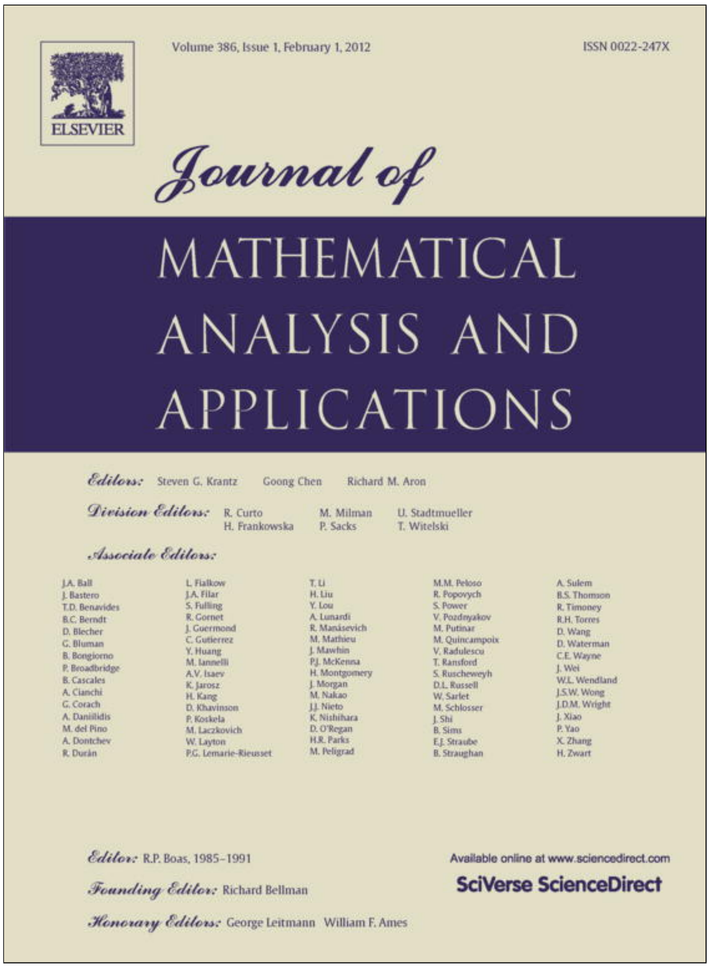
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# The divergence equation in rough spaces

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ABSTRACT

We aim at extending the existence theory for the equation $\operatorname{div} v = f$ in a bounded or exterior domain with homogeneous Dirichlet boundary conditions, to a class of solutions which need not have a trace at the boundary. Typically, the weak solutions that we shall consider will belong to some Besov space $B^s_{p,q}(\Omega)$ with $s \in (-1+1/p, 1/p)$. After generalizing the notion of a solution for this equation, we propose an explicit construction by means of the classical Bogovskiĭ formula. This construction enables us to keep track of a "marginal" information about the trace of solutions. In particular, it ensures that the trace is zero if $f$ is smooth enough. We expect our approach to be of interest for the study of rough solutions to systems of fluid mechanics.



## 1. Introduction

The divergence equation

$$\operatorname{div} v = f \quad \text{in } \Omega \quad \text{and} \quad v = 0 \quad \text{at } \partial\Omega, \tag{1.1}$$

where $f$ is a given function on $\Omega$ occurs in a number of mathematical problems. It is related to the study of the Helmholtz decomposition and of the Stokes system hence has close connections with the incompressible or compressible Navier–Stokes equations. It is also of interest in other fields where vector analysis plays an important role.

The divergence equation has been considered by a number of authors (see e.g. Galdi's book [12] and the references therein). In [3], M. Bogovskiĭ has proposed an explicit formula (after an old idea by Sobolev in [24]) for solving (1.1) in the case of a bounded star-shaped domain whenever the function $f$ is continuous and satisfies

$$\int_\Omega f\,dx = 0. \tag{1.2}$$

Arguing by density, this gives an explicit solution operator which is continuous from $L_p(\Omega)$ to $W^1_p(\Omega)$. The construction may be extended to more general domains and functional spaces. The starting point of our paper will be the following result which has been proved in e.g. [18]:

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**Theorem 1.** *Let $\Omega$ be a bounded domain with a Lipschitz boundary. Let $q \in [1,\infty]$, $p \in (1,\infty)$ and $-1+\frac{1}{p} < s < \frac{1}{p}$. There exists an explicit continuous mapping $\mathcal{L}$ from the set of functions $f$ in $B^s_{p,q}(\Omega)$ satisfying the compatibility condition* (1.2) *into $B^{1+s}_{p,q}(\Omega)$, such that $\mathcal{L}(f)$ satisfies* (1.1) *in the sense of distributions.*

The present paper aims at considering the less regular case where $f = \operatorname{div} k$ with $k$ in $B^s_{p,q}(\Omega;\mathbb{R}^n)$ and $s$ close to zero. In this framework, the solution to the divergence equation need not have a trace at the boundary. Nevertheless, we want to generalize the classical results keeping some marginal information about the trace in a very weak meaning. In addition, we want the constructed weak solutions to coincide with those of the above theorem if $k$ is smooth enough.

A different point of view concerning (1.1) has been presented recently in [5]. There, the authors consider a generalization of Bogovskiĭ formula on negative spaces. However the result therein involves the so-called spaces without boundary conditions, that is spaces for which extension by zero onto the whole $\mathbb{R}^n$ preserves regularity. This approach does not give any information for the behavior of solutions at the boundary, a question which is of fundamental importance from the PDEs point of view.

As pointed out above, in our context the meaning of the boundary condition in the divergence equation (1.1) is not obvious for a $B^s_{p,q}(\Omega)$ function with small $s$ need not have a trace at $\partial\Omega$. At the same time, it is well known that any vector field $u$ with coefficients and divergence in $L_p(\Omega)$ admits a normal trace at the boundary (see e.g. [11,27, Theorem 1.2]). It is based on the fact that if $u$ and $\partial\Omega$ are Lipschitz then Stokes formula implies that[1]

$$\int_{\partial\Omega} \varphi u \cdot \vec{n}\, d\sigma = \int_{\Omega} u \cdot \nabla\varphi\, dx + \int_{\Omega} \varphi \operatorname{div} u\, dx \quad \text{for all } \varphi \in \mathcal{C}^\infty_c(\overline{\Omega}).$$

The right-hand side makes sense whenever $u$ and $\operatorname{div} u$ are in $L_p(\Omega)$ and $\varphi \in W^1_{p'}(\Omega)$ with $1/p + 1/p' = 1$. As the trace operator on $\partial\Omega$ extends from $W^1_{p'}(\Omega)$ onto $W^{1/p}_{p'}(\partial\Omega)$ we thus deduce that $u \cdot \vec{n}|_{\partial\Omega}$ may be defined as an element of $W^{-1/p}_p(\partial\Omega) := (W^{1/p}_{p'}(\partial\Omega))^*$, that is as a continuous functional on $W^{1/p}_{p'}(\partial\Omega)$.

We shall first provide an *abstract* construction of solutions in connection with the description of functionals on $B^{-1-s}_{p',q'}(\Omega)$. Unfortunately, this simple construction does not supply any handy information on the solutions. This motivates us to propose another more explicit construction, so as to get a linear solution operator which is continuous in all the Besov spaces that we shall consider.

The paper is organized as follows. In the next section we reformulate (1.1) as a "generalized" divergence equation involving *distributions up to the boundary*. We expect this new approach to be of relevance for the study of boundary problems with very low regularity (see e.g. [19,25,26]) or for models of compressible fluid mechanics [10,20,21]. Next, we state our main result, Theorem 2. In Section 3, we recall basic definitions and auxiliary results for the Besov spaces, together with an interpolation result, Lemma 1, which, roughly, will enable us to reduce the study of (1.1) to the case $f = \operatorname{div} k$ with $k \in L_p(\Omega)$ or $k \in W^1_p(\Omega)$. An abstract functional analysis approach for solving the generalized divergence equation is presented in Section 4. The last two sections are devoted to solving the generalized divergence equation, explicitly. Section 5 is the core of the paper. There we prove Theorem 2 in the case of a bounded star-shaped domain. In Section 6, we consider more general domains. To simplify, we focus on the case of bounded or exterior domains. However, as the idea is to decompose the domain into a finite union of star-shaped domains, more complicated domains may be achieved by a similar method.

## 2. The main result

Let us first reformulate the divergence equation in terms of some functional $\mathcal{DIV}[k;\zeta]$ acting on smooth functions *up to the boundary of* $\Omega$, which contents both the information on the divergence of $k$, and some distribution $\zeta$ over the boundary.

**Definition 1.** Let $k = (k_1,\dots,k_n)$ be a distribution on $\Omega$ and $\zeta$, a distribution on $\partial\Omega$. We shall denote by $\mathcal{DIV}[k;\zeta]$ the linear functional defined on the set $\mathcal{C}^\infty_c(\overline{\Omega})$ of smooth functions with compact support in $\overline{\Omega}$, by

$$\mathcal{DIV}[k;\zeta](\varphi) := -\int_{\Omega} k \cdot \nabla\varphi\, dx + \int_{\partial\Omega} \varphi\zeta\, d\sigma.$$

In the smooth case, $\mathcal{DIV}[k;k\cdot\vec{n}]$ coincides with the definition of the divergence of $k$ in the distribution up to the boundary meaning and it is clear that finding a solution $v$ to $\operatorname{div} v = \operatorname{div} k$ with $v\cdot\vec{n}$ at the boundary is equivalent to

$$-\int_{\Omega} v \cdot \nabla\varphi\, dx = -\int_{\Omega} k \cdot \nabla\varphi\, dx + \int_{\partial\Omega} \varphi k \cdot \vec{n}\, d\sigma \quad \text{for all } \varphi \in \mathcal{C}^\infty_c(\overline{\Omega}).$$

[1] In all the paper $\vec{n}$ denotes the exterior normal vector at the boundary of $\Omega$.

In the rough context that we plan to investigate here, it is natural to decorrelate the normal trace of $k$ (which need not be defined) and $k$. More precisely, given some distributions $k$ and $\zeta$ on $\Omega$ and $\partial\Omega$, respectively, we aim at finding some vector field $v$ such that

$$\mathcal{DIV}[v;0]=\mathcal{DIV}[k;\zeta], \tag{2.1}$$

or in other words,

$$-\int_\Omega v\cdot\nabla\varphi\,dx=-\int_\Omega k\cdot\nabla\varphi\,dx+\int_{\partial\Omega}\varphi\zeta\,d\sigma\quad\text{for all }\varphi\in\mathcal{C}_c^\infty(\overline{\Omega}).$$

Note that if $(k,\zeta)$ is in $L_p(\Omega)\times W_p^{-\frac{1}{p}}(\partial\Omega)$, and if (2.1) admits a solution in $L_p(\Omega)$ then taking $\varphi$ with compact support implies that $\operatorname{div}(k-v)=0$ in $\mathcal{D}'(\Omega)$. Hence $(k-v)\cdot\vec{n}$ is defined on $\partial\Omega$ as an element of $W_p^{-\frac{1}{p}}(\partial\Omega)$ and we do have for all $\psi\in\mathcal{C}^\infty(\partial\Omega)$,

$$\bigl\langle(k-v)\cdot\vec{n},\psi\bigr\rangle_{W_p^{-\frac{1}{p}}(\partial\Omega),W_{p'}^{\frac{1}{p}}(\partial\Omega)}=\langle\zeta,\psi\rangle_{W_p^{-\frac{1}{p}}(\partial\Omega),W_{p'}^{\frac{1}{p}}(\partial\Omega)}.$$

In this paper, we aim at solving Eq. (2.1) whenever $k$ belongs to some Besov space $B^s_{p,q}(\Omega)$ with $-1+1/p<s<1/p$. Obviously, if $\Omega$ is a bounded domain then one may take $\varphi\equiv1$ as a test function, hence a necessary condition for solvability is that $\zeta$ satisfies the compatibility condition

$$\int_{\partial\Omega}\zeta\,d\sigma=0\quad\text{in the sense of distributions on }\partial\Omega. \tag{2.2}$$

This motivates our introducing the following functional framework and definition of a solution.

**Definition 2.** Let $1<p<\infty$, $-1+1/p<s<1/p$, $1\leqslant q\leqslant\infty$. The notation $\mathcal{B}^{s-1}_{p,q}(\Omega)$ denotes the set of all functionals $\mathcal{DIV}[k;\zeta]$ such that

$$k\in B^s_{p,q}\bigl(\Omega;\mathbb{R}^n\bigr)\quad\text{and}\quad\zeta\in B^{s-\frac{1}{p}}_{p,q}(\partial\Omega;\mathbb{R})\quad\text{with}\int_{\partial\Omega}\zeta\,d\sigma=0.$$

The space $\mathcal{B}^{s-1}_{p,q}(\Omega)$ is endowed with the following norm:

$$\bigl\|\mathcal{DIV}[k;\zeta]\bigr\|_{\mathcal{B}^{s-1}_{p,q}(\Omega)}=\inf_{k',\zeta'}\bigl(\bigl\|k'\bigr\|_{B^s_{p,q}(\Omega)}+\bigl\|\zeta'\bigr\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)}\bigr), \tag{2.3}$$

where the infimum is taken over the set of $(k',\zeta')$ such that $\mathcal{DIV}[k';\zeta']=\mathcal{DIV}[k;\zeta]$.

