A Lagrangian Approach for the Incompressible Navier-Stokes Equations with Variable Density

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Abstract

We investigate the Cauchy problem for the inhomogeneous Navier-Stokes equations in the whole n-dimensional space. Under some smallness assumption on the data, we show the existence of global-in-time unique solutions in a critical functional framework. The initial density is required to belong to the multiplier space of $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$. In particular, piecewise-constant initial densities are admissible data *provided the jump at the interface is small enough* and generate global unique solutions with piecewise constant densities. Using Lagrangian coordinates is the key to our results, as it enables us to solve the system by means of the basic contraction mapping theorem. As a consequence, conditions for uniqueness are the same as for existence. © 2012 Wiley Periodicals, Inc.

Introduction

We address the well-posedness issue for the the incompressible Navier-Stokes equations with variable density in the whole space $\mathbb{R}^n$:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla P = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{0.1}$$

Above $\rho = \rho(t,x) \in \mathbb{R}_+$ stands for the density, $u = u(t,x) \in \mathbb{R}^N$ for the velocity field, and $P = P(t,x) \in \mathbb{R}$ for the pressure field. The viscosity coefficient μ is a given positive real number. We supplement this system with the following boundary conditions:

- the velocity u tends to 0 at infinity,
- the density ρ tends to some positive constant ρ^* at infinity.

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The exact meaning of those boundary conditions will be given by the functional framework in which we shall solve the system. In what follows, we will take $\rho^* = 1$ to simplify the presentation.

This old system has elicited a renewed interest recently in the mathematics community. The existence of strong smooth solutions with positive density has been established in, for example, [10], whereas the theory of global weak solutions with finite energy has been performed in the book by P.-L. Lions [11] (see also the references therein and the monograph [3]). As pointed out in [5], it is possible to construct strong unique solutions for some classes of smooth enough data with vanishing density.

In the present paper, we aim at solving the above system in *critical* functional spaces, that is, in spaces that have the same invariance with respect to time and space dilations as the system itself (see, e.g., [6, 7] for more explanations about this now-classical approach). In this framework, it has been stated in [1, 6] that, for data (ρ_0, u_0) such that

$$(\rho_0 - 1) \in \dot{B}^{n/p}_{p,1}(\mathbb{R}^n), \quad u_0 \in \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \text{ with } \operatorname{div} u_0 = 0$$

and that, for a small enough constant c,

$$\|\rho_0 - 1\|_{\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)} + \mu^{-1}\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \leq c, \tag{0.2}$$

we have for any $p \in [1, 2n)$

- existence of a global solution $(\rho, u, \nabla P)$ with $\rho - 1 \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$, $u \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$, and $\partial_t u, \nabla^2 u, \nabla P \in L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$;
- uniqueness in the above space if in addition $p \leq n$.

These results have been somewhat extended in [2], where it has been noted that $\rho_0 - 1$ may be taken in a larger Besov space, with another Lebesgue exponent.

The above results are based on maximal regularity estimates in Besov spaces for the evolutionary Stokes system and on the Schauder-Tychonoff fixed point theorem. In effect, owing to the hyperbolicity of the density equation, there is a loss of one derivative in the stability estimates, thus precluding the use of the contraction mapping (or Banach fixed point) theorem. As a consequence, the conditions for uniqueness are *stronger* than those for existence.

Let us also point out that all the above results concerning existence *with* uniqueness require the density to be at least uniformly continuous. This condition has been somewhat weakened recently by P. Germain [9]. However, there initial densities with a jump across an interface cannot be considered.

In the present paper, we aim at solving system (0.1) in the Lagrangian coordinates. The main motivation is that the density is *constant* along the flow so that only the (parabolic type) equation for the velocity has to be considered. We shall show that, after performing this change of coordinates, solving (0.1) may be done by means of the Banach fixed point theorem. As a consequence, the condition for uniqueness need not be stronger than that for the existence, and the flow map is Lipschitz-continuous (for these new coordinates of course).

Our main result states the global-in-time existence of regular solutions to the inhomogeneous Navier-Stokes equations in $\mathbb{R}^n$ in the optimal Besov setting, under suitable smallness of the data. As regards the initial density, the admissible regularity is so low (small) that jumps across a C^1 interface may be considered. This is of particular interest from the viewpoint of physics, as it implies that the motion of a mixture of two incompressible fluids with slightly different densities can be modeled by the inhomogeneous Navier-Stokes equations. In addition, the regularity of the constructed velocity ensures that the C^1 regularity of the interface between fluids is conserved through the evolution.

We now come to the plan of the paper. In the next section, we present our main results and give some insight into the proof. Section 2 is devoted to solving the linearized system (0.1) in Lagrangian coordinates. This will enable us to define a map $\Phi : E_p \to E_p$ where E_p stands for the functional space in which the Lagrangian version of the momentum equation of (0.1) is going to be solved. Establishing that Φ fulfills the conditions of the contraction mapping theorem on a small enough ball of E_p is the main purpose of Section 3. In the Appendix we prove several important results concerning the Lagrangian coordinates and Besov spaces.

NOTATION: Throughout, the letter C stands for a generic constant (the meaning of which depends on the context), and we sometimes write $X \lesssim Y$ instead of $X \le CY$. Finally, for two matrices $A = (A_{ij})_{1\le i,j\le n}$ and $B = (B_{ij})_{1\le i,j\le n}$, we denote $A : B = \sum_{i,j} A_{ij} B_{ji}$.

1 Main Results and the Principle of the Proof

Let us first formally derive the Lagrangian equations corresponding to (0.1). (The reader can refer to the Appendix for the rigorous derivation in our functional setting.) Let X_u be the flow associated to the vector field u, that is, the solution to

$$X_u(t,y) = y + \int_0^t u(\tau, X_u(\tau,y))\,d\tau. \tag{1.1}$$

Let us recall that by Liouville's formula for transport equations, the divergence-free condition is equivalent to $|DX_u| \equiv 1$. In other words, the map (1.1) is measure preserving. Now, denoting

$$\bar\rho(t,y) := \rho(t, X_u(t,y)), \quad \bar P(t,y) := P(t, X_u(t,y)),$$
$$\bar u(t,y) = u(t, X_u(t,y)),$$

with $(\rho, u, \nabla P)$ a solution of (0.1), and using the chain rule and Lemma A.1 from the Appendix, we gather that $\bar\rho(t,\cdot) \equiv \rho_0$ and that $(\bar u, \nabla \bar P)$ satisfies

$$\begin{cases} \rho_0 \partial_t \bar u - \mu\, \mathrm{div}_y(A_u A_u^{\top} \nabla_y \bar u) + A_u^{\top} \nabla_y \bar P = 0 \quad \text{with } A_u = (D_y X_u)^{-1}, \\ \mathrm{div}_y(A_u \bar u) = 0. \end{cases} \tag{1.2}$$

Motivated by prior works (see, e.g., [1, 2, 6, 7]), we want to solve the above system in *critical* homogeneous Besov spaces. Let us recall that, for $1 \leq p \leq \infty$ and $s \leq n/p$, a tempered distribution u over $\mathbb{R}^n$ belongs to the homogeneous Besov space $\dot{B}^s_{p,1}(\mathbb{R}^n)$ if

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$\|u\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} := \sum_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L_p(\mathbb{R}^n)} < \infty.$$

Here $(\dot{\Delta}_j)_{j \in \mathbb{Z}}$ denotes a homogeneous dyadic resolution of unity in Fourier variables (see, e.g., [4, chap. 2] for more details).

Loosely speaking, a function belongs to $\dot{B}^s_{p,1}(\mathbb{R}^n)$ if it has s derivatives in $L_p(\mathbb{R}^n)$. In the present paper, we shall make extensive use of the following classical properties:

- the Besov space $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ is a Banach algebra embedded in the set of continuous functions going to 0 at infinity whenever $1 \leq p < \infty$;
- the usual product maps $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ in $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$ whenever $1 \leq p < 2n$.

From now on, we shall omit $\mathbb{R}^n$ in the notation for Besov spaces. We shall obtain the existence and uniqueness of a global solution $(\bar{u}, \nabla \bar{P})$ for (1.2) in the space

$$E_p := \big\{(\bar{u}, \nabla \bar{P}) : \bar{u} \in \mathcal{C}_b\big(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1}\big),\ \partial_t \bar{u}, \nabla^2 \bar{u}, \nabla \bar{P} \in L_1\big(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1}\big)\big\},$$

and we shall endow E_p with the norm

$$\|(\bar{u}, \nabla \bar{P})\|_{E_p} := \|\bar{u}\|_{L_\infty(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})} + \|\partial_t \bar{u}, \mu \nabla^2 \bar{u}, \nabla \bar{P}\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})}.$$

We shall also use the *local* version $E_p(T)$ of E_p, pertaining to functions defined on $[0, T) \times \mathbb{R}^n$. Writing out the exact definition and the corresponding norm is left to the reader.

