_1}(V^P \circ \psi_{\bar{u}}) = \frac{1}{V^P + \bar{u}(\mu)} [\bar{u}(2) \partial_{\bar{e}_2} V^P + \bar{u}(3) \partial_{\bar{e}_3} V^P].
$$

Now by the imbedding $W^1_p \subset L_\infty$ and (1.7) we have

$$
\|\partial_{\bar{e}_1}(V^P \circ \psi_{\bar{u}})\|_{L_\infty} \leq \frac{1}{\theta_3} \|\bar{u}\|_{L_\infty} \|\nabla V^P\|_{L_\infty} \leq C_{\mu, f} \|\bar{u}\|_{L_\infty} =: E(\bar{u}),
$$

(3.4)

where $\theta_3$ is the constant from (2.12) and

$$
C_{\mu, f} = \frac{C_{\omega(f)}}{\mu \theta_3},
$$

(3.5)

where $C$ is the constant from (1.7). Let us comment on the estimate (3.4). For given viscosity and $f|_{\Gamma_0}$ we have a given profile $V^P$ normalized by the condition $TF = 1$, hence the constant $\theta$ from (1.6) is also determined. Then for a given $\bar{u}$ small in $W^2_p$ the constant $\theta_3$ from (2.12) is also given, and so we have a given constant $C_{\mu, f}$. Now the required smallness of the r.h.s. of (3.4) determines the admissible magnitude of perturbation measured by $\|\bar{u}\|_{L_\infty}$.

In the remaining of this section we will write $V^P$ instead of $V^P \circ \psi_{\bar{u}}$. The fact that we consider the superposition does not influence the computations as we have (3.4). We apply the identities

$$
\int_{\Omega} V^P \partial_{\bar{e}_1}|u|^2 \, dz = \frac{1}{2} \int_{\Omega} V^P |u|^2 n^{(1)} \, d\sigma - \int_{\Omega} |u|^2 \partial_{\bar{e}_1} V^P \, dz
$$

(3.6)

and

$$
\int_{\Omega} (-\mu \Delta u - (\nu + \mu) \nabla \div u) \cdot v \, dz = \int_{\Omega} 2\mu \mathbf{D}(u) : \nabla v + \nu \div u \div v \, dz
$$

$$
- \int_{\Gamma} n \cdot [2\mu \mathbf{D}(u)] : v \, d\sigma - \int_{\Gamma} n \cdot [\nu(\div u) \mathbf{Id}] : v \, d\sigma.
$$

(3.7)
Now we multiply (3.1) by \( u \) and integrate. Using the above identities, with application of the boundary conditions (3.1) we arrive at

\[
\int_{\Omega} \left[ 2\mu \mathbf{D}(u) : \mathbf{D}(u) + \nu |\text{div}\,u|^2 \right] dz + \int_{\Gamma_{in}} \left( f - \frac{V^P}{2} \right) |u|^2 d\sigma + \int_{\Gamma_{out}} \left( \frac{f}{2} + V^P \right) |u|^2 d\sigma
\]

\[
- \gamma \int_{\Omega} w \text{div}\,u \, dz + \int_{\Omega} (u \cdot \nabla \bar{V}) \cdot u \, dz + \int_{\Omega} w \omega(f)u^{(1)} \, dz
\]

\[
= \int_{\Omega} F \cdot u \, dz + \int_{\Gamma} \{ B_1(u \cdot \tau_1) + B_2(u \cdot \tau_2) \} d\sigma + \int_{\Omega} |u|^2 \partial_{z_1} V^P d\sigma.
\]

(3.8)

Note that by (3.4) we have

\[
\left| \int_{\Omega} |u|^2 \partial_{z_1} V^P d\sigma \right| \leq E(\bar{u})\|u\|_{L^2}^2.
\]

The other terms on the r.h.s. are all ‘good’ terms. The \( \Gamma_{out} \) term on the l.h.s. is nonnegative and the \( \Gamma_{in} \) term will be positive for \( f \) large enough. To deal with the term \( \int_{\Omega} w \text{div}\,u \, dx \) we apply the continuity equation to express \( \text{div}\,u \) obtaining:

\[
- \int w \text{div}\,u \, dz = - \int Gw \, dz - \frac{1}{2} \int w^2 \partial_{z_1} (V^P + \bar{u}^{(1)}) \, dz
\]

\[
+ \frac{1}{2} \int \gamma \partial_{z_1} (V^P + \bar{u}^{(1)}) d\sigma - \frac{1}{2} \int \gamma \partial_{z_1} (V^P + \bar{u}^{(1)}) d\sigma.
\]

(3.9)

By (2.12) the integral over \( \Gamma_{out} \) will be nonnegative. To derive the \( W_1^2 \)-norm of \( u \) we apply the Korn inequality [22,26]:

\[
\int_{\Omega} \left[ 2\mu \mathbf{D}(u) : \mathbf{D}(u) + \nu |\text{div}\,u|^2 \right] dz + \int_{\Gamma_{in}} f (u \cdot \tau)^2 d\sigma \geq C_K \|u\|_{W_1^2}^2,
\]

(3.10)

where \( C_K = C_K(\mu, \nu, f, \Omega) \) and \( C_K \) is increasing with \( \mu \). A sketch of the proof of (3.10) one can find in Appendix A, note that in (3.10) we use only information at \( \Gamma_{in} \), a part of the boundary, but still it is sufficient to control the whole norm of \( W_1^2 \).

Combining (3.8), (3.9) and (3.10) we get

\[
C_K \|u\|_{W_2^1}^2 \leq \|G\|_{L^2} \|w\|_{L^2} + C \|u_{in}\|^2_{L^2(\Gamma_{in})} + \|F\|_{V^*} + \|B\|_{L^2(\Gamma)} \|u\|_{W_1^2}
\]

\[
+ E(\bar{u})\|w\|_{L^2}^2 + \|u\|_{L^2}^2 - \int_{\Omega} (u \cdot \nabla \bar{V}) \cdot u \, dz - \int_{\Omega} w \omega(f)u^{(1)} \, dz.
\]

(3.11)

We have to deal with the last two terms on the r.h.s. It is impossible to show they have a good sign, hence the only way is to estimate it directly with using the assumptions on the viscosity. We have

\[
\left| \int_{\Omega} (u \cdot \nabla \bar{V}) \cdot u \, dz \right| \leq C_P^2 \|\nabla \bar{V}\|_{L^\infty} \|u\|_{W_1^2}^2.
\]

(3.12)

where \( C_P \) is the constant from the Poincaré inequality in \( V \).

In the last term we apply the Cauchy inequality with \( \epsilon = \frac{C_K}{2\mu C_P} \), where \( C_K \) is the constant from (3.10). We get

\[
\left| \int_{\Omega} w \omega(f) \sqrt{\mu}u^{(1)} \, dz \right| \leq \frac{C_K}{2} \|u\|_{W_2^1}^2 + \frac{\mu C_P^2 \omega(f)}{2C_K \sqrt{\mu}} \|w\|_{L^2}^2.
\]

(3.13)
Now we have to verify that for given $\delta = \frac{C_K}{2\mu C_P}$ as a given constant (which does not enforce additional smallness assumptions).

