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Time-Periodic Solutions to the Full Navier–Stokes–Fourier System

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Abstract

We consider the full Navier–Stokes–Fourier system describing the motion of a compressible viscous and heat conducting fluid driven by a time-periodic external force. We show the existence of at least one weak time periodic solution to the problem under the basic hypothesis that the system is allowed to dissipate the thermal energy through the boundary. Such a condition is in fact necessary, as energetically closed fluid systems do not possess non-trivial (changing in time) periodic solutions as a direct consequence of the Second law of thermodynamics.

1. Introduction

Time periodic or time almost periodic processes are frequently observed in many real world applications of fluid mechanics. They are represented by the time periodic solutions of their associated mathematical models. Considerable effort has been exerted and a large variety of methods developed to prove the existence and to study the qualitative properties of time periodic solutions to evolutionary partial differential equations; see Vejvoda et al. [21].

Strangely enough, time periodic processes are forbidden for *energetically closed* fluid systems by the Second law of thermodynamics. Indeed the total entropy of such a system is always increasing in time and the mechanical energy is irreversibly converted into heat; see [5, Chapter 5, Section 5.2]. Thus the existence of time periodic processes is strictly conditioned by the ability of the system to exchange energy with its environment. Typical examples are reduced mathematical models, in which the thermodynamic effects are neglected; for instance, the compressible and incompressible Navier-Stokes system, where only the purely mechanical aspects of the fluid motion are taken into account. We refer to Galdi and Silvestre [7], Kobayashi [9], Iooss [8], Kučera [10], Maremonti [13], Yamazaki [22] and to [3], among many others, for relevant recent mathematical results. In addition, there is a very interesting result by MA et al. [12] for the *full* Navier–Stokes–Fourier system considered, unfortunately, in a rather unrealistic space dimension $N = 5$. The authors show the existence of non-trivial time periodic solutions to this apparently conservative system in $\mathbb{R}^N$, meaning there must be an energy leak at "infinity".

In this paper, we focus on the full *Navier–Stokes–Fourier system*, where the time evolution of the fluid density $\varrho = \varrho(t, x)$, the velocity field $\mathbf{u} = \mathbf{u}(t, x)$, and the absolute temperature $\vartheta = \vartheta(t, x)$ are governed by the following system of partial differential equations:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \varrho \mathbf{f} \tag{2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x - \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, \mathrm{dS}_x, \tag{4}$$

where $\mathbb{S}$ is the Newtonian viscous stress,

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \, \mathbb{I}, \tag{5}$$

$\mathbf{q}$ is the heat flux obeying Fourier's law

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta, \tag{6}$$

and where the thermodynamic quantities—the pressure p, the specific entropy s, and the specific internal energy e—are given numerical functions of the state variables ϱ, ϑ interrelated through Gibbs' equation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right). \tag{7}$$

Finally, the symbol $\sigma \geqq 0$ denotes the entropy production rate satisfying

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{8}$$

The fluid is confined to a smooth bounded domain $\Omega \subset \mathbb{R}^3$, where the velocity field satisfies the standard no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}. \tag{9}$$

A crucial feature of our problem is that the heat flux through the boundary is allowed, specifically,

$$\mathbf{q} \cdot \mathbf{n} = d(x)(\vartheta - \Theta_0), \quad \text{with } d \in L^\infty(\partial\Omega), \quad \Theta_0 \in L^1(\partial\Omega). \tag{10}$$

The system is driven by a time-periodic force

$$\mathbf{f} \in L^\infty(\mathbb{R}^1 \times \Omega; \mathbb{R}^3), \quad \mathbf{f}(t + L, \cdot) = \mathbf{f}(t, \cdot) \quad \text{for all } \ t \in \mathbb{R}^1. \tag{11}$$

Our approach is based on the theory of weak solutions to the evolutionary Navier–Stokes–Fourier system developed in the first part of the monograph [4], although many ideas used below are more related to the associated *stationary* problems, see [14–16] and [18,19]. Our main goal is to show that problem (1–11) possesses at least one time-periodic solution $\{\varrho, \mathbf{u}, \vartheta\}$ under certain restrictions imposed on the constitutive relations similar to those required by the existence theory (see [4, Chapter 3]). As already pointed out, in the light of the results presented in [5, Chapter 5, Section 5.2], the dissipative boundary condition (10) is necessary, as otherwise the energy of *any* solution to the Navier–Stokes–Fourier system blows up for $t \to \infty$ as long as the fluid is energetically isolated, meaning, $\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$. A similar conclusion holds for the stationary problem, where the only admissible solutions to the energetically isolated system are the so-called static states $\varrho = \tilde{\varrho}(x)$, $\mathbf{u} \equiv 0$, $\vartheta = \mathit{positive\ constant}$, corresponding to a potential force $\mathbf{f} = \nabla_x F(x)$; see [5, Chapter 5].

The proof of the existence of the time periodic solutions is based on *a priori bounds* derived in Section 2.4 below. It is interesting to note that the result is, in fact, "better" than for the isentropic case studied in [3], where certain restrictions have to be imposed on the value of the adiabatic coefficient. This rather surprising effect of thermodynamics included in the present problem has also been observed in the stationary case, see [18,19]. The fact that the transport coefficients depend effectively on the temperature apparently brings more dissipation into the system, which in turn "improves" the available a priori bounds.

The time periodic solutions are constructed by means of a direct method, where the approximation scheme is solved in the spaces of time-periodic functions. Accordingly, system (1–3) is replaced by an *elliptic* regularization, where the leading coefficients of the extra time derivatives are sent to zero. This approach allows us to overcome difficulties in proving sufficient smoothness of solutions to the approximative system. As soon as the approximate solutions are available, the main steps in the limit passage in the approximation scheme closely follow their counterparts in the existence theory for the initial-value problem developed in [4]. In order to conclude the introductory part, let us point out that we deal with a genuine large data problem, where the framework of weak solutions is the only one available.

The paper is organized as follows. In Section 2, we introduce the principal hypotheses imposed on the constitutive relations and state our main result. Section 2.4 is a collection of available a priori bounds on the family of time periodic solutions. Although this piece of information is never used directly in the subsequent sections, we feel it is quite useful for the reader as the estimates are free of additional technicalities imposed by the approximation scheme. Sections 3 and 4 form the heart of the paper. Here we introduce the elliptic regularization and solve it via a direct method. In this way a family of approximate solutions is obtained, together with uniform bounds guaranteed by the available a priori estimates. Finally, in Sections 5, 6, 7, 8 and 9, we pass to the limit in the family of approximate solutions, completing the construction of the desired time-periodic solutions to the original system.

2. Principal Hypotheses and Main Result

In this section, we list the principal hypotheses imposed on the constitutive relations and state our main result. A prototype of the pressure law used in this paper reads

$$p(\varrho, \vartheta) = a_1 \varrho^{5/3} + a_2 \varrho \vartheta + \frac{a}{3} \vartheta^4, \tag{12}$$

with the corresponding specific internal energy and specific entropy

$$\begin{aligned} e(\varrho, \vartheta) &= \frac{3}{2} a_1 \varrho^{2/3} + c_v \vartheta + \frac{a}{\varrho} \vartheta^4 \\ s(\varrho, \vartheta) &= c_v \ln \vartheta - a_2 \ln \varrho + \frac{4a}{3\varrho} \vartheta^3, \end{aligned} \tag{13}$$

with a, a_1, a_2 and c_v positive. Thus the pressure is a sum of three components: the cold (or degenerate) pressure due to the electron gas that may apply for degenerate gases, the standard perfect gas law and the contribution of radiation. Although the pressure law introduced below is more "sophisticated" and fits better in the underlying thermodynamic framework, one should always keep in mind its simplified version (12). The interested reader may consult [4, Chapter 1] for physical background and further extensions of the theory.

2.1. Constitutive Relations

We assume that the pressure p takes the form

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3} \vartheta^4, \quad \text{with } a > 0, \tag{14}$$

where $P \in C^1([0, \infty)) \cap C^2(0, \infty)$,

$$P'(Z) > 0 \text{ for all } Z \geqq 0, \quad \lim_{Z \to \infty} \frac{P'(Z)}{Z^{2/3}} = p_\infty > 0. \tag{15}$$

In accordance with Gibbs' equation (7), the specific internal energy can be taken as

$$e(\varrho, \vartheta) = \frac{3}{2} \vartheta \left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho} \vartheta^4, \tag{16}$$

while the specific entropy reads

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3\varrho} \vartheta^3, \quad S'(Z) = -\frac{3}{2} \frac{\frac{5}{3} P(Z) - Z P'(Z)}{Z^2}. \tag{17}$$

In addition to (14–17), we assume that the specific heat at constant volume is positive and uniformly bounded:

$$0 < \frac{\frac{5}{3} P(Z) - Z P'(Z)}{Z} < c \quad \text{for all } Z > 0. \tag{18}$$

From (14–18) it follows that we may write

$$
\begin{aligned}
p(\varrho,\vartheta) &= p_0(\varrho,\vartheta) + \frac{a}{3}\vartheta^4 \\
e(\varrho,\vartheta) &= e_0(\varrho,\vartheta) + \frac{a}{\varrho}\vartheta^4 \\
s(\varrho,\vartheta) &= s_0(\varrho,\vartheta) + \frac{4a}{3\varrho}\vartheta^3,
\end{aligned}
$$

with

$$
\begin{aligned}
c_1\varrho\vartheta \ &\leqq p_0(\varrho,\vartheta) \leqq c_2\varrho\vartheta, \quad \text{for } \ \varrho \leqq K_0\vartheta^{3/2} \\
c_3\varrho^{5/3} &\leqq p_0(\varrho,\vartheta) \leqq c_4 \begin{cases} \vartheta^{5/2}, & \text{for } \varrho \leqq K_0\vartheta^{3/2} \\ \varrho^{5/3}, & \text{for } \varrho > K_0\vartheta^{3/2}. \end{cases}
\end{aligned}
\tag{19}
$$

Further

$$
\begin{aligned}
&\frac{\partial p_0(\varrho,\vartheta)}{\partial\varrho} > 0 && \text{in } (0,\infty)^2 \\
&p_0(\varrho,\vartheta) = b_0\varrho^{5/3} + p_m(\varrho,\vartheta), && b_0 > 0, \\
&\text{with } \ \frac{\partial p_m(\varrho,\vartheta)}{\partial\varrho} \geqq 0 && \text{in } (0,\infty)^2.
\end{aligned}
\tag{20}
$$

For the specific internal energy defined by (16) it follows that

$$
\left.
\begin{aligned}
\frac{3}{2}p_\infty\varrho^{2/3} \leqq e_0(\varrho,\vartheta) \leqq c_5(\varrho^{2/3}+\vartheta) \\
\frac{\partial e_0(\varrho,\vartheta)}{\partial\varrho}\varrho \leqq c_5(\varrho^{2/3}+\vartheta)
\end{aligned}
\right\} \text{ in } (0,\infty)^2.
\tag{21}
$$

Moreover, for the specific entropy $s(\varrho,\vartheta)$ defined by (17) we have, due to the Gibbs relation (7),

$$
\begin{aligned}
\frac{\partial s_0(\varrho,\vartheta)}{\partial\varrho} &= \frac{1}{\vartheta}\Big(-\frac{p_0(\varrho,\vartheta)}{\varrho^2} + \frac{\partial e_0(\varrho,\vartheta)}{\partial\varrho}\Big) = -\frac{1}{\varrho^2}\frac{\partial p_0(\varrho,\vartheta)}{\partial\vartheta} \\
\frac{\partial s_0(\varrho,\vartheta)}{\partial\vartheta} &= \frac{1}{\vartheta}\frac{\partial e_0(\varrho,\vartheta)}{\partial\vartheta} = \frac{3}{2}\frac{\vartheta^{3/2}}{\varrho}\Big(\gamma P\Big(\frac{\varrho}{\vartheta^{3/2}}\Big) - \frac{\varrho}{\vartheta^{3/2}}P'\Big(\frac{\varrho}{\vartheta^{3/2}}\Big)\Big) > 0.
\end{aligned}
\tag{22}
$$

We also have for suitable choice of the additive constant in the definition of the specific entropy

$$
\begin{aligned}
|s_0(\varrho,\vartheta)| &\leqq c_6(1+|\ln\varrho|+|\ln\vartheta|) && \text{in } \ (0,\infty)^2 \\
|s_0(\varrho,\vartheta)| &\leqq c_7(1+|\ln\varrho|) && \text{in } \ (0,\infty)\times(1,\infty) \\
s_0(\varrho,\vartheta) \ &\geqq c_8 > 0 && \text{in } \ (0,1)\times(1,\infty) \\
s_0(\varrho,\vartheta) \ &\geqq c_9(1+\ln\vartheta) && \text{in } \ (0,1)\times(0,1).
\end{aligned}
\tag{23}
$$