The required regularity for the initial density ρ_0 is that it belongs to the *multiplier space* $\mathcal{M}(\dot{B}^{n/p-1}_{p,1})$ for $\dot{B}^{n/p-1}_{p,1}$, that is, the set of those distributions ρ_0 such that $\psi \rho_0$ is in $\dot{B}^{n/p-1}_{p,1}$ whenever ψ is in $\dot{B}^{n/p-1}_{p,1}$, endowed with the norm

$$\|\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})} := \sup \|\psi \rho_0\|_{\dot{B}^{n/p-1}_{p,1}}, \tag{1.3}$$

where the supremum is taken over those functions ψ in $\dot{B}^{n/p-1}_{p,1}$ with norm 1.

Let us now state our main result.

THEOREM 1.1. *Let $p \in [1, 2n)$ and u_0 be a divergence-free vector field with coefficients in $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$. Assume that the initial density ρ_0 belongs to the multiplier* *space* $\mathcal{M}(\dot B^{n/p-1}_{p,1})$. *There exists a constant* c *depending only on* p *and on* n *such that if*

$$\|\rho_0-1\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1})}+\mu^{-1}\|u_0\|_{\dot B^{n/p-1}_{p,1}}\leq c,$$

then system (1.2) *has a unique global solution* $(\overline{u},\nabla\overline{P})$ *in* E_p. *Moreover, we have*

$$\|(\overline{u},\nabla\overline{P})\leq C\|u_0\|_{\dot B^{n/p-1}_{p,1}}$$

for some constant C *depending only on* n *and on* p, *and the flow map* $(\rho_0,u_0)\longmapsto(\overline{u},\nabla\overline{P})$ *is Lipschitz-continuous from* $\mathcal{M}(\dot B^{n/p-1}_{p,1})\times\dot B^{n/p-1}_{p,1}$ *to* E_p.

In the case where only the density satisfies the smallness condition, we get the following local-in-time existence result:

THEOREM 1.2. *Under the above regularity assumptions, there exists a constant* c *depending only on* p *and on* n *such that if*

$$\|\rho_0-1\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1})}\leq c,$$

then there exists some $T>0$ *such that system* (1.2) *has a unique local solution* $(\overline{u},\nabla\overline{P})$ *in* $E_p(T)$ *and the flow map* $(\rho_0,u_0)\longmapsto(\overline{u},\nabla\overline{P})$ *is Lipschitz-continuous from* $\mathcal{M}(\dot B^{n/p-1}_{p,1})\times\dot B^{n/p-1}_{p,1}$ *to* $E_p(T)$.

The regularity given by Theorem 1.1 ensures that the map defined in (1.1) is defined globally (see the Appendix). Coming back to the Eulerian formulation, this will enable us to get the following result (we consider here only the case of small data to simplify the presentation):

THEOREM 1.3. *Under the above assumptions, system* (0.1) *has a unique global solution* $(\rho,u,\nabla P)$ *with* $\rho\in L_\infty(\mathbb{R}_+;\mathcal{M}(\dot B^{n/p-1}_{p,1}))$ *and* $(u,\nabla P)\in E_p$.

Let us make a few comments concerning the above assumptions.

- The condition $1\leq p<2n$ is a consequence of the product laws in Besov spaces. Let us emphasize that any space $L_\infty\cap B^{n/q-1}_{q,\infty}$ with q satisfying

$$\frac{1}{q}>\frac{1}{n}-\frac{1}{p}\quad\text{and}\quad\frac{1}{q}\geq\frac{1}{p}-\frac{1}{n}\tag{1.4}$$

  embeds in $\mathcal{M}(\dot B^{n/p-1}_{p,1})$, a consequence of basic continuity results for the paraproduct operator (see [4]). Hence the above statement improves those of [1, 2] as regards the uniqueness. In particular, one may take the initial velocity in a Besov space with a *negative* index of regularity, so that a highly oscillating “large” velocity may give rise to a unique global solution.
- In contrast to the results of [1, 2, 6], it is not clear that the above statements may be generalized so that the viscosity depends on the density, in which case the diffusion term in the momentum equation of (0.1) reads $\mathrm{div}(\mu(\rho)(\nabla u+\nabla^{\mathsf T}u))$. The stronger condition $\mu(\rho_0)-\mu(1)$ small in $\mathcal{M}(\dot B^{n/p}_{p,1})$ is required with our approach.

The space $\mathcal{M}(\dot{B}^{n/p-1}_{p,1})$ is in fact much larger than $L_\infty \cap B^{n/q-1}_{q,\infty}$ with q satisfying (1.4) (see, e.g., [12, chap. 4]). It contains characteristic functions of C^1-bounded domains whenever $p > n-1$ (see the proof in Lemma A.7). Hence our result applies to a mixture of fluids, which is of course of great physical interest. In addition, given that the constructed velocity field u is divergence-free and admits a C^1 flow X (again, see the Appendix), we deduce the following result, which emphasizes the range of Theorem 1.1 and gives a partial answer to an open problem posed by P.-L. Lions in [11, p. 34]. We just state the case of small velocities to simplify the presentation:

COROLLARY 1.4. *Assume that* $u_0 \in \dot{B}^{n/p-1}_{p,1}$ *with* $\operatorname{div} u_0 = 0$ *and* $n-1 < p < 2n$. *Let* Ω_0 *be a bounded* C^1 *domain of* $\mathbb{R}^n$. *There exist two constants* c (*depending only on* p *and* n) *and* c' (*depending only on* p, n, *and* Ω_0) *such that if*

$$\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \le c \quad \text{and} \quad \rho_0 = 1 + \sigma\chi_{\Omega_0} \ \text{with } |\sigma| \le c', \tag{1.5}$$

then system (0.1) *has a unique global solution* $(\rho, u, \nabla P)$ *with*

$$\rho \in L_\infty\big(\mathbb{R}_+; \mathcal{M}\big(\dot{B}^{n/p-1}_{p,1}\big)\big) \quad \text{and} \quad (u, \nabla P) \in E_p.$$

In addition, for all positive time t, *one has*

$$\rho(t) = 1 + \sigma\chi_{\Omega_t} \quad \text{where } \Omega_t = X_u(t, \Omega_0), \tag{1.6}$$

and the measure and the C^1 *regularity of* $\partial\Omega_t$ *are preserved.*

Let us give the main ideas of the proof of existence. Obviously it suffices to find a fixed point for the map $\Theta : (q, v) \mapsto (\rho, u)$ where (ρ, u) stands for the solution to the *linear* system

$$\begin{cases} \partial_t\rho + v \cdot \nabla\rho = 0, \\ q(\partial_t u + v \cdot \nabla u) - \mu\Delta u + \nabla P = 0, \\ \operatorname{div} u = 0, \\ u|_{|t=0} = u_0. \end{cases} \tag{1.7}$$

Although it is possible to prove uniform estimates in $L_\infty(\mathbb{R}_+; \mathcal{M}(\dot{B}^{n/p-1}_{p,1})) \times E_p$ for $(\rho, u, \nabla P)$, we do not know how to get stability estimates in the same space, owing to the hyperbolic nature of the density equation. As a consequence, the contraction mapping theorem does not apply.

In the present paper, we shall rather define the solution of the above system in the *Lagrangian coordinates corresponding to* v. For such coordinates, the density is time independent. So, given some reference vector field $\overline{v}$ and pressure field $\nabla\overline{Q}$ with $(\overline{v}, \nabla\overline{Q}) \in E_p$, one may define $(\overline{u}, \nabla\overline{P})$ to be the solution *in the Lagrangian coordinates* $y = X_v^{-1}(t, x)$ *pertaining to* v of the second equation in (1.7) (that X_v is a C^1-diffeomorphism over $\mathbb{R}^n$ is proved in the Appendix).

Let us give more details. We assume that $|DX_v| \equiv 1$ and set

$$\bar\rho(t,y) := \rho(t, X_v(t,y)), \quad \bar P(t,y) := P(t, X_v(t,y)),$$
$$\bar u(t,y) := u(t, X_v(t,y)),$$

where $(\rho, u, \nabla P)$ stands for a solution to (1.7). Then we have $\bar\rho(t,\cdot) \equiv \rho_0$ and (see the proof in the Appendix)

$$\begin{cases} \rho_0 \partial_t \bar u - \mu \operatorname{div}(A_v A_v^{\top} \nabla_y \bar u) + A_v^{\top} \nabla_y \bar P = 0, \\ \operatorname{div}_y (A_v \bar u) = 0, \end{cases} \tag{1.8}$$

with

$$A_v := (D_y X_v)^{-1} \quad \text{and} \quad X_v(t,y) := y + \int_0^t \bar v(\tau, y)\, d\tau. \tag{1.9}$$

Solving this linear system *globally* turns out to be possible under some smallness condition over $\bar v$ and $\rho_0 - 1$. This will enable us to define a self-map $\Phi : (\bar v, \nabla \bar Q) \mapsto (\bar u, \nabla \bar P)$ on E_p. Then it will only be a matter of checking that if the data ρ_0 and u_0 satisfy a suitable smallness condition, then the map Φ fulfills the assumptions of the standard Banach fixed point theorem. The key to that will be estimates for the Stokes system in E_p (see Proposition 2.1) and a "magic" algebraic relation involving the second equation of (1.8) (which has been used before in [13, 14, 15] in a different context).

