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A Note on a Model System with Sudden Directional Diffusion

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Abstract We study qualitative properties of solutions to a monodimensional problem

$$u_t - (u_x + \operatorname{sgn} u_x)_x = 0$$

with the Dirichlet boundary conditions. Such a system presents a key analytical challenge coming from the examination of models of anisotropic phenomena like crystal growth. Our analysis concentrates on the properties of facets—flat regions of solutions—typical for this type of problems.

Keywords Anisotropy · Parabolic systems · Sudden directional diffusion · Facets

1 Introduction

Anisotropy is a characteristic feature of a number of phenomena in natural sciences and engineering. Among the most spectacular examples are crystallization effects which by their nature give unusual pictures. We are able to find mathematical models which deliver ultimate challenges for mathematicians for coming decades, behind the mentioned above phenomena. Here, we want to analyze a model case of anisotropic systems arising from the theory of crystal growth, or more generally, from theories of phase transitions as well as from singular models of image processing. In descriptions of these systems, we find a special type of diffusion which distinguishes certain directions and creates strong nonlocal effects. We call this type of phenomenon the *sudden directional diffusion*, which is also known as 'singular diffusion', see e.g. [12].

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Materials science provides much motivation for studying equations like the weighted mean curvature flow, see the overview [7],

$$\beta V = -\operatorname{div}(\nabla_\zeta \gamma(\zeta)|_{\zeta=\mathbf{n}(x)}) \quad \text{on } \Gamma \subset \mathbb{R}^n. \tag{1.1}$$

Here, $\gamma : \mathbb{R}^n \to \mathbb{R}_+$ is the anisotropy function or the surface energy density, see details in [18].

More specifically, Angenent and Gurtin, [4], studied (1.1) with an additional forcing term on the r.h.s. in the context of thermodynamics of evolving interfaces, see also [15]. Equation (1.1) for graphs was derived by Spohn, [22], for the evolution of a relaxing crystal surface below the roughening temperature. These authors stressed the importance of singularities in γ leading to the appearance of terms like products of Dirac deltas in the equation.

Let us notice that if γ in (1.1) is the Euclidean norm, i.e. $\gamma(p) = \sqrt{p_1^2 + p_2^2}$, then this equation becomes the mean curvature flow,

$$\beta V = -\operatorname{div}_S \mathbf{n} \quad \text{on } \Gamma \subset \mathbb{R}^n. \tag{1.2}$$

If we write it for curves, which are level sets of a function u, then for a special choice of the kinetic coefficient β, we obtain the TV flow equation

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.3}$$

This problem has been studied quite extensively because of its relevance for the algorithms of image processing [2, 3, 21].

The TV flow in $\mathbb{R}^n$ with $n \geq 2$ is isotropic. Interestingly, the situation changes if $n = 1$, when u depends on a single real variable, because the one-dimensional equation (1.3) distinguishes the zero slope of u. Indeed, if we take $\gamma(p) = |p_1| + |p_2|$ and $\mathbf{n} = (-u_x, 1)/\sqrt{u_x^2 + 1}$ is the outer normal to the graph of u, then (1.1) becomes

$$u_t = \left(\frac{u_x}{|u_x|}\right)_x \tag{1.4}$$

for a proper choice of β. This equation, which is the 1-D version of (1.3), has been studied because of its relevance for the models of crystal growth, here u is the height of the crystal above the reference plane. This topic was studied in [9, 10, 12]; the last item is a review and introductory paper.

Despite the large body of the literature on (1.1), the theory is far from complete, even if we restrict our attention to planar curves. For the moment, we forget about any source of matter for the growing crystal. Two assumptions on γ are natural: γ is positive and one-homogeneous. A benign looking condition of convexity of γ is in fact a strong regularity assumption, without which we cannot proceed.

There is a literature dealing with (1.1) for special cases of anisotropy γ (called crystalline), starting with the pioneering papers by Angenent–Gurtin, [4], and J. Taylor, [23]. Various tools have been developed to study this problem. They include the variational approach, [5, 6, 9, 10], the viscosity methods, [8, 11] and more recently, the operator approach, [16, 19]. Moreover, in some cases a full problem coupled with the Laplace equation in the bulk is also studied, [13, 14].

Here, we focus our attention on a one-dimensional problem arising from (1.1), which can be written as follows,

$$\begin{aligned}
&u_t - \frac{d}{dx} L(u_x) = 0 && \text{in } I \times (0, T),\\
&u = D && \text{at } \partial I \times (0, T),\\
&u|_{t=0} = u_0 && \text{on } I,
\end{aligned} \tag{1.5}$$

where L is a multivalued maximal monotone graph. To be specific, we concentrate on the following choice of $L(\cdot)$

$$L(w) = \begin{cases} w+1 & \text{for } w>0, \\ [-1,1] & \text{for } w=0, \\ w-1 & \text{for } w<0, \end{cases} \quad \text{i.e.} \quad L(w) = w + \operatorname{sgn} w. \tag{1.6}$$

In (1.5) D denotes the boundary data, which in general can be time dependent, however, in most cases we consider constant (in time) D such that it generates a nontrivial equilibrium. Here, we will restrict our attention to a specific example with the key properties. We consider either finite or infinite interval I of $\mathbb{R}$.