The transport coefficients μ, η, and κ are supposed to be continuously differentiable functions of the absolute temperature ϑ satisfying

$$
0 < \underline{\mu}(1+\vartheta) \leqq \mu(\vartheta), \ |\mu'(\vartheta)| \leqq c \quad \text{for all } \vartheta \geqq 0 \tag{24}
$$

$$
0 \leqq \eta(\vartheta) \leqq \overline{\eta}(1+\vartheta) \quad \text{for all } \vartheta \geqq 0, \tag{25}
$$

and

$$
0 < \underline{\kappa}(1+\vartheta^3) \leqq \kappa(\vartheta) \leqq \overline{\kappa}(1+\vartheta^3) \quad \text{for all } \vartheta \geqq 0. \tag{26}
$$

2.2. Weak Solutions

When dealing with time-periodic problems, it is convenient to consider all quantities defined on a time "sphere"

$$S^1 = [0, T_{\rm per}]|_{\{0, T_{\rm per}\}}.$$

We will say that a triple $\{\varrho, \mathbf{u}, \vartheta\}$ is a time-periodic weak solution to the Navier–Stokes–Fourier system (1–10) if the following holds:

- the solution belongs to the class $\varrho \geqq 0$, $\vartheta > 0$ almost everywhere,

$$\varrho \in L^\infty(S^1; L^{5/3}(\Omega)),\ \ \vartheta \in L^\infty(S^1; L^4(\Omega)),\ \ \mathbf{u} \in L^2(S^1; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$
$$\vartheta^{3/2}, \ln \vartheta \in L^2(S^1; W^{1,2}(\Omega));$$

- equation of continuity (1) is satisfied in the sense of renormalized solutions,

$$\int_{S^1}\int_\Omega \big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + \big(b(\varrho) - b'(\varrho)\varrho\big)\,{\rm div}_x\mathbf{u}\varphi\big)\ {\rm d}x\ {\rm d}t = 0$$

for any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$, and any test function $\varphi \in C^\infty(S^1 \times \overline{\Omega})$;
- momentum equation (2) holds in the sense of distributions:

$$\int_{S^1}\int_\Omega \big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\boldsymbol{\varphi} + p(\varrho,\vartheta){\rm div}_x\boldsymbol{\varphi}\big)\ {\rm d}x\ {\rm d}t$$
$$= \int_{S^1}\int_\Omega \big(\mathsf{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} - \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\big)\ {\rm d}x\ {\rm d}t$$

for any $\boldsymbol{\varphi} \in C_c^\infty(S^1 \times \Omega; \mathbb{R}^3)$;
- entropy equation (3) and the boundary condition (10) are satisfied in the sense of the integral identity

$$\begin{aligned}\int_{S^1}\int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\psi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\psi + \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\psi\right)\ {\rm d}x\ {\rm d}t \\ = \int_{S^1}\int_{\partial\Omega} \frac{d}{\vartheta}(\vartheta - \Theta_0)\psi\ {\rm d}S_x\ {\rm d}t - \langle\sigma;\psi\rangle\end{aligned} \tag{27}$$

for any $\psi \in C^\infty(S^1 \times \overline{\Omega})$, where $\sigma \in \mathcal{M}^+(S^1 \times \overline{\Omega})$ is a non-negative measure satisfying

$$\sigma \geqq \frac{1}{\vartheta}\left(\mathsf{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right); \tag{28}$$

- the total energy balance

$$\int_{S^1}\left(\partial_t\psi\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)\ {\rm d}x\right)\ {\rm d}t$$
$$= \int_{S^1}\psi\left(\int_{\partial\Omega} d(\vartheta - \Theta_0)\ {\rm d}S_x - \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ {\rm d}x\right)\ {\rm d}t$$

holds for any $\psi \in C^\infty(S^1)$.

It is not difficult to see that the entropy production inequality (28) reduces to (8) as soon as the solution is smooth enough.

2.3. Main Result

Our aim is to show the following result:

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary of class $C^{2+\nu}$. Suppose that the thermodynamic functions p, e, and s satisfy hypotheses (14–18), while the transport coefficients μ, η, and κ comply with (24–26), and $d \in L^\infty(\partial\Omega)$, $\Theta_0 \in L^1(\partial\Omega)$,*

$$d(x) \geqq d_0 > 0, \quad \Theta_0(x) \geqq \overline{\Theta} > 0 \ \ \textit{for all } x \in \partial\Omega. \tag{29}$$

Finally, let $\mathbf{f} \in L^\infty(S^1 \times \Omega; \mathbb{R}^3)$.

Then for any $M_0 > 0$ the Navier–Stokes–Fourier system (1–10) possesses at least one time-periodic-solution $\{\varrho, \mathbf{u}, \vartheta\}$ in the sense specified in Section 2.2 above such that

$$\int_\Omega \varrho(t, \cdot)\, \mathrm{d}x = M_0 \ \ \textit{for all } t \in S^1.$$

The rest of the paper is devoted to the proof of Theorem 1. It is worth-noting that the result is in fact “better” than for the corresponding *isentropic* system established in [3], where the pressure satisfies $p(\varrho) = a\varrho^\gamma$, $\gamma > 5/3$ in contrast with (15). Indeed it is easy to check that Theorem 1 holds true also for the pressure, specific energy and specific entropy from (12) and (13).

2.4. A Priori Bounds

Before starting the technical part of the proof of Theorem 1, we establish a priori bounds available for (smooth) time-periodic solutions of problem (1–11).

Lemma 1. *Let $(\varrho, \mathbf{u}, s)$ be sufficiently smooth solutions to (1–4), then*

$$\begin{aligned} &\sup_{t\in S^1} \int_\Omega \left(\varrho \mathbf{u}^2 + \varrho^{5/3} + \vartheta^4\right) \mathrm{d}x \\ &\quad + \int_{S^1}\int_\Omega \left(|\nabla_x \mathbf{u}|^2 + \left(1 + \vartheta^3\right)\frac{|\nabla_x \vartheta|^2}{\vartheta^2} + \varrho^{5/3+1/9}\right) \mathrm{d}x\, \mathrm{d}t \leqq \mathit{Data}. \end{aligned} \tag{30}$$

The proof of this lemma is naturally split into two parts concerning bounds resulting from the energy estimates and improvement of integrability of the density.

2.4.1. Energy Estimates. To begin, observe that the total mass of the fluid is a constant of motion, meaning

$$\int_\Omega \varrho(t, \cdot)\, \mathrm{d}x = M_0, \quad \text{in particular, } \varrho \in L^\infty(S^1; L^1(\Omega)). \tag{31}$$

Next step is to integrate the entropy balance equation (3) over the time-space cylinder $S^1 \times \Omega$, or, equivalently, to take $\psi \equiv 1$ in (27), to obtain

$$\begin{aligned} &\int_{S^1}\int_\Omega \left(\frac{1}{2}\frac{\mu(\vartheta)}{\vartheta}\left|\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3}\mathrm{div}_x \mathbf{u}\, \mathbb{I}\right|^2 + \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta^2}\right) \mathrm{d}x\, \mathrm{d}t \\ &\quad + \int_{S^1}\int_{\partial\Omega} \frac{d}{\vartheta}\Theta_0\, \mathrm{dS}_x\, \mathrm{d}t \leqq \int_{S^1}\int_{\partial\Omega} d\, \mathrm{dS}_x\, \mathrm{d}t \leqq c. \end{aligned} \tag{32}$$

Consequently, by virtue of (24), (26), combined with the standard Korn inequality, we deduce that

$$\mathbf{u} \in L^2\big(S^1; W_0^{1,2}(\Omega; \mathbb{R}^3)\big), \tag{33}$$

$$\nabla_x \vartheta^{3/2} \in L^2\big(S^1 \times \Omega; \mathbb{R}^3\big), \quad \text{and} \quad \nabla_x \ln \vartheta \in L^2\big(S^1 \times \Omega; \mathbb{R}^3\big). \tag{34}$$

Inequality (32) is the heart of the analysis as it yields, almost for granted, important a priori bounds for the velocity field and the temperature gradient. Similar observation was exploited in [18, 19] for solving the stationary problem.

Next, we integrate the total energy balance (4) over S^1 to obtain

$$\int_{S^1} \int_{\partial\Omega} d(\vartheta - \Theta_0) \, \mathrm{d}S_x \, \mathrm{d}t = \int_{S^1} \int_{\Omega} \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t,$$

where, in accordance with (31), (33), and the embedding $W^{1,2} \hookrightarrow L^6$,

$$\left| \int_{S^1} \int_{\Omega} \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \right| \leqq c\big(1 + \|\varrho\|_{L^2(S^1; L^{6/5}(\Omega))}\big).$$

Consequently, by virtue of (29), $\|\vartheta\|_{L^1(S^1 \times \partial\Omega)} \leqq c\big(1 + \|\varrho\|_{L^2(S^1; L^{6/5}(\Omega))}\big)$. Thus, as a consequence of (34) and Poincaré's inequality,

$$\|\vartheta\|_{L^1(S^1; L^6(\Omega))} \leqq c\big(1 + \|\varrho\|_{L^2(S^1; L^{6/5}(\Omega))}\big). \tag{35}$$

Furthermore, by (31),

$$\|\varrho\|^2_{L^2(S^1; L^{6/5}(\Omega))} \leqq c \int_{S^1} \left(\int_{\Omega} \varrho^{5/3} \, \mathrm{d}x \right)^{1/2} \mathrm{d}t,$$

which, together with (35) implies that

$$\|\vartheta\|_{L^1(S^1; L^6(\Omega))} \leqq c \left(1 + \left(\int_{S^1} \left(\int_{\Omega} \varrho^{5/3} \, \mathrm{d}x \right)^{1/2} \mathrm{d}t \right)^{1/2} \right). \tag{36}$$

Next observe that, by virtue of hypotheses (14–17), there exist two positive constants c_1, c_2 such that

$$c_1\big(\varrho^{5/3} + \vartheta^4\big) \leqq \varrho e(\varrho, \vartheta) \leqq c_2\big(\varrho\vartheta + \varrho^{5/3} + \vartheta^4\big). \tag{37}$$

Denoting $E(t) = \int_{\Omega} \big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\big) \, \mathrm{d}x$ the total energy we deduce from the energy balance (4) that $E(t) \leqq E(s) + c\Big(1 + \int_{S^1} E(z) \, \mathrm{d}z\Big)$ for any $t \leqq s$. By means of the mean value theorem, we therefore obtain that

$$\sup_{t \in S^1} E(t) \leqq c \left(1 + \int_{S^1} E(s) \, \mathrm{d}s \right). \tag{38}$$

Seeing that, in accordance with (33),

$$\int_{S^1} \int_{\Omega} \frac{1}{2} \varrho |\mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \leqq c \|\varrho\|_{L^\infty(S^1; L^{3/2}(\Omega))},$$

formula (38) reads

$$\sup_{t\in S^1} E(t) \leqq c\left(1+\int_{S^1}\int_{\Omega} \varrho e(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\right) \leqq c\left(1+\int_{S^1}\int_{\Omega} \varrho^{5/3}\,\mathrm{d}x\,\mathrm{d}t+\int_{S^1}\int_{\Omega} \vartheta^4\,\mathrm{d}x\,\mathrm{d}t\right). \tag{39}$$

Now, we write $\|\vartheta\|^4_{L^4(\Omega)} \leqq \|\vartheta\|_{L^6(\Omega)}\|\vartheta\|^3_{L^4(\Omega)} \leqq c\|\vartheta\|_{L^6(\Omega)} \sup_{t\in S^1} E^{3/4}(t)$, therefore, in view of (39),

$$\sup_{t\in S^1} E(t) \leqq c\left[1+\int_{S^1}\int_{\Omega} \varrho^{5/3}\,\mathrm{d}x\,\mathrm{d}t+\left(\int_{S^1}\|\vartheta\|_{L^6(\Omega)}\,\mathrm{d}t\right)^4\right].$$

Using (36) we conclude that

$$\sup_{t\in S^1} E(t) \leqq c\left(1+\int_{S^1}\int_{\Omega} \varrho^{5/3}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{40}$$

2.4.2. Pressure Estimates. Having established the crucial relation (40), the remaining a priori bounds can be derived in the same way as in [3]. We multiply momentum equation (2) on

$$\mathcal{B}\Big[\varrho^\alpha - \{\varrho^\alpha\}_\Omega\Big] \text{ for a certain (small) } \alpha > 0,$$

where

$$\{g\}_\Omega = \frac{1}{|\Omega|}\int_\Omega g\,\mathrm{d}x, \tag{41}$$

and $\mathcal{B} \approx \mathrm{div}_x^{-1}$ is the operator constructing a vector field with zero traces and prescribed divergence, that is,

$$\mathrm{div}_x \mathcal{B}[g] = g \text{ in } \Omega, \quad \mathcal{B}[g]|_{\partial\Omega} = \mathbf{0}, \quad \text{for } \int_\Omega g\,\mathrm{d}x = 0, \tag{42}$$

see [1]. The mapping $\mathcal{B}\colon L^p(\Omega) \mapsto W_0^{1,p}(\Omega;\mathbb{R}^3)$ is bounded for any $1 < p < \infty$. If, moreover, $g = \mathrm{div}_x \mathbf{w}$, where $\mathbf{w} \in L^r(\Omega;\mathbb{R}^3)$ and $\mathbf{w}\cdot\mathbf{n} = 0$ in the sense of $W^{-\frac{1}{p'},p'}(\partial\Omega)$, then

$$\|\mathcal{B}[g]\|_{L^r(\Omega;\mathbb{R}^3)} \leqq c(r)\|\mathbf{w}\|_{L^r(\Omega;\mathbb{R}^3)} \quad \text{for any } 1 < r < \infty;$$

for the proof, see also [6].