Inserting (3.12) and (3.13) to (3.11) we get
\[
\frac{C_K}{2} \geq C_P^2 \left( \| \nabla \bar{V} \|_{L^\infty} + E(\bar{u}) \right) \| u \|^2_{W^1_2} \leq \left( E + C \frac{\omega(f)}{\sqrt{\mu}} \right) \| w \|^2_{L^2_2} + C \left( \| G \|_{L^2_2} + \| w_{\text{in}} \|_{L^2_2(\Gamma_{\text{in}})} \right) + \left( \| F \|_{V^*} + \| B \|_{L^2_2(\Gamma)} \right) \| u \|^2_{W^1_2}.
\] (3.14)

Now we have to verify that for given $\mu$ and $f|_{\Gamma_0}$, the constant on the l.h.s. of (3.14) will be positive provided the magnitude of the perturbation measured by $\| \tilde{u} \|_{W^1_2}$ will be sufficiently small. To this end we recall that $E$ is a small constant and $C$ is a data-dependent constant, not necessarily small. Now, $C_K$ is increasing with $\mu$, while $C_P$ does not depend on $\mu$. Moreover, as we already noticed, $E(\bar{u})$ can be made arbitrarily small taking $\| \tilde{u} \|_{W^1_2}$ small enough. Finally, (1.10) implies that for small values of the friction $f$ on $\Gamma_0$ it is enough to assume that the viscosity is large only compared to $f|_{\Gamma_0}$ to control $\| \nabla \bar{V} \|_{L^\infty}$. We conclude that the constant on the l.h.s. of (3.14) will be positive provided that $\mu$ and $f|_{\Gamma_0}$ satisfies (1.14) and $\| \tilde{u} \|_{W^1_2}$ is small enough. Note that constant by $\| w \|_{L^2_2}$ on the r.h.s. will also be small under the assumption (1.14). Here it is a good point to emphasize the necessity of sufficient magnitude of the viscosity coefficient for large $f$ at $\Gamma_0$. This assumption is somehow natural, although in the case $V^P = \text{const}$ it is not required [19]. We have to control (3.12)–(3.13), and largeness of dissipation may only give us this chance. In other words, we can expect only a laminar flow to be stable, and laminarity is described by relation (1.14) (recall also the discussion after formulation of Theorem 1).

Note that for the Dirichlet boundary condition [8], although the constant flow is considered, such an assumption is required, too.

To complete the proof of (3.2) we find a bound on $\| w \|_{L^\infty_2(L^2_2)}$. To this end notice that $w = S(G - \text{div} u)$, where $S : L^2_2(\Omega) \to L^\infty_2(L^2_2)$ is defined as
\[
w = S(v) \iff \begin{cases} (V^P + \bar{u}(1)) \partial_{\chi_1} w = v & \text{in } D'(\Omega), \\ w = w_{\text{in}} & \text{on } \Gamma_{\text{in}} \end{cases}
\] (3.15)

and satisfies the estimate
\[
\| S(v) \|_{L^\infty_2(L^2_2)} \leq C \left( \| w_{\text{in}} \|_{L^2_2(\Gamma_{\text{in}})} + \| v \|_{L^2_2} \right).
\] (3.16)

The construction of $S$ is quite straightforward. For a continuous $v$ we set
\[
S(v)(z) = w_{\text{in}}(0, z_2, z_3) + \int_0^{z_1} \frac{v}{V^P + \bar{u}(1)}(s, z_2, z_3) \, ds.
\] (3.17)

Next we show directly the estimate (3.16), which enables us to extend $S$ on $L^2_2(\Omega)$ using a standard density argument. Now (3.16) yields
\[
\| w \|_{L^\infty_2(L^2_2)} \leq C \left( \| G \|_{L^2_2} + \| u \|_{W^1_2^2} + \| w_{\text{in}} \|_{L^2_2(\Gamma_{\text{in}})} \right).
\] (3.18)

Combining (3.14) and (3.18) we conclude (3.2). \qed

In the next step we show higher bound on the vorticity of the velocity. To this end we take the vorticity of (3.1)1. Denoting $\alpha = \text{rot} u$ we get
\[
-\mu \Delta \alpha = \text{rot} \left[ F - V^P \partial_{\chi_1} u - u \cdot \nabla \bar{V} \right] \quad \text{in } \Omega,
\]
\[
\alpha \cdot \tau_2 = \left( 2\chi_1 - \frac{f}{v} \right) u \cdot \tau_1 + \frac{B_1}{v} \quad \text{on } \Gamma,
\]
\[
\alpha \cdot \tau_1 = \left( \frac{f}{v} - 2\chi_2 \right) u \cdot \tau_2 - \frac{B_2}{v} \quad \text{on } \Gamma,
\]
\[
\text{div } \alpha = 0 \quad \text{on } \Gamma,
\] (3.19)
where \( \chi_i \) is the curvature of the boundary in direction \( \tau_i \). The boundary conditions (3.19)2,3 are derived from differentiation of (3.1)4 in tangential directions and application of (3.1)3, see \([14,18]\). The above system gives the estimate \([26]\), Theorem 10.3 for unweighted spaces):

\[
\| \alpha \|_{W^1_p} \leq C \left[ \| F \|_{L_p} + \| \nabla \|_{W^1_{2\infty}} \| u \|_{W^1_p} + \| u \|_{W^{1-1/p}_{2\infty}(\Gamma')} + \| B \|_{W^{1-1/p}_{2\infty}(\Gamma')} \right] \\
\leq C \left[ \| F \|_{L_p} + \| B \|_{W^{1-1/p}_{2\infty}(\Gamma')} + \| u \|_{W^1_p} \right].
\]

Applying the interpolation inequality (A.7) to \( \| u \|_{W^1_p} \) and the energy estimate (3.2) we get

\[
\| \alpha \|_{W^1_p} \leq C(\epsilon) \left[ \| F \|_{L_p} + \| G \|_{W^1_p} + \| B \|_{W^{1-1/p}_{2\infty}(\Gamma')} + \| w_{\text{in}} \|_{W^1_p(\Gamma_{\text{in}})} \right] + \epsilon \| u \|_{W^2_p} \quad (3.20)
\]

for any \( \epsilon > 0 \). Now consider the Helmholtz decomposition of the velocity

\[
u = \nabla \phi + A, \quad (3.21)
\]

where \( \frac{\partial \phi}{\partial \nu} |_{\Gamma} = 0 \) and \( \text{div} \ A = 0 \). We see that the field \( A \) satisfies the system

\[
\begin{align*}
\text{rot} \ A &= \alpha \quad \text{in} \ \Omega, \\
\text{div} \ A &= 0 \quad \text{in} \ \Omega, \\
A \cdot n &= 0 \quad \text{on} \ \Gamma. \quad (3.22)
\end{align*}
\]

This is the classical rot–div system and from \([22]\) we have \( \| A \|_{W^2_p} \leq C \| \alpha \|_{W^1_p} \), what by (3.20) can be rewritten as

\[
\| A \|_{W^2_p} \leq C(\epsilon) \left[ \| F \|_{L_p} + \| G \|_{W^1_p} + \| B \|_{W^{1-1/p}_{2\infty}(\Gamma')} + \| w_{\text{in}} \|_{W^1_p(\Gamma_{\text{in}})} \right] + \epsilon \| u \|_{W^2_p} \quad (3.23)
\]

for any \( \epsilon > 0 \). Now we substitute the Helmholtz decomposition to (3.1)1. We get

\[
\nabla \left[ -(v + 2\mu) \Delta \phi + \gamma w \right] \\
= F - V^P \partial_{z_1} A + \mu \Delta A + (v + \mu) \nabla \text{div} A - A \cdot \nabla \nabla - V^P \partial_{z_1} \nabla \phi - \nabla \phi \cdot \nabla \nabla =: \tilde{F} \quad (3.24)
\]