Finally, for $F=\mathcal{DIV}[k;\zeta]$ in $\mathcal{B}^{s-1}_{p,q}(\Omega)$ we say that a vector field $v\in B^s_{p,q}(\Omega)$ fulfills the problem (1.1) in the weak sense if it satisfies (2.1).

Note that the space $\mathcal{B}^{s-1}_{p,q}(\Omega)$ may be identified with the quotient space of

$$\bigl\{(k,\zeta)\in B^s_{p,q}\bigl(\Omega;\mathbb{R}^n\bigr)\times B^{s-1/p}_{p,q}(\Omega;\mathbb{R})\text{ s.t. (2.2) is satisfied}\bigr\}$$

with the closed subspace of those $(k,\zeta)$ such that $\mathcal{DIV}[k;\zeta]\equiv0$, namely such that $\operatorname{div}k=0$ and $\zeta=k\cdot\vec{n}$. So in particular the quantity in (2.3) is a norm and $\mathcal{B}^{s-1}_{p,q}(\Omega)$ is a Banach space.

Let us now state our main result.

**Theorem 2.** *Let $\Omega$ be a bounded or exterior domain with a $C^{1,1}$-boundary. There exists an explicit linear operator $\mathcal{B}$ which is bounded from $\mathcal{B}^{s-1}_{p,q}(\Omega)$ to $B^s_{p,q}(\Omega;\mathbb{R}^n)$ whenever $1<p<\infty$, $1\leqslant q\leqslant\infty$ and $-1+1/p<s<1/p$, and such that for any $F=\mathcal{DIV}[k;\zeta]$ in $\mathcal{B}^{s-1}_{p,q}(\Omega)$, the vector field $v:=\mathcal{B}(F)$ fulfills* (2.1)*.*

*Furthermore, $\mathcal{B}(F)=\mathcal{L}(\operatorname{div}k)$ (where $\mathcal{L}$ is an operator meeting the properties of Theorem* 1) *whenever $k\in B^{1+s}_{p,q}(\Omega)$ and $\zeta=k\cdot\vec{n}$. In particular, $\mathcal{B}(F)$ vanishes at the boundary in this case.*

**Remark 1.** From the definition of operator $\mathcal{DIV}[k;\zeta]$, it is clear that the constructed solution $v$ to (2.1) fulfills

$$\operatorname{div}v=\operatorname{div}k\quad\text{in }\Omega,\qquad\vec{n}\cdot(k-v)=\zeta\quad\text{at }\partial\Omega. \tag{2.4}$$

Let us emphasize that the second part of the above theorem guarantees that we control much more information about our constructed solution.

**Remark 2.** As the construction of operator $\mathcal{B}$ coincides with that of [18] in the smooth case (that is $k \in B^{1+s}_{p,q}(\Omega)$ and $\zeta = k \cdot \vec{n}$), we shall focus on the proof of the first part of the statement. Let us also point out that higher order regularity estimates may be proved (again, the reader may refer to [18] for more details).

**Remark 3.** Another point which is worth pointing out is the regularity of the boundary. In [5,18], it is only required that $\partial\Omega$ is Lipschitz continuous. However, in the present paper, low regularity of data requires extra smoothness in order to solve some elliptic problems. As tracking the optimal regularity assumption is not the point here, we assumed that the boundary is in $C^{1,1}$.

We conclude this section with a few comments on the motivation for our approach. In a work in progress [7], we aim at analyzing the nonhomogeneous incompressible Navier–Stokes equation in bounded or exterior domains. This study strongly relies on the proof of low regularity estimates for the Stokes system:

$$\begin{aligned} &\partial_t u - \mu\Delta u + \nabla P = g \quad \text{in } \mathbb{R}_+ \times \Omega,\\ &\operatorname{div} u = f \quad \text{in } \mathbb{R}_+ \times \Omega,\\ &u = u_0 \quad \text{on } \{0\} \times \Omega, \qquad u = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega. \end{aligned} \tag{2.5}$$

Typically, we will have $g \in L_1(\mathbb{R}_+; B^s_{p,1}(\Omega))$, $u_0 \in B^s_{p,1}(\Omega)$ and $f \in L_1(\mathbb{R}_+; B^{s+1}_{p,1}(\Omega))$ for some $s \in (-1+1/p, 1/p)$ with, in addition,

$$\partial_t f = \mathcal{DIV}[R;\rho] \quad \text{with } R \in L_1\big(\mathbb{R}_+; B^s_{p,1}(\Omega)\big) \text{ and } \rho \in L_1\big(\mathbb{R}_+; B^{s-1/p}_{p,1}(\partial\Omega)\big).$$

Now, to reduce our study to the more classical case where $f \equiv 0$, it suffices to construct some solution $v$ to the divergence equation (1.1) such that $\nabla^2 v$ and $\partial_t v$ are in $L_1(\mathbb{R}_+; B^s_{p,1}(\Omega))$. Throughout this paper, it will appear clearly that, for time-dependent data, our explicit solution operator $\mathcal{B}$ commutes with the time derivative. Hence it will provide us with a solution to problem (1.1) with $f$ as in (2.5) such that

$$\|v_t, \nabla^2 v\|_{B^s_{p,1}(\Omega)} \leqslant C\big(\|f\|_{B^{1+s}_{p,1}(\Omega)} + \|R\|_{B^s_{p,1}(\Omega)} + \|\rho\|_{B^{s-1/p}_{p,1}(\partial\Omega)}\big).$$

Another motivation of our approach is related to the Neumann problem

$$\Delta v = \operatorname{div} k \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \vec{n}} = 0 \quad \text{at } \partial\Omega.$$

For general $k$ in $L_p(\Omega)$, this problem does not make sense so that one may rather consider the equation

$$\Delta v = \mathcal{DIV}[k;\zeta_2] \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \vec{n}} = \zeta_1 \quad \text{at } \partial\Omega,$$

that is

$$\int_\Omega \nabla v \cdot \nabla\phi\,dx = \int_\Omega k \cdot \nabla\phi\,dx + \int_{\partial\Omega} (\zeta_1 - \zeta_2)\phi\,d\sigma \quad \text{for all } \phi \in \mathcal{C}^\infty_c(\overline{\Omega}).$$

Here we see the main asset of our approach: the boundary data $\zeta$ may be put in $\mathcal{DIV}[k;\zeta]$ as well as in the boundary condition without any change of the weak formulation.

## 3. Notations and preliminaries

In this section, we introduce a few notation and recall classical results related to singular integrals, interpolation and Besov spaces. The reader will find more details and references in the textbooks [9,22,28].

Let us first recall that for any domain $\Omega$ with sufficiently smooth boundary, the Besov space $B^s_{p,q}(\Omega)$ stands for the restriction (in the distributional meaning) of functions in $B^s_{p,q}(\mathbb{R}^n)$ to $\Omega$. That is $f \in B^s_{p,q}(\Omega)$ means that there exists some $\widetilde{f} \in B^s_{p,q}(\mathbb{R}^n)$ such that for all smooth function $\varphi$ with compact support in $\Omega$ we have[2]

$$\int_\Omega f\varphi\,dx = \int_{\mathbb{R}^n} \widetilde{f}\varphi\,dx. \tag{3.1}$$

Setting

[2] In this paper, we shall use freely the notation $\int_\Omega f\varphi\,dx$ to denote $\langle f,\varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$.

$$\|f\|_{B^s_{p,q}(\Omega)} := \inf \|\widetilde{f}\|_{B^s_{p,q}(\mathbb{R}^n)},$$

where the infimum is taken over all the functions $\widetilde{f}$ such that (3.1) holds endows the set $B^s_{p,q}(\Omega)$ with a structure of Banach space.

We also recall that the Besov spaces are real interpolation spaces, namely

$$\left(B^{s_1}_{p,q_1}(\Omega), B^{s_2}_{p,q_2}(\Omega)\right)_{\theta,q} = B^{\theta s_2+(1-\theta)s_1}_{p,q}(\Omega), \tag{3.2}$$

whenever $1 \leqslant p, q, q_1, q_2 \leqslant \infty$, $s_1 \neq s_2$ and $\theta \in (0,1)$. A great deal of our results will be based on the following interpolation property (see e.g. [2, Chapter 4], [28, Chapter 3]):

$$\left(L_p(\Omega), W^1_p(\Omega)\right)_{s,q} = B^s_{p,q}(\Omega) \quad \text{for } 1 \leqslant q \leqslant \infty \text{ and } s \in (0,1). \tag{3.3}$$

The following density and duality results will be used a number of times (see e.g. [8,23] in the case of nonsmooth domains).

**Proposition 1.** *Let $\Omega$ be a Lipschitz domain.*

- *If $1 \leqslant p, q < \infty$ and $-1 + \frac{1}{p} < s < \frac{1}{p}$ then the set $\mathcal{C}^\infty_c(\Omega)$ of smooth functions with compact support in $\Omega$ is dense in $B^s_{p,q}(\Omega)$.*
- *If $1 < p < \infty$, $1 < q \leqslant \infty$ and $-1 + \frac{1}{p} < s < \frac{1}{p}$, then $B^s_{p,q}(\Omega)$ may be identified with the dual space of $B^{-s}_{p',q'}(\Omega)$, where $1/p' = 1 - 1/p$ and $1/q' = 1 - 1/q$.*
- *If $1 < p < \infty$ and $-1 + \frac{1}{p} < s < \frac{1}{p}$ then $B^s_{p,1}(\Omega)$ may be identified with the dual space of the completion $b^{-s}_{p',\infty}(\Omega)$ of $\mathcal{C}^\infty_c(\Omega)$ functions for the norm in $B^{-s}_{p',\infty}(\Omega)$.*

We shall also use that functions in $B^s_{p,q}(\Omega)$ with $s > 1/p$ have a trace at the boundary.

**Theorem 3** *(Trace theorem). If $\Omega$ is a Lipschitz bounded or exterior domain and $1/p < s < 1 + 1/p$ with $1 < p < \infty$ then, for all $q \in [1,\infty]$ the trace operator on $\partial\Omega$ extends continuously from $B^s_{p,q}(\Omega)$ to $B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$.*

In order to make the above statement more accurate, we have to explain what a Besov space on the boundary is. In fact, Besov spaces may be defined on any $r$-dimensional manifold $S$. For positive regularity indices, the idea is to use diffeomorphic maps after localization in order to reduce the definition to that of Besov spaces on $\mathbb{R}^r$ (see e.g. [14, Chapter 2], [22, Chapter 1], [29, Chapter 1] in the case of a smooth manifold, and [15, Definition 15.24] for only Lipschitz manifolds). If $s \in (0,1)$ and $p = q \in (1,\infty)$ then the Besov space $B^s_{p,p}(S)$ (which is alternately denoted by $W^s_p(S)$ in some places of the paper) may be endowed with the norm

$$\|u\|_{B^s_{p,p}(S)} = \|u\|_{L_p(S)} + \|u\|_{\dot{B}^s_{p,p}(S)}, \tag{3.4}$$

where $\|\cdot\|_{\dot{B}^s_{p,q}(S)}$ stands for the following homogeneous seminorm:

$$\|u\|_{\dot{B}^s_{p,p}(S)} = \left(\int_S\int_S \frac{|u(x)-u(y)|^p}{|x-y|^{r+sp}}\, d\sigma_x\, d\sigma_y\right)^{1/p}. \tag{3.5}$$

Besov spaces $B^s_{p,p}(S)$ with $-1 < s < 0$ may be defined by duality: we set

$$B^s_{p,p}(S) := \left(B^{-s}_{p',p'}(S)\right)^*.$$

The remaining spaces $B^s_{p,q}(S)$ for $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and $-1 < s < 1$ may be defined by interpolation according to the following relation:

$$\left(B^{s_1}_{p,p}(S), B^{s_2}_{p,p}(S)\right)_{\theta,q} = B^s_{p,q}(S), \tag{3.6}$$

taking $s_1 \in (-1,0)$ and $s_2 \in (0,1)$ such that $s_1 < s < s_2$, and $\theta \in (0,1)$ such that $s = \theta s_2 + (1-\theta)s_1$.