2 The Linear Theory

Our proof of existence for the linear system (1.8) will be based on the following a priori estimates for the Stokes system, the proof of which may be found in [7]:[1]

PROPOSITION 2.1. *Let $u_0 \in \dot B^s_{p,1}(\mathbb{R}^n)$ and $f \in L_1(0,T; \dot B^s_{p,1}(\mathbb{R}^n))$ with $p \in [1,\infty]$ and $s \in \mathbb{R}$. Let $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\nabla g \in L_1(0,T; \dot B^s_{p,1}(\mathbb{R}^n)), \quad \partial_t g = \operatorname{div} R \ \textit{ with } R \in L_1(0,T; \dot B^s_{p,1}(\mathbb{R}^n)),$$

and that the compatibility condition $g|_{t=0} = \operatorname{div} u_0$ on $\mathbb{R}^n$ is satisfied.

Then the system

$$\begin{cases} \partial_t u - \mu \Delta u + \nabla P = f & \text{in } (0,T) \times \mathbb{R}^n, \\ \operatorname{div} u = g & \text{in } (0,T) \times \mathbb{R}^n, \\ u|_{t=t_0} = u_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{2.1}$$

has a unique solution $(u, \nabla P)$ with

$$u \in \mathcal{C}([0,T); \dot B^s_{p,1}(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_1(0,T; \dot B^s_{p,1}(\mathbb{R}^n))$$

[1] Because homogeneous zeroth-order multipliers act on any homogeneous Besov space $\dot B^s_{p,1}$, one may take any index s and exponent $p \in [1,\infty]$ in Proposition 2.1.

and the following estimate is valid:

$$\|u\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} + \|\partial_t u, \mu\nabla^2 u, \nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} \leq C\big(\|f, \mu\nabla g, R\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n))} + \|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n)}\big) \tag{2.2}$$

where C is an absolute constant with no dependence on μ or T.

In order to apply the above statement, we rewrite system (1.8) as

$$\begin{cases} \partial_t\bar u - \mu\Delta\bar u + \nabla\bar P = \\ \quad (1-\rho_0)\partial_t\bar u + \mu\,\mathrm{div}((A_v A_v^\top - \mathrm{Id})\nabla\bar u) + (\mathrm{Id} - A_v^\top)\nabla\bar P, \\ \mathrm{div}\,\bar u = \mathrm{div}((\mathrm{Id} - A_v)\bar u), \\ \bar u|_{t=0} = u_0. \end{cases} \tag{2.3}$$

We assume that the vector field $\bar v$ from which A_v and DX_v are defined satisfies

$$\bar v \in \mathcal{C}(\mathbb{R}_+; \dot B^{n/p-1}_{p,1}), \quad \partial_t\bar v, \nabla^2\bar v \in L_1(\mathbb{R}_+; \dot B^{n/p-1}_{p,1}), \quad |DX_v| \equiv 1, \tag{2.4}$$

and that, for a small enough constant c,

$$\int_0^\infty \|D\bar v\|_{\dot B^{n/p}_{p,1}}\,dt \leq c. \tag{2.5}$$

Even though this system is linear, it cannot be solved directly by means of Proposition 2.1 for the right-hand side depends on the solution itself. So in order to prove the existence of $\bar u$, we shall look for a fixed point of the map

$$\Psi : (\bar w, \nabla\bar Q) \longmapsto (\bar u, \nabla\bar P)$$

where $(\bar w, \nabla\bar Q) \in E_p$ and $(\bar u, \nabla\bar P)$ stands for the solution of

$$\begin{cases} \partial_t\bar u - \mu\Delta\bar u + \nabla\bar P = f(\bar w, \nabla\bar Q), \\ \mathrm{div}\,\bar u = g(\bar w), \\ \bar u|_{t=0} = u_0. \end{cases} \tag{2.6}$$

Above, $g(\bar w) := \mathrm{div}((\mathrm{Id} - A_v)\bar w)$ and

$$f(\bar w, \nabla\bar Q) := (1-\rho_0)\partial_t\bar w + \mu\,\mathrm{div}((A_v A_v^\top - \mathrm{Id})\nabla\bar w) + (\mathrm{Id} - A_v^\top)\nabla\bar Q. \tag{2.7}$$

We claim that, if $\bar v$ satisfies (2.4) and the smallness condition (2.5), then for any $(\bar w, \nabla\bar Q)$ in the space E_p with $1 \leq p < 2n$, the above system has a unique solution $(\bar u, \nabla\bar P)$ in E_p and that, in addition, the map Ψ fulfills the required conditions for applying the contraction mapping theorem.

That $(\bar u, \nabla\bar P)$ exists will stem from Proposition 2.1 provided that $f(\bar w, \nabla\bar Q)$ and $g(\bar w)$ fulfill the required conditions. As regards $g(\bar w)$, this stems from the following "magic formula":

$$g(\bar w) = \mathrm{div}((\mathrm{Id} - A_v)\bar w) = D\bar w : (\mathrm{Id} - A_v), \tag{2.8}$$

a consequence of $|DX_v| \equiv 1$ (see Corollary A.3 in the Appendix).

Bounds for $g(\overline{w})$

Let us first check that $g(\overline{w}) \in L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$. As $D\overline{w} \in L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$ and as, according to (A.8), $\mathrm{Id} - A_v$ is in $L_\infty(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$, this is a consequence of the fact that $\dot{B}^{n/p}_{p,1}$ is a Banach algebra and that

$$g(\overline{w}) = D\overline{w} : (\mathrm{Id} - A_v).$$

In addition, we get

$$\|g(\overline{w})\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})} \lesssim \|D\overline{v}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})}\|D\overline{w}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})}. \tag{2.9}$$

Next, we see that $\partial_t(g(\overline{w})) = \mathrm{div}\, R^1(\overline{w}) + \mathrm{div}\, R^2(\overline{w})$ with

$$R^1(\overline{w}) := (\mathrm{Id} - A_v)\partial_t \overline{w} \quad \text{and} \quad R^2(\overline{w}) := -\partial_t A_v \overline{w}.$$

So, according to (A.7) and (A.9) and because the product operator maps $\dot{B}^{n/p}_{p,1} \times \dot{B}^{n/p-1}_{p,1}$ in $\dot{B}^{n/p-1}_{p,1}$ whenever $p < 2n$, we see that $R^1(\overline{w})$ and $R^2(\overline{w})$ belong to $L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$ and that

$$\|R^1(\overline{w})\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})} \lesssim \|D\overline{v}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})}\|\partial_t\overline{w}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})}, \tag{2.10}$$

$$\|R^2(\overline{w})\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})} \lesssim \|D\overline{v}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})}\|\overline{w}\|_{L_\infty(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})}. \tag{2.11}$$

Bounds for $f(\overline{w}, \nabla\overline{Q})$

That the first term of $f(\overline{w})$ belongs to $L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$ is a consequence of the definition of the multiplier space $\mathcal{M}(\dot{B}^{n/p-1}_{p,1})$; in addition, we have

$$\|(1-\rho_0)\partial_t\overline{w}\|_{\dot{B}^{n/p-1}_{p,1}} \leq \|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})}\|\partial_t\overline{w}\|_{\dot{B}^{n/p-1}_{p,1}}. \tag{2.12}$$

Next, according to (A.11), the second term of $f(\overline{w})$ belongs to $L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$ and

$$\|\mathrm{div}((A_vA_v^\top - \mathrm{Id})\nabla\overline{w})\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})} \lesssim \|D\overline{v}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})}\|D\overline{w}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})}. \tag{2.13}$$

Finally, inequality (A.8) and the fact that the product operator maps $\dot{B}^{n/p}_{p,1} \times \dot{B}^{n/p-1}_{p,1}$ in $\dot{B}^{n/p-1}_{p,1}$ if $p < 2n$ ensure that

$$\|(\mathrm{Id} - A_v^\top)\nabla\overline{Q}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})} \lesssim \|D\overline{v}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})}\|D\overline{Q}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})}. \tag{2.14}$$

So putting together (2.9) to (2.14), one may conclude from Proposition 2.1 that for any $(\overline{w}, \nabla\overline{Q})$ in E_p, system (2.6) has a unique solution $(\overline{u}, \nabla\overline{P})$ in E_p. In addition,

$$\|(\bar{u},\nabla\bar{P})\|_{E_p}\leq C\Big(\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \big(\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})}+\|D\bar{v}\|_{L_1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})}\big)\|(\bar{w},\nabla\bar{Q})\|_{E_p}\Big).$$

As a consequence, there exists a positive constant c (depending only on n and on p) such that if (2.5) is satisfied and

$$\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})}\leq c, \tag{2.15}$$

then

$$\|\Psi(\bar{w},\nabla\bar{Q})\|_{E_p}\leq C\|u_0\|_{\dot{B}^{n/p-1}_{p,1}}+\frac{1}{2}\|(\bar{w},\nabla\bar{Q})\|_{E_p}. \tag{2.16}$$

The Banach theorem thus entails that the linear map Ψ admits a unique fixed point in E_p, which we shall still denote by $(\bar{u},\nabla\bar{P})$. Let us emphasize that inequality (2.16) ensures that

$$\|(\bar{u},\nabla\bar{P})\|_{E_p}\leq 2C\|u_0\|_{\dot{B}^{n/p-1}_{p,1}}, \tag{2.17}$$

and that, by construction, $\operatorname{div} A_v\bar{u}=0$. Hence, according to Corollary A.3, $|DX_u|\equiv 1$. In other words, given $\bar{v}$ satisfying (2.4) and (2.5), system (2.3) admits a unique solution $(\bar{u},\nabla\bar{P})$ satisfying (2.4) and (2.17).