Since \( \Delta \phi = \text{div} \ u \), we can write

\[
-(v + 2\mu) \text{div} u + \gamma w = \tilde{H} \quad (3.25)
\]

We underline that we are now at the level of a priori estimates and (3.25) should be treated as the definition of \( \tilde{H} \). In fact we can think of \( \tilde{H} \) as a kind of effective viscous flux like in the theory of weak solutions to compressible Navier–Stokes equations \([6,11]\). Now (3.24) can be rewritten as \( \nabla \tilde{H} = \tilde{F} \). Combining the last equation with (3.1)2 we arrive at

\[
\tilde{\gamma} w + \left( V^P + \tilde{u}^{(1)} \right) \partial_{z_1} w = H, \quad (3.26)
\]

where \( \tilde{\gamma} = \frac{\gamma}{v + 2\mu} \) and

\[
H = \frac{\tilde{H}}{v + 2\mu} + G. \quad (3.27)
\]

Eq. (3.26) makes it possible to estimate \( \| w \|_{W^1_p} \) and \( \| \partial_{z_1} w \|_{W^1_p} \) in terms of \( H \). Next we can find the bound on \( H \) using interpolation and the energy estimate. The first step is in the following lemma:

**Lemma 7.** Let \( w \) solve (3.26) with \( H \in W^1_p \) and \( w|_{\Gamma_{\text{in}}} = w_{\text{in}} \). Then

\[
\| w \|_{W^1_p} + \| \partial_{z_1} w \|_{W^1_p} \leq C \left[ \| H \|_{W^1_p} + \| w_{\text{in}} \|_{W^1_p(\Gamma_{\text{in}})} \right]. \quad (3.28)
\]
The last term on the r.h.s. can be estimated by $\|\nabla\|_{H}$ and hence $2$ is complete.

To apply this method to $\partial z_i w$ we differentiate (3.26) with respect to $z_i$. If we assume that $w \in W^1_1$ then (3.26) implies $\partial z_i w \in W^1_1$, since $W^1_p$ is an algebra. Thus we differentiate (3.26) with respect to $z_i$, multiply by $|\partial z_i w|^{p-2} \partial z_i w$ and integrate. We have

$$\int_{\Omega} (V^p + \bar{u}^{(1)}) |\partial z_i w|^{p-2} \partial z_i w \partial z_i w \, dz$$

$$= \frac{1}{p} \int_{\Omega} (V^p + \bar{u}^{(1)}) \partial z_i |\partial z_i w|^p \, dz$$

$$= \int_{\Gamma} (V^p + \bar{u}^{(1)}) |\partial z_i w|^p d\sigma - \int_{\Omega} |\partial z_i w|^p \partial z_i (V^p + \bar{u}^{(1)}).$$

The boundary term on the l.h.s. is positive due to (2.12) and the constant in the last term on the r.h.s. is small (note that we take only the $z_1$ derivative of $V^p$). Hence the above implies

$$\|w\|_{L_p} \leq C \left[\|H\|_{L_p} + \|w_{in}\|_{L_p(\Gamma_{in})}\right].$$

In order to find a bound on $\partial z_i w$ we differentiate (3.26) with respect to $z_i$. To find a bound on $\partial z_i w_{in}$ we need some knowledge on $\partial z_i w_{in}$. To this end we can use (3.26), which, since $V^p + \bar{u}^{(1)} > 0$, can be rewritten on $\Gamma_{in}$ as

$$\partial z_i w_{in} = \frac{H - \bar{w}_{in}}{V^p + \bar{u}^{(1)}}.$$

Hence $\|\partial z_i w\|_{L_p(\Gamma_{in})} \leq C[\|H\|_{L_p(\Gamma_{in})} + \|w_{in}\|_{W^1_p(\Gamma_{in})}]$, and (3.31) implies (3.32) also for $i = 1$. From (3.29) and (3.32) we conclude

$$\|w\|_{W^1_p} \leq C \left[\|H\|_{L_p} + \|w_{in}\|_{L_p(\Gamma_{in})}\right].$$

The bound on $\|\partial z_i w\|_{W^1_p}$ results simply from the identity (3.26) and the fact that $W^1_p$ is an algebra. The proof of (3.28) is complete. \(\square\)

Now we need to find the bound on $\|H\|_{W^1_p}$, but this is straightforward. Interpolation inequality (A.7) yields

$$\|H\|_{L_p} \leq \delta \|\nabla H\|_{L_p} + C(\delta) \|H\|_{L^2},$$

for any $\delta > 0$. To estimate $\|H\|_{L^2}$ we use the fact that $\vec{H} = -(v + 2\mu) \text{div } u + \gamma w$ and the energy estimate (3.2). To find the bound on $\|\nabla H\|_{L_p}$ we use (3.23), (3.24), then (A.7) to estimate the term $\|u\|_{W^1_p}$ and finally (3.2). We obtain
Lemma 11 applied to the above system yields
\[ \| H \|_{W^2_p} \leq C(\delta) \left[ \| F \|_{L_p} + \| G \|_{W^1_p} + \| B \|_{W^{1-1/p}_p(\Gamma)} + \| w_{in} \|_{W^1_p(\Gamma_{in})} \right] + \delta \| u \|_{W^2_p}. \] (3.33)

We are now one short step from the main result of this section. It is given by the following:

**Theorem 2.** Let \((u, w)\) be a solution to (3.1) with \((F, G, B_k, w_{in}, \bar{u}) \in L_p \times W^1_p \times W^{1-1/p}_p(\Gamma) \times W^1_p(\Gamma_{in}) \times W^2_p\) such that \(\| \bar{u} \|_{W^2_p}\) is small enough. Assume that the friction \(f\) is large enough on \(\Gamma_{in}\) and the viscosity and the friction on \(\Gamma_0\) satisfy (1.14). Then
\[ \| u \|_{W^2_p} + \| w \|_{W^1_p} + \| \partial_z w \|_{W^1_p} \leq C \left[ \| F \|_{L_p} + \| G \|_{W^1_p} + \| B_k \|_{W^{1-1/p}_p(\Gamma_{in})} + \| w \|_{W^1_p(\Gamma_{in})} \right]. \] (3.34)

**Proof.** To close the estimate (3.34) it remains to find the bound on \(\| u \|_{W^2_p}\). To this end notice that in particular \(u\) satisfies the Lamé system:
\[ \begin{align*}
-\mu \Delta u - (v + \mu) \nabla u &= F - \gamma \nabla w - V^P \partial_z u - u \cdot \nabla \bar{V} & \text{in } \Omega, \\
n \cdot 2\mu \mathbf{D}(u) \cdot \tau_i + f u \cdot \tau_i &= B_i, & \text{on } \Gamma_i, \\
n \cdot u &= 0 & \text{on } \Gamma.
\end{align*} \] (3.35)

Lemma 11 applied to the above system yields
\[ \| u \|_{W^2_p} \leq C \left[ \| F \|_{L_p} + \| w \|_{W^1_p} + \| B \|_{W^{1-1/p}_p(\Gamma)} + \| u \|_{W^1_p} \right]. \]

Applying the interpolation inequality (A.7) to the term \(\| u \|_{W^2_p}\) and then the energy estimate (3.2) we get
\[ \| u \|_{W^2_p} \leq C \left[ \| F \|_{L_p} + \| G \|_{W^1_p} + \| w \|_{W^1_p} + \| B \|_{W^{1-1/p}_p(\Gamma)} + \| w_{in} \|_{L_2(\Gamma_{in})} \right]. \] (3.36)

Combining this estimate with (3.28) and (3.33) with appropriate \(\delta\) we conclude (3.34). \(\square\)

Note that Theorem 2 gives more than we need to solve (3.1), namely the bound in \(W^1_p\) of \(\partial_z w\). We will use this result to show the contraction property for the operator \(T\) in Section 4.