The proof of the continuity results for the solution to the divergence equation will be partly based on Theorem 2 in [4] which reads as follows:

**Theorem 4** *(Calderon–Zygmund). Let $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a measurable function, homogeneous of degree $-n$ with respect to the second variable. Assume in addition that*

(1) *for almost every $x \in \mathbb{R}^n$, the function $z \mapsto K(x,z)$ is integrable over the unit sphere and satisfies $\int_{|z|=1} K(x,z)\, dz = 0$;*

(2) *there exist $C \geqslant 0$ and $r$ in $(1,\infty)$ such that*

$$\int_{|z|=1} \left|K(x,z)\right|^r dz \leqslant C \quad \text{for all } x \in \Omega.$$

*Then for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$ the principal value $Tf(x)$ of*

$$\int K(x, x-y) f(y)\, dy$$

*exists and, for all $p \in [r', \infty)$ with $\frac{1}{r} + \frac{1}{r'} = 1$, operator $T$ extends continuously from $L_p(\mathbb{R}^n)$ to itself.*

We shall make an extensive use of the following result pertaining to the Neumann problem for the Laplace equation. It is a consequence of Theorem 3 of [16] and of Lemma 2.1 in [13].

**Proposition 2.** *Let $\Omega$ be a $C^{1,\varepsilon}$ bounded domain with $0 < \varepsilon < 1$. Then for any $f \in C^{0,\eta}(\Omega)$ with $0 < \eta < 1$ and $g \in C^{0,\varepsilon}(\partial\Omega)$ such that*

$$\int_\Omega f\, dx = \int_{\partial\Omega} g\, d\sigma$$

*the Neumann problem*

$$\Delta u = f \quad \text{in } \Omega,$$
$$\partial_{\vec{n}} u = g \quad \text{on } \partial\Omega$$

*has a solution $u$ in $C^{1,\varepsilon}(\Omega)$ satisfying*

$$\|u\|_{C^{1,\varepsilon}(\Omega)} \leqslant C\big(\|f\|_{C^{0,\eta}(\Omega)} + \|g\|_{C^{0,\varepsilon}(\partial\Omega)}\big).$$

To some extent, the following lemma will enable us to interpret the set $\mathcal{B}^{s-1}_{p,q}(\Omega)$ as an interpolation space between $L_p(\Omega;\mathbb{R}^n) \times W_p^{-\frac{1}{p}}(\partial\Omega;\mathbb{R})$ and $W^1_p(\Omega;\mathbb{R}^n)$.

**Lemma 1.** *Let $\Omega$ be a $C^{1,\frac{1}{p}}$ bounded domain of $\mathbb{R}^n$. Let $X$ be the subset of $W^1_p(\Omega;\mathbb{R}^n) \times W_p^{1-\frac{1}{p}}(\partial\Omega;\mathbb{R})$ defined by*

$$X = \big\{(f,\zeta) \in W^1_p\big(\Omega;\mathbb{R}^n\big) \times W_p^{1-\frac{1}{p}}(\partial\Omega;\mathbb{R}):\ \zeta = \vec{n}\cdot f \text{ at } \partial\Omega\big\}. \tag{3.7}$$

*Then*

$$\big(L_p\big(\Omega;\mathbb{R}^n\big) \times W_p^{-\frac{1}{p}}(\partial\Omega;\mathbb{R}), X\big)_{s,q} = B^s_{p,q}\big(\Omega;\mathbb{R}^n\big) \times B^{s-\frac{1}{p}}_{p,q}(\partial\Omega;\mathbb{R}), \tag{3.8}$$

*whenever $0 < s < \frac{1}{p}$ and $1 \leqslant q \leqslant \infty$.*

**Proof.** It is clear that

$$\big(L_p\big(\Omega;\mathbb{R}^n\big) \times W_p^{-\frac{1}{p}}(\partial\Omega;\mathbb{R}), W^1_p\big(\Omega;\mathbb{R}^n\big) \times W_p^{1-\frac{1}{p}}(\partial\Omega;\mathbb{R})\big)_{s,q} = B^s_{p,q}\big(\Omega;\mathbb{R}^n\big) \times B^{s-\frac{1}{p}}_{p,q}(\partial\Omega;\mathbb{R}).$$

We have to show that in the case $0 < s < 1/p$ changing the space $W^1_p(\Omega;\mathbb{R}^n) \times W_p^{1-\frac{1}{p}}(\partial\Omega;\mathbb{R})$ into its subspace $X$ yields the same interpolation space. For the time being, let us assume that $q < \infty$. Then, arguing by density, we see that it suffices to prove that any couple $(F, f)$ with $F$ a smooth vector field on $\overline{\Omega}$ and $f$ a smooth function on $\partial\Omega$ is the limit of a sequence of functions in $X$ *for the norm of* $B^s_{p,q}(\Omega;\mathbb{R}^n) \times B^{s-\frac{1}{p}}_{p,q}(\partial\Omega;\mathbb{R})$.

If $f = F\cdot\vec{n}$, the result is, of course, obvious. So let us focus on the approximation of any couple $(0, f)$ with $f$ a (say) $C^1$ function on $\partial\Omega$. First we want to construct a vector field $Ef$ over $\Omega$ with normal trace $f$ at $\partial\Omega$. This may be achieved by solving the following problem:

$$\Delta P = \alpha \quad \text{in } \Omega, \qquad \frac{\partial P}{\partial\vec{n}} = f \quad \text{on } \partial\Omega \quad \text{with } \alpha = \frac{1}{|\Omega|}\int_{\partial\Omega} f\, d\sigma. \tag{3.9}$$

According to Proposition 2, the above problem has a solution $P$ in $C^{1,\frac{1}{p}}(\overline{\Omega})$. Hence the function $Ef := \nabla P$ is in $C^{0,s}(\overline{\Omega})$ and, obviously, we have $\vec{n}\cdot Ef = f$ at the boundary.

As a consequence of the fact that extensions by 0 of elements in $B^{\sigma}_{p,q}(\Omega)$ with $0<\sigma<1/p$, belong to $B^{\sigma}_{p,q}(\mathbb{R}^n)$, one may construct (see e.g. [28, Chapter 3] or [6, Proposition 3]) a sequence of smooth functions $\chi_k : \overline{\Omega}\to[0,1]$ such that

$$\chi_k|_{\partial\Omega} = 1 \quad\text{and}\quad \chi_k(x)=0 \quad \text{if } \operatorname{dist}(\partial\Omega, x) > \frac{1}{k}; \tag{3.10}$$

and that

$$\chi_k \to 0 \quad \text{in } B^s_{p,q}(\Omega). \tag{3.11}$$

So bearing in mind that $Ef$ is $C^{0,s}$ on $\overline{\Omega}$ and using product laws in Besov spaces, one may conclude that

$$\|\chi_k Ef\|_{B^s_{p,q}(\Omega)} \xrightarrow[k\to+\infty]{} 0. \tag{3.12}$$

Therefore

$$(0,f) = \lim_{k\to\infty}(\chi_k Ef, f) \quad \text{in } B^s_{p,q}(\Omega)\times B^{s-\frac{1}{p}}_{p,q}(\partial\Omega).$$

Since each $(\chi_k Ef, f)$ belongs to $X$, this completes the proof of the lemma in the case $q<\infty$. The case $q=\infty$ follows from the case $q<\infty$ and the reiteration theorem (see e.g. [2]). □

## 4. The abstract approach

Here we present an abstract proof of solvability for the generalized divergence equation (2.1) in the case of a general *Lipschitz* bounded or exterior domain. This is intimately connected with the characterization of functionals on the Besov space $B^{\sigma}_{a,b}(\Omega)$ in the case $1<a<\infty$, $\frac{1}{a}<\sigma<1+\frac{1}{a}$ and $1\leqslant b<\infty$. So let us first give this characterization.

**Theorem 5.** *Let $\Omega$ be a* bounded *Lipschitz domain. If $1<a<\infty$, $1\leqslant b<\infty$ and $\frac{1}{a}<\sigma<1+\frac{1}{a}$ then $L$ is a linear functional on $B^{\sigma}_{a,b}(\Omega)$ if and only if there exists some vector field $w$ with coefficients in $B^{1-\sigma}_{a',b'}(\Omega)$ such that*

$$L(\varphi) = \int_{\Omega} w\cdot\nabla\varphi\,dx + L(1)M(\varphi) \quad \textit{for all } \varphi\in B^{\sigma}_{a,b}(\Omega), \tag{4.1}$$

*where $M$ stands for the map $\varphi\mapsto\frac{1}{|\Omega|}\int_\Omega \varphi\,dx$.*

*In addition, we have*

$$\|w\|_{B^{1-\sigma}_{a',b'}(\Omega)} \approx \big\|L-L(1)M\big\|_{(B^{\sigma}_{a,b}(\Omega))^*}. \tag{4.2}$$

**Proof.** Let us first observe that the map $\Phi:\varphi\mapsto\nabla\varphi$ is continuous from $B^{\sigma}_{a,b}(\Omega)$ to $B^{\sigma-1}_{a,b}(\Omega)$. This easily follows from the analogous property in the $\mathbb{R}^n$ case and the definition of Besov spaces by restriction. By taking advantage of the trace and duality properties stated in Section 3, this implies that the right-hand side of (4.1) defines a linear functional on $B^{\sigma}_{a,b}(\Omega)$.

Conversely, let $L$ be a linear functional over $B^{\sigma}_{a,b}(\Omega)$. It is clear that, as the domain $\Omega$ is connected, the map $\Phi$ is bijective from the subset $\bar{B}^{\sigma}_{a,b}(\Omega)$ of $B^{\sigma}_{a,b}(\Omega)$ functions with zero average, to $E:=\Phi(\bar{B}^{\sigma}_{a,b}(\Omega))$.

We claim that the inverse map $\Phi^{-1}$ is also continuous. Indeed, let us admit for a while that for $\sigma>1$ and $1\leqslant a,b\leqslant\infty$ we have

$$\|\cdot\|_{B^{\sigma}_{a,b}(\Omega)} \approx \|\cdot\|_{L_a(\Omega)} + \|\nabla\cdot\|_{B^{\sigma-1}_{a,b}(\Omega)}. \tag{4.3}$$

Then the continuity of $\Phi^{-1}$ may be shown by contradiction. In fact, if $\Phi^{-1}$ were not continuous then, according to (4.3), one might find some sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $\bar{B}^{\sigma}_{a,b}(\Omega)$ such that

$$\forall j\in\mathbb{N},\quad \|\varphi_j\|_{L_a(\Omega)}=1 \quad\text{and}\quad \nabla\varphi_j\to 0 \quad \text{in } B^{\sigma-1}_{a,b}(\Omega). \tag{4.4}$$

In particular, the sequence $(\varphi_j)_{j\in\mathbb{N}}$ is bounded in $W^{\varepsilon}_a(\Omega)$ for all small enough $\varepsilon$. Therefore, taking advantage of the compactness of the embedding of $W^{\varepsilon}_a(\Omega)$ in $L_a(\Omega)$, one may assume in addition that $\varphi_j$ converges *strongly* in $L_a(\Omega)$ to some function $\varphi$. Of course, this implies that $\nabla\varphi_j$ converges to $\nabla\varphi$ in $\mathcal{D}'(\Omega)$. Hence $\varphi\equiv 0$, a contradiction with (4.4).

We have proved that $\Phi$ is a bicontinuous isomorphism from $\bar{B}^{\sigma}_{a,b}(\Omega)$ to $E \subset B^{\sigma-1}_{a,b}(\Omega)$. Hence $L \circ \Phi^{-1}$ belongs to $E^*$ and, by virtue of the Hahn–Banach theorem, may be continued into a linear functional $\widetilde{L}$ on $B^{\sigma-1}_{a,b}(\Omega)$ with the same norm as $L \circ \Phi^{-1}$. Now, we notice that $-1+1/a < \sigma - 1 < 1/a$. Hence, according to Proposition 1, there exists some vector field $w$ with coefficients in $B^{1-\sigma}_{a',b'}(\Omega)$ such that

$$\forall \psi \in B^{\sigma-1}_{a,b}\big(\Omega;\mathbb{R}^n\big), \quad \widetilde{L}(\psi) = \int_{\Omega} w \cdot \nabla\psi \, dx.$$

Therefore, as for any $\varphi \in B^{\sigma}_{a,b}(\Omega;\mathbb{R})$, the function $\varphi - M(\varphi)$ belongs to $\bar{B}^{\sigma}_{a,b}(\Omega)$, one may write

$$L(\varphi) - L(1)M(\varphi) = L \circ \Phi^{-1}(\nabla\varphi) = \widetilde{L}(\nabla\varphi) = \int_{\Omega} w \cdot \nabla\varphi \, dx.$$

Finally, the above calculations show that, up to some irrelevant multiplicative constant the norm of $L \circ \Phi^{-1}$ in $E^*$ is the same as the norm of $w$ in $B^{\sigma}_{a,b}(\Omega)$. Hence (4.2) holds true. $\square$

**Remark 4.** A similar result holds in the limit case $b = \infty$ if we consider linear functionals over the completion of $\mathcal{C}^{\infty}_c(\Omega)$ functions for the $\|\cdot\|_{B^{\sigma}_{a,\infty}(\Omega)}$ norm.