We get, due to (40),

$$\int_{S^1}\int_\Omega \varrho^{\frac{5}{3}+\alpha}\,\mathrm{d}x\,\mathrm{d}t \leqq c\left(1+\sum_{j=1}^5 |I_j|\right),$$

where

$$I_1 = \int_{S^1}\int_\Omega \mathbb{S} : \nabla_x \mathcal{B}\Big[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha \,\mathrm{d}x\Big]\,\mathrm{d}x\,\mathrm{d}t$$
$$I_2 = \int_{S^1}\int_\Omega [\varrho\mathbf{u}\otimes\mathbf{u}] : \nabla_x \mathcal{B}\Big[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha \,\mathrm{d}x\Big]\,\mathrm{d}x\,\mathrm{d}t$$
$$I_3 = \int_{S^1}\int_\Omega \varrho\mathbf{f}\cdot \mathcal{B}\Big[\varrho^\alpha - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha \,\mathrm{d}x\Big]\,\mathrm{d}x\,\mathrm{d}t$$
$$I_4 = \int_{S^1}\int_\Omega \varrho\mathbf{u}\cdot \mathcal{B}\Big[\mathrm{div}_x(\varrho^\alpha\mathbf{u})\Big]\,\mathrm{d}x\,\mathrm{d}t$$
$$I_5 = \int_{S^1}\int_\Omega \varrho\mathbf{u}\cdot \mathcal{B}\Big[\varrho^\alpha \mathrm{div}_x\mathbf{u} - \frac{1}{|\Omega|}\int_\Omega \varrho^\alpha \mathrm{div}_x\mathbf{u}\,\mathrm{d}x\Big]\,\mathrm{d}x\,\mathrm{d}t.$$

To begin, we observe that I_3 is bounded, provided α is small enough. Next, by virtue of Hölder's inequality

$$|I_1| \leqq c\int_{S^1}\left(1 + \|\vartheta\|_{L^4(\Omega)}\|\nabla_x\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^9)}\right)\,\mathrm{d}t \leqq c\left(1 + \sup_{t\in S^1} E^{1/4}(t)\right).$$

Similarly, in accordance with (33) (for α sufficiently small),

$$|I_2| + |I_4| \leqq c \sup_{t\in S^1}\|\varrho\|_{L^{5/3}(\Omega)}\int_{S^1}\|\mathbf{u}\|^2_{L^6(\Omega;\mathbb{R}^3)}\,\mathrm{d}t \leqq c\sup_{t\in S^1} E^{3/5}(t).$$

Finally, by the same token,

$$|I_5| \leqq \sup_{t\in S^1}\|\varrho\|_{L^{5/3}(\Omega)}\int_{S^1}\|\mathbf{u}\|_{L^6(\Omega;\mathbb{R}^3)}\|\mathrm{div}_x\mathbf{u}\|_{L^2(\Omega)}\,\mathrm{d}t \leqq c\sup_{t\in S^1} E^{3/5}(t).$$

Combining (40) with the estimates obtained in this section, we conclude that

$$\sup_{t\in S^1} E(t) \leqq c. \tag{43}$$

Relation (43) closes the circle of a priori bounds that are the same as for the initial-value problem, see [4, Chapter 2]. In Section 9 we will show that we may take $\alpha = \frac{1}{9}$. Lemma 1 is proved. □

3. The Full Approximation

We aim to solve the following approximation of the original problem. Assume $N \in \mathbb{N}$, $\tau, l, \varepsilon, \delta > 0$. We look for unknown triplet $(\varrho, \mathbf{u}_N, \vartheta)$, fulfilling the following approximation:

Continuity equation:

$$\begin{aligned} \partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}_N) - \varepsilon\Delta\varrho + \varepsilon\varrho = \varepsilon h \quad \text{in } S^1\times\Omega,\\ \frac{\partial\varrho}{\partial\mathbf{n}} = 0 \quad \text{at } S^1\times\partial\Omega, \end{aligned} \tag{44}$$

where $h = \frac{M_0}{|\Omega|}$.

Momentum equation:

$$
\begin{aligned}
&\int_{S^1}\int_\Omega \Big(l\partial_t \mathbf{u}_N \cdot \mathbf{w}^i + \partial_t(\varrho \mathbf{u}_N)\cdot \mathbf{w}^i - \varrho \mathbf{u}_N \otimes \mathbf{u}_N : \nabla_x \mathbf{w}^i\Big)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_{S^1}\int_\Omega \mathsf{S}(\vartheta, \nabla_x \mathbf{u}_N) : \nabla_x \mathbf{w}^i \, \mathrm{d}x\, \mathrm{d}t \\
&\quad - \int_{S^1}\int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\big)\mathrm{div}_x \mathbf{w}^i \, \mathrm{d}x\, \mathrm{d}t \\
&= \int_{S^1}\int_\Omega \Big(-\varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u}_N \mathbf{w}^i + \frac{1}{2}\varepsilon(h-\varrho)\mathbf{u}_N \cdot \mathbf{w}^i + \varrho \mathbf{f}\cdot \mathbf{w}^i\Big)\, \mathrm{d}x\, \mathrm{d}t
\end{aligned} \tag{45}
$$

for $i = 1, \ldots, N$, where $\{\mathbf{w}^i(t,x)\}_{i\in\mathbb{N}}$ form a basis in the Hilbert space $L^2(S^1; W_0^{1,2}(\Omega;\mathbb{R}^3))$, orthonormal with respect to the scalar product

$$
(\mathbf{w}^i, \mathbf{w}^j)_{L^2(S^1;W_0^{1,2}(\Omega;\mathbb{R}^3))} = \int_{S^1}\int_\Omega \nabla_x \mathbf{w}^i : \nabla_x \mathbf{w}^j \, \mathrm{d}x\, \mathrm{d}t. \tag{46}
$$

We take the basis functions sufficiently smooth, so finite combination of basis vectors is always smooth in time and space.

We take $\mathbf{w}^i(t,x) = a^k(t)\mathbf{b}^l(x)$ with $i = i(k,l)$, functions $a^k(\cdot)$ are sin/cos giving the basis over the time circle, and $\mathbf{b}^l(\cdot)$ are the basis in $W_0^{1,2}(\Omega;\mathbb{R}^3)$. We keep in mind that the index i hides two integers k and l; this will be important in Section 6 within the procedure of the limit.

Energy equation:

$$
\begin{aligned}
&-\tau\partial_t^2 \Phi(\ln\vartheta) + l\partial_t\vartheta + \tau\Phi(\ln\vartheta) + \partial_t(\varrho e) - \mathrm{div}_x \nabla\Phi(\ln\vartheta) + \mathrm{div}_x(\varrho e \mathbf{u}_N) \\
&\quad = \mathsf{S}(\vartheta, \nabla_x \mathbf{u}_N) : \nabla_x \mathbf{u}_N - p(\varrho,\vartheta)\mathrm{div}_x \mathbf{u}_N + \varepsilon\delta(\Gamma\varrho^{\Gamma-2} + 2)|\nabla_x\varrho|^2 + \delta\vartheta^{-1}
\end{aligned} \tag{47}
$$

in $S^1 \times \Omega$,

$$
\Big(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\Big)\frac{\partial\vartheta}{\partial\mathbf{n}} + d(\vartheta - \Theta_0) = 0 \tag{48}
$$

at $S^1 \times \partial\Omega$, where

$$
\Phi(w) = \int_0^w [\kappa(e^z)e^z + \delta e^{(B+1)z} + \delta]\, \mathrm{d}z. \tag{49}
$$

The energy equation has just an auxiliary character, in the final result, so instead we will consider the entropy equation. We now rewrite (47) in the form of the entropy; we divide (47) by ϑ and use (44).

Entropy equation:

$$
\begin{aligned}
&-\tau\partial_t\Big(\frac{\Phi'(\ln\vartheta)\partial_t(\ln\vartheta)}{\vartheta}\Big) - \tau\frac{\Phi'(\ln\vartheta)(\partial_t\vartheta)^2}{\vartheta^3} + l\partial_t \ln\vartheta + \partial_t(\varrho s) \\
&\quad + \tau\frac{\Phi(\ln\vartheta)}{\vartheta} + [\mathrm{div}_x(\varrho \mathbf{u}_N) + \partial_t\varrho]\frac{1}{\varrho\vartheta}(\varrho e + p - \varrho\vartheta s) + \mathrm{div}_x(\varrho s \mathbf{u}_N)
\end{aligned}
$$

$$-\mathrm{div}_x\left((\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1})\frac{\nabla_x\vartheta}{\vartheta}\right)=\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N$$
$$+(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1})\frac{|\nabla_x\vartheta|^2}{\vartheta^2}+\delta\vartheta^{-2}+\frac{\varepsilon\delta}{\vartheta}(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2 \quad (50)$$

in $S^1\times\Omega$, where the specific entropy fulfills (17).

4. Existence for Fixed Parameters

In this part we show the existence of a solution for fixed approximation parameters, particularly as the approximation level of the Galerkin method is fixed. Namely we consider system (44, 45, 47, 48).

The proof will follow from an application of the Leray–Schauder theorem to the following map

$$\mathcal{T}:\mathbb{R}^N\times W^{1,p}(S^1\times\Omega)\mapsto\mathbb{R}^N\times W^{1,p}(S^1\times\Omega) \quad (51)$$

and

$$\mathcal{T}(\widetilde{\mathbf{u}}^N,\ln\widetilde{\vartheta})=(\mathbf{u}^N,\ln\vartheta) \quad (52)$$

such that:

Momentum equation:

$$\int_{S^1}\int_\Omega\left(l\partial_t\mathbf{u}_N\cdot\mathbf{w}^i+\partial_t(\varrho\widetilde{\mathbf{u}}_N)\cdot\mathbf{w}^i-(\varrho\widetilde{\mathbf{u}}_N\otimes\widetilde{\mathbf{u}}_N):\nabla_x\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{S^1}\int_\Omega\mathsf{S}(\widetilde{\vartheta},\nabla_x\mathbf{u}_N):\nabla_x\mathbf{w}^i\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{S^1}\int_\Omega(p(\varrho,\widetilde{\vartheta})+\delta(\varrho^\Gamma+\varrho^2))\mathrm{div}_x\mathbf{w}^i\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{S^1}\int_\Omega\left(-\varepsilon\nabla_x\varrho\cdot\nabla_x\widetilde{\mathbf{u}}_N\mathbf{w}^i+\frac{1}{2}\varepsilon(h-\varrho)\widetilde{\mathbf{u}}_N\cdot\mathbf{w}^i+\varrho\mathbf{f}\cdot\mathbf{w}^i\right)\mathrm{d}x\,\mathrm{d}t \quad (53)$$

for $i=1,\dots,N$;

Energy equation:

$$-\tau\partial_t^2\Phi(\ln\vartheta)+l\partial_t\widetilde{\vartheta}+\partial_t(\widetilde{\varrho e})+\tau\Phi(\ln\vartheta)+\mathrm{div}_x(\varrho\widetilde{e}\widetilde{\mathbf{u}}_N)-\mathrm{div}_x\nabla\Phi(\ln\vartheta)$$
$$=\mathsf{S}(\widetilde{\vartheta},\nabla_x\widetilde{\mathbf{u}}_N):\nabla_x\widetilde{\mathbf{u}}_N-p(\varrho,\widetilde{\vartheta})\mathrm{div}_x\widetilde{\mathbf{u}}_N+\varepsilon\delta(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2+\delta\widetilde{\vartheta}^{-1} \quad (54)$$

in $S^1\times\Omega$,

$$\frac{\partial\Phi(\ln\vartheta)}{\partial\mathbf{n}}+d(\widetilde{\vartheta}-\Theta_0)=0 \quad (55)$$

at $S^1\times\partial\Omega$, where $\Phi(w)=\int_0^w[\kappa(e^z)e^z+\delta e^{(B+1)z}+\delta]\,\mathrm{d}z$, and ϱ solves

Continuity equation:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x(\varrho \widetilde{\mathbf{u}}_N) - \varepsilon \Delta_x \varrho + \varepsilon \varrho = \varepsilon h \quad \text{in} \quad S^1 \times \Omega, \\ \frac{\partial \varrho}{\partial \mathbf{n}} = 0 \quad \text{at } S^1 \times \partial\Omega. \end{aligned} \tag{56}$$

First let us observe that by definition $\mathbf{u}^N$ is a smooth function in space and time, since it is a finite combination of basis vectors. Subsequently, ϱ is also smooth.