Equation (1.5) was proposed by Spohn, [22], to describe the evaporation phenomena in crystal growth below the roughening temperature. In [22] the unknown u represents the height of crystal. The periodic boundary condition is imposed there, but this does seem an issue when we are interested in flat parts, called 'facets', where $u_x = 0$. More precisely, in general, facets have slopes corresponding to the jump discontinuities of L.

We present here the qualitative properties of solutions to (1.5). We stress that we relegate the technical issues of the existence of a solution to another paper, see [20]. A stationary version of (1.5) has been considered in [17]. We noticed here that there is a general existence theory based on the nonlinear semigroup theory (or the subdifferential theory), [9, 10, 12] or viscosity methods, [8]. Our theory is different from the preceding ones, because we get more regularity of solutions by assuming more regularity of the initial data. We also notice that for a class of initial conditions the problem is reduced to a free boundary problem as noted in [22]. Moreover, a self-similar shrinking solution is constructed there.

The most interesting phenomenon to study is the creation and evolution of facets, i.e. flat regions of the solutions, see [10, 12]. The example of L presented in (1.6) combines classical isotropic diffusion related to the heat equation, $u_t - u_{xx} = 0$, $L(w) = w$, and what we call the *sudden directional diffusion* associated to the total variation flow, (1.4), see [16]. We know that the one-dimensional total variation flow is responsible for creation and propagation of facets. Here, these are pieces of solutions with the zero slope.

Indeed, we are able to derive the equation for the evolution of facets appearing in a simple class of solutions, see Proposition 1. We do not strive for optimal generality but readability. Thus, we present our analysis for solutions, which have a finite number of inflection points at a.e. time instance $t > 0$.

We expect to see the competition between these two types of diffusion, one leads to the rounding off corners and the infinite speed of propagation of signals. The other is responsible for the creation of facets, corners and mostly for a finite speed of propagation of disturbances like facets. The only instances, when the signal speed is infinite, is when facets are created. We expect that this competition will lead to establishing a critical size of persistent facets. Putting it differently, (1.5) with L given by (1.6) is an interpolation between the heat equation and the total variation flow.

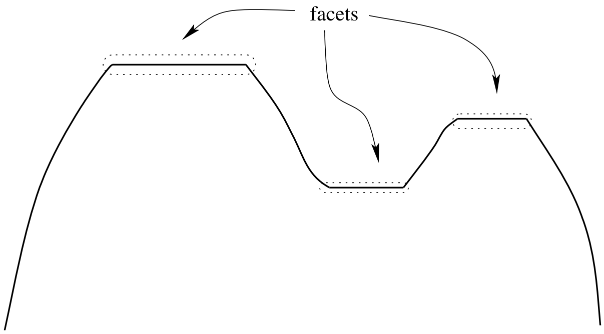
The combination of these two types of diffusion leads us, nonetheless, to a relatively high regularity of solutions. Here, by a weak solution u to (1.5) we mean such a function that

$$u_x \in L_\infty(0, T; BV(I)), \tag{1.7}$$

and the following identity holds

$$-\int_0^T \int_I u\phi_t + \int_0^T \int_I \sigma\phi_x = \int_I u_0 \phi(\cdot, 0) \tag{1.8}$$

Fig. 1 Generic shape



for all $\phi \in C^{\infty}(I \times [0,T))$ such that $\phi|_{\partial I \times (0,T)} =$ and $\phi|_{t=T} = 0$; $\sigma(x,t) \in L(u_x(x,t))$, where $L(u_x)$ is treated as a composition of two maximal multivalued function. To clarify our notation, by $L_\infty(\Omega)$ we mean the Banach space of pointwisely (almost everywhere) bounded function, $BV(I)$ denotes the space of function with a bounded variation, i.e. for a smooth function f, we have $\|f\|_{BV(I)} = \int_I |f'|$.

For this notion of a solution we are able to show the following existence result.

Theorem 1 *Let $u_0 \in L_1(I)$, $u_{0,x} \in BV(I)$ and $D \in C^1((0,T) \times \partial I)$, then there exists a unique solution to system* (1.5).

We do not present the proof of the existence here, it will appear in [20]. We apply there the same technique as used in [19, Sect. 3.1], which is based on the regularization of the non-linear term. Its advantage is that it simplifies the regularity considerations of the subsequent section.

Let us list the key results of the paper. In Theorem 2, we show that the solutions to (1.5) are indeed in a better class of regularity. Next, in Theorem 3, we improve the meaning of the solution. In Sect. 4, we give some examples of solutions to the system. Section 5—the heart of the paper—presents the main results, we prove that facets must appear for suitable regular solutions. Finally, in Theorem 5, we point examples of initial data admitting various behavior of facets. Namely, we are able to find such shapes that facets shrink or expand.

Summing up all partial results presented in our contribution, we obtain a picture of typical solution to (1.5). A key element are facets resulting from the sudden directional diffusion, preferring in our model the direction of vector [0, 1]. In addition, using notation of the theory of crystal growth, convex and concave facets are stable. But zero curvature facets, which are embedded into graphs of monotone functions, are unstable/disappearing. In the last case, we observe the behavior typical for the linear diffusion, but such phenomena never happen for the TV flow.