**Proof of (4.3).** Let $\chi$ be in $\mathcal{C}^{\infty}_c(\mathbb{R}^n)$ and satisfy $\chi \equiv 1$ near the origin. Arguing by density (see Proposition 1), it suffices to establish the inequality for smooth functions compactly supported in $\Omega$. For such a function, one may write

$$v = \chi(D)v - A(D)\nabla v$$

where $\chi(D)v$ stands for the inverse Fourier transform of $\chi\widehat{v}$ and $A(D) := (-\Delta)^{-1}(\mathrm{Id} - \chi(D))\,\mathrm{div}$. In particular, $\chi(D)$ maps $L_a(\mathbb{R}^n)$ in any Besov space $B^{\sigma}_{a,b}(\mathbb{R}^n)$ while, being homogeneous of degree $-1$ away from the origin, the multiplier $A(D)$ maps $B^{\sigma-1}_{a,b}(\mathbb{R}^n)$ in $B^{\sigma}_{a,b}(\mathbb{R}^n)$ (see [1, Chapter 2]). So we get the desired inequality. $\square$

**Corollary 1.** *Let $\Omega$ be a bounded or exterior Lipschitz domain. Let $p \in (1,\infty)$, $q \in [1,\infty]$ and $s \in (-1+1/p, 1/p)$. There exists a constant $C$ such that for any $F = \mathcal{DIV}[k;\zeta]$ in $\mathcal{B}^{s-1}_{p,q}(\Omega)$ Eq. (2.1) has a solution $v$ in $B^{s}_{p,q}(\Omega)$ such that*

$$\|v\|_{B^{s}_{p,q}(\Omega)} \leqslant C \big\|\mathcal{DIV}[k;\zeta]\big\|_{\mathcal{B}^{s-1}_{p,q}(\Omega)}.$$

**Proof.** The exterior domain case follows from the bounded case. It is only a matter of following the arguments of the second part of Section 6. Indeed, the regularity of the domain is used only to apply the result in the bounded case proved in the first part of Section 6.

So let us focus on the case where the domain $\Omega$ is bounded. If $k \in B^{s}_{p,q}(\Omega)$ and $\zeta \in B^{s-1/p}_{p,q}(\Omega)$ with $-1+1/p < s < 1/p$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ then $\mathcal{DIV}[k;\zeta]$ is a functional on $B^{1-s}_{p',q'}(\Omega)$ as an immediate consequence of the continuity of function $\Phi$, of the trace theorem and of duality properties for Besov spaces. Indeed we have

$$\bigg|\int_{\Omega} k \cdot \nabla\varphi \, dx\bigg| \leqslant C\|k\|_{B^{s}_{p,q}(\Omega)}\|\nabla\varphi\|_{B^{-s}_{p',q'}(\Omega)} \leqslant C\|k\|_{B^{s}_{p,q}(\Omega)}\|\varphi\|_{B^{1-s}_{p',q'}(\Omega)},$$
$$\bigg|\int_{\partial\Omega} \zeta\varphi \, d\sigma\bigg| \leqslant C\|\zeta\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)}\|\varphi\|_{B^{\frac{1}{p}-s}_{p',q'}(\partial\Omega)} \leqslant C\|\zeta\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)}\|\varphi\|_{B^{1-s}_{p',q'}(\Omega)}.$$

Here it is crucial that $1-s > 1/p'$ (in order to apply the trace theorem) and that $-1+1/p < s < 1/p$ (so that $B^{s}_{p,q}(\Omega)$ may be identified with the dual space of $B^{-s}_{p',q'}(\Omega)$).

Given that the compatibility property (2.2) implies that $\mathcal{DIV}[k;\zeta](1) = 0$, the result thus stems from Theorem 5 (or from the remark that follows if $q = \infty$). $\square$

**Remark 5.** Note that the required regularity for the domain is weaker in Corollary 1 than in Theorem 2. However, Corollary 1 does not give much information on the solution. In particular, as the proof is not explicit, it does not supply any linear solution operator nor regularity estimates.

## 5. The bounded star-shaped case

This section is the core of the proof of Theorem 2. Here, in the case where $\Omega$ is a bounded star-shaped domain with respect to some ball $B(x_0,R)$, we give an explicit solution to problem (2.1) based on the following formula that has been introduced by M. Bogovskiĭ in [3]:

$$v(x)=\int_\Omega f(y)\frac{x-y}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)\big(|x-y|+r\big)^{n-1}\,dr\,dy. \tag{5.1}$$

Above, $\omega$ stands for a smooth function with support in $B(x_0,R)$ and average 1.

The main statement of this section reads:

**Theorem 6.** *Let $\Omega$ be a bounded $C^{1,1}$ star-shaped domain with respect to some ball. Let $p\in(1,\infty)$, $q\in[1,\infty]$ and $s\in(-1+1/p,1/p)$. There exists a continuous linear operator:*

$$\mathcal{B}:\mathcal{B}^{s-1}_{p,q}(\Omega)\to B^s_{p,q}\big(\Omega;\mathbb{R}^n\big)$$

*such that for any $F=\mathcal{DIV}[k;\zeta]$ in $\mathcal{B}^{s-1}_{p,q}(\Omega)$, the vector field $v:=\mathcal{B}(F)$ fulfills* (2.1)*.*

It is well known that in the smooth case, formula (5.1) does provide a solution to the divergence equation (1.1):

**Lemma 2.** *Assume that $f$ is bounded and continuous on the bounded Lipschitz domain $\Omega$. If in addition $f$ has average 0 then the vector-field $v$ given by* (5.1) *is continuous and bounded on $\mathbb{R}^n$, satisfies $\operatorname{div} v=f$ in $\Omega$ and is supported in the closure of the set*

$$\Sigma=\big\{x\in\mathbb{R}^n\colon\ x=\lambda x_1+(1-\lambda)x_2,\ x_1\in\operatorname{Supp} f,\ x_2\in B(x_0,R),\ \lambda\in[0,1]\big\}.$$

**Proof.** We here provide a complete proof as it will be a model for solving the generalized divergence equation (see Lemma 3 below). First, we reformulate (5.1) into

$$v(x)=\int_\Omega (x-y)f(y)\int_1^\infty \omega\big(y+r(x-y)\big)r^{n-1}\,dr\,dy. \tag{5.2}$$

Note that as $\omega$ is compactly supported and $\Omega$ is bounded, there exists a constant $M$ such that $\omega(y+r(x-y))=0$ for $r\geqslant M/|x-y|$. Therefore there exists $C>0$ such that

$$\left|\int_1^\infty \omega\big(y+r(x-y)\big)r^{n-1}\,dr\right|\leqslant C|x-y|^{-n}. \tag{5.3}$$

Hence the above formula defines a continuous locally bounded function on $\mathbb{R}^n$ whenever $f$ is a continuous bounded function over $\Omega$. In addition, if $x$ is not in $\Sigma$ then for all $y\in\operatorname{Supp} f$ and $r>1$ we have $y+r(x-y)\notin B(x_0,R)$, hence $v(x)=0$.

Next, we have to check that $\operatorname{div} v=f$ in $\Omega$, namely

$$-\int_\Omega v\cdot\nabla\varphi\,dx=\int_\Omega f\varphi\,dx\quad\text{for all }\varphi\in\mathcal{C}^\infty_c(\Omega). \tag{5.4}$$

According to (5.2) and Lebesgue's dominated convergence theorem, we have

$$-\int_\Omega v\cdot\nabla\varphi\,dx=\lim_{M\to+\infty}\mathcal{I}_M$$

with

$$\mathcal{I}_M:=-\int_\Omega\int_\Omega f(y)\nabla\varphi(x)\cdot\left(\int_1^M (x-y)\omega\big(y+r(x-y)\big)r^{n-1}\,dr\right)dy\,dx.$$

Now, performing an integration by parts with respect to $x$ yields

$$\mathcal{I}_M = \int_\Omega \int_\Omega f(y)\varphi(x)\left(\int_1^M \frac{d}{dr}\big(r^n\omega\big(y+r(x-y)\big)\big)\,dr\right)dx\,dy$$
$$= \int_\Omega \int_\Omega f(y)\varphi(x)M^n\omega\big(y+M(x-y)\big)\,dx\,dy - \int_\Omega \int_\Omega f(y)\varphi(x)\omega(x)\,dx\,dy.$$

Note that the last term vanishes for $f$ has total mass 0. So extending $\varphi$ on $\mathbb{R}^n$ by 0 and performing the change of variable $x' = y + M(x-y)$, we discover that

$$\mathcal{I}_M = \int_\Omega f(y)\left(\int_{\mathbb{R}^d} \varphi\big(y+M^{-1}\big(x'-y\big)\big)\omega\big(x'\big)\,dx'\right)dy.$$

As $\omega$ has average 1 by assumption, it is clear that the inner integral converges uniformly to $\varphi(y)$ when $M$ goes to infinity. Hence (5.4) is satisfied. □

Note that if $f = \operatorname{div} k$ then integrating by parts in (5.1) yields in the principal value meaning,

$$v(x) = -\int_\Omega k(y)\cdot\nabla_y\left[\frac{x-y}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)\big(|x-y|+r\big)^{n-1}\,dr\right]dy$$
$$+ \int_{\partial\Omega} (k\cdot\vec{n})(y)\frac{x-y}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)\big(|x-y|+r\big)^{n-1}\,dr\,d\sigma_y.$$

**Convention.** From now on, we agree that all formulae (as the above one for instance) involving singular kernels have to be understood in the principal value meaning, that is as in Theorem 4.

In the framework we want to consider, namely $k \in B^s_{p,q}(\Omega)$ with $s < 1/p$, the vector field $k$ need not have a trace so that the meaning of the second term of the above formula is unclear. To overcome this difficulty, the idea is to decorrelate $k$ and its normal trace $k\cdot\vec{n}$: we shall define *two* operators $I$ and $J$ acting on vector fields of $\Omega$ and functions of $\partial\Omega$, respectively, as follows

$$Ik(x) := -\int_\Omega k(y)\cdot\nabla_y\left[\frac{x-y}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)\big(|x-y|+r\big)^{n-1}\,dr\right]dy,$$
$$J\zeta(x) := \int_{\partial\Omega} \zeta(y)\frac{x-y}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)\big(|x-y|+r\big)^{n-1}\,dr\,d\sigma_y. \tag{5.5}$$

**Lemma 3.** *Let $\Omega$ be a $C^{1,\varepsilon}$ bounded domain with $\varepsilon \in (0,1]$. Let $k \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^n)$ and $\zeta \in C^{0,\varepsilon}(\partial\Omega;\mathbb{R})$ with zero average. Assume in addition that the sets $\Sigma$ pertaining to $k$ and $\zeta$ (see the definition in the previous lemma) are included in $\overline{\Omega}$.*

*Then $Ik$ and $J\zeta$ are defined a.e. on $\mathbb{R}^n$, vanish on $\mathbb{R}^n \setminus \overline{\Omega}$, and*

$$v := Ik + J\zeta \quad \text{satisfies } -\int_\Omega v\cdot\nabla\varphi\,dx = \mathcal{DIV}[k;\zeta](\varphi) \text{ for all } \varphi \in \mathcal{C}^\infty(\overline{\Omega}). \tag{5.6}$$

**Proof.** The fact that $Ik$ and $J\zeta$ are defined almost everywhere is a consequence of Theorem 4. This will be justified below in Lemma 4. In addition, arguing as in the proof of Lemma 2, we see that $Ik$ and $J\zeta$ vanish outside $\overline{\Omega}$.