More precisely:

Lemma 2. *Let* $\widetilde{\mathbf{u}}_N \in Lin\{\mathbf{w}^i\}_{i=1}^N$, $\varepsilon > 0$. *Then there exists a unique solution to* (56). *Moreover,* $\varrho \in C^\infty(S^1; W^{2,p}(\Omega))$ *(provided* $\Omega \in C^2$*) and the mapping* $\widetilde{\mathbf{u}}_N \mapsto \varrho$ *is continuous and compact from* $\mathbb{R}^N \mapsto W^{1,p}(S^1 \times \Omega)$. *In addition,* $\varrho \geqq 0$.

Proof. We give just the main steps of the proof:

Step 1 Fix the density $\overline{\varrho}$ in the nonlinear term and construct the solution to the linear problem by means of the Galerkin method.

Step 2 Show that the mapping $\overline{\varrho} \mapsto \varrho$ has a fixed point in $W^{1,p}(S^1 \times \Omega)$, via the Schauder fixed point theorem.

Step 3 Show that the solution fulfills $\int_\Omega \varrho(t, \cdot)\,\mathrm{d}x = M_0$. To this aim, integrate over Ω the equation:

$$\partial_t \int_\Omega \varrho \,\mathrm{d}x + \varepsilon \int_\Omega \varrho \,\mathrm{d}x = \varepsilon M_0, \quad \text{so} \int_\Omega \varrho(x,t)\,\mathrm{d}x = M_0. \tag{57}$$

Step 4 Show that the density is non-negative (if h is so). We examine the set $\{\varrho < 0\}$. Assume that the set $\{\varrho = 0\}$ is a regular submanifold; then we have

$$\partial_t \int_{\{\varrho<0\}} \varrho \,\mathrm{d}x + 0 - \varepsilon \int_{\partial\{\varrho<0\}} \frac{\partial \varrho}{\partial n}\,\mathrm{d}S_x + \varepsilon \int_{\{\varrho<0\}} \varrho \,\mathrm{d}x = \varepsilon \int_{\{\varrho<0\}} h \,\mathrm{d}x. \tag{58}$$

Since $\frac{\partial \varrho}{\partial n}|_{\partial\{\varrho<0\}} \geqq 0$, we find $\varepsilon \int_{S^1} \int_\Omega \varrho \chi_{\{\varrho<0\}}\,\mathrm{d}x\,\mathrm{d}t \geqq \varepsilon h |\{\varrho < 0\}|$, so $|\{\varrho < 0\}| = 0$. In case $\{\varrho = 0\}$ is not a regular submanifold, we may construct a sequence of $\varepsilon_n \to 0^-$ for which $\{\varrho = \varepsilon_n\}$ is a regular submanifold and pass with ε_n to zero.

Step 5 It follows that the solution is unique (for the zero right-hand side it must be zero). □

Next consider the momentum equation.

Lemma 3. *For any* $N \in \mathbb{N}$, $l > 0$, $\varepsilon > 0$, $\delta > 0$, $\varrho \in W^{1,p}(S^1 \times \Omega)$, $\widetilde{\mathbf{u}}_N \in Lin\{\mathbf{w}^i\}_{i=1}^N$, $\widetilde{\vartheta} \in W^{1,p}(S^1 \times \Omega)$ *there is a unique solution to* (53). *Moreover,* $\mathbf{u}_N \in C^\infty(S^1 \times \overline{\Omega}; \mathbb{R}^N)$.

Proof. We solve the system of linear algebraic equations, its solvability being guaranteed by Korn's inequality and the Brower fixed point theorem. The regularity follows from the regularity of the basis functions, the uniqueness is straightforward. □

Last problem is the energy equation.

Lemma 4. *For l, ε, τ and $\delta > 0$, $\varrho \in W^{1,2p}(S^1 \times \Omega)$, $\widetilde{\mathbf{u}}_N \in Lin\{\mathbf{w}^i\}_{i=1}^N$ and $\widetilde{\vartheta} \in W^{1,p}(S^1 \times \Omega)$ there is a unique solution to (54) such that $\ln\vartheta$ and $\vartheta \in W^{2,p}(S^1 \times \Omega)$. In addition $\vartheta > 0$ on $\overline{S^1 \times \Omega}$.*

Proof. Solving the system for $\ln\vartheta$, setting $\vartheta := e^{\ln\vartheta}$ implies strict positiveness of the temperature, which is a very important step in our examination; it guarantees that our final temperature will be positive almost everywhere. The following is the scheme of solvability in order to control uniqueness and continuity. We solve the system

$$\begin{aligned} -\tau\partial_t^2 Z - \operatorname{div}_x \nabla Z + \tau Z &= -l\partial_t\widetilde{\vartheta} - \partial_t(\varrho\widetilde{e}) - \operatorname{div}_x(\varrho\widetilde{e}\widetilde{\mathbf{u}}_N) \\ &+ \mathsf{S}(\widetilde{\vartheta}, \nabla_x\widetilde{\mathbf{u}}_N) : \nabla_x\widetilde{\mathbf{u}}_N - p(\varrho,\widetilde{\vartheta})\operatorname{div}_x\widetilde{\mathbf{u}}_N + \varepsilon\delta(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2 + \delta\widetilde{\vartheta}^{-1} \end{aligned} \tag{59}$$

in $S^1 \times \Omega$,

$$\frac{\partial Z}{\partial \mathbf{n}} + d(\widetilde{\vartheta} - \Theta_0) = 0 \tag{60}$$

at $\partial S^1 \times \Omega$, and then we define

$$\ln\theta = \Phi^{-1}(Z). \tag{61}$$

Due to the definition $|\frac{d}{dt}\Phi^{-1}| \leqq \delta^{-1}$—see (49)—and the sought $\ln\vartheta$ is well defined. The regularity of ϑ is straightforward. □

Hence our operator $\mathcal{T}$ is a compact operator from $\mathbb{R}^N \times W^{1,p}(S^1 \times \Omega)$ into itself. Its continuity is also straightforward. Moreover, the solution has the regularity from previous lemmas. To conclude, we have to show that the possible fixed points

$$\lambda\mathcal{T}(\mathbf{u}_N, \vartheta) = (\mathbf{u}_N, \vartheta) \text{ for } \lambda \in [0, 1], \tag{62}$$

are bounded in $\mathbb{R}^N \times W^{1,p}(S^1 \times \Omega)$. To this aim, we show

Lemma 5. *Let $(\mathbf{u}_N, \vartheta)$ be a solution to (62). Then for $\lambda \in [0, 1]$*

$$\begin{aligned} &(1-\lambda)\int_{S^1}\int_\Omega \mathsf{S}(\vartheta, \nabla_x\mathbf{u}_N) : \nabla_x\mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega \Phi(\ln\vartheta)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \lambda\int_{S^1}\int_{\partial\Omega} d(\vartheta - \Theta_0)\,\mathrm{d}S_x\,\mathrm{d}t + \varepsilon\delta\lambda\int_{S^1}\int_\Omega \Big(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma + 2\varrho^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \lambda\int_{S^1}\int_\Omega \Big(\varrho\mathbf{f}\cdot\mathbf{u}_N + \varepsilon\delta\frac{\Gamma}{\Gamma-1}h\varrho^{\Gamma-1} + 2\varepsilon\delta h\varrho + \delta\vartheta^{-1}\Big)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{63}$$

and

$$\int_{S^1}\int_{\Omega}\Big[\big(\kappa(\vartheta)+\delta\vartheta^{B}+\delta\vartheta^{-1}\big)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}+\tau\Phi'(\ln\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^2}\Big]\,\mathrm{d}x\,\mathrm{d}t$$

$$+\lambda\int_{S^1}\int_{\Omega}\Big(\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N+\delta\vartheta^{-2}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\lambda\varepsilon\delta\int_{S^1}\int_{\Omega}\Big(\frac{\Gamma}{\Gamma-1}\varrho^{\Gamma}+2\varrho^2\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\varepsilon\delta\lambda\int_{S^1}\int_{\Omega}\frac{1}{\vartheta}|\nabla_x\varrho|^2(\Gamma\varrho^{\Gamma-2}+2)\,\mathrm{d}x\,\mathrm{d}t+\tau\int_{S^1}\int_{\Omega}(\vartheta^{B+1}+\kappa(\vartheta)\vartheta)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\lambda\int_{S^1}\int_{\partial\Omega}d\Big(\frac{\Theta_0}{\vartheta}+\vartheta\Big)\,\mathrm{d}S_x\,\mathrm{d}t\leqq C\Big(1+\lambda\Big|\int_{S^1}\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}t\Big|\Big),\quad (64)$$

where C is independent of approximative parameters and of λ.

Proof. Our system reads

$$\int_{S^1}\int_{\Omega}\Big(l\partial_t\mathbf{u}_N\cdot\mathbf{w}^i+\lambda\partial_t(\varrho\mathbf{u}_N)\cdot\mathbf{w}^i+\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{w}^i\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\lambda\int_{S^1}\int_{\Omega}\big((\varrho\mathbf{u}_N\otimes\mathbf{u}_N):\nabla_x\mathbf{w}^i+(p(\varrho,\vartheta)+\delta(\varrho^{\Gamma}+\varrho^2))\mathrm{div}_x\mathbf{w}^i\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\lambda\int_{S^1}\int_{\Omega}\Big(-\varepsilon\nabla_x\varrho\cdot\nabla_x\mathbf{u}_N\mathbf{w}^i+\frac{1}{2}\varepsilon(h-\varrho)\mathbf{u}_N\cdot\mathbf{w}^i+\varrho\mathbf{f}\cdot\mathbf{w}^i\Big)\,\mathrm{d}x\,\mathrm{d}t\quad (65)$$

for $i=1,\dots,N$,

$$\begin{aligned}&-\tau\partial_t^2\Phi(\ln\vartheta)+l\lambda\partial_t\vartheta+\lambda\partial_t(\varrho e)+\tau\Phi(\ln\vartheta)\\&\quad+\lambda\mathrm{div}_x(\varrho e\mathbf{u}_N)-\mathrm{div}_x\nabla\Phi(\ln\vartheta)=\lambda\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N\\&\quad-\lambda p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}_N\\&\quad+\lambda\varepsilon\delta(\Gamma\varrho^{\Gamma-2}+2)|\nabla_x\varrho|^2+\lambda\delta\vartheta^{-1}\end{aligned}\quad (66)$$

in $S^1\times\Omega$,

$$\big(\kappa(\vartheta)+\delta\vartheta^{B}+\delta\vartheta^{-1}\big)\frac{\partial\vartheta}{\partial\mathbf{n}}+\lambda d(\vartheta-\Theta_0)=0\quad (67)$$

at $S^1\times\partial\Omega$, where $\Phi(w)=\int_0^w[\kappa(e^z)e^z+\delta e^{(B+1)z}+\delta]\,\mathrm{d}z$, and ϱ solves

$$\begin{aligned}\partial_t\varrho+\mathrm{div}_x(\varrho\mathbf{u}_N)-\varepsilon\Delta_x\varrho+\varepsilon\varrho=\varepsilon h\quad&\text{in}\quad S^1\times\Omega,\\ \frac{\partial\varrho}{\partial\mathbf{n}}=0\quad&\text{at } S^1\times\partial\Omega.\end{aligned}\quad (68)$$