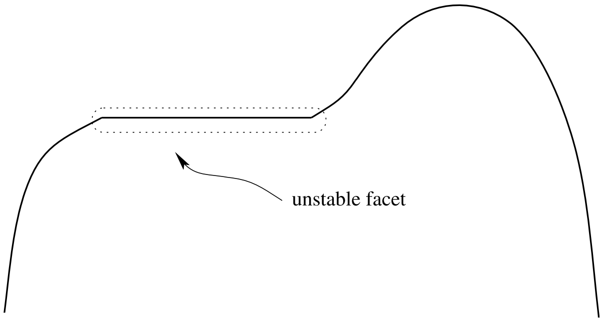
Returning to the physical interpretation, our results say that facets must appear in the front of propagation of the studied interface (local maximum/minimum of the graph)—see Fig. 1.

Facets may never appear in the middle of a the graph of a monotone function, see Fig. 2.

2 Extra Regularity

We observe that (1.6) gives a somehow better dissipation than the standard heat equation. So, we want to describe this "better" information in order to improve smoothness of solutions

Fig. 2 Nonadmissible shape



given by Theorem 1. Here, boundary data $D = \{A, B\}$ are constant/time-independent, i.e. the solution is fixed at the ends of the interval I.

Theorem 2 *Let $A, B \in \mathbb{R}$ and $u_0 \in H^1(I)$ satisfy the boundary condition* (1.5_2), *then u, a unique solution to* (1.5), *belongs to $W_2^{2,1}(I \times (0, T))$, i.e.* (*see* [1]),

$$\int_0^T \int_I [u^2 + u_{xx}^2 + u_t^2] < \infty.$$

In addition, $\sup_{t\in[\delta,T]} \|u_t(\cdot, t)\|_{L_2(I)}$, $\sup_{t\in[\delta,T]} \|u_{xx}(\cdot, t)\|_{L_2(I)}$ *are bounded for any* $\delta > 0$.

Proof First, we use u_t as a test function in (1.5) and after integration by parts in $\int_I (L(u_x))_x u_t$ we obtain that

$$\begin{aligned}\int_0^T \int_I u_t^2 \, dx \, dt + \int_I \int_0^T L(u_x) u_{xt} - \int_0^T L(u_x) u_t \Big|_{\partial I} \\ = \int_I \frac{1}{2} u_x^2 + |u_x| \, dx \Big|_{t=T} \\ = \int_I \frac{1}{2} u_{0,x}^2 + |u_{0,x}| \, dx.\end{aligned} \tag{2.1}$$

We keep in mind here that $L(p) = \frac{d}{dp}(\frac{1}{2}p^2 + |p|)$ and that the value of u_t at the boundary is zero, because D is time independent.

If we use a test function u_{xx} in (1.5), then (after similar calculation) we get

$$\int_I \frac{1}{2} u_x^2 dx \Big|_{t=T} + \int_0^T \int_I u_{xx}^2 + \delta(u_x) u_{xx}^2 \, dx \, dt = \int_I \frac{1}{2} u_{0,x}^2 \, dx. \tag{2.2}$$

Note that the term $\int_0^T \int_I \delta(u_x) u_{xx}^2 \, dx \, dt$, involving products of squares of Dirac deltas, requires further explanations. However, whatever is the interpretation of this term, it is non-negative, hence it may be dropped yielding an inequality estimating $\|u_x\|^2(T)$.

At the rigorous level of regularization, we proceed as follows. Let us suppose that σ_ε is a regularization of sgn, i.e. σ_ε is smooth. For all $y \in \mathbb{R}$, we have $\sigma_\varepsilon(y) \in [-1, 1]$ and $\sigma_\varepsilon(y) = \mathrm{sgn}(y)$, provided that $|y| > \varepsilon$ and $0 \le \sigma_\varepsilon' \le \frac{2}{\varepsilon}$. If we multiply both sides of the regularized equation

$$u_t^\varepsilon - u_{xx}^\varepsilon - \sigma_\varepsilon(u_x^\varepsilon)_x = 0 \tag{2.3}$$

by u^ε_{xx}, then after integration by parts, we will obtain

$$\frac{1}{2}\int_I u^\varepsilon_x|_T\,dx + \int_0^T\int_I (u^\varepsilon_{xx})^2\,dx\,dt + \int_0^T\int_I \sigma'_\varepsilon(u^\varepsilon_x)(u^\varepsilon_{xx})^2\,dx\,dt = \frac{1}{2}\int_I u^2_{0,x}\,dx.$$

Now, after dropping the positive term $\int_0^T\int_I \sigma'_\varepsilon(u^\varepsilon_x)(u^\varepsilon_{xx})^2\,dx\,dt$, we may pass to the limit with $\varepsilon \to 0$, to get the desired result.

As a result (2.1) and (2.2) yield $u \in W^{2,1}_2(I \times (0,T))$. Moreover,

$$\|u\|^2_{W^{2,1}_2(I\times(0,T))} \le C\bigg(\int_I u^2_{0,x} + |u_{0,x}|\,dx + D^2\bigg). \tag{2.4}$$

In order to show the second part of the theorem, we observe that we can not evaluate the initial time derivative of u, because

$$u_t|_{t=0} = (L(u_{0,x}))_x = u_{0,xx} + \delta(u_{0,x})u_{0,xx}. \tag{2.5}$$

At the moment, the term $\delta(u_{0,x})u_{0,xx}$ is not well defined, in general. So it is better to remove the initial datum to find nice properties of the system. Let us define $\eta : [0,\infty) \to [0,1]$ such that $\eta(0) = 0$, $\eta' \ge 0$, $\eta(t) = 1$ for $t > \delta$ and $\eta' \le 2/\delta$.