Let us now check (5.6). First, let us notice that if we assume that $\zeta = \vec{n}\cdot k$ at the boundary then we are in the classical setting and the previous lemma gives the result. Indeed we saw that in this case $\operatorname{div} v = \operatorname{div} k$ in $\Omega$ and $v$ is a continuous function supported in $\overline{\Omega}$ so that Stokes formula ensures that

$$-\int_\Omega v\cdot\nabla\varphi\,dx = -\int_\Omega k\cdot\nabla\varphi\,dx + \int_{\partial\Omega}\varphi k\cdot\vec{n}\,d\sigma \quad \text{for all } \varphi \in \mathcal{C}^\infty(\overline{\Omega}).$$

Hence it suffices to consider the case $k \equiv 0$ and to prove that $v := J(\zeta)$ satisfies

$$-\int_\Omega v \cdot \nabla\varphi \, dx = \int_{\partial\Omega} \varphi\zeta \, d\sigma \quad \text{for all } \varphi \in \mathcal{C}^\infty(\overline{\Omega}). \tag{5.7}$$

For that, let us introduce the solution $P$ to

$$\Delta P = 0 \quad \text{on } \Omega, \qquad \frac{\partial P}{\partial \vec{n}} = \zeta \quad \text{at } \partial\Omega.$$

This may be solved as $\zeta$ has 0 average and, because $\partial\Omega$ is $C^{1,\varepsilon}$ the function $P$ is $C^1$ up to the boundary (see Proposition 2). Therefore, using the Stokes formula, we see that (5.7) is satisfied if and only if

$$-\int_\Omega v \cdot \nabla\varphi \, dx = \int_\Omega \nabla P \cdot \nabla\varphi \, dx \quad \text{for all } \varphi \in \mathcal{C}^\infty(\overline{\Omega}). \tag{5.8}$$

Let us fix some $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ and denote by $\mathcal{J}$ the left-hand side of (5.8). Using the definition of $J\zeta$ we may write

$$\mathcal{J} = -\int_{\partial\Omega}\int_\Omega \zeta(y)\nabla\varphi(x)\cdot(x-y)\int_1^\infty \omega\big(y+r(x-y)\big)r^{n-1}\,dr\,dx\,d\sigma_y.$$

Let us notice that as $\Omega$ is bounded and $\omega$, compactly supported, there exists some constant $C$ such that for all $M \geqslant 1$,

$$\forall x \in \Omega,\ \forall y \in \partial\Omega, \quad |x-y|\int_1^M \big|\omega\big(y+r(x-y)\big)\big|r^{n-1}\,dr \leqslant C\max\big(1, |x-y|^{1-n}\big). \tag{5.9}$$

Hence $\mathcal{J}$ is well defined and the dominated convergence theorem ensures that $\mathcal{J}$ is the limit of $\mathcal{J}_M$ when $M$ goes to $+\infty$, with

$$\mathcal{J}_M := -\int_{\partial\Omega}\int_\Omega \zeta(y)\nabla\varphi(x)\cdot(x-y)\int_1^M \omega\big(y+r(x-y)\big)r^{n-1}\,dr\,dx\,d\sigma_y.$$

In fact, given that $\nabla P$ is bounded on $\Omega$ (because it is continuous up to the boundary) and keeping in mind inequality (5.9), we see that one may assume with no loss of generality that $\varphi \in \mathcal{C}_c^\infty(\Omega)$. This will be important to justify the computations in the sequel.

Now, the Stokes formula and the definition of $P$ imply that, with the summation convention over repeated indices,

$$\mathcal{J}_M = \int_\Omega\int_\Omega \partial_j\varphi(x)\partial_i P(y)\frac{\partial}{\partial y^i}\left(\big(x^j-y^j\big)\int_1^M \omega\big(y+r(x-y)\big)r^{n-1}\,dr\right)dy\,dx.$$

Therefore

$$\mathcal{J}_M = \int_\Omega \partial_i P(y)\int_1^M\int_\Omega \partial_j\varphi(x)\big(\delta_{i,j}\omega\big(y+r(x-y)\big)+(r-1)\big(x^j-y^j\big)\partial_i\omega\big(y+r(x-y)\big)\big)r^{n-1}\,dx\,dr\,dy.$$

Let us make the change of variable $x' = y + r(x-y)$ in the inner integral. We readily get $\mathcal{J}_M = \mathcal{J}_M^1 + \mathcal{J}_M^2 + \mathcal{J}_M^3$ with

$$\mathcal{J}_M^1 = \int_\Omega \partial_i P(y)\int_1^M\int_{\mathbb{R}^n} \partial_i\varphi\left(y+\frac{x'-y}{r}\right)\frac{\omega(x')}{r}\,dx'\,dr\,dy,$$

$$\mathcal{J}_M^2 = \int_\Omega \partial_i P(y)\int_1^M\int_{\mathbb{R}^n} \nabla\varphi\left(y+\frac{x'-y}{r}\right)\cdot\left(\frac{x'-y}{r}\right)\partial_i\omega(x')\,dx'\,dr\,dy,$$

$$\mathcal{J}_M^3 = -\int_\Omega \partial_i P(y)\int_1^M\int_{\mathbb{R}^n} \nabla\varphi\left(y+\frac{x'-y}{r}\right)\cdot\left(\frac{x'-y}{r^2}\right)\partial_i\omega(x')\,dx'\,dr\,dy.$$

In order to handle $\mathcal{J}_M^3$, let us notice that

$$\frac{\partial}{\partial r}\left[\varphi\left(y+\frac{x-y}{r}\right)\right]=-\nabla\varphi\left(y+\frac{x-y}{r}\right)\cdot\left(\frac{x-y}{r^2}\right).$$

Therefore, an explicit integration with respect to $r$, followed by an integration by parts with respect to $x$ yields

$$\mathcal{J}_M^3=\int_\Omega\int_{\mathbb{R}^n}\nabla P(y)\cdot\nabla\varphi(x)\omega(x)\,dx\,dy-\frac{1}{M}\int_\Omega\int_{\mathbb{R}^n}\nabla P(y)\cdot\nabla\varphi\left(y+\frac{x-y}{M}\right)\omega(x)\,dx\,dy.$$

Using the fact that the functions $\nabla P$ and $\nabla\varphi$ are bounded, that $\varphi$ is compactly supported and that $\int_{\mathbb{R}^n}\omega(x)\,dx=1$, we thus get

$$\mathcal{J}_M^3=\int_\Omega\nabla P(y)\cdot\nabla\varphi(y)\,dy+\int_\Omega\int_{\mathbb{R}^n}\nabla P(y)\cdot\big(\nabla\varphi(x)-\nabla\varphi(y)\big)\omega(x)\,dx\,dy+\mathcal{O}\left(\frac{1}{M}\right).$$

Next, performing an integration by parts with respect to $x$ in $\mathcal{J}_M^2$ yields

$$\begin{aligned}\mathcal{J}_M^2=&-\int_1^M\int_\Omega\int_{\mathbb{R}^n}\partial_i\varphi\left(y+\frac{x-y}{r}\right)\frac{\omega(x)}{r}\partial_i P(y)\,dx\,dy\,dr\\&-\sum_{i,j}\int_1^M\int_\Omega\int_{\mathbb{R}^n}\left(\frac{x^j-y^j}{r^2}\right)\partial_{ij}^2\varphi\left(y+\frac{x-y}{r}\right)\partial_i P(y)\omega(x)\,dx\,dy\,dr.\end{aligned}$$

The first part of the r.h.s. compensates $\mathcal{J}_M^1$. Hence, putting together all the previous computations, we get

$$\begin{aligned}\mathcal{J}_M-\int_\Omega\nabla P(y)\cdot\nabla\varphi(y)\,dy=&\int_\Omega\int_{\mathbb{R}^n}\omega(x)\partial_i P(y)\left(\partial_i\varphi(x)-\partial_i\varphi(y)-\int_1^M\nabla\partial_i\varphi\left(y+\frac{x-y}{r}\right)\cdot\left(\frac{x-y}{r^2}\right)dr\right)dy\,dx\\&+\mathcal{O}\left(\frac{1}{M}\right).\end{aligned}$$

Now, we notice that

$$\int_1^M\nabla\partial_i\varphi\left(y+\frac{x-y}{r}\right)\cdot\left(\frac{x-y}{r^2}\right)dr=\partial_i\varphi(x)-\partial_i\varphi\left(y+\frac{x-y}{M}\right).$$

Therefore

$$\mathcal{J}_M-\int_\Omega\nabla P(y)\cdot\nabla\varphi(y)\,dy=\int_\Omega\int_{\mathbb{R}^n}\omega(x)\nabla P(y)\cdot\left(\nabla\varphi\left(y+\frac{x-y}{M}\right)-\nabla\varphi(y)\right)dy\,dx+\mathcal{O}\left(\frac{1}{M}\right).$$

Hence

$$\lim_{M\to+\infty}\mathcal{J}_M=\int_\Omega\nabla P(y)\cdot\nabla\varphi(y)\,dy,$$

which completes the proof of the lemma. □

In the sequel, we focus on the proof of continuity results for operators $I$ and $J$. The result in general Besov spaces will be achieved by interpolating between low and high regularity properties. Let us start with the study of low regularity properties.

**Lemma 4.** *For any $p\in(1,\infty)$, operator $I$ extends continuously from $L_p(\Omega)$ to $L_p(\Omega)$ and operator $J$ extends continuously from $W_p^{-\frac{1}{p}}(\partial\Omega)$ to $L_p(\Omega)$.*

**Proof.** Let us first concentrate on operator $I$. As $\mathcal{C}_c^\infty(\Omega)$ is dense in $L_p(\Omega)$, it suffices to prove that

$$\|Ik\|_{L_p(\Omega)} \leqslant C\|k\|_{L_p(\Omega)} \quad \text{for all } k \in \mathcal{C}_c^\infty(\Omega). \tag{5.10}$$

For that, we have to analyze the following kernel:

$$A^{ij}(x,y) := \partial_{y^j}\left[\frac{x^i-y^i}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)\big(|x-y|+r\big)^{n-1}\,dr\right]. \tag{5.11}$$

We aim at isolating the pure singular part of the kernel $A^{ij}(x,y)$. Now, from Leibniz formula and the fact that

$$\big(|x-y|+r\big)^{n-1} = r^{n-1} + \sum_{1\leqslant k\leqslant n-1}\binom{n-1}{k} r^{n-1-k}|x-y|^k, \tag{5.12}$$

we gather that[3] $A(x,y)$ may be split into $A_1(x,y)+A_2(x,y)+A_3(x,y)$ with

$$A_1(x,y) := \partial_{y^j}\left(\frac{x^i-y^i}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)r^{n-1}\,dr\right),$$

$$A_2(x,y) := \sum_{1\leqslant k\leqslant n-1}\binom{n-1}{k}\int_0^\infty \partial_{y^j}\left(\frac{x^i-y^i}{|x-y|^n}\omega\left(x+r\frac{x-y}{|x-y|}\right)\right)|x-y|^k r^{n-1-k}\,dr,$$

$$A_3(x,y) := \frac{x^i-y^i}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)(n-1)\big(|x-y|+r\big)^{n-2}\partial_{y^j}|x-y|\,dr.$$

Let us first analyze the kernels $A_2$ and $A_3$ which are easier to deal with. Owing to the boundedness of $\Omega$, we notice that these two terms are of the form

$$A_2(x,y) = \frac{\mathcal{B}_2(x,y)}{|x-y|^{n-1}} \quad \text{and} \quad A_3(x,y) = \frac{\mathcal{B}_3(x,y)}{|x-y|^{n-1}}, \tag{5.13}$$

where $\mathcal{B}_2$ and $\mathcal{B}_3$ are bounded on $\Omega\times\Omega$.

Next, in order to analyze the singular kernel $A_1$, we make the following computation:

$$\begin{aligned}
A_1(x,y) &= \partial_{y^j}\left(\big(x^i-y^i\big)\int_0^\infty \omega\big(x+s(x-y)\big)s^{n-1}\,ds\right)\\
&= -\delta_{ij}\int_0^\infty \omega\big(x+s(x-y)\big)s^{n-1}\,ds - \big(x^i-y^i\big)\int_0^\infty \partial_j\omega\big(x+s(x-y)\big)s^n\,ds\\
&= -\frac{\delta_{ij}}{|x-y|^n}\int_0^\infty \omega\left(x+r\frac{x-y}{|x-y|}\right)r^{n-1}\,dr - \frac{x^i-y^i}{|x-y|^{n+1}}\int_0^\infty \partial_j\omega\left(x+r\frac{x-y}{|x-y|}\right)r^n\,dr.
\end{aligned}$$

From the last line, it is now clear that $A_1(x,y)=K(x,x-y)$, where the singular kernel $K$ is homogeneous of degree $-n$ with respect to the second variable and satisfies

$$\int_{|z|=1} K(x,z)\,d\sigma_z = -\int_0^\infty r^{n-1}\int_{|z|=1}\partial_{z^j}\big(z^i\omega(x+rz)\big)\,d\sigma_z\,dr = 0.$$

It is also clear that

$$\sup_{x\in\Omega}\int_{|z|=1}\big|K(x,z)\big|^q\,dz < \infty \quad \text{for all } q\in[1,\infty).$$

Hence, Theorem 4 implies that $A_1$ is the kernel of a continuous operator on $L_p(\Omega)$.

[3] In what follows we omit the indices $i$ and $j$ for notational simplicity.

We conclude that

$$Ik(x) = \sum_{j=1}^{n} \int_{\Omega} \left(k^j(y) K^j_{CZ}(x,y) + k^j(y) K^j_{int}(x,y)\right) dy,$$

where $K^j_{CZ}$ is the kernel of a Calderon–Zygmund operator and $K^j_{int}$ satisfies

$$\left|K^j_{int}(x,y)\right| \leqslant C|x-y|^{1-n}.$$

Therefore,

$$\left\| \int_{\mathbb{R}^n} k^j(y) T^j_{CZ}(\cdot,y)\,dy \right\|_{L_p(\Omega)} \leqslant C \left\|k^j\right\|_{L_p(\Omega)} \tag{5.14}$$

and, as the domain $\Omega$ is bounded,

$$\left\| \int_{\Omega} k^j(y) T^j_{int}(\cdot,y)\,dy \right\|_{L_p(\Omega)} \leqslant C \left\|k^j\right\|_{L_p(\Omega)}. \tag{5.15}$$

So putting inequalities (5.14) and (5.15) together, we end up with (5.10).