### 3.2. Solution of the linear system

With the estimates that we obtained we are ready to solve the system (3.1). First we define the weak solution and show its existence. Next, applying the estimate (3.34) we show its regularity under the appropriate regularity of the data.

#### 3.2.1. Weak solution

By the weak solution to (3.1) we mean a couple \((u, w) \in V \times L_\infty(L_2)\) such that
\[ \int_{\Omega} \left\{ v \cdot \left( V^P \partial_z u + u \cdot \nabla \bar{V} \right) + 2\mu \mathbf{D}(u) : \nabla v + \nu \nabla u \nabla v - \gamma w \nabla v \right\} dz + \int_{\Gamma} f(u \cdot \tau_i)(v \cdot \tau_i) d\sigma \]
\[ = \int_{\Omega} F \cdot v dz + \int_{\Gamma} B_i(v \cdot \tau_i) d\sigma \] (3.37)
is satisfied \(\forall v \in V\) and (3.1)_2 is satisfied in \(D'(\Omega)\), i.e. for all \(\phi \in \tilde{C}^\infty(\Omega), \phi|_{\Gamma_{out}} = 0:\)
\[ -\int_{\Omega} \left( V^P + \bar{u}^{(1)} \right) w \partial_z \phi d\sigma - \int_{\Omega} \partial_z \left( V^P + \bar{u}^{(1)} \right) w \phi d\sigma = \int_{\Omega} \phi(G - \nabla u) d\sigma + \int_{\Gamma_{in}} \left( V^P + \bar{u}^{(1)} \right) w_{in} \phi d\sigma. \] (3.38)

To find the weak solution we apply the Galerkin method. Hence we introduce an orthonormal basis of \(\omega_k \subset V\) and finite dimensional spaces \(V_N = \sum_{i=1}^N \alpha_i \omega_i: \alpha_i \in \mathbb{R}) \subset V\). We look for the approximations of the velocity of the form \(u^N = \sum_{i=1}^N c_i^N \omega_i\). Taking into account the continuity equation we have to define the approximations of the density in an appropriate way. Namely, we set \(w^N = S(G^N - \nabla u^N)\), where \(S\) is defined in (3.15).
Now we proceed with the Galerkin scheme. Taking \( F = F^N \), \( u = u^N = \sum_i c_i^N \omega_i \), \( v = \omega_k \), \( k = 1 \ldots N \) and \( w = w^N = S(G^N - \text{div} \ u^N) \) in (3.37), where \( F^N \) and \( G^N \) are orthogonal projections of \( F \) and \( G \) on \( V^N \), we arrive at a system of \( N \) equations
\[
B^N(u^N, \omega_k) = 0, \quad k = 1 \ldots N,
\]
where \( B^N : V^N \to V^N \) is defined as
\[
B^N(\xi^N, v^N) = \int_\Omega \{ v^N V P \partial_z \xi^N + \xi^N \cdot \nabla \overline{V} + 2\mu D(\xi^N) : \nabla v^N + \text{div} \xi^N \text{div} v^N \} \, dz - \gamma \int_\Omega S(G^N - \text{div} \xi^N) \text{div} v^N \, dz + \int_\Gamma [ f (\xi^N \cdot \tau_j) - B_j](v^N \cdot \tau_j) \, d\sigma - \int_\Omega F^N \cdot v^N \, dz.
\]
Now, if \( u^N \) satisfies (3.39) for \( k = 1 \ldots N \) and \( w^N = S(G^N - \text{div} u^N) \), then a pair \((u^N, w^N)\) satisfies (3.37)–(3.38) for \((v, \phi) \in (V^N \times C^\infty(\Omega))\), \( \phi|_{\Gamma_{\text{out}}} = 0 \). We will call such a pair an approximate solution to (3.37)–(3.38). To solve the system (3.39) we apply the following well-known result (the proof can be found in [24]):

**Lemma 8.** Let \( X \) be a finite dimensional Hilbert space and let \( P : X \to X \) be a continuous operator satisfying
\[
\exists M > 0: \quad (P(\xi), \xi) > 0 \quad \text{for} \quad \|\xi\| = M.
\]
Then there is at least one \( \xi^* \) such that \( \|\xi^*\| \leq M \) and \( P(\xi^*) = 0 \).

We define \( P^N : V^N \to V^N \) as
\[
P^N(\xi^N) = \sum_k B^N(\xi^N, \omega_k) \omega_k \quad \text{for} \quad \xi^N \in V^N.
\]
In order to apply Lemma 8 we show that \( (P(\xi^N), \xi^N) > 0 \) on some sphere in \( V^N \) with radius dependent on the norms of the data. To this end we follow the proof of the energy estimate for (3.1). This is in fact a standard approach in the Galerkin method: the energy estimate combined with Lemma 8 gives the existence of the approximate solutions, hence we skip the details here. Except from the existence of the approximate solution \( u^N \), Lemma 8 gives the estimate
\[
\|u^N\|_{W^1_2} \leq C(\text{DATA}),
\]
which combined with (3.16) gives
\[
\|u^N\|_{W^1_2} + \|w^N\|_{L_\infty(L^2)} \leq C(\text{DATA}).
\]
Thus
\[
u^N \rightharpoonup u \quad \text{in} \quad W^1_2 \quad \text{and} \quad w^N \rightharpoonup w \quad \text{in} \quad L_\infty(L^2)
\]
for some \((u, w) \in W^1_2 \times L_\infty(L^2)\). We easily verify that \((u, w)\) is a weak solution. First, passing to the limit in (3.37) for \((u^N, w^N)\) we see that \( u \) satisfies (3.37) with \( w \). On the other hand, taking the limit in (3.38) we verify that \( w = S(G - \text{div} u) \). We conclude that \((u, w)\) satisfies (3.37)–(3.38), thus we have the weak solution. The boundary condition on \( w \) is guaranteed to hold by the definition of \( S(3.15) \). This completes the proof of existence of weak solution.

### 3.2.2. Strong solution

The following result gives strong solution to the linear system (3.1) for the data of appropriate regularity.