Then, differentiating (1.5_1) with respect to t and testing the result by $u_t\eta(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_I u_t^2\eta\,dx + \int_I (L \mathbin{\bar\circ} u_x)_t u_{xt}\,dx = \frac{1}{2}\int_I \eta' u_t^2\,dx. \tag{2.6}$$

Here, we indicate the need of making sense out of the nonlinear term, denoted by $L \mathbin{\bar\circ} u_x$. We present the definition of $L \mathbin{\bar\circ} u_x$ in a simple case only. A general discussion is contained in [19]. We notice however, that it is a selection of the composition of two multifunctions.

Thus, integrating (2.6) with respect to time yields,

$$\int_I u_t^2\eta\,dx\bigg|_{t=T} + 2\int_0^T\int_I u^2_{xt} + \delta(u_x)u^2_{xt}\,dx\,dt = \int_0^T\int_I \eta' u_t^2\,dx\,dt, \tag{2.7}$$

because $u_t^2\eta|_{t=0} = 0$ by definition of η. Since the r.h.s. is bounded by (2.1), after dropping the third term on the l.h.s., which is positive, we find

$$\sup_{t\in[\delta,T]}\int_I u_t^2 dx \le \frac{2}{\delta}\|u_t\|^2_{L_2(0,T;L_2(I))}, \tag{2.8}$$

hence,

$$\sup_{t\in[\delta,T]}\int_I u^2_{xx}\,dx \le \frac{2}{\delta}\|u_t\|^2_{L_2(0,T;L_2(I))}.$$

We shall emphasize that the above bounds hold for all $t \in (0,T]$, not only for almost all. Theorem 2 is proved. □

3 Almost Classical Solutions

The theory built in [16, 19] allows us to extend the meaning of solutions constructed there to system (1.5). Theorems 1 and 2 lead us to the following result below. However, in order to avoid a complete discussion of what is the proper smoothness of solutions to (1.5), we restrict ourselves to the case of convex initial datum. We shall see in Theorem 3 below that this property is preserved. General setting requires a new language and delicate separate examinations.

Theorem 3 *Let us suppose that the assumptions of Theorem 2 are fulfilled. In addition let u_0 be a convex function. Then, the corresponding solution to (1.5) with L given by (1.6) satisfies this system as an almost classical solution, i.e.*

$$\begin{aligned} &u_t - \frac{d}{dx} L \,\bar{\circ}\, u_x = 0 && \text{in } I \times (0,T), \\ &u = D && \text{at } \partial I \times (0,T), \\ &u|_{t=0} = u_0 && \text{at } I. \end{aligned} \tag{3.1}$$

The definition of the almost classical solutions is based on the introduction of the composition $\bar{\circ}$, see [16] for details. Here, we present the definition which covers the needs of our analysis, namely the case of u, being a convex function.

Definition 1 Let L be a multivalued operator given by (1.6) and $u_x \in H^1(I)$ be a derivative of a convex function. Then, the multivalued operator

$$L \,\bar{\circ}\, u_x \tag{3.2}$$

is defined as follows:

if $u_x(x_0) \neq 0$, then

$$L \,\bar{\circ}\, u_x(x_0) = L(u_x(x_0)); \tag{3.3}$$

if $u_x(x_0) = 0$, then one of the two possibilities must hold:

(i) there exists only a single point $x_0 \in I$, such that $u_x(x_0) = 0$, then $L \,\bar{\circ}\, u_x(x_0)$ is a set,

$$L \,\bar{\circ}\, u_x(x_0) = [-1, 1]; \tag{3.4}$$

(ii) there exists a single interval $[\xi_-, \xi_+] \subset I$ such that $x_0 \in [\xi_-, \xi_+]$ and

$$u_x(s) = 0 \quad \text{for } s \in [\xi_-, \xi_+], \tag{3.5}$$

furthermore u_x is strictly monotone over $I \setminus [\xi_-, \xi_+]$; in this case, if $\{\xi_-, \xi_+\} \cap \partial I \neq \emptyset$, i.e. $[\xi_-, \xi_+]$ touches the boundary. Then, we set

$$\begin{aligned} &L \,\bar{\circ}\, u_x(s) = 1 \quad \text{for } s \in [\xi_-, \xi_+] \quad \text{if } \xi_- \in \partial I \text{ and } \xi_+ \notin \partial I; \\ &L \,\bar{\circ}\, u_x(s) = -1 \quad \text{for } s \in [\xi_-, \xi_+] \quad \text{if } \xi_+ \in \partial I \text{ and } \xi_- \notin \partial I; \\ &L \,\bar{\circ}\, u_x(s) = \frac{2}{\xi_+ - \xi_-}(s - \xi_+) + 1 \quad \text{for } s \in (\xi_-, \xi_+) \quad \text{if } [\xi_-, \xi_+] = I. \end{aligned} \tag{3.6}$$

Otherwise, if $[\xi_-, \xi_+] \subset \operatorname{int} I$, then

$$L \,\bar{\circ}\, u_x(s) = \frac{2}{\xi_+ - \xi_-}(s - \xi_+) + 1 \quad \text{for } s \in (\xi_-, \xi_+). \tag{3.7}$$

By definition, $L \,\bar{\circ}\, u_x =: \sigma$ is an element of composition of two multifunctions, $L \circ u_u$. In the examples below, we frequently specify a selection σ, which is absolutely continuous. Thus by the uniqueness theorem, it is sufficient that (1.5) is satisfied pointwise a.e.

4 Important Examples

We start with the simplest examples. We begin with an analogue of a special solution known for the heat equation.