Let us now go to the second part of the lemma. Here also, as $\mathcal{C}^\infty(\partial\Omega)$ is dense in $W_p^{-\frac{1}{p}}(\partial\Omega)$, it suffices to prove that

$$\|J\zeta\|_{L_p(\Omega)} \leqslant C\|\zeta\|_{W_p^{-\frac{1}{p}}(\partial\Omega)} \quad \text{for all } \zeta \in \mathcal{C}^\infty(\partial\Omega). \tag{5.16}$$

So let us fix some function $\zeta \in \mathcal{C}^\infty(\partial\Omega)$. For bounding the $L^p$ norm of $J\zeta$ we write that

$$\|J\zeta\|_{L_p(\Omega)} = \sup_{\|\pi\|_{L_{p'}(\Omega)} \leqslant 1} \int_{\Omega} \pi J\zeta\,dx.$$

Thus we consider

$$\int_{\Omega}\int_{\partial\Omega} \zeta(y) \frac{x^i-y^i}{|x-y|^n} \int_0^\infty \omega\left(x + r\frac{x-y}{|x-y|}\right)\left(|x-y|+r\right)^{n-1} dr\,d\sigma_y\,\pi(x)\,dx.$$

The problem to estimate the above integral lays in the fact that we only want to use the norm of $\zeta$ in $W_p^{-\frac{1}{p}}(\partial\Omega)$. Now, let us introduce the operator $K$ defined by

$$K\pi(y) := \int_{\Omega} \frac{x^i-y^i}{|x-y|^n} \int_0^\infty \omega\left(x + r\frac{x-y}{|x-y|}\right)\left(|x-y|+r\right)^{n-1} dr\,\pi(x)\,dx.$$

We claim that this operator extends to a bounded map from $L_{p'}(\Omega)$ to $W^1_{p'}(\Omega)$. Indeed, first, as the kernel is of the type considered in (5.13), operator $K$ is obviously bounded from $L_{p'}(\Omega)$ to itself. Second, we notice that the kernel

$$\nabla_y \left[ \frac{x^i-y^i}{|x-y|^n} \int_0^\infty \omega\left(x + r\frac{x-y}{|x-y|}\right)\left(|x-y|+r\right)^{n-1} dr \right] \tag{5.17}$$

defines a bounded map on $L_{p'}(\Omega)$ because the above kernel is almost the same as the one that we considered in the definition of $Ik$. The only change occurs in the argument of $\omega$: $x$ plays the role of $y$ and conversely. However, this change is of lower order since

$$\left|\omega\left(x + r\frac{x-y}{|x-y|}\right) - \omega\left(y - r\frac{y-x}{|y-x|}\right)\right| \leqslant C|x-y|.$$

Therefore the proof from (5.11) till (5.15) may be repeated and we end up with

$$\|K\pi\|_{W^1_{p'}(\Omega)} \leqslant C\|\pi\|_{L_{p'}(\Omega)}. \tag{5.18}$$

By virtue of the trace theorem (see Theorem 3), we can thus write that

$$\begin{aligned}\|J\zeta\|_{L_p(\Omega)} &= \sup_{\|\pi\|_{L_{p'}(\Omega)}\leqslant 1} \int_{\partial\Omega} \zeta K\pi\, d\sigma \\ &\leqslant \sup_{\|\pi\|_{L_{p'}(\Omega)}\leqslant 1} \|\zeta\|_{W_p^{-\frac{1}{p}}(\partial\Omega)} \|K\pi\|_{W_{p'}^{1-\frac{1}{p'}}(\partial\Omega)} \\ &\leqslant \sup_{\|\pi\|_{L_{p'}(\Omega)}\leqslant 1} \|\zeta\|_{W_p^{-\frac{1}{p}}(\partial\Omega)} \|K\pi\|_{W^1_{p'}(\Omega)}.\end{aligned}$$

Bounding the last term according to (5.18) completes the proof of (5.16). □

We now want to study the continuity properties of the operator $(k,\zeta)\mapsto v := Ik + J\zeta$ for $k\in W^1_p(\Omega)$. Then $k$ has a trace at $\partial\Omega$ and it is thus relevant to restrict to the case where $\zeta = k\cdot\vec{n}$ (so that one may use formula (5.1)).

**Lemma 5.** *Assume that $f = \operatorname{div} k$ with $k\in W^1_p(\Omega)$. Then $v$ given by* (5.1) *belongs to $W^1_p(\Omega)$ and satisfies*

$$\|v\|_{W^1_p(\Omega)} \leqslant C\|\operatorname{div} k\|_{L_p(\Omega)}. \tag{5.19}$$

**Proof.** First let us observe that $\operatorname{div} k\in L_p(\Omega)$ hence, by virtue of (5.1),

$$\nabla v(x) = \int_{\Omega} \operatorname{div} k(y)\nabla_x\left(\frac{x-y}{|x-y|^n}\int_0^\infty \omega\left(x + r\frac{x-y}{|x-y|}\right)\left(|x-y|+r\right)^{n-1}\right) dr\,dy.$$

To prove the result, one may argue as for estimating $Ik$ in the previous lemma. The only difference concerns the part with differentiation of $\omega$ as we now have to deal with the term

$$\partial_k\omega\left(x + r\frac{x-y}{|x-y|}\right)\left(\delta_{jk} + r\partial_{x_j}\left(\frac{x^k-y^k}{|x-y|}\right)\right).$$

The only definitely new term is generated by $\delta_{jk}$. But it is obviously of lower order. We skip the end of the proof as it is almost the same as for $Ik$. □

We are now ready to prove Theorem 6 in the case $s>0$.

**Lemma 6.** *Let $1<p<\infty$, $s\in(0,1/p)$ and $q\in[1,\infty]$. Consider an element $F = \mathcal{DIV}[k;\zeta]$ of $\mathcal{B}^{-1+s}_{p,q}(\Omega)$. Then the function $v := Ik + J\zeta$ belongs to $B^s_{p,q}(\Omega)$, satisfies*

$$\|v\|_{B^s_{p,q}(\Omega)} \leqslant C\left\|\mathcal{DIV}[k;\zeta]\right\|_{\mathcal{B}^{-1+s}_{p,q}(\Omega)}$$

*and is a solution to* (1.1) *in the weak sense.*

**Proof.** Arguing by density and knowing that $Ik + J\zeta$ is a solution in the smooth case, it suffices to prove the estimate. It will be achieved by taking advantage of Lemma 1 and of the continuity properties stated in Lemmas 4 and 5.

If $k\in W^1_p(\Omega)$, then $\vec{n}\cdot k\in W_p^{1-\frac{1}{p}}(\partial\Omega)$, so that $(k,\vec{n}\cdot k)$ belongs to the space $X$ defined in Lemma 1. Now Lemma 5 implies that

$$\mathcal{B}:\begin{cases} X\to W^1_p(\Omega;\mathbb{R}^n),\\ (k,k\cdot\vec{n})\mapsto v := Ik + J(k\cdot\vec{n})\end{cases}$$

and Lemma 4 says that

$$\mathcal{B}:\begin{cases} L_p(\Omega;\mathbb{R}^n)\times W_p^{-\frac{1}{p}}(\Omega;\mathbb{R})\to L_p(\Omega;\mathbb{R}^n),\\ (k,\zeta)\mapsto v := Ik + J\zeta.\end{cases}$$

Therefore, Lemma 1 implies that

$$\mathcal{B}: B^s_{p,q}\big(\Omega;\mathbb{R}^n\big)\times B^{s-\frac{1}{p}}_{p,q}(\partial\Omega;\mathbb{R})\to B^s_{p,q}\big(\Omega;\mathbb{R}^n\big)$$

for all $s\in(0,1/p)$ and $q\in[1,\infty]$. □

Let us now establish the continuity properties of operator $\mathcal{B}$ for *negative* $s$.

**Lemma 7.** *Let* $1 < p < \infty$, $-1 + 1/p < s < 0$, $q \in [1, \infty]$. *Let* $F := \mathcal{DIV}[k; \zeta]$ *be in* $\mathcal{B}^{-1+s}_{p,q}(\Omega)$. *Then* $v := Ik + J\zeta$ *belongs to* $B^s_{p,q}(\Omega)$, *satisfies* (2.1) *and*

$$\|v\|_{B^s_{p,q}(\Omega)} \leqslant C \big\|\mathcal{DIV}[k; \zeta]\big\|_{\mathcal{B}^{-1+s}_{p,q}(\Omega)}. \tag{5.20}$$

**Proof.** It is enough to show the estimate. For $1 \leqslant i, j \leqslant n$, let us state

$$a^i(x, y) := \frac{x^i - y^i}{|x - y|^n} \int_0^\infty \omega\left(x + r\frac{x - y}{|x - y|}\right)\big(|x - y| + r\big)^{n-1}\, dr \quad \text{and} \quad A^{i,j}(x, y) = \partial_{y^j} a^i(x, y).$$

Arguing by duality (see Proposition 1), one may assert that

$$\big\|v^i\big\|_{B^s_{p,q}(\Omega)} = \sup_{\|\pi\|_{B^{-s}_{p',q'}(\Omega)}=1} \left(-\sum_j \int_\Omega \int_\Omega k^j(y) A^{i,j}(x, y)\pi(x)\, dy\, dx + \int_\Omega \int_{\partial\Omega} \zeta(y) a^i(x, y)\pi(x)\, d\sigma_y\, dx\right). \tag{5.21}$$

Let us consider the first term[4]

$$K_1(k, \pi)(y) := -\int_\Omega \int_\Omega k(y) A(x, y)\pi(x)\, dy\, dx. \tag{5.22}$$

Let $E\pi : y \mapsto \int_\Omega A(x, y)\pi(x)\, dx$. We want to prove that

$$E : B^{-s}_{p',q'}(\Omega) \to B^{-s}_{p',q'}(\Omega). \tag{5.23}$$

It will be shown by interpolation. First, take $\pi \in L_{p'}(\Omega)$, then the results from the proof of Lemma 4 immediately lead to the following estimate

$$\|E\pi\|_{L_{p'}(\Omega)} \leqslant C\|\pi\|_{L_{p'}(\Omega)}. \tag{5.24}$$

Next we want to show the same inequality for higher regularity. For simplicity we restrict ourselves to the case where $\pi$ belongs to the completion $W^1_{p';0}(\Omega)$ of $\mathcal{C}^\infty_c(\Omega)$ functions for the $W^1_{p'}(\Omega)$ norm (as it will sufficient for our purpose). Let us remark that

$$A^{i,j}(x, y) = -\partial_{x^j} a(x, y) + \frac{x^i - y^i}{|x - y|^n} \int_0^\infty \partial_j \omega\left(x + r\frac{x - y}{|x - y|}\right)\big(|x - y| + r\big)^{n-1}\, dr.$$

Hence $\partial_{y^l} E\pi = E_1\pi + E_2\pi$ with

$$E_1\pi(y) := -\partial_{y^l} \int_\Omega \partial_{x^j} a(x, y)\pi(x)\, dx,$$

$$E_2\pi(y) := \int_\Omega \pi(x)\partial_{y^l}\left[\frac{x^i - y^i}{|x - y|^n} \int_0^\infty \partial_j \omega\left(x + r\frac{x - y}{|x - y|}\right)\big(|x - y| + r\big)^{n-1}\, dr\right] dx.$$

Since $\partial_j\omega$ is a smooth function, mimicking the proof of Lemma 4 leads immediately to the following estimate:

$$\|E_2\pi\|_{L_{p'}(\Omega)} \leqslant C\|\pi\|_{L_{p'}(\Omega)}.$$

Let us now look at $E_1\pi$. For $\pi \in W^1_{p';0}(\Omega)$, integrating by parts does not generate any boundary term. Therefore

$$E_1\pi(y) = \int_\Omega \partial_{y^l} a(x, y)\partial_{x^j}\pi(x)\, dx = \int_\Omega A(x, y)\partial_{x^j}\pi(x)\, dx.$$

Then, arguing as in the proof of Lemma 5, we obtain

$$\|E_1\pi\|_{L_{p'}(\Omega)} \leqslant C\|\nabla\pi\|_{L_{p'}(\Omega)}.$$

Thus we proved that operator $E$ maps $L_{p'}(\Omega)$ to $L_{p'}(\Omega)$ and $W^1_{p';0}(\Omega)$ to $W^1_{p'}(\Omega)$.

[4] We omit the indices $i$ and $j$ for notational simplicity.