**Theorem 3.** Let \((F, G, B_k, w_{in}, \tilde{u}) \in (L_p \times W^1_p \times W^{1-1/p}(\Omega) \times W^1_p(\Gamma_{in}) \times W^2_p)\) with \( \|\tilde{u}\|_p \) small enough. Assume further that \( f \) is large enough on \( \Gamma_{in} \) and \( \mu \) and \( f \) fulfills (1.14). Then there exists a unique solution \((u, w) \in W^2_p \times W^1_p\) to the system (3.1) and the estimate (3.34) holds.
Proof. To show appropriate regularity of the weak solution for the regular data it is enough to apply the estimate (3.34) provided that we handle the singularities of the boundary at the junctions of the wall $\Gamma_0$ with inlet $\Gamma_{i0}$ and outlet $\Gamma_{o1}$. To this end we apply the result on the elliptic regularity of the Lamé system with slip boundary conditions, Lemma 11 in Appendix A. Notice that we can apply this method since we work in the fixed domain $\Omega$ due to the appropriate choice of the change of variables. Otherwise we would end up in a free boundary problem and the solution of the linear system would be highly nontrivial. □

4. Contraction

In this section we show the contraction property for the operator $T$ defined in (2.32). However, first of all we notice that Theorem 3 gives the solution of the linear system (3.1) provided that $\|\bar{u}\|_{W_p^2}$ is small enough, and so this constraint must hold if we want to have $T(\bar{u}, \bar{w})$ well defined. We start this section with showing a stronger result, namely that $T : B_R \to B_R$ for some ball $B_R \in W_p^2 \times W_p^1$ with $R$ depending on the data, which can be arbitrarily small for the data small enough. This property will also be needed to show the contraction. Eventually, we prove Theorem 1, the main result of the paper.

Lemma 9. There exists $R > 0$ depending on the size of the data measured by $D_0$ (1.13) such that, provided the data is small enough, $T(B_R) \subset B_R$, where $B_R$ is a ball of radius $R$ in $W_p^2 \times W_p^1$.

Proof. The estimates (2.26) and (3.34) imply that
\[
\|T(\bar{u}, \bar{w})\|_{W_p^2 \times W_p^1} \leq C(\|\bar{u}\|_{W_p^2}^2 + \|\bar{w}\|_{W_p^1}^3) + \delta, \tag{4.1}
\]
where $\delta = E(D_0)$ can be arbitrarily small for $D_0$ defined in (1.13) small enough. Obviously we can set $C > 1$. Then assume the data measured by $D_0$ to be sufficiently small that $\delta < \frac{1}{8C}$. For such $\delta$ we have
\[
\|T(\bar{u}, \bar{w})\|_{W_p^2 \times W_p^1} < 4\delta \quad \Rightarrow \quad \|T(\bar{u}, \bar{w})\|_{W_p^2 \times W_p^1} < 4\delta.
\]
Hence $T(B_R) \subset B_R$. □

Now we show the contraction property for $T$. To this end consider $(u_1, w_1) = T(\bar{u}_1, \bar{w}_1), (u_2, w_2) = T(\bar{u}_2, \bar{w}_2)$. By the definition of $T(2.32)$, the difference $(u, \bar{w}) := (u_1 - u_2, w_1 - w_2)$ satisfies the system
\[
u \cdot \nabla_z \bar{F} + V^P \partial_{z_1} u - \mu \Delta_z u - (\mu + v) \nabla_z \nabla_z u + \gamma \nabla_z w = \bar{F}(\bar{u}_1, \bar{w}_1) - \bar{F}(\bar{u}_2, \bar{w}_2),
\]
\[
(V^P + \bar{u}_2^{(1)}) \partial_{z_1} w + \nabla_z u = \bar{G}(\bar{u}_1, \bar{w}_1) - \bar{G}(\bar{u}_2, \bar{w}_2) - (\bar{u}_1^{(1)} - \bar{u}_2^{(1)}) \partial_{z_1} w_1, \tag{4.2}
\]
\[
n \cdot 2\mu D(u) \cdot \tau_k + f(u \cdot \tau_k) = n \cdot 2\mu [R(u_2, \bar{D}) - R(u_1, \bar{D})] \cdot \tau_k, \quad u |_{\Gamma} = 0, \quad \bar{w} |_{\Gamma} = 0.
\]
In fact there are additional terms in (4.2) resulting from the fact that in the system $(u_1, w_1) = T(\bar{u}_1, \bar{w}_1)$ we take superposition with $\psi_{\bar{u}_1 + \delta 0}$, while in the system $(u_2, w_2) = T(\bar{u}_2, \bar{w}_2)$ with $\psi_{\bar{u}_2 + \delta 0}$. We omit these terms for the sake of simplicity, let us explain briefly how they are dealt with. A simple but representative example of such term is $(V^P \partial_{\bar{u}_1 + \delta 0} - V^P \partial_{\bar{u}_2 + \delta 0}) \partial_{z_1} w_1$ in (4.2). By the definition of $\psi$ the $L_p$ norm of this term can be estimated with $\|\bar{u}_1 - \bar{u}_2\|_{L^p_\Gamma} \|w_1\|_{L^p_\Gamma}$. The other terms of this kind can be estimated in a similar way.

In order to show the contraction property for $T$ we can apply (3.34) provided that we have good bounds on the r.h.s. of (4.1). A result we need is given by the following:

Lemma 10. We have
\[
\|\bar{F}(\bar{u}_1, \bar{w}_1) - \bar{F}(\bar{u}_2, \bar{w}_2)\|_{L_p} + \|\bar{G}(\bar{u}_1, \bar{w}_1) - \bar{G}(\bar{u}_2, \bar{w}_2)\|_{W_p^1}
\]
\[
+ \|\bar{u}_1^{(1)} - \bar{u}_2^{(1)}\|_{W_p^1} + \|n \cdot 2\mu [R(u_2, \bar{D}) - R(u_1, \bar{D})] \cdot \tau_k\|_{W_p^{1+1/p} \cap \Gamma})
\]
\[
\leq E(D_0, \|\bar{u}_1\|_{W_p^2}, \|\bar{w}_1\|_{W_p^1}) [\|\bar{u}_1 - \bar{u}_2\|_{W_p^2} + \|\bar{w}_1 - \bar{w}_2\|_{W_p^1} + \|u_1 - u_2\|_{W_p^1}]. \tag{4.2}
\]
**Proof.** We have

\[
\tilde{F}(\bar{u}_1, \bar{v}_1) - \tilde{F}(\bar{u}_2, \bar{v}_2) = \bar{u}_1 \cdot \nabla_x \bar{u}_1 - \bar{u}_2 \cdot \nabla_x \bar{u}_2 + (\bar{u}_1 - \bar{u}_2) \cdot \nabla_x u_0 + u_0 \cdot \nabla_x (\bar{u}_1 - \bar{u}_2) \\
+ \delta \pi'(\bar{w}_1) \nabla_x \bar{w}_1 - \delta \pi'(\bar{w}_2) \nabla_x \bar{w}_2 + (\bar{w}_1 - \bar{w}_2)(F + \omega(f)\hat{\epsilon}_1) + \tilde{R},
\]

where

\[
\delta \pi'(w_i) := \pi'(w_i + 1) - \pi'(w_i)
\]

and \(\tilde{R}\) denotes all the differences between corresponding commutators in \(\tilde{F}\). We can afford using such abbreviation and estimate \(\tilde{R}\) without any additional computation if we just notice that the commutators are linear with respect to the functions and hence if certain estimate in terms of the function holds for a commutator, then the same estimate in terms of the difference of the commutators, for example

\[
|R(u, \partial_1)| \leq E\|u\|_{W^p_\delta} \quad \Rightarrow \quad |R(u_1, \partial_1) - R(u_2, \partial_1)| \leq E\|u_1 - u_2\|_{W^p_\delta}.
\]