4.1 A Translative Solution

We present a special solution, which is analogous to the solution known for the heat equation, namely $u(x,t)=x^2+2t$. Let us take $I=\mathbb{R}$ and any $\alpha>0$. We consider the following function,

$$u^\alpha(x,t)=\begin{cases}\frac{1}{2\alpha}(|x|-\alpha)^2+\frac{t}{\alpha} & \text{for } |x|>\alpha,\\ \frac{t}{\alpha} & \text{for } x\in[-\alpha,\alpha].\end{cases} \tag{4.1}$$

Then, we see that

$$u_t^\alpha\equiv\frac{1}{\alpha} \quad \text{and} \quad L\,\bar{\circ}\,u_x^\alpha=\begin{cases}\frac{1}{\alpha}(x-\alpha)-1 & \text{for } x<-\alpha,\\ \frac{x}{\alpha} & \text{for } x\in[-\alpha,\alpha],\\ \frac{1}{\alpha}(x-\alpha)+1 & \text{for } x>\alpha.\end{cases} \tag{4.2}$$

We had to use the definition of the composition $\bar{\circ}$ in order to find the value of u^α on the interval $[-\alpha,\alpha]$.

The regularity of u^α (resp. $L\,\bar{\circ}\,u_x^\alpha$) given by (4.1) (resp. (4.2)) permits us to write

$$u_t^\alpha-(L\,\bar{\circ}\,u_x^\alpha)_x=0. \tag{4.3}$$

So, taking $\sigma=L\,\bar{\circ}\,u_x^\alpha$, we find that u^α is a solution to (1.8) with a suitable initial datum. We were able to integrate by part in the second term of (1.8), because $L\,\bar{\circ}\,u_x^\alpha\in H^1_{(loc)}(\mathbb{R})$.

We notice that for each α the graph of u^α is a wave traveling upward with a constant speed, without changing the shape. We also see that the graph of u^α has a constant curvature. Thus, the facet, a part of the graph with the zero slope, must be nontrivial so that its curvature can match the curvature of the remaining part of the graph. Here, a nontrivial facet is the graph of u^α over $[-\alpha,\alpha]$.

Note that $u^\alpha\notin C^\infty$, however in the setting associated to the nonlinearity (1.6), it should be treated as the best possible smooth function. This point opens many interesting questions, which can be a motivation for our further studies.

4.2 A Monotone Solution

Let us consider a case of monotone initial data. We assume that

$$u(0,t)=A<B=u(1,t) \quad \text{and} \quad u_{0,x}\geq 0. \tag{4.4}$$

Theorem 4 *Let the initial and boundary data to problem* (1.5) *fulfill* (4.4), *then the solution to* (1.5) *is given as the solution to the heat problem*

$$u_t-u_{xx}=0, \qquad u(0,t)=A<B=u(1,t) \quad \text{and} \quad u|_{t=0}=u_0. \tag{4.5}$$

Proof We look at the solution to (4.5). Since $u_{0,x}\geq 0$, the maximum principle yields $u_x(x,t)>0$ for $t>0$ and $x>0$. Hence, we are tempted to write $L\,\bar{\circ}\,u_x=u_x+1$ for $u_x>0$. However, strictly speaking, our definition does not cover such a situation, for it is set up for convex functions. Nonetheless, we may set $\sigma=u_x+1$, defining a selection of $L\circ u_x$, which is absolutely continuous, hence $\sigma_x=u_{xx}$. We can insert our σ into definition (1.8) to see that we get a weak form of the original problem. The smoothness of u for $t>0$ proves our claim. The uniqueness of weak solutions completes the proof of Theorem 4. □

One point which is hidden in the above "simple" result is a quite unusual behavior of a class of facets. Namely, let us consider the case of a flat region contained in a monotone part of initial datum. For example, we take

$$u_0 = \begin{cases} 2x & \text{for } [0, \frac{1}{4}), \\ \frac{1}{2} & \text{for } [\frac{1}{4}, \frac{3}{4}), \\ 2x + \frac{1}{2} & \text{for } [\frac{3}{4}, 1]. \end{cases} \tag{4.6}$$

We observe that the flat region $[\frac{1}{4}, \frac{3}{4}]$ immediately disappears. Such a phenomenon is characteristic for the heat equation. But it never happens in case of the total variation flow, where such facet is stable locally in time.

5 Facets

In Sect. 4.1, we saw a special case of a traveling wave solution, however, it does not satisfy any fixed in time boundary conditions. Here, we would like to take into account the influence of the data as well as to derive the equation for the evolution of the facets. We do not intend to present a complete theory, but rather to give a number of intriguing examples.

5.1 Evolution of Facets

By the regularity result, i.e. Theorem 2, we know that $u_x(\cdot, t) \in C^{1/2}[-a, a]$ (see [1]), for a.e. $t > 0$, hence the sets $\{u_x(\cdot, t) = 0\}$ are closed. The Sobolev imbedding gives $W_2^1(\mathbb{R}) \subset C^{1/2}(\mathbb{R})$. We shall see that in fact for almost all $t > 0$ the derivative u_x cannot vanish at isolated points. In other words, facets appear instantly from the initial data and they cannot disappear. Indeed, we assume that we have a sufficiently smooth solution and particular boundary data do not matter.