Remark that if we restrict our attention to those $\sigma$ which are in $(0, 1/p')$ then the information about the traces is lost, so that one may write (see e.g. [22,28])

$$\big(L_{p'}(\Omega), W^1_{p';0}(\Omega)\big)_{\sigma,q} = B^{\sigma}_{p',q'}(\Omega) = \big(L_{p'}(\Omega), W^1_{p'}(\Omega)\big)_{\sigma,q'}.$$

Therefore, given that $-s \in (0, 1/p')$, one may conclude that

$$\|E\pi\|_{B^{-s}_{p',q'}(\Omega)} \leqslant C\|\pi\|_{B^{-s}_{p',q'}(\Omega)}. \tag{5.25}$$

This completes the proof of (5.23).

It is now easy to estimate the term $K_1(k,\pi)$ defined in (5.22): we have

$$\big|K_1(k,\pi)\big| \leqslant C\|k\|_{B^s_{p,q}(\Omega)}\|E\pi\|_{B^{-s}_{p',q'}(\Omega)}.$$

So finally

$$\big|K_1(k,\pi)\big| \leqslant C\|k\|_{B^s_{p,q}(\Omega)}\|\pi\|_{B^{-s}_{p',q'}(\Omega)}.$$

To complete the proof of the lemma, we now have to investigate the second term of (5.21), namely

$$K_2(\zeta,\pi)(y) := \int_{\Omega}\int_{\partial\Omega} \zeta(y)a(x,y)\pi(x)\,dx\,d\sigma_y.$$

Let us introduce the function

$$e\pi(y) := \int_{\Omega} a(x,y)\pi(x)\,dx.$$

As the domain $\Omega$ is bounded and the kernel $a$ satisfies $|a(x,y)| \leqslant C|x-y|^{1-n}$, it is obvious that we have the estimate

$$\|e\pi\|_{L_{p'}(\Omega)} \leqslant C\|\pi\|_{L_{p'}(\Omega)}.$$

Let us now have a look at the gradient of $e$. Note that

$$\partial_{y^l} e\pi(y) = \int_{\Omega} \partial_{y^l} a(x,y)\pi(x)\,dx = \int_{\Omega} A(x,y)\pi(x)\,dx = E\pi(y).$$

But we already proved that $E$ satisfies (5.25). Hence

$$e\pi \in B^{1-s}_{p',q'}(\Omega) \quad \text{with } \|e\pi\|_{B^{1-s}_{p',q'}(\Omega)} \leqslant C\|\pi\|_{B^s_{p',q'}(\Omega)}.$$

And because $1-s > \frac{1}{p'}$ we are allowed to apply the trace theorem (see Theorem 3), which leads to the conclusion that

$$e\pi|_{\partial\Omega} \in B^{1-s-\frac{1}{p'}}_{p',q'}(\partial\Omega) \quad \text{with } \|e\pi|_{\partial\Omega}\|_{B^{1-s-\frac{1}{p'}}_{p',q'}(\partial\Omega)} \leqslant C\|\pi\|_{B^s_{p',q'}(\Omega)}.$$

Given that $K_2(\zeta,\pi) = \int_{\partial\Omega} \zeta(y)e\pi(y)\,d\sigma_y$ and that $(s-\frac{1}{p}) + (1-s-\frac{1}{p'}) = 0$, we get

$$\big|K_2(\zeta,\pi)\big| \leqslant C\|\zeta\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)}\|e\pi\|_{B^{1-s-\frac{1}{p'}}_{p',q'}(\partial\Omega)} \leqslant C\|\zeta\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)}\|\pi\|_{B^s_{p',q'}(\Omega)}.$$

Now we return to (5.21) and may conclude that

$$\|v\|_{B^s_{p,q}(\Omega)} \leqslant \sup_{\|\pi\|_{B^{-s'}_{p',q'}(\Omega)}=1} \big(K_1(k,\pi) + K_2(\zeta,\pi)\big) \leqslant C\Big(\|k\|_{B^s_{p,q}(\Omega)} + \|\zeta\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)}\Big). \tag{5.26}$$

This completes the proof of the estimate. The fact that $v$ is a solution in the weak sense follows from Lemma 3. □

**End of the proof of Theorem 6.** The remaining case $s = 0$ may be achieved by interpolation by putting together Lemmas 6 and 7. This completes the proof of the theorem. □

## 6. The general case

This section is dedicated to the proof of Theorem 2 in general bounded or exterior domains with $C^{1,1}$ boundary.

In the bounded case, the idea is to decompose the original domain into a finite union of star-shaped domains after the method proposed by Galdi in [12, Lemma 3.4], or Maz'ya and Poborchi in [17, Lemma 1]. We start with a decomposition of $\Omega$ into a finite union of $C^{1,1}$ open sets $\Omega_i$, which are star-shaped with respect to some ball. Then we enlarge each $\Omega_i$ to an open set $G_i$ so that $\Omega_i = G_i \cap \Omega$ and that $\{G_1, \dots, G_m\}$ is a covering of $\overline{\Omega}$.

**Lemma 8.** *There exist smooth functions $\chi_i$, $\theta_i^j$ and $\phi_i^j$ compactly supported in $G_i$, $\Omega_i$ and $\overline{\Omega}$ respectively such that for any functional $F$ on $\mathcal{C}^\infty(\overline{\Omega})$ satisfying*

$$\int_\Omega F\,dx := F(1) = 0 \tag{6.1}$$

*we may write $F = \sum_{i=1}^m F_i$ with $F_i = \chi_i F + \sum_{j=1}^{m_i} \theta_i^j F(\phi_i^j)$ and*

$$\int_\Omega F_i\,dx = 0. \tag{6.2}$$

**Proof.** This may be proved inductively. Let us introduce a partition of unity subordinate to $\{G_1, \dots, G_m\}$ and $\Omega$, that is $m$ functions $\psi_1, \dots, \psi_m$ such that

$$\sum_{i=1}^m \psi_i \equiv 1 \quad \text{on } \overline{\Omega} \quad \text{and} \quad \psi_i \in \mathcal{C}_c^\infty(G_i).$$

Then, keeping in mind (6.1), we decompose $F$ into $F_1 + H_1$ with

$$F_1 := \psi_1 F - \theta_1 \int_\Omega F\psi_1\,dx \quad \text{and} \quad H_1 := (1-\psi_1)F - \theta_1 \int_\Omega F(1-\psi_1)\,dx,$$

where $\theta_1$ stands for some smooth function such that

$$\operatorname{Supp}\theta_1 \Subset \Omega_1 \cap (\Omega_2 \cup \dots \cup \Omega_m) \quad \text{and} \quad \int_\Omega \theta_1\,dx = 1.$$

Owing to (6.1), we have

$$\int_\Omega F_1\,dx = 0 \quad \text{and} \quad \int_\Omega H_1\,dx = 0.$$

Hence one may repeat the construction starting from $H_1$ instead of $F$ and the domain $\Omega \setminus \Omega_1$ with the covering $\{G_2, \dots, G_m\}$. Within a finite number of steps, we get the desired decomposition. The easy verifications are left to the reader. □

Now we are in a good position to prove Theorem 2 in the case of a general $C^{1,1}$ bounded domain. First we apply Lemma 8 with $F = \mathcal{DIV}[k;\zeta]$ in order to reduce our study to the case of a star-shaped domain in which case Theorem 6 applies. We get

$$\mathcal{DIV}[k;\zeta] = \sum_{i=1}^m F_i \quad \text{with } F_i := \chi_i\,\mathcal{DIV}[k;\zeta] + \sum_j \theta_i^j\,\mathcal{DIV}[k;\zeta]\big(\phi_i^j\big).$$

In consequence, in order to solve the original problem, it suffices to set $v = \sum_{i=1}^m v_i$ where the vector fields $v_1, \dots, v_m$ satisfy suitable estimates and

$$-\int_\Omega v_i \cdot \nabla\varphi\,dx = \int_\Omega F_i\varphi\,dx \quad \text{for all } \varphi \in \mathcal{C}^\infty(\overline{\Omega}). \tag{6.3}$$

For doing that, let us notice that, thanks to the support properties of functions $\chi_i$ and $\theta_i^j$, we have for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$,

$$\int_{\Omega_i} F_i \varphi\, dx = \left( -\int_{\Omega_i} \chi_i k \cdot \nabla\varphi\, dx + \int_{\partial\Omega_i} \chi_i \zeta \varphi\, dx \right) + \int_{\Omega_i} \varphi \left( -k \cdot \nabla\chi_i + \sum_j \theta_i^j\, \mathcal{DIV}[k;\zeta]\big(\phi_i^j\big) \right) dx.$$

The first term in the right-hand side may be seen as the *singular* part of the functional $F_i$ whereas the second part may be identified with an element of $B^s_{p,q}(\Omega)$ (as Besov spaces on bounded domains are stable by multiplication by a smooth function, see [1,22], we see that $k \cdot \nabla\chi_i \in B^s_{p,q}(\Omega_i)$).

Therefore, in order to solve (6.3), it suffices to split $v_i$ into $v_i^1 + v_i^2$ with $v_i^1$ satisfying

$$\mathcal{DIV}\big[v_i^1; 0\big] = \mathcal{DIV}[\chi_i k; \chi_i \zeta]$$

and $v_i^2$ satisfying

$$\begin{aligned} &\operatorname{div} v_i^2 = -k \cdot \nabla\chi_i + \sum_j \theta_i^j\, \mathcal{DIV}[k;\zeta]\big(\phi_i^j\big) \quad \text{in } \Omega_i, \\ &v_i^2 = 0 \quad \text{on } \partial\Omega_i. \end{aligned}$$

Of course, as $\Omega_i$ is star-shaped, we expect to construct the rough part $v_i^1$ by means of Theorem 6 whereas Theorem 1 should help us to construct the smooth part $v_i^2$. However this may be done only under the compatibility conditions

$$\int_{\partial\Omega_i} \chi_i \zeta\, d\sigma = 0 \quad \text{and} \quad \int_{\Omega_i} \left( -k \cdot \nabla\chi_i + \sum_j \theta_i^j\, \mathcal{DIV}[k;\zeta]\big(\phi_i^j\big) \right) dx = 0.$$

These conditions need not be satisfied so we are required to construct a corrector. Let

$$\alpha_i := \frac{1}{|\partial\Omega_i|} \int_{\partial\Omega_i} \chi_i \zeta\, d\sigma.$$

Then we solve the following Neumann problem

$$\begin{aligned} &\Delta P_i = \alpha_i \frac{|\partial\Omega_i|}{|\Omega_i|} \quad \text{in } \Omega_i, \\ &\partial_{\vec n} P_i = \alpha_i \quad \text{on } \partial\Omega_i. \end{aligned} \tag{6.4}$$

As the boundary of $\Omega_i$ is $C^{1,1}$, Proposition 2 implies that $\nabla P_i \in C^{0,1-\varepsilon}(\Omega_i)$ for any $\varepsilon \in (0,1)$. So in particular $\nabla P_i \in B^s_{p,q}(\Omega_i)$. Now, using the definition of $F_i$ and $\nabla P_i$, we may write for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega}_i)$,

$$\int_{\Omega_i} F_i \varphi\, dx = \mathcal{DIV}[\chi_i k - \nabla P_i; \chi_i \zeta - \alpha_i](\varphi) + \int_{\Omega_i} f_i \varphi\, dx$$

with

$$f_i := \frac{|\partial\Omega_i|}{|\Omega_i|}\alpha_i - k \cdot \nabla\chi_i + \sum_j \theta_i^j\, \mathcal{DIV}[k;\zeta]\big(\phi_i^j\big).$$

Note that by construction and by virtue of (6.2), we have

$$\mathcal{DIV}[\chi_i k - \nabla P_i; \chi_i \zeta - \alpha_i](1) = 0 \quad \text{and} \quad \int_{\Omega_i} f_i = 0.$$

Now, on the one hand, as $\Omega_i$ is star-shaped, Theorem 6 ensures the existence of some solution $v_i^1 \in B^s_{p,q}(\Omega_i)$ to

$$\mathcal{DIV}\big[v_i^1; 0\big] = \mathcal{DIV}[\chi_i k - \nabla P_i; \chi_i \zeta - \alpha_i] \tag{6.5}$$

such that

$$\|v_i^1\|_{B^s_{p,q}(\Omega_i)} \leqslant C\big(\|\chi_i k - \nabla P_i\|_{B^s_{p,q}(\Omega_i)} + \|\chi_i \zeta - \alpha_i\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega_i)}\big).$$

On the other hand, Theorem 1 provides a solution $v_i^2$ in $B^{s+1}_{p,q}(\Omega_i)$ to

$$\operatorname{div} v_i^2 = f_i \quad \text{in } \Omega_i, \qquad v_i^2 = 0 \quad \text{on } \partial\Omega_i \tag{6.6}$$

satisfying

$$\|v_i^2\|_{B^{1+s}_{p,q}(\Omega_i)} \leqslant C\left(\|\Delta P_i - k\cdot\nabla\chi_i\|_{B^s_{p,q}(\Omega_i)} + \sum_j \bigl|\mathcal{DIV}[k;\zeta]\bigl(\phi_i^j\bigr)\bigr|\|\theta_i^j\|_{B^s_{p,q}(\Omega)}\right).$$

So finally, setting $v_i := v_i^1 + v_i^2$ and using the definition of $\Delta P_i$ and the stability of Besov spaces under multiplication by smooth functions, we get

$$\|v_i\|_{B^s_{p,q}(\Omega_i)} \leqslant C\bigl(\|k\|_{B^s_{p,q}(\Omega_i)} + \|\zeta\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega_i)}\bigr) \tag{6.7}$$

and we see that (6.3) is satisfied for all test functions in $\mathcal{C}^\infty(\overline{\Omega}_i)$.