3 Inhomogeneous Navier-Stokes Equations

Let us denote by $\widetilde{E}_p^R$ the closed subset of E_p containing all the couples $(\bar{v},\nabla\bar{Q})$ such that

$$|DX_v|\equiv 1 \quad\text{and}\quad \|(\bar{v},\nabla\bar{Q})\|_{E_p}\leq R.$$

According to the previous section, if one takes $(\bar{v},\nabla\bar{Q})$ in E_p with $\bar{v}$ satisfying $|DX_v|\equiv 1$ and (2.5), then (1.8) admits a solution $(\bar{u},\nabla\bar{P})$ in the same space such that $|DX_u|\equiv 1$. Let $\Phi(\bar{v},\nabla\bar{Q})$ denote this solution.[2] We claim that if u_0 is small enough with respect to μ in $\dot{B}^{n/p-1}_{p,1}$, if the density ρ_0 satisfies (2.15), and if R is small enough, then Φ admits a unique fixed point in $\widetilde{E}_p^R$ as a consequence of the contraction mapping theorem.

3.1 Stability of $\widetilde{E}_p^R$ by Φ

Assume that (2.15) is satisfied and let us take $R=c\mu$. Then (2.5) holds true whenever $(\bar{v},\nabla\bar{Q})$ is in $\widetilde{E}_p^R$. Therefore $(\bar{u},\nabla\bar{P}):=\Phi(\bar{v},\nabla\bar{Q})$ satisfies (2.17). So it is clear that if

$$2C\|u_0\|_{\dot{B}^{n/p-1}_{p,1}}\leq c\mu, \tag{3.1}$$

with C as in (2.17), then $(\bar{u},\nabla\bar{P})$ is in $\widetilde{E}_p^R$, too.

[2] Of course, it is independent of $\nabla\bar{Q}$. However, prescribing the pressure is needed so as to define a map from a subset of E_p to itself.

3.2 Contraction Properties

In this part, we show that under conditions (2.15) and (3.1) (with a greater constant C and smaller constant c if needed), the map $\Phi : \widetilde{E}_p^R \to \widetilde{E}_p^R$ is $\frac{1}{2}$-Lipschitz.

So we are given $(\overline{v}_1, \nabla \overline{Q}_1)$ and $(\overline{v}_2, \nabla \overline{Q}_2)$ in $\widetilde{E}_p^R$, and denote

$$(\overline{u}_1, \nabla \overline{P}_1) := \Phi(\overline{v}_1, \nabla \overline{Q}_1) \quad \text{and} \quad (\overline{u}_2, \nabla \overline{P}_2) := \Phi(\overline{v}_2, \nabla \overline{Q}_2).$$

Let X_1 and X_2 be the flows associated to $\overline{v}_1$ and $\overline{v}_2$. Set $A_i = (DX_i)^{-1}$ for $i = 1, 2$. The equations satisfied by $\delta u := \overline{u}_2 - \overline{u}_1$ and $\nabla \delta P := \nabla \overline{P}_2 - \nabla \overline{P}_1$ read

$$\begin{cases} \partial_t \delta u - \mu \Delta \delta u + \nabla \delta P = \delta f := \delta f_1 + \delta f_2 + \delta f_3 + \mu \operatorname{div} \delta f_4 + \mu \operatorname{div} \delta f_5, \\ \operatorname{div} \delta u = \delta g := \operatorname{div}((\mathrm{Id} - A_2)\delta u + (A_1 - A_2)\overline{u}_1), \end{cases}$$

with

$$\delta f_1 := (1-\rho_0)\partial_t \delta u, \quad \delta f_2 := (\mathrm{Id} - A_2^{\mathsf{T}})\nabla \delta P, \quad \delta f_3 := (A_1 - A_2)^{\mathsf{T}} \nabla \overline{P}_1,$$
$$\delta f_4 := (A_2 A_2^{\mathsf{T}} - A_1 A_1^{\mathsf{T}})\nabla \overline{u}_1, \quad \delta f_5 := (A_2 A_2^{\mathsf{T}} - \mathrm{Id})\nabla \delta u.$$

Once again, bounding $(\delta u, \nabla \delta P)$ will stem from Proposition 2.1, which ensures that if $\partial_t \delta g = \operatorname{div} \delta R$, then

$$\|(\delta u, \nabla \delta P)\|_{E_p} \lesssim \|\delta f\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})} + \mu \|\delta g\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})} + \|\delta R\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})}. \tag{3.2}$$

Consequently, we have to bound $\delta f_1, \delta f_2, \delta f_3$ in $L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$ and $\delta f_4, \delta f_5$ in $L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$. First, from the definition of the multiplier space $\mathcal{M}(\dot{B}^{n/p-1}_{p,1})$, we readily have

$$\|\delta f_1\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})} \le \|\rho_0 - 1\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})} \|\partial_t \delta u\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})}. \tag{3.3}$$

Next, using inequalities (A.8) and (A.11) and product laws in Besov space yields

$$\|\delta f_2\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})} \lesssim \|D\overline{v}_2\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})} \|D\delta P\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})}, \tag{3.4}$$

$$\|\delta f_5\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})} \lesssim \|D\overline{v}_2\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})} \|D\delta u\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})}. \tag{3.5}$$

Inequality (A.13) ensures that

$$\|\delta f_3\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})} \lesssim \|D\delta v\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})} \|D\overline{P}_1\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})}, \tag{3.6}$$

whereas inequalities (A.13) and (A.14) yield

$$\|\delta f_4\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})} \lesssim \|D\delta v\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})} \|D\overline{u}_1\|_{L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})}. \tag{3.7}$$

In order to bound δg in $L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$, we shall use the fact that, by construction,

$$\operatorname{div} \overline{u}_i = \operatorname{div}((\mathrm{Id} - A_i)\overline{u}_i) = D\overline{u}_i : (\mathrm{Id} - A_i).$$

Hence

$$\delta g = D\delta u : (\mathrm{Id} - A_2) - D\overline{u}_1 : (A_2 - A_1).$$

Now, easy computations based on (A.8) and (A.13) yield

$$\|D\delta u : (\mathrm{Id}-A_2)\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})} \lesssim \|D\overline{v}_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}\|D\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}, \tag{3.8}$$

$$\|D\overline{u}_1 : (A_2 - A_1)\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})} \lesssim \|D\overline{u}_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}\|D\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}. \tag{3.9}$$

Finally, to bound $\partial_t\delta g$, we decompose it into $\mathrm{div}(\delta R_1+\delta R_2+\delta R_3+\delta R_4)$ with

$$\delta R_1 = -\partial_t A_2\,\delta u, \qquad \delta R_2 = (\mathrm{Id}-A_2)\partial_t\delta u,$$
$$\delta R_3 = \partial_t(A_1 - A_2)\overline{u}_1, \quad \delta R_4 = (A_1 - A_2)\partial_t\overline{u}_1.$$

Using (A.7), (A.9), and product laws in Besov spaces, we see that

$$\|\delta R_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1})} \lesssim \|D\overline{v}_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}\|\delta u\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p-1}_{p,1})}, \tag{3.10}$$

$$\|\delta R_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1})} \lesssim \|D\overline{v}_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}\|\partial_t\delta u\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1})}. \tag{3.11}$$

In order to bound δR_3, it suffices to take advantage of (A.15). We get

$$\|\delta R_3\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1})} \lesssim \|D\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}\|\overline{u}_1\|_{L_\infty(\mathbb{R}_+;\dot B^{n/p-1}_{p,1})}. \tag{3.12}$$

Finally, using again (A.14), we see that

$$\|\delta R_4\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1})} \lesssim \|D\delta v\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}\|\partial_t\overline{u}_1\|_{L_1(\mathbb{R}_+;\dot B^{n/p-1}_{p,1})}. \tag{3.13}$$

One can now plug inequalities (3.3) into (3.13) in (3.2). We end up with

$$\|(\delta u, \nabla\delta P)\|_{E_p} \le C\big(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1})} + \|D\overline{v}_2\|_{L_1(\mathbb{R}_+;\dot B^{n/p}_{p,1})}\big)\|(\delta u, \nabla\delta P)\|_{E_p}$$
$$+ C\mu^{-1}\|(\overline{u}_1, \nabla\overline{P}_1)\|_{E_p}\|(\delta v, \nabla\delta Q)\|_{E_p}.$$

So we see that if (2.5) and (2.15) are satisfied for $\overline{v}_1$, $\overline{v}_2$, and ρ_0 with a small enough constant c, then we have

$$\|(\delta u, \nabla\delta P)\|_{E_p} \le 2CR\mu^{-1}\|(\delta v, \nabla\delta Q)\|_{E_p}.$$

Hence the map $\Phi : \widetilde{E}^R_p \mapsto \widetilde{E}^R_p$ is $\frac{1}{2}$-Lipschitz whenever R and the data have been chosen so that (2.15) and (3.1) are satisfied and $4CR \le \mu$. This completes the proof of existence of a unique solution to system (1.2) in $\widetilde{E}^R_p$.