Proposition 1 *Let us suppose that u is a solution to (1.5) with initial condition $u_0 \in W_2^2(I)$ satisfying the compatibility condition. If for all $t > 0$ $u(\cdot, t)$ has only a finite number of inflection points, then if function $u(\cdot, t)$ attains a local minimum at $x_0 \notin \partial I$, then there is $[\xi^-, \xi^+]$ with $\xi^- < \xi^+$ and $x_0 \in [\xi^-, \xi^+]$, hence $u_x = 0$ there. The same assertion holds for local maxima.*

Proof We want to show that for $t > 0$, a local minimum of the function $u(\cdot, t)$ can not be attained just at an isolated point but it must be on a flat region, i.e. a facet $[\xi^-, \xi^+]$ with $\xi^- < \xi^+$ for $t > 0$.

Let us assume that our claim is not true and for $t > 0$ a minimizer is a single point, say $m(t)$, then the function $u(\cdot, t)$ is strictly decreasing for $m(t) - \epsilon < x < m(t)$ and strictly increasing for $m(t) + \epsilon > x > m(t)$. Then, we integrate (1.5) over $[m(t)^-, m(t)^+]$ ($m(t)^\pm$ means limits $x \to m(t)^\pm$)

$$\int_{m(t)^-}^{m(t)^+} u_t \, dx - \int_{m(t)^-}^{m(t)^+} L \,\bar{\circ}\, u_x \, dx = 0. \tag{5.1}$$

In order to compute the second term in the l.h.s. of (5.1), we shall recall that by Theorem 2, $u_x(\cdot, t) \in C^{1/2}(I)$ for each $t > 0$, in other words u_x is continuous, so at a minimum the derivative must be zero. This leads to the following conclusion,

$$L \,\bar{\circ}\, u_x(m(t)^-) = -1 \quad \text{and} \quad L \,\bar{\circ}\, u_x(m(t)^+) = 1. \tag{5.2}$$

Thus, $\int_{m(t)^-}^{m(t)^+}(L\,\bar{\circ}\,u_x)_x\,dx=2$ and

$$\int_{m(t)^-}^{m(t)^+} u_t\,dx=2, \quad \text{but by assumption } m(t)^+-m(t)^-=0. \tag{5.3}$$

Thus $u_t(\cdot,t)$ can not be integrable, but we have already proved $u_t(\cdot,t)\in L_2(I)$—Theorem 2. It follows that $m(t)^+-m(t)^->0$, so $[\xi^-,\xi^+]$ is not degenerated for all $t>0$. □

5.2 Preservation of Convexity

We also notice that convexity of initial data is preserved.

Proposition 2 *Let us suppose that u is a unique solution to (1.5) with initial data u_0. If u_0 is convex and it satisfies the boundary conditions so $u(\cdot,t)$ are for almost all $t>0$.*

Proof We aim at proving that for $t>0$, solution $u(\cdot,t)$ is a convex function. We show it via the approximation, since u is a limit of solutions to (2.3). It is sufficient to show that for all $t>0$ and $\epsilon>0$ approximate solutions u^ϵ are convex.

Take system (2.3). Since u^ϵ at the boundary is fixed, then $u^\epsilon_t=0$ at ∂I, so $u^\epsilon_{xx}+\sigma'_\epsilon(u^\epsilon_x)u^\epsilon_{xx}=0$ at ∂I, too. As a result, we find that $u^\epsilon_{xx}=0$ at ∂I.

Making use of the fact that u^ϵ are smooth, we are allowed to differentiate twice (2.3). This yields

$$\begin{aligned}
&u^\epsilon_{xxt}-\frac{\partial^2}{\partial x^2}(1+\sigma'_\epsilon(u^\epsilon_x))u^\epsilon_{xx}=0 && \text{in } I\times(0,T),\\
&u^\epsilon_{xx}=0 && \text{at } \partial I\times(0,T),\\
&u^\epsilon_{xx}|_{t=0}=u^\epsilon_{0,xx}\ge 0 && \text{at } I.
\end{aligned} \tag{5.4}$$

Although u^ϵ is smooth, we cannot claim that the set $\{(x,t)\in I\times[0,T]\colon u^\epsilon_{xx}(x,t)=0\}$ is regular. However, the Sard theorem guarantees us that there exists a sequence $\delta_k\to 0^-$, such that sets $\{(x,t)\in I\times[0,T]\colon u^\epsilon_{xx}(x,t)=\delta_k\}$ are regular submanifolds, where $\delta_k<0$. We follow the notation $f_-=\min\{f,0\}$. Next, we restate (5.4) as follows

$$(u^\epsilon_{xx}-\delta_k)_t-\frac{\partial^2}{\partial x^2}(1+\sigma'_\epsilon(u^\epsilon_x))(u^\epsilon_{xx}-\delta_k)=\delta_k\frac{\partial^2}{\partial x^2}(1+\sigma'_\epsilon(u^\epsilon_x)). \tag{5.5}$$

Next, we integrate (5.5) over the set $A_k=\{(x,t)\in I\times[0,T]\colon u^\epsilon_{xx}(x,t)<\delta_k\}$

$$\int_{\partial A_k}(u^\epsilon_{xx}-\delta_k)n_t\,d\sigma-\int_{\partial A_k}\frac{\partial}{\partial x}(1+\sigma'_\epsilon(u^\epsilon_x))(u^\epsilon_{xx}-\delta_k)n_x\,d\sigma=\delta_k\int_{A_k}\frac{\partial^2}{\partial x^2}(1+\sigma'_\epsilon(u^\epsilon_x)).$$

Here, (n_x,n_t) is the outer normal vector to A_k.