Step 1 Total energy estimate

a) Use as test function $\mathbf{u}_N$ in (65):

$$\int_{S^1}\int_\Omega \mathsf{S}(\vartheta, \nabla_x \mathbf{u}_N) : \nabla_x \mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t$$
$$= \lambda \int_{S^1}\int_\Omega \Big(\big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2) \big) \mathrm{div}_x \mathbf{u}_N + \varrho \mathbf{f}\cdot \mathbf{u}_N \Big) \,\mathrm{d}x\,\mathrm{d}t. \quad (69)$$

b) Integrate (66) over $S^1 \times \Omega$. This reads

$$\tau \int_{S^1}\int_\Omega \Phi(\ln\vartheta)\,\mathrm{d}x\,\mathrm{d}t + \lambda \int_{S^1}\int_{\partial\Omega} d(\vartheta - \Theta_0)\,\mathrm{d}S_x\,\mathrm{d}t$$
$$= \lambda \int_{S^1}\int_\Omega \Big(\mathsf{S}(\vartheta, \nabla_x \mathbf{u}_N) : \nabla_x \mathbf{u}_N - p(\varrho,\vartheta)\mathrm{div}_x \mathbf{u}_N + \delta \vartheta^{-1} \Big) \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \lambda\delta\varepsilon \int_{S^1}\int_\Omega |\nabla_x \varrho|^2 (\Gamma \varrho^{\Gamma-2} + 2)\,\mathrm{d}x\,\mathrm{d}t. \quad (70)$$

We use (68) to get its renormalized version for $\beta = 2$ and Γ

$$\varepsilon\beta \int_{S^1}\int_\Omega \Big(\frac{1}{\beta-1}\varrho^\beta + \varrho^{\beta-2}|\nabla_x \varrho|^2 \Big) \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{S^1}\int_\Omega \varrho^\beta \mathrm{div}_x \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t = \varepsilon \frac{\beta}{\beta-1} \int_{S^1}\int_\Omega h\varrho^{\beta-1}\,\mathrm{d}x\,\mathrm{d}t. \quad (71)$$

c) Thus, summing up (69–71) yields the total energy balance

$$(1-\lambda)\int_{S^1}\int_\Omega \mathsf{S}(\vartheta, \nabla_x \mathbf{u}_N) : \nabla_x \mathbf{u}_N \,\mathrm{d}x\,\mathrm{d}t + \tau \int_{S^1}\int_\Omega \Phi(\ln\vartheta)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{S^1}\int_{\partial\Omega} \lambda d(\vartheta - \Theta_0)\,\mathrm{d}S_x\,\mathrm{d}t + \varepsilon\delta\lambda \int_{S^1}\int_\Omega \Big(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma + 2\varrho^2 \Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \lambda \int_{S^1}\int_\Omega \Big(\varrho \mathbf{f}\cdot \mathbf{u}_N + \varepsilon\delta \frac{\Gamma}{\Gamma-1} h\varrho^{\Gamma-1} + 2\varepsilon\delta h\varrho + \delta\vartheta^{-1} \Big)\,\mathrm{d}x\,\mathrm{d}t. \quad (72)$$

Step 2 Entropy estimate

We recall the entropy identity

$$- \tau\partial_t \Big(\frac{\Phi'(\ln\vartheta)\partial_t(\ln\vartheta)}{\vartheta} \Big) - \tau \frac{\Phi'(\ln\vartheta)(\partial_t\vartheta)^2}{\vartheta^3} + l\lambda\partial_t \ln\vartheta + \lambda\partial_t(\varrho s)$$
$$+ \tau \frac{\Phi(\ln\vartheta)}{\vartheta} + \lambda[\mathrm{div}_x(\varrho\mathbf{u}_N) + \partial_t\varrho] \frac{1}{\varrho\vartheta}(\varrho e + p - \varrho\vartheta s) + \mathrm{div}_x(\varrho s \mathbf{u}_N)$$
$$- \mathrm{div}_x \Big(\kappa(\vartheta) + \delta(\vartheta^B + \vartheta^{-1}) \frac{\nabla_x \vartheta}{\vartheta} \Big) = \lambda \frac{1}{\vartheta} \mathsf{S}(\vartheta, \nabla_x \mathbf{u}_N) : \nabla_x \mathbf{u}_N$$
$$+ \lambda\delta\vartheta^{-2} + (\kappa(\vartheta) + \delta(\vartheta^B + \vartheta^{-1})) \frac{|\nabla_x\vartheta|^2}{\vartheta^2} + \lambda \frac{\varepsilon\delta}{\vartheta}(\Gamma\varrho^{\Gamma-2} + 2)|\nabla_x\varrho|^2, \quad (73)$$

in $S^1 \times \Omega$. We integrate (73) over $S^1 \times \Omega$ to get

$$\begin{aligned}
&\int_{S^1}\int_\Omega \left(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\right)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \tau\int_{S^1}\int_\Omega \Phi'(\ln\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^3}\,\mathrm{d}x\,\mathrm{d}t + \lambda\int_{S^1}\int_{\partial\Omega}\frac{1}{\vartheta}d\Theta_0\,\mathrm{d}S_x\,\mathrm{d}t\\
&\quad + \lambda\int_{S^1}\int_\Omega\left(\frac{1}{\vartheta}\mathbf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N + \delta\vartheta^{-2}\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad + \varepsilon\delta\lambda\int_{S^1}\int_\Omega\frac{1}{\vartheta}|\nabla\varrho|^2\left(\Gamma\varrho^{\Gamma-2}+2\right)\mathrm{d}x\,\mathrm{d}t\\
&= \lambda\int_{S^1}\int_{\partial\Omega} d\Theta_0\,\mathrm{d}S_x\,\mathrm{d}t + \tau\int_{S^1}\int_\Omega\frac{\Phi(\ln\vartheta)}{\vartheta}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \lambda\int_{S^1}\int_\Omega\frac{1}{\varrho\vartheta}\left(\varrho e(\varrho,\vartheta) + p(\varrho,\vartheta) - \varrho\vartheta s(\varrho,\vartheta)\right)[\mathrm{div}_x(\varrho\mathbf{u}) + \partial_t\varrho]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{74}$$

We need to estimate the last term on the right-hand side. Using (44), we get that it is equal to

$$\varepsilon\int_\Omega\frac{1}{\varrho\vartheta}\left(\varrho e(\varrho,\vartheta) + p(\varrho,\vartheta) - \varrho\vartheta s(\varrho,\vartheta)\right)(h - \varrho + \Delta_x\varrho)\,\mathrm{d}x.$$

We will try to find parts of the integral above having a "good" sign and put them to the left-hand side. The rest will be estimated using the left-hand side of (74) and (72), see [18] for similar calculations. We have

$$\begin{aligned}
&-\varepsilon\int_{S^1}\int_\Omega\frac{1}{\varrho\vartheta}\left(\varrho e(\varrho,\vartheta) + p(\varrho,\vartheta) - \varrho\vartheta s(\varrho,\vartheta)\right)\Delta_x\varrho\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \varepsilon\int_{S^1}\int_\Omega|\nabla_x\varrho|^2\frac{\partial}{\partial\varrho}\left(\frac{e(\varrho,\vartheta)}{\vartheta} + \frac{p(\varrho,\vartheta)}{\varrho\vartheta} - s(\varrho,\vartheta)\right)\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \varepsilon\int_{S^1}\int_\Omega\nabla_x\varrho\cdot\nabla_x\vartheta\frac{\partial}{\partial\vartheta}\left(\frac{e(\varrho,\vartheta)}{\vartheta} + \frac{p(\varrho,\vartheta)}{\varrho\vartheta} - s(\varrho,\vartheta)\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad = \varepsilon\int_{S^1}\int_\Omega|\nabla_x\varrho|^2\frac{1}{\varrho\vartheta}\frac{\partial p(\varrho,\vartheta)}{\partial\varrho}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad - \varepsilon\int_{S^1}\int_\Omega\nabla_x\varrho\cdot\nabla_x\vartheta\frac{1}{\vartheta^2}\left(e(\varrho,\vartheta) + \varrho\frac{\partial e(\varrho,\vartheta)}{\partial\varrho}\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Thus we consider the first term on the left-hand side, while the other term can be bounded from above by

$$\begin{aligned}
&\varepsilon\int_{S^1}\int_\Omega\nabla_x\varrho\cdot\nabla_x\vartheta\frac{1}{\vartheta^2}\left(\varrho^{2/3}+\vartheta\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqq \frac{\varepsilon\delta}{4}\int_{S^1}\int_\Omega\frac{1}{\vartheta}\left(|\nabla_x\varrho|^2 + |\nabla_x\varrho|^2\varrho^{\Gamma-2}\right)\mathrm{d}x\,\mathrm{d}t\\
&\qquad + C(\delta)\varepsilon\int_{S^1}\int_\Omega\left(\frac{|\nabla_x\vartheta|^2}{\vartheta^3} + \frac{|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

and all terms are estimated using terms on the left-hand side, provided $\Gamma \geqq 10/3$. The other terms can be treated exactly as in [18] and we get

$$\begin{aligned}
&\int_{S^1}\int_{\Omega}\left(\kappa(\vartheta)+\delta\vartheta^{B}+\delta\vartheta^{-1}\right)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_{S^1}\int_{\Omega}\tau\Phi'(\ln\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^3}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\lambda\int_{S^1}\int_{\Omega}\left(\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N+\delta\vartheta^{-2}\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad+\lambda\int_{S^1}\int_{\partial\Omega}\frac{1}{\vartheta}d\Theta_0\,\mathrm{d}S_x\,\mathrm{d}t+\varepsilon\delta\lambda\int_{S^1}\int_{\Omega}\frac{1}{\vartheta}|\nabla_x\varrho|^2\left(\Gamma\varrho^{\Gamma-2}+2\right)\mathrm{d}x\,\mathrm{d}t\\
&\leqq C\left(1+\tau\int_{S^1}\int_{\Omega}\frac{\Phi(\ln\vartheta)}{\vartheta}\,\mathrm{d}x\,\mathrm{d}t+\varepsilon\lambda\int_{S^1}\int_{\Omega}\varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned}\tag{75}$$

where C is independent of approximation parameters.

Step 3 Estimates

Thus, summing up the energy inequality (72) and the entropy inequality (75) we end up with

$$\begin{aligned}
&\int_{S^1}\int_{\Omega}\left(\left(\kappa(\vartheta)+\delta\vartheta^{B}+\delta\vartheta^{-1}\right)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}+\tau\Phi'(\ln\vartheta)\frac{(\partial_t\vartheta)^2}{\vartheta^2}\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad+\lambda\int_{S^1}\int_{\Omega}\left(\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N+\delta\vartheta^{-2}\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{1}{2}\varepsilon\delta\lambda\int_{S^1}\int_{\Omega}\left(\frac{\Gamma}{\Gamma-1}\varrho^{\Gamma}+2\varrho^2\right)\mathrm{d}x\,\mathrm{d}t+\ \tau\int_{S^1}\int_{\Omega}\left(\vartheta^{B+1}+\kappa(\vartheta)\vartheta\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad+\lambda\varepsilon\delta\int_{S^1}\int_{\Omega}\frac{1}{\vartheta}|\nabla_x\varrho|^2(\Gamma\varrho^{\Gamma-2}+2)\,\mathrm{d}x\,\mathrm{d}t+\lambda\int_{S^1}\int_{\partial\Omega}d\left(\frac{\Theta_0}{\vartheta}+\vartheta\right)\mathrm{d}S_x\,\mathrm{d}t\\
&\leqq C\left(1+\lambda\left|\int_{S^1}\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}t\right|+\lambda\varepsilon\left|\int_{S^1}\int_{\Omega}\varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\right|\right).
\end{aligned}\tag{76}$$

Again, C in (76) is independent from approximation parameters. As

$$\begin{aligned}
\varepsilon\int_{\Omega}\varrho s(\varrho,\vartheta)\,\mathrm{d}x &\leqq \frac{\varepsilon\delta}{4}\int_{\Omega}\varrho^2\,\mathrm{d}x\\
&\quad+\frac{1}{2}\int_{\partial\Omega}d\left(\frac{\Theta_0}{\vartheta}+\vartheta\right)\mathrm{d}S_x+\int_{\Omega}\kappa(\vartheta)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x+C,
\end{aligned}$$

after integrating over the time period we get (64) provided $\varepsilon \ll \delta$. □

Estimating the other term on the right-hand side as below

$$\begin{aligned}
\left|\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{u}_N\,\mathrm{d}x\right| &\leqq C\|\mathbf{u}_N\|_{L^6(\Omega;\mathbb{R}^3)}\|\varrho\|_{L^{6/5}(\Omega)}\\
&\leqq \frac{1}{2}\int_{\Omega}\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N\,\mathrm{d}x+C\|\varrho\|^2_{L^{6/5}(\Omega)}\\
&\leqq \frac{1}{2}\int_{\Omega}\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N\,\mathrm{d}x+\frac{\varepsilon\delta}{2}\int_{\Omega}\varrho^2\,\mathrm{d}x+C(\varepsilon,\delta),
\end{aligned}$$

we get bounds of the approximative solution, that is, solutions to (62). Using them together with the bootstrap method applied on our system we finish the proof of the existence of a solution. Note that we can estimate the right-hand side in (64) by a constant, which is independent of τ, N and l.