3.3 Stability Estimates in E_p

In this part, we want to prove stability estimates in E_p for the solutions to (1.2). This will ensure both uniqueness and that the flow map is Lipschitz.

We consider two initial divergence-free velocity fields $u_{0,1}$ and $u_{0,2}$ in $\dot B^{n/p-1}_{p,1}$, and densities $\rho_{0,1}$ and $\rho_{0,2}$ in $\mathcal{M}(\dot B^{n/p-1}_{p,1})$ satisfying (2.15) and (3.1). We want to compare two solutions $(\overline{u}_1, \nabla\overline{P}_1)$ and $(\overline{u}_2, \nabla\overline{P}_2)$ in E_p of system (1.2), corresponding to data $(\rho_{0,1}, u_{0,1})$ and $(\rho_{0,2}, u_{0,2})$.

The proof is similar to that of the contractivity of Φ: we have to bound

$$\delta U(t) := \|\delta u\|_{L_\infty(0,t;\dot B^{n/p-1}_{p,1})} + \|\partial_t \delta u, \mu\nabla^2\delta u, \nabla\delta P\|_{L_1(0,t;\dot B^{n/p-1}_{p,1})}$$

with $(\delta u, \nabla\delta P) := (\overline{u}_2 - \overline{u}_1, \nabla\overline{P}_2 - \nabla\overline{P}_1)$ in terms of

$$\|\delta u_0\|_{\dot B^{n/p-1}_{p,1}} \quad \text{and} \quad \|\delta\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1})}.$$

Now, the system for $(\delta u, \nabla\delta P)$ reads

$$\begin{cases} \partial_t\delta u - \mu\Delta\delta u + \nabla\delta P = \delta f_0 + \delta f_1 + \delta f_2 + \delta f_3 + \mu\,\mathrm{div}(\delta f_4 + \delta f_5), \\ \mathrm{div}\,\delta u = \mathrm{div}((\mathrm{Id} - A_2)\delta u + (A_1 - A_2)\overline{u}_1) \\ \qquad\quad = D\delta u : (\mathrm{Id} - A_2) + D\overline{u}_1 : (A_1 - A_2) \end{cases}$$

with $\delta f_0 := \delta\rho_0\,\partial_t\overline{u}_1$ and where δf_i for $i \in \{1, \dots, 5\}$ has been defined in the previous subsection. Of course, now the matrices A_1 and A_2 correspond to the vector fields $\overline{u}_1$ and $\overline{u}_2$.

Using Proposition 2.1, we gather that for all $t \in [0, T)$

$$\begin{aligned} \delta U(t) \lesssim & \sum_{i=0}^{3} \|\delta f_i\|_{L_1(0,t;\dot B^{n/p-1}_{p,1})} + \mu\sum_{i=4}^{5}\|\delta f_i\|_{L_1(0,t;\dot B^{n/p}_{p,1})} \\ & + \sum_{i=1}^{4}\|\delta R_i\|_{L_1(0,t;\dot B^{n/p-1}_{p,1})} + \mu\|D\delta u : (\mathrm{Id} - A_2)\|_{L_1(0,t;\dot B^{n/p}_{p,1})} \\ & + \mu\|D\overline{u}_1 : (A_2 - A_1)\|_{L_1(0,t;\dot B^{n/p}_{p,1})} + \|\delta u_0\|_{\dot B^{n/p-1}_{p,1}}. \end{aligned}$$

From the definition of the multiplier space $\mathcal{M}(\dot B^{n/p-1}_{p,1})$, we readily have

$$\|\delta f_0\|_{\dot B^{n/p-1}_{p,1}} \le \|\delta\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1})}\|\partial_t\overline{u}_1\|_{\dot B^{n/p-1}_{p,1}}.$$

The other terms may be bounded as in the previous subsection. So we eventually conclude that for all $t \in [0, T)$

$$\delta U(t) \le \frac{1}{2}\delta U(t) + C\big(\|\delta\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1})} + \|\delta u_0\|_{\dot B^{n/p-1}_{p,1}}\big)$$

whenever, for $i = 1, 2$,

$$\sup_{t\in[0,T)}\|\overline{u}_i(t)\|_{\dot B^{n/p-1}_{p,1}} + \int_0^T \big(\mu^{-1}\|\partial_t\overline{u}_i, \nabla\overline{P}_i\|_{\dot B^{n/p-1}_{p,1}} + \|D\overline{u}_i\|_{\dot B^{n/p}_{p,1}}\big)\,dt$$

is small enough. This completes the proof of stability estimates.

3.4 Proof of the Local-in-Time Existence Result

Here we explain how the arguments of the previous subsections have to be modified so as to handle large initial velocities.

Let us first notice that the computations that have been performed in Section 2 also hold *locally* on $[0, T)$ whenever $\bar{v}$ satisfies

$$\begin{gathered}\bar{v} \in \mathcal{C}_b([0,T); \dot{B}^{n/p-1}_{p,1}), \quad \partial_t \bar{v}, \nabla^2 \bar{v} \in L_1(0,T; \dot{B}^{n/p-1}_{p,1}),\\ |DX_v| \equiv 1 \ \text{ on } [0,T) \times \mathbb{R}^n,\end{gathered} \tag{3.14}$$

and

$$\int_0^T \|D\bar{v}\|_{\dot{B}^{n/p}_{p,1}}\, dt \le c. \tag{3.15}$$

This ensures that, under condition (2.15), system (1.8) may be solved locally in $E_p(T)$. Of course, inequality (2.17) is still satisfied (for the norm in $E_p(T)$). However, it is not accurate enough to solve the nonlinear system if u_0 is too large. To overcome this, we shall apply the contraction mapping theorem in some suitable neighborhood of the solution $(u_L, \nabla P_L)$ to the "free" Stokes system, that is,

$$\begin{cases} \partial_t u_L - \mu \Delta u_L + \nabla P_L = 0 & \text{in } [0,T) \times \mathbb{R}^n,\\ \operatorname{div} u_L = 0 & \text{in } [0,T) \times \mathbb{R}^n,\\ u_L|_{t=0} = u_0 & \text{on } \mathbb{R}^n. \end{cases} \tag{3.16}$$

Setting $(\bar{u}, \nabla \bar{P}) := \Phi(\bar{v}, \nabla \bar{Q})$, we want to show that if T is small enough (a condition that will be expressed in terms of the free solution only) then $(\widetilde{u}, \nabla \widetilde{P}) := (\bar{u} - u_L, \nabla(\bar{P} - P_L))$ is small. For that, we shall apply Proposition 2.1 to the system satisfied by $(\widetilde{u}, \nabla \widetilde{P})$, namely,

$$\begin{cases} \partial_t \widetilde{u} - \mu \Delta \widetilde{u} + \nabla \widetilde{P} = f(\bar{u}, \nabla \bar{P}),\\ \operatorname{div} \widetilde{u} = g(\bar{u}),\\ \widetilde{u}|_{t=0} = 0, \end{cases} \tag{3.17}$$

where $f(\bar{u}, \nabla \bar{P})$ and $g(\bar{u})$ have been defined in Section 2.

On the one hand, we shall bound $f(\bar{u}, \nabla \bar{P})$ in $L_1(0,T; \dot{B}^{n/p-1}_{p,1})$ and $g(\bar{u})$ in $L_1(0,T; \dot{B}^{n/p}_{p,1})$ exactly as in Section 2; on the other hand, decomposing $\partial_t(g(\bar{u}))$ into $R^1(\bar{u}) + R^2(\bar{u})$, we see that the bound (2.11) for $R^2(\bar{u})$ is not accurate enough as it involves $\|\bar{u}\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1})}$, which need not be small for T going to 0, if u_0 is large. So we shall instead write

$$R^2(\bar{u}) = -\partial_t A_v u_L - \partial_t A_v \widetilde{u},$$

and use product laws and inequality (A.10) to get

$$\begin{aligned}\|R^2(\bar{u})\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1})} \lesssim{}& \|D\bar{v}\|_{L_2(0,T;\dot{B}^{n/p-1}_{p,1})} \|u_L\|_{L_2(0,T;\dot{B}^{n/p}_{p,1})}\\ &+ \|D\bar{v}\|_{L_1(0,T;\dot{B}^{n/p}_{p,1})} \|\widetilde{u}\|_{L_\infty(0,T;\dot{B}^{n/p-1}_{p,1})}.\end{aligned}$$

Finally, using Proposition 2.1 and decomposing everywhere $\overline{u}$ and $\nabla\overline{P}$ in $u_L+\widetilde{u}$ and $\nabla P_L+\nabla\widetilde{P}$, we get

$$\begin{aligned}
&\|(\widetilde{u},D\widetilde{P})\|_{E_p(T)}\\
&\quad\le C(\|1-\rho_0\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1})}+\|D\overline{v}\|_{L_1(0,T;\dot B^{n/p}_{p,1})})(\|(\widetilde{u},D\widetilde{P})\|_{E_p(T)}\\
&\qquad+\|\partial_t u_L,\mu D^2u_L,DP_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1})})\\
&\qquad+C\|\overline{v}\|_{L_2(0,T;\dot B^{n/p}_{p,1})}\|u_L\|_{L_2(0,T;\dot B^{n/p}_{p,1})}.
\end{aligned}$$