Since the set ∂A_k is regular and on $\partial A_k\setminus I\times\{0,T\}$ the function u^ϵ_{xx} equals δ_k, we get

$$\int_{\partial A_k}(u^\epsilon_{xx}-\delta_k)n_t\,d\sigma=\int_I(u^\epsilon_{xx}-\delta_k)_-\,dx,$$

because the initial datum $u^\epsilon_{0,xx}$ is non-negative and $\delta_k<0$. Moreover,

$$\int_{\partial A_k}\frac{d}{dx}(1+\sigma'_\epsilon(u^\epsilon_x))(u^\epsilon_{xx}-\delta_k)n_x\,d\sigma<0,$$

because of $(1+\sigma'_\epsilon(u^\epsilon_x))>0$ and $\partial I\cap(\partial A_k\cap I\times\{t\})=\emptyset$ for $t\in(0,T)$.

Thus, we obtain

$$\left|\int_I (u^\epsilon_{xx} - \delta_k)_- \, dx\right| \le |\delta_k| \int_0^T \int_I \left|\frac{\partial}{\partial x}(1 + \sigma'_\epsilon(u^\epsilon_x))\right| dx\, dt. \tag{5.6}$$

After letting $k \to \infty$ ($\delta_k \to 0^-$), we get

$$\left|\int_I (u^\epsilon_{xx})_- \, dx\right| = 0. \qquad \text{Thus} \quad u^\epsilon_{xx}(x,t) \ge 0 \quad \text{for } t > 0. \tag{5.7}$$

Next, after passing with ϵ to 0, we complete the proof of Proposition 2. □

The above results allow us to describe precisely the solution to the system. Here, we want to analyze rigorously the evolution of facets. We consider a symmetric convex case, analogical to the one from Sect. 4.1.

We postulate the following form of the solution to (1.5) with $A = B = 1$,

$$u(x,t) = \begin{cases} h(t) & \text{for } |x| \le a(t), \\ u^H & \text{for } |x| \in (a(t), 1], \end{cases} \tag{5.8}$$

where

$$u^H(a(t),t) = h(t), \qquad u^H_x(a(t),t) = 0, \qquad h'(t) = \frac{1}{a(t)},$$
$$u^H_t - u^H_{xx} = 0 \quad \text{on } (a(t),1), \qquad u|_{t=0} = u_0.$$

By definition of the composition, we find $L \,\bar{\circ}\, u_x|_{|x|<a(t)} = \frac{1}{a(t)} x$.

We shall see that u given by (5.8) indeed fulfills the weak formulation (1.8), see Theorem 3. Taking $\sigma = L \,\bar{\circ}\, u_x$ gives us

$$(u_t, \phi)_{L^2} - (\sigma_x, \phi)_{L^2} = 0,$$

because $L \,\bar{\circ}\, u_x$ belongs at least to H^1 (there is no jump at $|x| = a(t)$). Thus, u given by (5.8) is a weak solution to (1.5).

5.3 Competing Types of Diffusion

The last example presents a case where the two types of diffusion compete: the sudden directional diffusion and the isotropic one. Below, we will construct a solution with a facet whose length, i.e. its curvature does not coincide with the one-sided second derivative of the solution at the end of facets. The construction is performed on a bounded interval $I = (-a, a)$. We impose on end points of I the homogeneous Dirichlet boundary condition. The initial condition u_0, to be constructed, has a facet $(-b, b)$, $b \in (0, a)$, i.e. $u_0(x) = A_0 < 0$ on $(-b, b)$. For the sake of simplicity, we assume that u_0 is even and $u_0|_{(b,a)}$ is not only strictly increasing but also $\frac{du_0}{dx}(x) > 0$ on (b, a). However, if at any point x_0 the derivative $\frac{du_0}{dx}(x_0)$ vanishes, then we will have to deal with facet creation, which is not our goal here.

We will use the versatile Galerkin method to represent the solution on $(b(t), a)$. We stress that the length of the facet is expected to change. As a result, the Galerkin basis will depend on time. We can write,

$$u(x,t) = \begin{cases} A_0(t) & \text{on } (-b(t), b(t)), \\ \alpha_0(t) + \sum_{k=1}^\infty \alpha_k(t) \frac{\varphi_k(x,t)}{\|\varphi_k\|_{L^2(b(t),a)}} & \text{on } (-a, -b(t)) \cup (a, b(t)), \end{cases} \tag{5.9}$$

where $\{\frac{\varphi_k(x,t)}{\|\varphi_k\|_{L^2(b(t),a)}}\}_{k=1}^\infty$ is an orthonormal basis of $L^2(b(t), a)$. We will take φ_k to be the eigenfunctions of the Laplace operator with proper boundary conditions. Since we assumed the zero Dirichlet data at $x=\pm a$, we set $\varphi_k(a,t)=0$. On the other hand, due to the regularity result of Theorem 2, we expect that $\frac{\partial\varphi_k}{\partial x}(b(t),t)=0$. In other words, φ_k, for all $k=1,2,\ldots$, satisfy

$$\begin{cases} \frac{\partial^2\varphi_k}{\partial x^2}=\lambda_k\varphi_k & \text{in } (b(t),a),\\ \frac{\partial\varphi_k}{\partial x}=0 & \text{at } x=b(t),\\ \varphi_k=0 & \text{at } x=a. \end{cases}$$