Applying this reasoning to all the commutators in \(\tilde{F}\) we conclude that

\[
\|
\]

We estimate the remaining parts. Obviously we have

\[
\leq E\|u_0, \phi_{u_0+u_0}\|_L^p \|\bar{u}_1 - \bar{u}_2\|_{W^p_\delta} \leq E\|u_0, \|\bar{w}_1\|_{L^p_\delta}, \|\bar{w}_2\|_{L^p_\delta}\|\bar{u}_1 - \bar{u}_2\|_{W^p_\delta},
\]

where \(E(\cdot, \cdot)\) depends also on \(\phi\) since we have commutators as the gradients w.r.t. \(x\). In the last step we used the fact that \(E(u_0) = E(D_0)\) and (2.18). A similar remark applies do the estimates below. Let us proceed with the remaining terms. A little bit closer examination shows that the \(i\)-th coordinate

\[
(u_1 \cdot \nabla x u_1 - u_2 \cdot \nabla x u_2)(i)
\]

\[
= (u_1 \cdot \nabla x u_1 - u_2 \cdot \nabla x u_2)(i)
\]

\[
= \sum_j \left[ (u_1 - u_2)(j) \partial_j \bar{u}_2 + R(u_1(j), \partial_j) \right] + \sum_j \left[ (u_1(j) \partial_j (u_1 - u_2)(j) + R((u_1 - u_2)(j), \partial_j)) \right],
\]

and so by a direct computation we get

\[
\|
\]

It remains to estimate

\[
\delta \pi'(\bar{w}_1) \nabla_x \bar{w}_1 - \delta \pi'(\bar{w}_2) \nabla_x \bar{w}_2
\]

\[
= \delta \pi'(\bar{w}_1) \nabla_x (\bar{w}_1 - \bar{w}_2) + \left[ \delta \pi'(\bar{w}_1) - \delta \pi'(\bar{w}_2) \right] \nabla_x \bar{w}_2
\]

\[
= \delta \pi'(\bar{w}_1) \left[ \nabla_x (\bar{w}_1 - \bar{w}_2) + R(\bar{w}_1 - \bar{w}_2, \nabla) \right] + \left[ \delta \pi'(\bar{w}_1) - \delta \pi'(\bar{w}_2) \right] \left[ \nabla_x \bar{w}_2 + R(\bar{w}_2, \nabla) \right].
\]

It follows easily that

\[
\|\delta \pi'(\bar{w}_1) \nabla_x \bar{w}_1 - \delta \pi'(\bar{w}_2) \nabla_x \bar{w}_2\|_L^p \leq E\|\bar{w}_1\|_{L^p_\delta}, \|\bar{w}_2\|_{L^p_\delta}\|\bar{w}_1 - \bar{w}_2\|_{L^p_\delta}.
\]

Combining (4.3), (4.4), (4.5) and (4.6) we conclude

\[
\|\tilde{F}(\bar{u}_1, \bar{v}_1) - \tilde{F}(\bar{u}_2, \bar{v}_2)\|_L^p \leq E\|\bar{w}_1\|_{L^p_\delta}, \|\bar{w}_2\|_{L^p_\delta}\|\bar{u}_1 - \bar{u}_2\|_{L^p_{\delta}} + \|\bar{v}_1 - \bar{v}_2\|_{L^p_{\delta}}.
\]

Now we estimate the difference in \(G\). We have

\[
\tilde{G}(\bar{u}_1, \bar{v}_1) - \tilde{G}(\bar{u}_2, \bar{v}_2)
\]

\[
= (\bar{w}_2 - \bar{w}_1) \operatorname{div} u_0 + \bar{v}_1 \operatorname{div}(\bar{u}_1 - \bar{u}_2) + \operatorname{div} \bar{u}_2(\bar{w}_1 - \bar{w}_2) + R(\bar{u}_2, \operatorname{div}) - R(\bar{u}_1, \operatorname{div}).
\]

The first term
\[
\| (\bar{u}_2 - \bar{w}_1) \, \text{div} \, u_0 \|_{W^1_p} \leq \| \nabla (\bar{u}_2 - \bar{w}_1) \, \text{div} \, u_0 \|_{L^p} + \| (\bar{u}_2 - \bar{w}_1) \nabla^2 u_0 \|_{L^p} \\
\leq \| \text{div} \, u_0 \|_{L^\infty} \| \bar{w}_1 - \bar{w}_2 \|_{W^1_p} + \| u_0 \|_{W^2_p} \| \bar{w}_1 - \bar{w}_2 \|_{L^\infty} \\
\leq C \| u_0 \|_{W^2_p} \| \bar{w}_1 - \bar{w}_2 \|_{W^1_p}.
\]

The second
\[
\| \bar{w}_1 \, \text{div} (\bar{u}_1 - \bar{u}_2) \|_{W^1_p} \leq \| (\nabla \bar{w}_1) \, \text{div} (\bar{u}_1 - \bar{u}_2) \|_{L^p} + \| w_1 \nabla^2 (\bar{u}_1 - \bar{u}_2) \|_{L^p} \leq C \| w_1 \|_{W^1_p} \| \bar{u}_1 - \bar{u}_2 \|_{W^1_p}.
\]

Similarly we show
\[
\| \text{div} \bar{u}_2 (\bar{w}_1 - \bar{w}_2) \|_{W^1_p} \leq C \| \bar{u}_2 \|_{W^2_p} \| \bar{w}_1 - \bar{w}_2 \|_{W^1_p}.
\]

Now we have to estimate the difference of the commutators in \( W^1_p \). It turns out to be straightforward as we have
\[
R(\bar{u}_1, \text{div}) - R(\bar{u}_2, \text{div}) = \sum_i \partial_{z_i} (\bar{u}_1 - \bar{u}_2)^{(i)} \phi_i^{(i)} + \sum_{i \neq j} \partial_{z_j} (\bar{u}_1 - \bar{u}_2)^{(i)} \phi_i^{(j)}
\]
and so, identically as in (2.30), we show that
\[
\| R(u_1, \text{div}) - R(u_2, \text{div}) \|_{W^1_p} \leq E(\| \bar{u}_i \|_{W^2_p}) \| u_1 - u_2 \|_{W^2_p}.
\]
Combining the above results we conclude
\[
\| \bar{G}(\bar{u}_1, \bar{w}_1) - \bar{G}(\bar{u}_2, \bar{w}_2) \|_{W^1_p} \leq \left[ E(\| \bar{u}_i \|_{W^2_p}, \| \bar{w}_i \|_{W^2_p}) \right] (\| \bar{u}_1 - \bar{u}_2 \|_{W^2_p} + \| \bar{w}_1 - \bar{w}_2 \|_{W^1_p}).
\] (4.8)
To treat the last term of the r.h.s. of (4.1) we observe that
\[
\| (\bar{u}_1^{(1)} - \bar{u}_2^{(1)}) \|_{W^1_p} \leq C \| \partial_{z_1} w_1 \|_{W^1_p} \| \bar{u}_1 - \bar{u}_2 \|_{W^2_p}.
\] (4.9)
It remains to estimate the boundary terms \( n \cdot [R(u_1, D) - R(u_2, D)] \cdot \tau_k \). To this end it is enough to notice that
\[
\{ R(u_1, D) - R(u_2, D) \}_{i,j} = R(\| u_1 - u_2 \|^{(i)}, \partial_{x_i}) - R(\| u_1 - u_2 \|^{(j)}, \partial_{x_j}) =: R_{i,j}.
\]
Applying the trace theorem and repeating the proof of (4.8) we can show that
\[
\| R_{i,j} \|_{W^{1-1/p(G)}_p} \leq C \| R_{i,j} \|_{W^1_p} \leq E(\phi) \| u_1 - u_2 \|_{W^2_p},
\]
and so
\[
\| n \cdot [R(u_1, D) - R(u_2, D)] \cdot \tau_k \|_{W^{1-1/p(G)}_p} \leq E(\| u_i \|_{W^2_p}) \| u_1 - u_2 \|_{W^2_p}.
\] (4.10)
Combining (4.7), (4.8), (4.9) and (4.10) we conclude (4.2). \( \square \)