Therefore, if (2.15) and (3.15) are satisfied with c small enough and if $\widetilde{v}:=\overline{v}-u_L$, then we get

$$\begin{aligned}
&\|(\widetilde{u},D\widetilde{P})\|_{E_p(T)}\\
&\quad\le Cc\|\partial_t u_L,\mu D^2u_L,DP_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1})}+C\|u_L\|^2_{L_2(0,T;\dot B^{n/p}_{p,1})}\\
&\qquad+C\|\widetilde{v}\|_{L_2(0,T;\dot B^{n/p}_{p,1})}\|u_L\|_{L_2(0,T;\dot B^{n/p}_{p,1})}.
\end{aligned}$$

This inequality together with the interpolation inequality

$$\|\widetilde{v}\|_{L_2(0,T;\dot B^{n/p}_{p,1})}\|\le\|\widetilde{v}\|^{1/2}_{L_1(0,T;\dot B^{n/p+1}_{p,1})}\|\widetilde{v}\|^{1/2}_{L_\infty(0,T;\dot B^{n/p-1}_{p,1})}$$

ensures that Φ maps $(u_L,\nabla P_L)+\widetilde{E}^R_p(T)$ (where $\widetilde{E}^R_p(T)$ is the "local" version of $\widetilde{E}^R_p$) into itself whenever T satisfies

$$\begin{aligned}
&Cc\|\partial_t u_L,\mu D^2u_L,DP_L\|_{L_1(0,T;\dot B^{n/p-1}_{p,1})}\\
&\qquad+C\|u_L\|^2_{L_2(0,T;\dot B^{n/p}_{p,1})}\le\frac{R}{2},\\
&\quad C\mu^{-1/2}\|u_L\|_{L_2(0,T;\dot B^{n/p}_{p,1})}\le\frac12.
\end{aligned}\tag{3.18}$$

Of course, for (3.15) to be satisfied, it suffices to take $R=c\mu/2$ and to assume that T is so small that

$$\|Du_L\|_{L_1(0,T;\dot B^{n/p}_{p,1})}\le\frac{c}{2}.\tag{3.19}$$

So if (3.18) and (3.19) are satisfied (conditions that depend only on the data), then one may conclude that Φ maps $(u_L,\nabla P_L)+\widetilde{E}^R_p(T)$ into itself.

The proof of the contraction properties for Φ in this context follows the same lines: We consider $(\overline{u}_i,\nabla\overline{P}_i)=\Phi(\overline{v}_i,\nabla\overline{Q}_i)$ with $(\overline{v}_i,\nabla\overline{Q}_i)$ in $(u_L,\nabla P_L)+\widetilde{E}^R_p(T)$ for $i=1,2$; then we bound all the terms δf_i, δg_i, and δR_i as in the case of small initial velocity except for δR_3, since it involves $\|\overline{u}_1\|_{L_\infty(0,T;\dot B^{n/p-1}_{p,1})}$, which need not be small for T going to 0. For this latter term, we notice that, according to (A.16),

$$\|\delta R_3\|_{L_1(0,T;\dot{B}^{n/p-1}_{p,1})} \lesssim \|\delta v\|_{L_2(0,T;\dot{B}^{n/p}_{p,1})}\|\overline{u}_1\|_{L_2(0,T;\dot{B}^{n/p}_{p,1})}.$$

So we eventually get

$$\begin{aligned}
&\|(\delta u, \nabla\delta P)\|_{E_p(T)}\\
&\quad\le C(\|1-\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})} + \|D\overline{v}_2\|_{L_1(0,T;\dot{B}^{n/p}_{p,1})})\|(\delta u, \nabla\delta P)\|_{E_p(T)}\\
&\qquad + C(\mu^{-1}\|(\widetilde{u}_1, \nabla\widetilde{P}_1)\|_{E_p(T)} + \mu^{-1/2}\|u_L\|_{L_2(0,T;\dot{B}^{n/p}_{p,1})})\|(\delta v, \nabla\delta Q)\|_{E_p(T)}).
\end{aligned}$$

Note that our assumptions on $(\widetilde{u}_1, \nabla\widetilde{P}_1)$ and on the free solution ensures that if R and T have been chosen small enough, then the factor of the last term is smaller than, say, $\frac{1}{2}$. So the contraction mapping theorem applies. This completes the proof of the existence part of Theorem 1.2. Proving the stability and uniqueness follows from similar arguments. The details are left to the reader.

3.5 Proof of Theorem 1.3

Given data (ρ_0, u_0) satisfying the assumptions of Theorem 1.3, one may construct a global solution $(\overline{u}, \nabla\overline{P})$ to system (1.2) in E_p. If X_u denotes the "flow" to $\overline{u}$, which is defined according to (1.9), then the results of the appendix ensure that, for all $t \in \mathbb{R}_+$, $X_u(t,\cdot)$ is a C^1-diffeomorphism of $\mathbb{R}^n$. In particular, one may set

$$\rho(t,\cdot) := \rho_0 \circ X_u^{-1}(t,\cdot), \quad P(t,\cdot) := \overline{P}(t,\cdot) \circ X_u^{-1}(t,\cdot),$$
$$u(t,\cdot) := \overline{u}(t,\cdot) \circ X_u^{-1}(t,\cdot),$$

and the algebraic relations that are derived in the Appendix show that $(\rho, u, \nabla P)$ satisfies system (0.1). In addition, given that $X_u(t,\cdot)$ is measure preserving and that $DX_u(t) - \mathrm{Id}$ belongs to $\dot{B}^{n/p}_{p,1}$, the map $a \mapsto a \circ X_u^{\pm 1}(t)$ is continuous from $\dot{B}^{s}_{p,1}$ to itself if $s \in \{n/p - 1, n/p\}$ (see, e.g., [8, chap. 2]). This implies the following:

- The Eulerian velocity u is in $\mathcal{C}_b(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$.
- For any $\phi \in \dot{B}^{n/p-1}_{p,1}$, we have $\phi \circ X_u^{\pm 1}(t) \in \dot{B}^{n/p-1}_{p,1}$. So, given that $\rho_0 \in \mathcal{M}(\dot{B}^{n/p-1}_{p,1})$, we have

  $$\phi\rho(t) = ((\phi \circ X_u(t))\rho_0) \circ X_u^{-1}(t) \in \dot{B}^{n/p-1}_{p,1}.$$

  Hence $\rho \in L_\infty(\mathbb{R}_+; \mathcal{M}(\dot{B}^{n/p-1}_{p,1}))$.
- The chain rule ensures that

  $$\nabla P = (A_u^\top \cdot \nabla\overline{P}) \circ X_u^{-1}.$$

  So combining product laws and the invariance of $\dot{B}^{n/p-1}_{p,1}$ by right-composition, we get $\nabla P \in L_1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1})$.

- The chain rule also ensures that

$$\nabla u = (A_u^{\mathsf{T}} \cdot \nabla \overline{u}) \circ X_u^{-1}.$$

So using the fact that $\dot{B}^{n/p}_{p,1}$ is a Banach algebra, and using the invariance of $\dot{B}^{n/p}_{p,1}$ by right-composition, we get $\nabla u \in L_1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$.

In order to prove uniqueness, we consider $(\rho_1, u_1, \nabla P_1)$ and $(\rho_2, u_2, \nabla P_2)$, two solutions of (0.1) corresponding to the same data (ρ_0, u_0), and perform the Lagrangian change of variable pertaining to the flow of u_1 and u_2, respectively. The obtained functions $(\overline{u}_1, \nabla \overline{P}_1)$ and $(\overline{u}_2, \nabla \overline{P}_2)$ both satisfy (1.2) with the same ρ_0 and u_0. Hence they coincide as a consequence of the uniqueness part of Theorem 1.1.