Theorem 2. *Let p be sufficiently large, N be fixed and $\tau > 0$, then there exists at least one solution to* (44, 45, 47) *such that*

$$(\varrho, \mathbf{u}^N, \ln \vartheta) \in \left(W^{1,p}(S^1 \times \Omega) \cap L^p(S^1; W^{2,p}(\Omega)) \right) \times Lin\{\mathbf{w}^i\}_{i=1}^N \times W^{2,p}(S^1 \times \Omega), \tag{77}$$

$\vartheta := e^{\ln \vartheta}$ is strictly positive, and $\varrho \geqq 0$.

5. Limit $\tau \to 0$

From (64) we have the following estimates independent of τ and N:

$$\begin{aligned} &\|\nabla_x \mathbf{u}_N\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^9))} + \|\nabla_x \vartheta\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))} + \|\nabla_x \vartheta^{B/2}\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))} \\ &\quad + \|\vartheta\|_{L^1(S^1;L^{3B}(\Omega))} + \|\nabla_x \vartheta^{-\frac{1}{2}}\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))} + \|\vartheta^{-2}\|_{L^2(S^1;L^2(\Omega))} \\ &\qquad + \|\varrho\|_{L^\Gamma(S^1;L^{3\Gamma}(\Omega))} \leqq C(\varepsilon, \delta). \end{aligned} \tag{78}$$

The continuity equation yields

$$\|\nabla_x^2 \varrho\|_{L^q(S^1;L^q(\Omega;\mathbb{R}^9))} + \|\partial_t \varrho\|_{L^q(S^1;L^q(\Omega))} \leqq C(\varepsilon, \delta) \tag{79}$$

for some $1 < q < 2$. In case $C = C(\varepsilon, \delta, N)$ we are allowed to take any $q < \infty$.

Moreover, due to the fact that the velocity is contained in a finite dimensional space, it is evidently relatively compact. To conclude, we consider the energy equation. We have to find some other extra information which may depend on l.

Testing the energy equation by $\Phi(\ln \theta)$ and $\partial_t \Phi(\ln \vartheta)$, we can get

$$\begin{aligned} &\tau \|\partial_t \Phi(\ln \vartheta)\|^2_{L^2(S^1;L^2(\Omega))} + \tau \|\Phi(\ln \vartheta)\|^2_{L^2(S^1;L^2(\Omega))} \\ &\quad + \|\nabla_x \Phi(\ln \vartheta)\|^2_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))} + l \|\Phi'(\ln \vartheta) \left(\frac{(\partial_t \vartheta)^2}{\vartheta} + \vartheta (\partial_t \vartheta)^2 \right) \|_{L^1(S^1;L^1(\Omega))} \\ &\quad \leqq C(\varepsilon, \delta, N). \end{aligned} \tag{80}$$

Additionally, $\Phi'(\ln \vartheta)\vartheta^{-1} \geqq const$, thus the form of Φ delivers us the following bound

$$l \|\partial_t \vartheta\|_{L^2(S^1;L^2(\Omega))} + \|\nabla_x \vartheta\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))} \leqq C(\varepsilon, \delta, N). \tag{81}$$

So we control the strong and pointwise convergence of the temperature, thus we can pass with $\tau \to 0^+$.

Additionally high regularity of the right-hand side of (59)—we remember that N is finite and $\mathbf{u}_N$ and ϱ are smooth—allows us to divide the equation by ϑ getting the entropy form.

Theorem 3. *Let p be sufficiently large, N fixed, then there exists at least one solution to* (44,45,47, 48) *with $\tau = 0$ such that*

$$(\varrho, \mathbf{u}^N, \ln \vartheta) \in (W^{1,p}(S^1 \times \Omega) \cap L^p(S^1; W^{2,p}(\Omega))) \times Lin\{\boldsymbol{w}^i\}_{i=1}^N \\ \times (W^{1,p}(S^1 \times \Omega) \cap L^p(S^1; W^{2,p}(\Omega))) \tag{82}$$

and $\vartheta := e^{\ln \vartheta}$ is strictly positive. Moreover, the entropy equality (50) *with $\tau = 0$ holds true together with* (63) *and* (64) *with $\tau = 0$, $\lambda = 1$. In addition,*

$$\int_{S^1} \int_\Omega \left(\kappa(\vartheta) + \delta \vartheta^B + \delta \vartheta^{-1}\right) \frac{|\nabla_x \vartheta|^2}{\vartheta^2} \,\mathrm{d}x \,\mathrm{d}t \\ + \int_{S^1} \int_\Omega \left(\frac{1}{\vartheta} \mathsf{S}(\vartheta, \nabla_x \mathbf{u}_N) : \nabla_x \mathbf{u}_N + \delta \vartheta^{-2}\right) \mathrm{d}x \,\mathrm{d}t + \int_{S^1} \int_{\partial\Omega} \frac{\Theta_0 d}{\vartheta} \,\mathrm{d}S_x \,\mathrm{d}t \\ + \varepsilon\delta \int_{S^1} \int_\Omega \frac{1}{\vartheta} |\nabla_x \varrho|^2 \left(\Gamma \varrho^{\Gamma-2} + 2\right) \mathrm{d}x \,\mathrm{d}t \leqq C\Big(1 + \varepsilon \int_{S^1} \int_\Omega \varrho s(\varrho, \vartheta) \,\mathrm{d}x \,\mathrm{d}t\Big), \tag{83}$$

where C is independent of N, l, δ, ε, depends only on data for the original problem, and

$$\int_{S^1} \int_\Omega \left[\left(\kappa(\vartheta) + \delta \vartheta^B + \delta \vartheta^{-1}\right) \frac{|\nabla_x \vartheta|^2}{\vartheta^2}\right] \mathrm{d}x \,\mathrm{d}t \\ + \int_{S^1} \int_\Omega \left(\frac{1}{\vartheta} \mathsf{S}(\vartheta, \nabla_x \mathbf{u}_N) : \nabla_x \mathbf{u}_N + \delta \vartheta^{-2}\right) \mathrm{d}x \,\mathrm{d}t \\ + \varepsilon\delta \int_{S^1} \int_\Omega \left(\frac{\Gamma}{2(\Gamma-1)} \varrho^\Gamma + \varrho^2 + \frac{1}{\vartheta} |\nabla_x \varrho|^2 (\Gamma \varrho^{\Gamma-2} + 2)\right) \mathrm{d}x \,\mathrm{d}t \\ + \int_{S^1} \int_{\partial\Omega} d\left(\frac{\Theta_0}{\vartheta} + \vartheta\right) \mathrm{d}S_x \,\mathrm{d}t \leqq C(\varepsilon, \delta), \tag{84}$$

where $C(\varepsilon, \delta)$ is independent of N, l and it depends only on δ, ε and the data of the original problem.

6. Limit $N \to \infty$

We are ready to pass to the limit in the momentum equation. However, we divide this operation into two steps. First we recall that basis vectors in the Galerkin approximation are in the form $\boldsymbol{w}^i(t, x) = a^k(t)\mathbf{b}^l(x)$. Our approach requires that we first pass to the limit in the time approximation. Let us denote $\mathbf{u}_N = \mathbf{u}_{N_t, N_x}$, that is $1 \leqq k \leqq N_t$ and $1 \leqq l \leqq N_x$. By the estimate (84) we find a subsequence

$$\mathbf{u}_{N_t, N_x} \rightharpoonup \mathbf{u}_{N_x} \text{ in } L^2(S^1; \mathbb{R}^{3N_x}) \quad \text{as} \quad N_t \to \infty. \tag{85}$$

Moreover, looking once more at the momentum equation (65) we easily get the estimate of the time derivative of $\mathbf{u}_N$ in $L^1(S^1; \mathbb{R}^{3N_x})$ which gives the estimate of $\mathbf{u}_N$ in $L^\infty(S^1; \mathbb{R}^{3N_x})$, hence we get the estimate of the time derivative in any

$L^q(S^1;\mathbb{R}^{3N_x})$, $q < \infty$. The other terms are harmless and the limit fulfills the following identity

$$\int_{S^1}\int_\Omega \Big(l\partial_t \mathbf{u}_{N_x}\cdot\phi\mathbf{w}^i + \partial_t(\varrho\mathbf{u}_{N_x})\cdot\phi\mathbf{w}^i + \mathsf{S}(\vartheta,\nabla_x\mathbf{u}_{N_x}):\phi\nabla_x\mathbf{w}^i\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{S^1}\int_\Omega (\varrho\mathbf{u}_{N_x}\otimes\mathbf{u}_{N_x}):\phi\nabla_x\mathbf{w}^i + \Big(p(\varrho,\vartheta)+\delta\Big(\varrho^\Gamma+\varrho^2\Big)\Big)\mathrm{div}_x\phi\mathbf{w}^i\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{S^1}\int_\Omega\Big(-\varepsilon\nabla_x\varrho\cdot\nabla_x\mathbf{u}_{N_x}\phi\mathbf{w}^i + \frac{1}{2}\varepsilon(h-\varrho)\mathbf{u}_{N_x}\cdot\phi\mathbf{w}^i + \varrho\mathbf{f}\cdot\phi\mathbf{w}^i\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{86}$$

for any $\phi\in C^\infty(S^1)$ and $i=1,\dots,N_x$.

From now on we write $\mathbf{u}_{N_x}=\mathbf{u}_N$ and we analyze the limit $N\to\infty$, that is, we pass to the limit in the space approximation. Note that we have bounds (78), (79), (83) and (84), in addition equality (86) is valid.

Step 1 Total energy estimate

a) Use as test function $\mathbf{u}_N\psi$ in (65), where $\psi\in C_c^\infty(S^1)$:

$$-\int_{S^1}\Big(\int_\Omega\Big(l\frac{1}{2}|\mathbf{u}_N|^2+\frac{1}{2}\varrho|\mathbf{u}_N|^2\Big)\,\mathrm{d}x\Big)\partial_t\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_\Omega \mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N\,\mathrm{d}x\,\psi\,\mathrm{d}t$$
$$=\int_{S^1}\int_\Omega\Big(\big(p(\varrho,\vartheta)+\delta(\varrho^\Gamma+\varrho^2)\big)\mathrm{div}_x\mathbf{u}_N+\varrho\mathbf{f}\cdot\mathbf{u}_N\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t. \tag{87}$$

b) Integrate (66) with $\tau=0$ over Ω, multiply by a smooth in time function ψ and integrate over S^1. This reads

$$-\int_{S^1}\int_\Omega(l\vartheta+\varrho e(\varrho,\vartheta))\,\mathrm{d}x\,\partial_t\psi\,\mathrm{d}t+\int_{S^1}\int_{\partial\Omega}d(\vartheta-\Theta_0)\,\mathrm{d}S_x\,\psi\,\mathrm{d}t$$
$$=\int_{S^1}\int_\Omega\Big(\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N-p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}_N$$
$$+\delta\vartheta^{-1}+\delta\varepsilon|\nabla_x\varrho|^2(\Gamma\varrho^{\Gamma-2}+2)\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t. \tag{88}$$