**Proof of Theorem 1.** With the results of the previous section we can apply the Banach fixed point theorem to the operator \( T \). Namely, (3.34) applied to (4.1) together with (4.2) yields
\[
\| T(u_1, w_1) - T(u_2, w_2) \|_{W^{2,2}_p \times W^{1,1}_p} \leq \frac{E}{1-E} \| (u_1, w_1) - (u_2, w_2) \|_{W^{2,2}_p \times W^{1,1}_p},
\] (4.11)
where \( E \) is the constant from (4.2). We have \( \frac{E}{1-E} < 1 \) for \( (\bar{u}, \bar{w}) \in B(0, R) \in W^2_p \times W^1_p \) with \( R \) sufficiently small. Hence the Banach fixed point theorem gives existence of a unique fixed point within the ball \( B(0, R) \in W^2_p \times W^1_p \).

By Lemma 9 we have \( R = R(D_0) \) where \( D_0 \) is defined in (1.13). By the definition of \( T \), the fixed point \( (u, w) \) solves the system (2.23).

Now we recall Section 2 and conclude that the original coordinate system is \( x = \psi_{u+u_0}(z) \), and in the \( x \) variable our solution satisfies the system (2.8). It follows that \( v = \bar{v} + u + u_0 \) and \( \rho = w + 1 \) solves (1.1) and the estimate (1.15) holds. Theorem 1 is proved. \( \square \)
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Appendix A

Proof of Lemma 1. As explained in Section 1, we assume the pressure of the form \( \Pi = \omega x \). Then on each \( x_1 \)-cut of \( \Omega \) (i.e. on each set \( \Omega_0 \times \{ x_1 \} \) \( V^P \) can be found as a solution to the elliptic problem (1.9) which we recall here:

\[
\mu \Delta v = \Pi_{x_1} = \omega < 0 \quad \text{in } \Omega_0, \\
\mu \frac{\partial v}{\partial n} + f v = 0 \quad \text{on } \partial \Omega_0. 
\]

(A.1)

Testing (1.9) with \( v_\gamma = v \chi_{v<0} \) we get

\[
- \int_{\Omega} \mu |\nabla v_\gamma|^2 - \int_{\Omega} f v_\gamma = \int_{\Omega} \omega v_\gamma \geq 0. 
\]

(A.2)

The last inequality results from nonpositivity of \( v_\gamma \) and the direction of the flow which implies \( \omega < 0 \) (recall the remark after (1.9)). Clearly (A.2) implies \( v \geq 0 \). We want to show sharp inequality. To this end consider \( \tilde{v} \) satisfying

\[
\Delta \tilde{v} = \omega/\mu < 0 \quad \text{in } \Omega_0, \\
\tilde{v}|_{\partial \Omega_0} = 0 \quad \text{on } \partial \Omega_0.
\]

We have \( \tilde{v} \geq 0 \), and, by the maximum principle applied to \( v - \tilde{v} \) we get \( \inf_{\Omega_0} v = \inf_{\Gamma} v \). Assume that \( \inf_{\Gamma} v = v(x_0) = 0 \) for some \( x_0 \in \Gamma \). Then, since \( f \geq 0 \) and \( \frac{\partial \tilde{v}}{\partial n} \leq 0 \), we must have

\[
\frac{\partial (v - \tilde{v})}{\partial n}(x_0) = -\frac{\partial \tilde{v}}{\partial n}(x_0) - f(v(x_0)) \geq 0.
\]

But since \( v(x_0) = \inf_{\Omega_0} v = \inf_{\Gamma} v \), by the Hopf lemma we must have \( \frac{\partial (v - \tilde{v})}{\partial n}(x_0) < 0 \). The application of the Hopf lemma is possible since \( \Omega_0 \) is a \( C^2 \) subset of \( \mathbb{R}^2 \), hence \( v \) is a classical solution to (A.1), i.e. \( v \in C^2(\Omega) \cap C^1(\bar{\Omega}) \). We conclude that \( v \geq \theta > 0 \) on \( \Gamma \), hence \( v \geq \theta \) in \( \bar{\Omega} \). This completes the proof. \( \square \)

Lemma 11 (Lamé system with slip boundary conditions). Let \( \Omega = (0, L) \times \Omega_0 \) with \( \Omega_0 \subset \mathbb{R}^2 \) of class \( C^2 \). Let \( \mu > 0, v + 2\mu > 0, F \in L^p(\Omega) \) and \( B \in W^{1-1/p}_p(\Gamma) \). Then there exists \( u \in W^2_p(\Omega) \) solving

\[
-\mu \Delta u - (v + \mu)\nabla \text{div} u = F \quad \text{in } \Omega, \\
n \cdot 2\mu \mathbf{D}(u) \cdot \tau_i + fu \cdot \tau_i = B_i, \quad i = 1, 2 \quad \text{on } \Gamma, \\
n \cdot u = 0 \quad \text{on } \Gamma.
\]

(A.3)

Moreover, the following estimate holds:

\[
\|u\|_{W^2_p} \leq C \left[ \|F\|_{L^p} + \|B\|_{W^{1-1/p}_p(\Gamma)} \right]. 
\]

(A.4)

The construction is possible provided the natural compatibility condition on \( B \) at \( (\Gamma_{in} \cup \Gamma_{out}) \cap \tilde{\Gamma}_0 \).

Proof. First let us explain the compatibility condition. Let \( n, \tau_1 \) be normal vectors to the curve \( \tilde{\Gamma}_{in} \cap \tilde{\Gamma}_0 \), then the compatibility condition requires \( B \big|_{\tilde{\Gamma}_{in} \cap \tilde{\Gamma}_0} = 0 \). This constraint however holds as an elementary consequence of the geometry of \( \Gamma \) and condition (A.3) since \( B \in W^{1-1/p}_p(\Gamma) \). The same holds for \( \Gamma_{out} \cap \tilde{\Gamma}_0 \).