Appendix

Let us first derive algebraic relations involving changes of coordinates. We are given a C^1 *measure-preserving* diffeomorphism X over $\mathbb{R}^n$. For $H : \mathbb{R}^n \to \mathbb{R}^m$, we agree that $\overline{H}(y) = H(x)$ with $x = X(y)$. With this convention, the chain rule can be written

$$D_y \overline{H}(y) = D_x H(X(y)) \cdot D_y X(y) \quad \text{with } (D_y X)_{ij} := \partial_{y_j} X^i, \tag{A.1}$$

or, denoting $\nabla_y = D_y^{\mathsf{T}}$,

$$\nabla_y \overline{H}(y) = (\nabla_y X(y)) \cdot \nabla_x H(X(y)).$$

Hence we have

$$D_x H(x) = D_y \overline{H}(y) \cdot A(y) \quad \text{with } A(y) := (D_y X(y))^{-1} = D_x X^{-1}(x). \tag{A.2}$$

LEMMA A.1. *Let H be a vector field over $\mathbb{R}^n$. If we denote $\overline{H} = H \circ X$, then the following relation holds true*:

$$\operatorname{div}_x H(x) = \operatorname{div}_y (A\overline{H})(y) = \operatorname{div}_y (\operatorname{adj}(D_y X)\overline{H})(y) \quad \textit{with } x = X(y), \tag{A.3}$$

where $\operatorname{adj}(D_y X)$ *stands for the adjugate of $D_y X$, that is, the transpose of the cofactor matrix of $D_y X$.*

PROOF. This lemma stems from the following series of computations (based on integrations by parts, (A.2), and the fact that X is measure preserving), which hold for any scalar test function q:

$$\begin{aligned}
\int q(x) \operatorname{div}_x H(x)\, dx &= -\int D_x q(x) \cdot H(x) dx, \\
&= -\int D_x q(X(y)) \cdot H(X(y)) dy \\
&= -\int D_y \overline{q}(y) \cdot A(y) \cdot \overline{H}(y) dy, \\
&= \int \overline{q}(y) \operatorname{div}_y (AH)(y) dy.
\end{aligned}$$

As X is measure preserving, we have $A = \operatorname{adj}(D_y X)$, whence the desired result. □

Remark A.2. Combining (A.2) and (A.3), we deduce that if $a : \mathbb{R}^n \to \mathbb{R}$ then

$$\overline{\Delta_x a} = \overline{\operatorname{div}_x \nabla_x a} = \operatorname{div}_y(A\overline{\nabla_x a}) = \operatorname{div}_y(AA^{\mathsf{T}}\nabla_y \overline{a}).$$

If v is a time-dependent vector field with coefficients in $L_1(0,T;C^{0,1})$, we recall that it then has, by virtue of the Cauchy-Lipschitz theorem, a unique C^1 flow X_v satisfying

$$X_v(t,y) = y + \int_0^t v(\tau, X_v(\tau,y))\,d\tau \quad \text{for all } t \in [0,T),$$

and that $X_v(t,\cdot)$ is a C^1-diffeomorphism over $\mathbb{R}^n$.

Lemma A.1 enables us to deduce the following "magic" relation, which is the cornerstone of the proof of our main results:

COROLLARY A.3. *Let v and w be two time-dependent vector fields with coefficients in $L_1(0,T;C^{0,1})$. Let X_v and X_w be the corresponding flows. Denote $A_v := (DX_v)^{-1}$ and $A_w := (DX_w)^{-1}$. Let us introduce the Lagrangian coordinates y_v and y_w pertaining to v and w, respectively, defined by*

$$x = X_v(y_v) = X_w(y_w).$$

Assume in addition that

$$|DX_v| \equiv 1 \quad \text{and} \quad \operatorname{div}(A_v \overline{w}_v) = 0 \ \text{ with } \overline{w}_v := w \circ X_v.$$

Then $|DX_w| \equiv 1$ and for any C^1 vector field H, one has

$$\operatorname{div} H(x) = (D\overline{H}_v : A_v)(y_v) = \operatorname{div}(A_w \overline{H}_w)(y_w)$$
$$\text{with } \overline{H}_v := H \circ X_v \text{ and } \overline{H}_w := H \circ X_w.$$

PROOF. With the above notation, the chain rule ensures that

$$D_x H(x) = D_{y_v}\overline{H}_v(y_v) \cdot A_v(y_v).$$

Hence taking the trace yields the left equality.

Next, according to Lemma A.1 and to our assumption over v and w, we have

$$0 = \operatorname{div}(A_v \overline{w}_v)(y_v) = \operatorname{div}_x w(x).$$

Hence the Liouville theorem ensures that $|DX_w| \equiv 1$. Finally, applying Lemma A.1 with X_w completes the proof. □

LEMMA A.4. *There exist n^2 at least quadratic polynomials $P_{ij} : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ of degree $n-1$ such that*

$$\operatorname{Id} - \operatorname{adj}(\operatorname{Id} + C) = (C - (\operatorname{Tr} C)\operatorname{Id}) + P_2(C),$$

where $P_2(C)$ is the $n \times n$ matrix with entries $P_{ij}(C)$.

PROOF. It suffices to use the fact that, by definition of the differential of adj, we have

$$\mathrm{Id} - \mathrm{adj}(\mathrm{Id} + C) = \mathrm{adj}(\mathrm{Id}) - \mathrm{adj}(\mathrm{Id} + C) = -d\,\mathrm{adj}(\mathrm{Id})(C) + P_2(C).$$

Now,

$$\mathrm{adj}(\mathrm{Id} + C) = (\mathrm{Id} + C)^{-1}\det(\mathrm{Id} + C),$$

and the differential of the reciprocal operator at Id is $C \mapsto -C$, while

$$d\det(\mathrm{Id})(C) = (\mathrm{Tr}\, C)\,\mathrm{Id}.$$

So $d\ \mathrm{adj}(\mathrm{Id})(C) = (\mathrm{Tr}\, C)\,\mathrm{Id} - C$. □

We now want to establish some a priori estimates for the flow that will be needed in our main results. The first difficulty that has to be faced is that when implementing the iterative process for solving (1.2), we are given the velocity field $\overline{v}$ *in Lagrangian coordinates*. Therefore, it first has to be checked whether the "flow" $X_v(t,\cdot)$ defined by

$$X_v(t,y) := y + \int_0^t \overline{v}(\tau,y)\,d\tau \tag{A.4}$$

is a C^1-diffeomorphism over $\mathbb{R}^n$. This property is required for constructing the Eulerian vector field v by setting $v(t,\cdot) := v \circ X_v^{-1}(t,\cdot)$.

So let us assume that we are given some vector field $\overline{v}$ over $[0,T) \times \mathbb{R}^n$ with

$$\overline{v} \in \mathcal{C}_b\big([0,T);\dot{B}^{n/p-1}_{p,1}\big), \quad \partial_t\overline{v} \in L_1\big([0,T);\dot{B}^{n/p-1}_{p,1}\big),$$
$$D\overline{v} \in L_1\big([0,T);\dot{B}^{n/p}_{p,1}\big).$$

Differentiating (A.4) with respect to the space variable yields

$$DX_v(t,y) := \mathrm{Id} + \int_0^t D\overline{v}(\tau,y)\,d\tau. \tag{A.5}$$

As $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ is embedded in the set $\mathcal{C}_0(\mathbb{R}^n)$ of continuous functions going to 0 at infinity, we deduce that X_v is a C^1 function over $\mathbb{R}_+ \times \mathbb{R}^n$. However, in general, $X_v(t,\cdot)$ need not be a C^1-diffeomorphism over $\mathbb{R}^n$ for all $t \in \mathbb{R}_+$. So we assume that the smallness condition (2.5) is satisfied with c small enough. Then, using embedding we see that it guarantees that

$$\|DX_v(t,\cdot) - \mathrm{Id}\|_{L_\infty(\mathbb{R}^n)} \le \frac{1}{2} \quad \text{for all } t \in \mathbb{R}_+.$$

Hence, for any $t \in \mathbb{R}_+$, the map $X_v(t,\cdot)$ is a local diffeomorphism. In order to show that it is a *global* diffeomorphism, we introduce the solution Y_v to the ordinary differential equation

$$\overline{v}(t,Y_v(t,x)) + DX_v(t,Y_v(t,x))\frac{d}{dt}Y_v(t,x) = 0. \tag{A.6}$$

Under (2.5), the matrix DX_v is invertible at every point and $(DX_v)^{-1} - \mathrm{Id}$ belongs to $L_\infty(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})$. Indeed, one may write

$$(DX_v)^{-1} - \mathrm{Id} = \sum_{k\geq 1} (\mathrm{Id} - DX_v)^k.$$

Hence, using the assumptions over $\overline{v}$ and the product laws in Besov spaces (here we need that $1 \leq p < 2n$), (A.6) may be seen as an ordinary differential equation in the Banach space $\dot{B}^{n/p-1}_{p,1}$, which may be solved on $[0, T)$ according to the Cauchy-Lipschitz theorem. In particular, differentiating $X_v(t, \cdot) \circ Y_v(t, \cdot)$ with respect to time, we easily gather that $X_v(t, \cdot) \circ Y_v(t, \cdot) = \mathrm{Id}$ for all $t \in [0, T)$. Therefore, one may eventually conclude that $X_v(t, \cdot)$ is a C^1-diffeomorphism over $\mathbb{R}^n$, with inverse $Y_v(t, \cdot)$.

Let us now derive some "flow estimates" that will be needed for constructing the maps Φ and Ψ. For completeness, the statements here are slightly more general than needed; that $|DX| \equiv 1$ is not assumed.