We use (68) to get its renormalized version for $\beta=2$ and Γ in the form

$$-\int_{S^1}\int_\Omega\frac{1}{\beta-1}\varrho^\beta\,\mathrm{d}x\,\partial_t\psi\,\mathrm{d}t+\varepsilon\beta\int_{S^1}\int_\Omega\Big(\frac{1}{\beta-1}\varrho^\beta+\varrho^{\beta-2}|\nabla_x\varrho|^2\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_\Omega\varrho^\beta\mathrm{div}_x\mathbf{u}\,\mathrm{d}x\,\psi\,\mathrm{d}t=\varepsilon\frac{\beta}{\beta-1}\int_{S^1}\int_\Omega h\varrho^{\beta-1}\,\mathrm{d}x\,\psi\,\mathrm{d}t. \tag{89}$$

c) Thus, summing up (87)–(89) yields the total energy balance

$$-\int_{S^1}\int_\Omega\Big(\frac{l}{2}|\mathbf{u}_N|^2+l\vartheta+\frac{1}{2}\varrho|\mathbf{u}_N|^2+\varrho e+\delta\Big(\frac{\varrho^\Gamma}{\Gamma-1}+\varrho^2\Big)\Big)\,\mathrm{d}x\,\partial_t\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_{\partial\Omega}d(\vartheta-\Theta_0)\,\mathrm{d}S_x\,\psi\,\mathrm{d}t+\varepsilon\delta\int_{S^1}\int_\Omega\Big(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma+2\varrho^2\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t$$
$$=\int_{S^1}\int_\Omega\Big(\varrho\mathbf{f}\cdot\mathbf{u}_N+\varepsilon\delta\frac{\Gamma}{\Gamma-1}h\varrho^{\Gamma-1}+2\varepsilon\delta h\varrho+\delta\vartheta^{-1}\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t. \tag{90}$$

Step 2 Entropy estimate

We integrate (73) with $\tau = 0$ over Ω, multiply by the same function ψ and integrate over S^1 to get

$$\int_{S^1}\int_{\Omega}\Big(l\ln\vartheta + \varrho s\Big)\,\mathrm{d}x\,\partial_t\psi\,\mathrm{d}t + \int_{S^1}\int_{\partial\Omega}\frac{1}{\vartheta}d\Theta_0\,\mathrm{d}S_x\,\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_{\Omega}\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_{\Omega}\Big(\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N+\delta\vartheta^{-2}\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t$$
$$+\varepsilon\delta\int_{S^1}\int_{\Omega}\frac{1}{\vartheta}|\nabla\varrho|^2\big(\Gamma\varrho^{\Gamma-2}+2\big)\,\mathrm{d}x\,\psi\,\mathrm{d}t = \int_{S^1}\int_{\partial\Omega}d\Theta_0\,\mathrm{d}S_x\,\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_{\Omega}\frac{1}{\varrho\vartheta}\big(\varrho e(\varrho,\vartheta)+p(\varrho,\vartheta)-\varrho\vartheta s(\varrho,\vartheta)\big)[\mathrm{div}_x(\varrho\mathbf{u})+\partial_t\varrho]\,\mathrm{d}x\,\psi\,\mathrm{d}t. \tag{91}$$

Next, we repeat considerations from Section 4—after (74), getting

$$\int_{S^1}\int_{\Omega}\big(l\ln\vartheta+\varrho s\big)\,\mathrm{d}x\,\partial_t\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_{\Omega}\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_{\Omega}\Big(\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N+\delta\vartheta^{-2}\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t$$
$$+\int_{S^1}\int_{\partial\Omega}\frac{1}{\vartheta}d\Theta_0\,\mathrm{d}S_x\,\psi\,\mathrm{d}t+\varepsilon\delta\int_{S^1}\int_{\Omega}\frac{1}{\vartheta}|\nabla_x\varrho|^2\big(\Gamma\varrho^{\Gamma-2}+2\big)\,\mathrm{d}x\,\psi\,\mathrm{d}t$$
$$\leqq C\Big(1+\varepsilon\int_{S^1}\int_{\Omega}\varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\psi\,\mathrm{d}t\Big), \tag{92}$$

where C is independent of approximation parameters.

Step 3 Estimates

Thus summing up the energy equality (90) and the entropy inequality (92) we end up with

$$
\begin{aligned}
&-\int_{S^1}\int_\Omega\Big(\frac{l}{2}|\mathbf{u}_N|^2+l\vartheta+\frac{1}{2}\varrho|\mathbf{u}_N|^2\\
&\quad+\varrho e\delta\Big(\frac{1}{\Gamma-1}\varrho^\Gamma+\varrho^2\Big)-l\ln\vartheta-\varrho s\Big)\,\mathrm{d}x\,\partial_t\psi\,\mathrm{d}t\\
&\quad+\int_{S^1}\int_\Omega\big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\big)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\psi\,\mathrm{d}t\\
&\quad+\int_{S^1}\int_\Omega\Big(\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N+\delta\vartheta^{-2}\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t\\
&\quad+\frac{1}{2}\varepsilon\delta\int_{S^1}\int_\Omega\Big(\frac{\Gamma}{\Gamma-1}\varrho^\Gamma+2\varrho^2\Big)\,\mathrm{d}x\,\psi\,\mathrm{d}t\\
&\quad+\int_{S^1}\varepsilon\delta\int_\Omega\frac{1}{\vartheta}|\nabla_x\varrho|^2(\Gamma\varrho^{\Gamma-2}+2)\,\mathrm{d}x\,\psi\,\mathrm{d}t+\int_{S^1}\int_{\partial\Omega}d\Big(\frac{\Theta_0}{\vartheta}+\vartheta\Big)\,\mathrm{d}S_x\,\psi\,\mathrm{d}t\\
&\quad\leqq C\Big(1+\Big|\int_{S^1}\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}_N\,\mathrm{d}x\,\psi\,\mathrm{d}t\Big|+\varepsilon\Big|\int_{S^1}\int_\Omega\varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\psi\,\mathrm{d}t\Big|\Big).
\end{aligned}
\tag{93}
$$

The above estimates are not enough to guarantee strong convergence of the velocity and the temperature, more specifically, the compactness of the velocity and the temperature in time; we need an additional piece of information. Let us define

$$
E(t)=\int_\Omega\Big(\frac{l}{2}|\mathbf{u}_N|^2+l(\vartheta-\ln\vartheta)+\frac{1}{2}\varrho|\mathbf{u}_N|^2+\varrho(e-s)+\delta\big(\frac{\varrho^\Gamma}{\Gamma-1}+\varrho^2\big)\Big)\,\mathrm{d}x.
\tag{94}
$$

Note that

$$
\begin{aligned}
E(t)\geqq\int_\Omega\Big(&\frac{l}{2}|\mathbf{u}_N|^2+\frac{1}{2}\varrho|\mathbf{u}_N|^2+\frac{l}{2}(\vartheta+|\ln\vartheta|)+\delta\Big(\frac{\varrho^\Gamma}{\Gamma-1}+\varrho^2\Big)\\
&+\frac{\varrho^\gamma}{\gamma-1}+\varrho|\ln\varrho|+\frac{1}{2}\vartheta^4+\varrho|\ln\vartheta|\Big)(t)\,\mathrm{d}x.
\end{aligned}
\tag{95}
$$

We get from (93), taking a sequence $\psi_n\to\psi_{t,s}$ with

$$
\psi_{t,s}(\tau)=\begin{cases}0, 0<\tau<t\\ 1, t\leqq\tau\leqq s<T_{\mathrm{per}}\\ 0, s<\tau\leqq T_{\mathrm{per}},\end{cases}
$$

that

$$
\sup_{t\in S^1}E(t)\leqq C(\varepsilon,\delta)\Big(1+\int_{S^1}E(s)\,\mathrm{d}s\Big).
\tag{96}
$$

Due to (78) it suffices to consider just two terms, that is, $\varrho|\mathbf{u}_N|^2$ and ϑ^4. But $\varrho|\mathbf{u}_N|^2 \leqq |\sqrt{\varrho}\mathbf{u}_N||\sqrt{\varrho}\mathbf{u}_N|$, $\sqrt{\int_\Omega \varrho|\mathbf{u}_N|^2\,\mathrm{d}x} \leqq CE^{1/2}(t)$, $\sqrt{\varrho} \in L^{2\Gamma}(S^1 \times \Omega)$, $\mathbf{u}_N \in L^2(S^1; L^6(\Omega; \mathbb{R}^3))$. Hence for $\Gamma > 3/2$ we easily find

$$\int_{S^1}\int_\Omega \varrho|\mathbf{u}_N|^2\,\mathrm{d}x\,\mathrm{d}t \leqq C(\varepsilon,\delta)\sup_{t\in S^1} E^{\frac{1}{2}}(t).$$

Next we consider the temperature. As the heat conductivity $\kappa(\vartheta) \sim (1+\vartheta^3)$ and the temperature in $L^1(S^1; L^9(\Omega))$, we find

$$\begin{aligned}\int_{S^1}\int_\Omega \vartheta^4\,\mathrm{d}x\,\mathrm{d}t &\leqq \int_{S^1}\left(\int_\Omega \vartheta^9\,\mathrm{d}x\right)^{1/9}\left(\int_\Omega \vartheta^{3\cdot\frac{9}{8}}\,\mathrm{d}x\right)^{\frac{8}{9}}\mathrm{d}t\\ &\leqq C(\varepsilon,\delta)\sup_{t\in S^1} E^{3/4}(t).\end{aligned} \tag{97}$$

Thus we get

$$\sup_{t\in S^1} E(t) \leqq C(\varepsilon,\delta). \tag{98}$$

Together with (78) we also have

$$\|\vartheta\|_{L^B(S^1;L^{3B}(\Omega))} \leqq C(\varepsilon,\delta). \tag{99}$$

Hence we may improve the bounds in the continuity equation to

$$\begin{aligned}\|\nabla_x^2\varrho\|_{L^{3/2}(S^1;L^{3/2}(\Omega;\mathbb{R}^9))} + \|\nabla_x\varrho\|_{L^3(S^1;L^3(\Omega;\mathbb{R}^3))}\\ +\|\partial_t\varrho\|_{L^{3/2}(S^1;L^{3/2}(\Omega))} \leqq C(\varepsilon,\delta).\end{aligned} \tag{100}$$

The bound on $\|\nabla_x\varrho\|_{L^3(S^1;L^3(\Omega;\mathbb{R}^3))}$ can be obtained provided $\Gamma \geqq 30$ $\left(\frac{3}{10}+\frac{1}{30}=\frac{1}{3}\right)$.

We may now use these estimates to control the time derivative of the velocity. As the main terms in the estimate of $(l+\varrho)\partial_t\mathbf{u}$ are $\varrho_t\mathbf{u}$ and $\mathrm{div}_x(\varrho|\mathbf{u}|^2)$, we get that particularly

$$l\|\partial_t\mathbf{u}\|_{(L^{30}(S^1;W_0^{1,5}(\Omega;\mathbb{R}^3)))^*} \leqq C(\varepsilon,\delta). \tag{101}$$

The space in (101) is not optimal, but sufficient. Note that due to (78) and (98), ϱ is bounded in $L^{5/3\,\Gamma}(S^1 \times \Omega)$. This point is crucial since it allows us to pass to the limit in the term with ϱ^Γ.

The limit $N \to \infty$ shows the first serious difficulty appearing in our analysis. This is a problem related to definition of the term $\mathsf{S}(\vartheta, \nabla_x\mathbf{u}_N) : \nabla_x\mathbf{u}_N$ in the limit. Since we have no uniform control of this sequence, we shift considerations on the level of the entropy equation (50). Then we consider $\frac{\mathsf{S}(\vartheta,\nabla_x\mathbf{u}_N):\nabla_x\mathbf{u}_N}{\vartheta}$, which is bounded by (64) in L^1 space. This information is weak but sufficient.

Finally, we need compactness in time of the temperature. Since compactness in space follows from boundedness of the gradient of the temperature, we find some information about the time derivative of the temperature in some negative space. From (50) with $\tau = 0$ we find

$$\Big(\frac{l+\varrho}{\vartheta} + \vartheta^2\Big)\partial_t\vartheta = -s\partial_t\varrho + \partial_t\varrho$$
$$- [\mathrm{div}_x(\varrho\mathbf{u}_N) + \partial_t\varrho]\frac{1}{\varrho\vartheta}(\varrho e + p - \varrho\vartheta s) + \mathrm{div}_x(\varrho s\mathbf{u}_N)$$
$$- \mathrm{div}_x\Big(\kappa(\vartheta) + \delta(\vartheta^B + \vartheta^{-1})\frac{\nabla_x\vartheta}{\vartheta}\Big)$$
$$+ \frac{1}{\vartheta}\mathsf{S}(\vartheta, \nabla_x\mathbf{u}_N) : \nabla_x\mathbf{u}_N + \big(\kappa(\vartheta) + \delta(\vartheta^B + \vartheta^{-1})\big)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}$$
$$+ \delta\vartheta^{-2} + \frac{\varepsilon\delta}{\vartheta}(\Gamma\varrho^{\Gamma-2} + 2)|\nabla_x\varrho|^2 \quad \text{in} \quad S^1 \times \Omega. \tag{102}$$

By our previous analysis we are able to show that at least

$$\Big(\frac{l+\varrho}{\vartheta} + \vartheta^2\Big)\partial_t\vartheta \in L^1(S^1; (W_0^{3,3}(\Omega))^*). \tag{103}$$

This information is sufficient to obtain the pointwise convergence of the approximative temperature for $N \to \infty$.