We immediately see that

$$\varphi_{k+1}(x,t)=\cos\left[\left(\frac{\pi}{2}+\pi k\right)\frac{(x-b)}{(a-b)}\right],\quad x\in(b(t),a),\ k=0,1,\ldots$$

and

$$\|\varphi_{k+1}\|^2_{L^2(b(t),a)}=\frac{1}{2}(a-b(t)),\qquad \lambda_{k+1}=-\pi^2\left(k+\frac{1}{2}\right)^2,\quad k=0,1,\ldots.$$

We want to glue together the pieces in (5.9). We notice that the zero Dirichlet conditions at $x=a$ imply that

$$\alpha_0(t)=0.$$

Moreover, we observe that $\alpha_1<0$ is consistent with $A_0<0$

Continuity at $x=b(t)$ of u given by (5.9) yields

$$A_0(t)=\sum_{k=1}^{\infty}\frac{\sqrt{2}\alpha_k(t)}{\sqrt{a-b(t)}}.$$

We notice that

$$L\,\bar{\circ}\,u_x=\frac{x}{b(t)}\quad\text{on } (-b(t),b(t)).$$

Thus,

$$u_t-(L\,\bar{\circ}\,u_x)_x=0$$

leads to

$$\frac{d}{dt}A_0=\frac{1}{b(t)}>0. \tag{5.10}$$

Now, we are going to study the equation on (b,a). We insert the series into the equation, while calculating u_t, we keep in mind that $b=b(t)$. Subsequently, we take the inner product of the equation with $\varphi_k/\|\varphi_k\|$. We keep in mind that $(\varphi_k,\varphi_i)=\frac{1}{2}(a-b)\delta_{ik}$. This yields,

$$\begin{aligned}&\sqrt{a-b}\frac{d}{dt}\left(\frac{\alpha_k}{\sqrt{a-b}}\right)-\sqrt{2}\frac{d}{dt}b\sum_{j\neq k}^{\infty}\frac{\alpha_j}{\sqrt{a-b}}(a-b)^{-\frac{3}{2}}\frac{2j-1}{(k-j)(j+k-1)}\\ &\quad-\frac{\alpha_k}{\sqrt{a-b}}\frac{\sqrt{2}}{2k+1}\frac{\frac{d}{dt}b}{(a-b)^{\frac{3}{2}}}=-\sqrt{2}\pi^2\frac{\alpha_k}{\sqrt{a-b}}\frac{(k+\frac{1}{2})^2}{\sqrt{a-b}}.\end{aligned} \tag{5.11}$$

In particular, we obtain from (5.10)

$$0<\frac{1}{b}=2\sum_{k=1}^{\infty}\frac{d}{dt}\left(\frac{\alpha_k}{\sqrt{a-b}}\right). \tag{5.12}$$

After having multiplied (5.11) by $\sqrt{a-b}$ and summing up, we arrive at

$$\sum_{k=1}^{\infty} \frac{d}{dt}\left(\frac{\alpha_k}{\sqrt{a-b}}\right) - \sqrt{2}\sum_{k=1}^{\infty}\sum_{j\neq k}^{\infty} \frac{\alpha_j}{\sqrt{a-b}} \frac{\frac{d}{dt}b}{(a-b)^2} \frac{2j-1}{(k-j)(j+k-1)}$$
$$- \sqrt{2}\frac{\frac{d}{dt}b}{(a-b)^2}\sum_{k=1}^{\infty}\frac{\alpha_k}{\sqrt{a-b}} = -\sqrt{2}\pi^2\sum_{k=1}^{\infty}\frac{\alpha_k}{\sqrt{a-b}}\frac{(k+\frac{1}{2})^2}{a-b}. \tag{5.13}$$

After inserting (5.12) and interchanging the summation in the double sum, we reach the following conclusion

$$\frac{1}{2b} - \frac{\sqrt{2}\frac{d}{dt}b}{(a-b)^2}\sum_{j=1}^{\infty}\frac{\alpha_j}{(2j-1)(2j+1)j} = -\frac{\sqrt{2}\pi^2}{(a-b)^{3/2}}\sum_{k=1}^{\infty}\alpha_k\left(k+\frac{1}{2}\right)^2. \tag{5.14}$$

In this way, we have shown the following result:

Theorem 5 *Let us suppose that the initial data u_0 is sufficiently smooth to allow twice the term-by-term differentiation of the series* (5.9$_1$). *Moreover, u_0 has a single central facet, it is even and $u_{0,x}(x) > 0$ for all $x \in [b, a]$. Then,*

(a) *Formula* (5.14) *is the equation for the evolution of the facet end point b.*
(b) *If we suppose that $\alpha_1(0) < 0$, and $\alpha_k(0) = 0$ for $k = 2, \ldots,$ then*

$$\frac{d}{dt}b(0) < 0 \quad \Leftrightarrow \quad 9\pi^2 < \sqrt{2}\frac{(a-b)^{3/2}}{b\alpha_1}$$

and

$$\frac{d}{dt}b(0) > 0 \quad \Leftrightarrow \quad 9\pi^2 > \sqrt{2}\frac{(a-b)^{3/2}}{b\alpha_1}.$$

Thus, we see that the behavior of the length of the facet depends not only on itself but also on the 'height of the bump', i.e. the difference of the values of the function at the endpoints of the interval of monotonicity of u. Moreover, φ_1 has an inflection point, so it is neither convex, nor concave.

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