Under the assumptions on \( \mu \) and \( v \) (A.3) is elliptic so we easily get a weak solution. The only problem we encounter showing regularity of the weak solution under appropriate regularity of the data are the singularities of the boundary on the junctions of \( \Gamma_0 \) with \( \Gamma_{in} \) and \( \Gamma_{out} \). These can be dealt with using symmetry arguments. But first we have to reformulate slightly the system (A.3). Having the weak solution, and the bound in \( W^1_2 \) for the velocity, we are able to
consider only the case \( f \equiv 0 \) since the term \( f \mathbf{u} \cdot \mathbf{\tau} \) can be retrieved by interpolation. The next step is to have \( B = 0 \) on \( \Gamma_{in} \). For this purpose we construct extensions \( V^2, V^3 \in W^p_k \) such that

\[
V^k|_{\Gamma_{in}} = 0 \quad \text{and} \quad V^k_x|_{\Gamma_{in}} = B_{k-1}, \quad k = 2, 3.
\]

Then \( \tilde{u} = (u^{(1)}, u^{(2)} - V^2, u^{(3)} - V^3) \) fulfills the system (A.3) with a new r.h.s. \( \tilde{F} \in L_p \), but with

\[
\tilde{u}^{(1)}|_{\Gamma_{in}} = 0 \quad \text{and} \quad \tilde{B}_k|_{\Gamma_{in}} = 0.
\]

We have thus reduced the problem to the case \( f \equiv 0, B \equiv 0 \). Assume that \( \Gamma_{in} \subset \{ x_1 = 0 \} \), then we define the operator \( E_{as}^v \) extending a vector field defined for \( \{ x : x_1 \geq 0 \} \) on the whole space as

\[
E_{as}^v(u)(x) = \begin{cases} 
  u(x), & x_1 \geq 0, \\
  \tilde{u}(\tilde{x}), & x_1 < 0,
\end{cases} \tag{A.5}
\]

where \( \tilde{x} = (-x_1, x_2, x_3) \) and \( \tilde{u}(\tilde{x}) = [-u^{(1)}(x), u^2(x), u^{(3)}(x)] \). Then we have

\[
\Delta E_{as}^v(v) + \nabla \text{div} E_{as}^v(v) = \Delta v + \nabla \text{div} v,
\]

and on the plane \( x_1 = 0 \) the extension \( E_{as}^v \) preserves the slip boundary conditions for \( f \equiv 0 \) and \( B \equiv 0 \), since

\[
n \cdot D(E_{as}^v(v)) \cdot \tau|_{x_1=0} = \partial_{x_1} E_{as}^v(v)^{(1)} + \partial_{x_1} E_{as}^v(v)^{(i)} = 0
\]

and \( n \cdot E_{as}^v(v)|_{x_1=0} = 0, \partial_{x_1} E_{as}^v(v)^{(i)} = 0 \) for \( i = 2, 3 \).

In order to localize the system we are obliged to look at the behavior of boundary conditions at \( \Gamma_0 \) after the extension (A.5). Let us check it for a point \( x \in \Gamma_0 \) s.t. \( n(x) = \hat{\nu}_2 \). Then the conditions (A.3)\(_{2,3}\) read

\[
\begin{cases} 
  u_{x_1}^{(2)} + u_{x_2}^{(1)} = B_1, \\
  u_{x_3} + u_{x_2}^{(3)} = B_2, \\
  u^{(2)} = 0.
\end{cases}
\]

Hence we have

\[
\begin{cases} 
  (u^{(2)} - V^2)_{x_1} + u_{x_2}^{(1)} = B_1 - V^2_{x_1}, \\
  (u^{(2)} - V^2)_{x_3} + (u^{(3)} - V^3)_{x_2} = B_3 - V^2_{x_3} - V^3_{x_2}, \\
  u^{(2)} - V^2 = -V^2.
\end{cases}
\]

Now by the definition of \( V^2 \) we have \( B_1 - V^2_{x_1} \rightarrow 0 \) for \( x \in \Gamma_0, x \rightarrow \Gamma_{in} \). Hence the extension (A.5) keeps the continuity in the condition (A.3)\(_{2,1}\), \( i = 1 \). The continuity in the condition for \( i = 2 \) is kept anyway since the reflexion is symmetric in the tangential components on \( \Gamma_{in} \). Finally let us observe that by (2.9) we can differentiate \( u^{(2)} - V^2 \) w.r.t. \( x_1 \) after the extension and obtain the required \( W^{2-1/p}_p \) regularity since \( V^2_{x_1}|_{x_1=0} = 0 \).

Now we are allowed to localize the problem in a vicinity of \( \Gamma_{in} \) getting the full regularity of \( E_{as}^v(v) \), we apply here the theory for smooth domains [2,3].

Application of analogous antisymmetric extension on \( \Gamma_{out} \) completes the proof. \( \square \)

**Lemma 12 (Interpolation inequality).** For \( f \in W^1_p(\Omega), p > 3 \):

\[
\| f \|_{L_p} \leq \varepsilon \| \nabla f \|_{L_p} + C(\varepsilon, p, \Omega)\| f \|_{L_2}. \tag{A.7}
\]

**Proof.** The interpolation inequality in \( L_p \) [1, Theorem 2.11] and the imbedding \( W^1_p \subset L_\infty \) for \( p > 3 \) yields

\[
\| f \|_{L_p} \leq C(p)\| f \|_{L_\infty}^{\theta} \| f \|_{L_2}^{1-\theta} \leq C(p)\left( \| f \|_{L_p} + \| \nabla f \|_{L_p} \right)^\theta \| f \|_{L_2}^{1-\theta},
\]

what entails (A.7) after application of the Cauchy inequality. \( \square \)

**Lemma 13.** Let \( \Omega \) be bounded with sufficiently smooth boundary and \( \Gamma_{part} \) be an open regular subset of \( \partial \Omega \). Then

\[
\| u \|_{W^1_2} \leq C(\| D(u) \|_{L_2} + \| u \|_{\Gamma_{part}} \| L_2(\Gamma_{part})}) \tag{A.8}
\]

for \( u \in W^1_2(\Omega) \).
Proof. The known result based on properties of the kernel of $D(\cdot)$ \cite{[10,27]} yields
\[ \|u\|_{W^{1,2}_2} \leq C_1 \|D(u)\|_{L^2} + C_2 \|u\|_{L^2}. \] (A.9)
In order to obtain (A.8) we shall prove that
\[ \|u\|_{L^2} \leq \frac{1}{2C_2} \|\nabla u\|_{L^2} + M(\|D(u)\|_{L^2} + \|u|_{\Gamma_{\text{part}}|_{L^2(\Gamma_{\text{part}})}}). \] (A.10)
Compactness argument and features of $D$ imply (A.10). We use that fact that the only solution to the system $D(u^*) = 0$, $u|_{\Gamma_{\text{part}}} = 0$ is $u^* \equiv 0$. To show it we use the fact that if $D(u) = 0$ then
\[ u = A \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} + B \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} + C \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}. \] (A.11)
Hence $u$ is an affine map and it is easy to verify that if at least one of the coefficients $A$, $B$, $C$ is nonzero then the rank of matrix of $u$ is 2, hence $\text{Ker} u$ is a line. On the other hand, $\Gamma_{\text{part}}$ is a two dimensional submanifold as it is an open, regular subset of $\partial \Omega$. But $\Gamma_{\text{part}} \subset \text{Ker} u$, hence we conclude that $A = B = C = 0$. \qed

References