LEMMA A.5. *Let $p \in [1, +\infty)$. Under assumption (2.5) for $\overline{v}$, we have*

$$\|\mathrm{Id} - \mathrm{adj}(DX(t))\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|D\overline{v}\|_{L_1(0,t;\dot{B}^{n/p}_{p,1})}, \tag{A.7}$$

$$\|\mathrm{Id} - A(t)\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|D\overline{v}\|_{L_1(0,t;\dot{B}^{n/p}_{p,1})}, \tag{A.8}$$

$$\|\partial_t(\mathrm{adj}(DX))(t)\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|D\overline{v}(t)\|_{\dot{B}^{n/p}_{p,1}}, \tag{A.9}$$

$$\|\partial_t(\mathrm{adj}(DX))(t)\|_{\dot{B}^{n/p-1}_{p,1}} \lesssim \|D\overline{v}(t)\|_{\dot{B}^{n/p-1}_{p,1}} \quad \textit{if } p < 2n, \tag{A.10}$$

$$\|\mathrm{adj}(DX(t))A^{\mathsf{T}}(t) - \mathrm{Id}\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|D\overline{v}\|_{L_1(0,t;\dot{B}^{n/p}_{p,1})}. \tag{A.11}$$

PROOF. According to Lemma A.4 and to (A.5), one may write

$$\mathrm{Id} - \mathrm{adj}(DX(t)) = \int_0^t (D\overline{v} - \mathrm{div}\,\overline{v}\,\mathrm{Id})\,d\tau + P_2\left(\int_0^t D\overline{v}\,d\tau\right)$$

where the coefficients of P_2 are at least quadratic polynomials of degree $n-1$. Given that $\dot{B}^{n/p}_{p,1}$ is a Banach algebra and that (2.5) holds, we readily get the result.

In order to prove the second estimate, we just use the fact that, under assumption (2.5), we have

$$A(t) = (\mathrm{Id} + C(t))^{-1} = \sum_{k\in\mathbb{N}} (-1)^k (C(t))^k \quad \text{with } C(t) = \int_0^t D\overline{v}\,d\tau, \tag{A.12}$$

and that $\dot{B}^{n/p}_{p,1}$ is a Banach algebra.

In order to prove the third inequality, we use the fact that, according to Lemma A.4, we have

$$\partial_t(\mathrm{adj}(DX)) = \frac{\partial}{\partial t}\left(\mathrm{Id} + \int_0^t \big(\mathrm{div}\,\overline{v}\,\mathrm{Id} - D\overline{v}\big)\,d\tau + P_2\left(\int_0^t D\overline{v}\,d\tau\right)\right).$$

Hence

$$\partial_t(\operatorname{adj}(DX))(t) = (\operatorname{div}\overline{v}(t)\,\mathrm{Id} - D\overline{v}(t)) + dP_2\left(\int_0^t D\overline{v}\,d\tau\right)\cdot D\overline{v}(t).$$

As the coefficients of dP_2 are polynomials of n^2 variables that vanish at 0, we get

$$\|\partial_t(\operatorname{adj}(DX))(t)\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|D\overline{v}(t)\|_{\dot{B}^{n/p}_{p,1}}\left(1 + \left\|\int_0^t D\overline{v}\,d\tau\right\|_{\dot{B}^{n/p}_{p,1}}\right);$$

hence (A.9). Proving (A.10) is similar: it is only a matter of using the continuity of the product from $\dot{B}^{n/p-1}_{p,1} \times \dot{B}^{n/p}_{p,1}$ to $\dot{B}^{n/p-1}_{p,1}$, if $p < 2n$.

For proving the last inequality, we use the decomposition

$$\operatorname{adj}(DX)A^{\mathsf{T}} - \mathrm{Id} = (\operatorname{adj}(DX) - \mathrm{Id})A^{\mathsf{T}} + (A - \mathrm{Id})^{\mathsf{T}}.$$

So combining inequalities (A.7) and (A.8) and the fact that $\dot{B}^{n/p}_{p,1}$ is a Banach algebra, we get the result. □

LEMMA A.6. *Let $\overline{v}_1$ and $\overline{v}_2$ be two vector fields satisfying* (2.5) *and* $\delta v := \overline{v}_2 - \overline{v}_1$. *Then we have for all $p \in [1, +\infty)$,*

$$\|A_2 - A_1\|_{L_\infty(0,t;\dot{B}^{n/p}_{p,1})} \lesssim \|D\delta v\|_{L_1(0,t;\dot{B}^{n/p}_{p,1})}, \tag{A.13}$$

$$\|\operatorname{adj}(DX_2) - \operatorname{adj}(DX_1)\|_{L_\infty(0,t;\dot{B}^{n/p}_{p,1})} \lesssim \|D\delta v\|_{L_1(0,t;\dot{B}^{n/p}_{p,1})}, \tag{A.14}$$

$$\|\partial_t(\operatorname{adj}(DX_2) - \operatorname{adj}(DX_1))\|_{L_1(0,t;\dot{B}^{n/p}_{p,1})} \lesssim \|D\delta v\|_{L_1(0,t;\dot{B}^{n/p}_{p,1})}, \tag{A.15}$$

$$\|\partial_t(\operatorname{adj}(DX_2) - \operatorname{adj}(DX_1))\|_{L_2(0,t;\dot{B}^{n/p-1}_{p,1})} \lesssim \|D\delta v\|_{L_2(0,t;\dot{B}^{n/p-1}_{p,1})} \quad \text{if } p < 2n. \tag{A.16}$$

PROOF. In order to prove the first inequality, we use the fact that, for $i = 1, 2$, we have

$$A_i = (\mathrm{Id} + C_i)^{-1} = \sum_{k\geq 0}(-1)^k C_i^k \quad \text{with } C_i(t) = \int_0^t D\overline{v}_i\,d\tau.$$

Hence

$$A_2 - A_1 = \sum_{k\geq 1}(C_2^k - C_1^k) = \left(\int_0^t D\delta v\,d\tau\right)\sum_{k\geq 1}\sum_{j=0}^{k-1} C_1^j C_2^{k-1-j}.$$

So using the fact that $\dot{B}^{n/p}_{p,1}$ is a Banach algebra, it is easy to conclude (A.13).

The second inequality is a consequence of Lemma A.4 and the Taylor formula, which ensures that, denoting $\delta C := C_2 - C_1$,

$$\operatorname{adj}(DX_2) - \operatorname{adj}(DX_1) = (\operatorname{Tr}\delta C)\,\mathrm{Id} - \delta C + dP_2(C_1)(\delta C) + \frac{1}{2}d^2P_2(C_1)(\delta C, \delta C) + \cdots$$

where the coefficients of P_2 are polynomials of degree $n-1$. As the sum is finite and $\dot{B}^{n/p}_{p,1}$ is a Banach algebra, we get (A.14).

In order to prove the last two estimates, one may differentiate the above relation with respect to t. Keeping in mind the definition of C_1 and C_2, we get

$$\partial_t(\operatorname{adj}(DX_2)-\operatorname{adj}(DX_1)) = (\operatorname{div}\delta v)\operatorname{Id}-D\delta v + dP_2(C_1)\cdot\partial_t\delta C + d^2P_2(C_1)(\partial_t C_1,\delta C)+\cdots.$$

Then using the product laws in Besov spaces yields the desired inequalities. □

Finally, we have to justify that the multiplier space $\mathcal{M}(\dot{B}^{n/p-1}_{p,1})$ contains characteristic functions of C^1-bounded domains if $p>n-1$. This is a consequence of the following lemma:

LEMMA A.7. *Let Ω be the half-space $\mathbb{R}^n_+$ or a bounded domain of $\mathbb{R}^n$ with C^1-boundary. Assume that $s\in\mathbb{R}$ and $p,q\in[1,\infty]$ are such that*

$$-1+\frac{1}{p}<s<\frac{1}{p}\cdot \tag{A.17}$$

Then the characteristic function $\mathbb{1}_\Omega$ of Ω belongs to the space $\mathcal{M}(\dot{B}^s_{p,q}(\mathbb{R}^n))$.

PROOF. This result, which belongs to mathematical folklore, is closely related to the fact that under condition (A.17), functions in $\dot{B}^s_{p,q}(\Omega)$ extended by 0 on the whole space belong to $\dot{B}^s_{p,q}(\mathbb{R}^n)$. In the case where Ω is the half-space $\mathbb{R}^n_+$, the lemma has been proved in [7, prop. 3].

If Ω is a bounded C^1 domain, then one may find a finite number N of $\mathcal{C}^\infty_c(\mathbb{R}^n)$ functions ϕ_i and C^1-diffeomorphisms ψ_i so that for any $u\in\dot{B}^s_{p,q}(\mathbb{R}^n)$,

$$u\mathbb{1}_\Omega=\sum_{i=1}^N u\phi_i\mathbb{1}_\Omega \quad\text{and}\quad (u\phi_i\mathbb{1}_\Omega)\circ\psi_i=\mathbb{1}_{\mathbb{R}+}\cdot((u\phi_i)\circ\psi_i).$$

Now, because condition (A.17) is satisfied, the space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is stable by multiplication by smooth compactly supported functions and by a C^1 change of variables (see, e.g., [8, chap. 2]). Therefore the functions $(u\phi_i)\circ\psi_i$ belong to $\dot{B}^s_{p,q}(\mathbb{R}^n)$, too. Using again the stability of this space by multiplication by $\mathbb{1}_{\mathbb{R}+}$, one may thus conclude that $(u\phi_i\mathbb{1}_\Omega)\circ\psi_i\in\dot{B}^s_{p,q}(\mathbb{R}^n)$. Hence $u\phi_i\mathbb{1}_\Omega$ is in $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for all $i\in\{1,\dots,N\}$. This completes the proof of the lemma. □

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