In addition, we observe that if $v_i$ is extended by 0 on $\Omega$ then inequality (6.7) is still valid with $\Omega$ instead of $\Omega_i$ and that, thanks to the support properties of functions $\chi_i$ and $\theta_i^j$, (6.3) is satisfied *for all test functions in* $\mathcal{C}^\infty(\overline{\Omega})$. So finally, $v := \sum_{i=1}^m v_i$ is a solution to our problem. This completes the proof of Theorem 2 in the case of a general $C^{1,1}$ bounded domain.

Let us finally prove Theorem 2 in the case of an exterior domain with $C^{1,1}$ boundary. Reducing the problem to the case of a bounded domain is the key idea. For that, we introduce an extension $\widetilde{k}$ of $k$ over $\mathbb{R}^n$ such that

$$\|\widetilde{k}\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant C\|k\|_{B^s_{p,q}(\Omega)}$$

and first solve the following problem

$$\Delta P = \operatorname{div}\widetilde{k} \quad \text{in } \mathbb{R}^n \tag{6.8}$$

by the fundamental solution. As $\widetilde{k} \to \nabla P$ is a Calderon–Zygmund operator and $1 < p < \infty$, we do have $\nabla P \in B^s_{p,q}(\mathbb{R}^n)$ (see e.g. [1]), and

$$\|\nabla P\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant C\|k\|_{B^s_{p,q}(\Omega)}. \tag{6.9}$$

Let $K$ be a compact $C^{1,1}$ subdomain of $\overline{\Omega}$ surrounding $\partial\Omega$ (that is such that $\operatorname{dist}(\Omega\setminus K, \partial\Omega) > 0$). We consider

$$w := \chi\nabla P,$$

where $\chi : \mathbb{R}^n \to [0,1]$ is a smooth function such that $\chi \equiv 1$ in a neighborhood of $\Omega \setminus K$ and $\chi \equiv 0$ in a neighborhood of $\mathbb{R}^n \setminus \Omega$.

Note that $\nabla\chi$ has compact support (included in the interior of $K$). Hence one may check (see e.g. [1,22]) that

$$w \in B^s_{p,q}\bigl(\mathbb{R}^n\bigr) \quad \text{and} \quad \|w\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant C\|k\|_{B^s_{p,q}(\Omega)}. \tag{6.10}$$

Let us resume the proof of the theorem. We want to find a vector field $v$ satisfying

$$-\int_\Omega v\cdot\nabla\varphi\,dx = -\int_\Omega k\cdot\nabla\varphi\,dx + \int_{\partial\Omega} \zeta\varphi\,d\sigma \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\overline{\Omega}).$$

Setting $v = w + z$, we see that it amounts to finding some vector field $z$ so that

$$-\int_\Omega z\cdot\nabla\varphi\,dx = \int_\Omega (\chi\nabla P - k)\cdot\nabla\varphi\,dx + \int_{\partial\Omega}\zeta\varphi\,d\sigma \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\overline{\Omega}). \tag{6.11}$$

Now, we notice that for all $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega})$, one may write

$$\begin{aligned}\int_\Omega (\chi\nabla P - k)\cdot\nabla\varphi\,dx &= -\int_\Omega (1-\chi)k\cdot\nabla\varphi\,dx + \int_\Omega \chi(\nabla P - k)\cdot\nabla\varphi\,dx\\ &= -\int_\Omega (1-\chi)k\cdot\nabla\varphi\,dx + \int_\Omega \nabla\chi\cdot(k - \nabla P)\varphi\,dx + \int_\Omega (\nabla P - k)\cdot\nabla(\chi\varphi)\,dx.\end{aligned}$$

Observe that $\chi\varphi$ is in $\mathcal{C}_c^\infty(\Omega)$. Therefore, as (6.8) is satisfied and $\operatorname{div} k = \operatorname{div}\widetilde{k}$ in $\Omega$, the last term above vanishes and one may look for the solution $z$ under the form $z = z_1 + z_2$ with

$$\mathcal{DIV}[z_1; 0] = \mathcal{DIV}\big[(1-\chi)k; \zeta\big] \quad \text{and} \quad \operatorname{div} z_2 = \nabla\chi \cdot (k - \nabla P) \quad \text{with } z_2|_{\partial\Omega} = 0. \tag{6.12}$$

By construction (recall that $\chi \equiv 0$ on $\partial\Omega$ and that $\operatorname{div}\widetilde{k} = \Delta P$ in $\mathbb{R}^n$) we have guaranteed the compatibility conditions

$$\int_\Omega \mathcal{DIV}\big[(1-\chi)k; \zeta\big]\,dx = 0 \quad \text{and} \quad \int_\Omega \nabla\chi \cdot (k - \nabla P)\,dx = \int_{\mathbb{R}^n} \nabla(\chi - 1) \cdot (\widetilde{k} - \nabla P)\,dx = 0.$$

In fact, the support properties of function $\chi$ ensure that the integrants vanish away from $K$. So if we extend $\zeta$ by 0 on $\partial K \setminus \partial\Omega$, the above compatibility conditions rewrite in the following more accurate form:

$$\int_K \mathcal{DIV}\big[(1-\chi)k; \zeta\big]\,dx = 0 \quad \text{and} \quad \int_K \nabla\chi \cdot (k - \nabla P)\,dx = 0 \tag{6.13}$$

and one may consider the two problems described in (6.12) in the *bounded* domain $K$ rather than in the unbounded domain $\Omega$.

Now, applying Theorems 1 and 2 to the case of the bounded domain $K$, we get two vector fields $z_1$ and $z_2$ satisfying

$$\begin{aligned} &-\int_K z_1 \cdot \nabla\varphi\,dx = \int_K (\chi - 1)k \cdot \nabla\varphi\,dx + \int_{\partial K} \zeta\varphi\,d\sigma \quad \text{for all } \varphi \in \mathcal{C}^\infty(K), \\ &\operatorname{div} z_2 = \nabla\chi \cdot (k - \nabla P) \quad \text{in } K \quad \text{and} \quad z_2|_{\partial K} = 0, \end{aligned}$$

and such that

$$\begin{aligned} &\|z_1\|_{B^s_{p,q}(K)} \leqslant C\Big(\big\|(1-\chi)k\big\|_{B^s_{p,q}(K)} + \|\zeta\|_{B^{s-\frac{1}{p}}_{p,q}(\partial K)}\Big), \\ &\|z_2\|_{B^{s+1}_{p,q}(K)} \leqslant C\big\|\nabla\chi \cdot (k - \nabla P)\big\|_{B^s_{p,q}(K)}. \end{aligned}$$

Note that if we extend $z_1$ and $z_2$ by 0 on $\Omega$, and use (6.9) and the product laws in Besov spaces then the above two inequalities imply that

$$\|z_1\|_{B^s_{p,q}(\Omega)} \leqslant C\Big(\|k\|_{B^s_{p,q}(\Omega)} + \|\zeta\|_{B^{s-\frac{1}{p}}_{p,q}(\partial\Omega)}\Big), \tag{6.14}$$

$$\|z_2\|_{B^{s+1}_{p,q}(\Omega)} \leqslant C\|k\|_{B^s_{p,q}(\Omega)}. \tag{6.15}$$

In addition, the support properties of the function $\chi$ ensure that we have

$$\forall \varphi \in \mathcal{C}^\infty_c(\overline{\Omega}), \quad \begin{cases} -\int_\Omega z_1 \cdot \nabla\varphi\,dx = \int_\Omega (\chi - 1)k \cdot \nabla\varphi\,dx + \int_{\partial\Omega} \zeta\varphi\,d\sigma, \\ -\int_\Omega z_2 \cdot \nabla\varphi\,dx = \int_\Omega \varphi\nabla\chi \cdot (k - \nabla P)\,dx. \end{cases}$$

So finally, setting $v := w + z_1 + z_2$ and using (6.10), (6.14), (6.15), we get a vector field satisfying (2.1) and the desired estimate.

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## References

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren Math. Wiss., vol. 243, Springer, 2011.
[2] C. Bennett, R. Sharpley, Interpolation of Operators, Pure Appl. Math., vol. 129, Academic Press, Inc., Boston, MA, 1988.
[3] M. Bogovskiĭ, Solutions of some problems of vector analysis, associated with the operators div and grad, in: Tr. Semin. S.L. Soboleva, vol. 1, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1980, pp. 5–40.
[4] A. Calderón, A. Zygmund, On singular integrals, Amer. J. Math. 78 (1956) 289–309.
[5] M. Costabel, A. McIntosch, On Bogovskiì and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains, Math. Z. 265 (2010) 297–320.
[6] R. Danchin, P.B. Mucha, A critical functional framework for he inhomogeneous Navier–Stokes equations, J. Funct. Anal. 256 (3) (2009) 881–927.
[7] R. Danchin, P.B. Mucha, Critical functional framework and maximal regularity in action on systems of incompressible flows, in preparation.
[8] R. DeVore, R. Sharpley, Besov spaces on domains in $R^d$, Trans. Amer. Math. Soc. 335 (2) (1993) 843–864.
[9] J. Duoandikoetxea, Fourier Analysis, Amer. Math. Soc., Providence, RI, 2001.
[10] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford Lecture Ser. Math. Appl., vol. 26, Oxford University Press, 2004.

[11] D. Fujiwara, H. Morimoto, An $L_r$-theorem of the Helmholtz decomposition of vector fields, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (3) (1977) 685–700.
[12] G. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, vol. I. Linearized Steady Problems, Springer Tracts Nat. Philos., vol. 38, Springer-Verlag, New York, 1994.
[13] D. Gilbarg, L. Hörmander, Intermediate Schauder estimates, Arch. Ration. Mech. Anal. 74 (1980) 297–318.
[14] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, RI, 1968.
[15] G. Leoni, A First Course in Sobolev Spaces, Grad. Stud. Math., vol. 105, Amer. Math. Soc., Providence, RI, 2009.
[16] G. Lieberman, Intermediate Schauder estimates for oblique derivative problems, Arch. Ration. Mech. Anal. 93 (2) (1986) 129–134.
[17] V. Maz'ya, S. Poborchi, Differentiable Functions on Bad Domains, World Scientific Publishing Co., Inc., River Edge, NJ, 1997.
[18] D. Mitrea, M. Mitrea, S. Monniaux, The Poisson problem for the exterior derivative operator with Dirichlet boundary condition in nonsmooth domains, Commun. Pure Appl. Anal. 7 (6) (2008) 1295–1333.
[19] P.B. Mucha, On weak solutions to the Stefan problem with Gibbs–Thomson correction, Differential Integral Equations 20 (2007) 769–792.
[20] P.B. Mucha, M. Pokorný, On a new approach to the issue of existence and regularity for the steady compressible Navier–Stokes equations, Nonlinearity 19 (8) (2006) 1747–1768.
[21] A. Novotný, I. Straškraba, Mathematical Theory of Compressible Flows, Oxford Sci. Publ., 2004.
[22] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, de Gruyter Ser. Nonlinear Anal. Appl., vol. 3, Walter de Gruyter & Co., Berlin, 1996.
[23] V. Rychkov, On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains, J. London Math. Soc. 60 (1) (1999) 237–257.
[24] S. Sobolev, Applications of Functional Analysis to Mathematical Physics, Amer. Math. Soc. Transl. Monogr., vol. 7, 1964.
[25] Ph. Souplet, Optimal regularity conditions for elliptic problems via $L^p_\delta$-spaces, Duke Math. J. 127 (1) (2005) 175–192.
[26] M.E. Taylor, Tools for PDE. Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials, Math. Surveys Monogr., vol. 81, Amer. Math. Soc., Providence, RI, 2000.
[27] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis, Stud. Math. Appl., vol. 2, North-Holland Publishing Co., 1977.
[28] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland Math. Library, vol. 18, North-Holland Publishing Co., 1978.
[29] H. Triebel, Theory of Function Spaces. III, Monogr. Math., vol. 100, Birkhäuser Verlag, Basel, 2006.