We can easily pass to the limit in the continuity and momentum equations. However, we cannot pass to the limit in the internal energy balance, due to the presence of the term $\mathsf{S}(\vartheta_{(N)}, \nabla_x\mathbf{u}_N) : \nabla_x\mathbf{u}_N$ which is not equi-integrable with respect to N. Hence, instead of examination the energy equation (47) we consider the entropy equation (50). However, we cannot pass to the limit directly since

$$\frac{\mathsf{S}(\vartheta_{(N)}, \nabla_x\mathbf{u}_N) : \nabla_x\mathbf{u}_N}{\vartheta_{(N)}} + \frac{\big(\kappa(\vartheta_{(N)}) + \delta({\vartheta_{(N)}}^B + {\vartheta_{(N)}}^{-1})\big)|\nabla_x\vartheta_{(N)}|^2}{{\vartheta_{(N)}}^2},$$
$$\frac{\mathsf{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{u})}{\vartheta} + \frac{\big(\kappa(\vartheta) + \delta(\vartheta^B + \vartheta^{-1})\big)|\nabla_x\vartheta|^2}{\vartheta^2} \qquad \in L^1(S^1; L^1(\Omega)). \tag{104}$$

Thus, the only information obtained from the uniform estimate of the above sequence is (up to a subsequence)

$$\frac{\mathsf{S}(\vartheta_{(N)}, \nabla_x\mathbf{u}_N) : \nabla_x\mathbf{u}_N}{\vartheta_{(N)}} + \frac{\big(\kappa(\vartheta_{(N)}) + \delta({\vartheta_{(N)}}^B + {\vartheta_{(N)}}^{-1})\big)|\nabla_x\vartheta_{(N)}|^2}{{\vartheta_{(N)}}^2} \rightharpoonup \sigma \tag{105}$$

weakly in $\mathcal{M}(S^1 \times \Omega)$. Additionally, we deduce that (for a chosen subsequence)

$$\lim_{N\to\infty} \frac{\mathsf{S}(\vartheta_{(N)}, \nabla_x\mathbf{u}_N) : \nabla_x\mathbf{u}_N}{\vartheta^{(N)}} + \frac{\big(\kappa(\vartheta_{(N)}) + \delta({\vartheta_{(N)}}^B + {\vartheta_{(N)}}^{-1})\big)|\nabla_x\vartheta_{(N)}|^2}{{\vartheta_{(N)}}^2}$$
$$\geqq \frac{\mathsf{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{u}}{\vartheta} + \frac{\big(\kappa(\vartheta) + \delta(\vartheta^B + \vartheta^{-1})\big)|\nabla_x\vartheta|^2}{\vartheta^2}.$$

Hence σ is a positive measure.

Finally, we pass to the limit in the total energy balance (90). We have shown

Theorem 4. *There exists a solution*

$$(\varrho, \mathbf{u}, \vartheta) \in (W^{1,\frac{3}{2}}(S^1 \times \Omega) \cap L^{\frac{3}{2}}(S^1; W^{2,p}(\Omega))) \times L^2(S^1; W_0^{1,2}(\Omega; \mathbb{R}^3)) \times L^2(S^1; W^{1,2}(\Omega))$$

to the following problem:

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) - \varepsilon \Delta_x \varrho + \varepsilon \varrho = \varepsilon h \quad \text{in } S^1 \times \Omega \\ &\frac{\partial \varrho}{\partial \mathbf{n}} = 0 \quad \text{at } S^1 \times \partial\Omega; \end{aligned} \tag{106}$$

$$\begin{aligned} &\int_{S^1}\int_\Omega \Big(l \partial_t \mathbf{u} \cdot \boldsymbol{\varphi} + \partial_t(\varrho \mathbf{u}) \cdot \boldsymbol{\varphi} - (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla_x \boldsymbol{\varphi} + \mathsf{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_{S^1}\int_\Omega \big(p(\varrho, \vartheta) + \delta(\varrho^\Gamma + \varrho^2) \big) \operatorname{div}_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{S^1}\int_\Omega \Big(- \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u} \, \boldsymbol{\varphi} + \frac{1}{2} \varepsilon (h - \varrho) \mathbf{u} \cdot \boldsymbol{\varphi} + \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{107}$$

for any $\boldsymbol{\varphi} \in C_c^\infty(S^1 \times \Omega; \mathbb{R}^3)$*;*

$$\begin{aligned} &\int_{S^1}\int_\Omega -(l \ln \vartheta + \varrho s) \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t - \int_{S^1}\int_\Omega \varrho s \mathbf{u} \cdot \nabla_x \psi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_{S^1}\int_\Omega [\operatorname{div}_x(\varrho \mathbf{u}) + \partial_t \varrho] \frac{1}{\varrho \vartheta} (\varrho e + p - \varrho \vartheta s) \psi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_{S^1}\int_\Omega \big(\kappa(\vartheta) + \delta(\vartheta^B + \vartheta^{-1}) \big) \frac{\nabla_x \vartheta}{\vartheta} \cdot \nabla_x \psi \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_{S^1}\int_{\partial\Omega} \frac{1}{\vartheta} (\vartheta - \theta_0) d \psi \, \mathrm{d}S_x \, \mathrm{d}t = \langle \sigma, \psi \rangle_{\mathcal{M}(S^1 \times \Omega)} \\ &\quad + \int_{S^1}\int_\Omega \delta \vartheta^{-2} \psi \, \mathrm{d}x \, \mathrm{d}t + \int_{S^1}\int_\Omega \frac{\varepsilon \delta}{\vartheta} (\Gamma \varrho^{\Gamma-2} + 2) |\nabla_x \varrho|^2 \psi \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{108}$$

for any $\psi \in C^\infty(S^1 \times \Omega)$*, where* σ *is a positive measure such that*

$$\sigma \geqq \frac{1}{\vartheta} \mathsf{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \big(\kappa(\vartheta) + \delta(\vartheta^B + \vartheta^{-1}) \big) \frac{|\nabla_x \vartheta|^2}{\vartheta^2}, \tag{109}$$

together with the total energy balance

$$\begin{aligned} &- \int_{S^1} \partial_t \psi \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + l \left(\frac{1}{2} |\mathbf{u}|^2 + \vartheta \right) \right. \\ &\quad \left. + \delta \left(\frac{1}{\Gamma - 1} \varrho^\Gamma + \varrho^2 \right) \right] \mathrm{d}x \, \mathrm{d}t \\ &= - \int_{S^1} \psi \Big(\int_{\partial\Omega} d(\vartheta - \Theta_0) \, \mathrm{d}S_x + \int_\Omega (\varrho \mathbf{f} \cdot \mathbf{u} + \delta \vartheta^{-1}) \, \mathrm{d}x \Big) \, \mathrm{d}t \\ &\quad + \varepsilon \delta \int_{S^1}\int_\Omega \Big(\frac{\Gamma}{\Gamma - 1} h \varrho^{\Gamma-1} h - \frac{\Gamma}{\Gamma - 1} \varrho^\Gamma + 2 h \varrho - 2 \varrho^2 \Big) \, \mathrm{d}x \, \psi \, \mathrm{d}t \end{aligned} \tag{110}$$

for any $\psi \in C^{\infty}(S^1)$. *In addition, we have* (63), (64) *with* $\tau = 0, \lambda = 1$ *and* (92) *with* $\mathbf{u}$ *instead of* $\mathbf{u}_N$.

7. Higher Regularity of Density $\varepsilon > 0$

In this section we show additional regularity of the density which will allow us to pass to the limit $\varepsilon \to 0$. The structure of the pressure implies that $p(\varrho, \theta) \sim \varrho^{5/3} + \varrho\theta + \theta^4$.

Introduce

$$\begin{aligned}\mathcal{E} = \sup_{t \in S^1} E(t) = \sup_{t \in S^1} \int_{\Omega} \frac{l}{2} \Bigg[|\mathbf{u}_N|^2 + l(\vartheta - \ln \vartheta) \\ + \frac{1}{2}\varrho|\mathbf{u}_N|^2 + \varrho(e - s) + \delta\left(\frac{\varrho^{\Gamma}}{\Gamma - 1} + \varrho^2\right)\Bigg] \,\mathrm{d}x.\end{aligned} \tag{111}$$

We prove

Theorem 5. *Let* $(\varrho, \mathbf{u}, \vartheta)$ *be a solution given by Theorem* 4, *then*

$$\mathcal{E} + \int_{S^1}\int_{\Omega} \left(\varrho^{\frac{5}{3}+1} + \delta(\varrho^{\Gamma+1} + \varrho^3)\right) \mathrm{d}x\,\mathrm{d}t \leqq C(\delta), \tag{112}$$

where $C(\delta)$ *is independent of* ε.

Proof. In order to prove inequality (112) we find a bound on $\mathcal{E}$. Noting first that integrating (93), we conclude

$$\mathcal{E} \leqq C\Big(1 + \int_{S^1} E(s)\,\mathrm{d}s + \int_{S^1}\int_{\Omega} |\varrho\mathbf{f}\cdot\mathbf{u}|\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_{S^1}\int_{\Omega} \varrho s(\varrho, \vartheta)\,\mathrm{d}x\,\mathrm{d}t\Big). \tag{113}$$

The structure of $\mathcal{E}$ allows us to put the last two terms in the right-hand side of (113) into the left-hand side, with the constant independent of approximation parameters.

The most rigorous terms on the right-hand side are the terms with powers of ϱ and ϑ^4. To estimate the density we apply the Bogovskii operator, see (42) that is, for each $t \in S^1$ we test the momentum equation by

$$\mathbf{\Phi} = \mathcal{B}[\varrho - M_0/|\Omega|].$$

Then

$$\begin{aligned}&\int_{S^1}\int_{\Omega} \big(p(\varrho, \vartheta) + \delta(\varrho^{\Gamma} + \varrho^2)\big)\varrho\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{S^1}\int_{\Omega} \big(p(\varrho, \vartheta) + \delta(\varrho^{\Gamma} + \varrho^2)\big)M_0\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_{\Omega} (l + \varrho)\mathbf{u}\cdot\partial_t\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{S^1}\int_{\Omega} (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_{\Omega} \mathsf{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$+\int_{S^1}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t-\varepsilon\int_{S^1}\int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\,\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{1}{2}\varepsilon\int_{S^1}\int_\Omega (h-\varrho)\mathbf{u}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t. \tag{114}$$

We consider here only a few terms, the most difficult ones. First take

$$\int_{S^1}\int_\Omega(\varrho\mathbf{u}\otimes\mathbf{u}):\nabla_x\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \leqq C\|\mathbf{u}\|^2_{L^2(S^1;L^6(\Omega;\mathbb{R}^3))}\|\varrho\|^2_{L^\infty(S^1;L^3(\Omega))}$$
$$\leqq C\Big(1+\varepsilon\int_{S^1}\int_\Omega(\varrho|\ln\varrho|+\varrho|\ln\vartheta|)\,\mathrm{d}x\,\mathrm{d}t\Big)^2\mathcal{E}^{2/\Gamma}\leqq C(1+\mathcal{E}^{2/\Gamma+\eta}), \tag{115}$$

for arbitrarily small $\eta>0$, since

$$\int_{S^1}\int_\Omega(\varrho|\ln\varrho|+\varrho|\ln\vartheta|)\,\mathrm{d}x\,\mathrm{d}t\leqq C(\delta)(1+\mathcal{E}^\eta), \tag{116}$$

(recall that the total mass is fixed). Before considering the other terms, we multiply the continuity equation by the density, and after integration over time and space we have

$$\varepsilon\int_{S^1}\int_\Omega\big(|\nabla\varrho|^2+\varrho^2\big)\,\mathrm{d}x\,\mathrm{d}t\leqq\varepsilon\int_{S^1}\int_\Omega h\varrho\,\mathrm{d}x\,\mathrm{d}t+\Big|\int_{S^1}\int_\Omega \mathrm{div}_x(\varrho\mathbf{u})\varrho\,\mathrm{d}x\,\mathrm{d}t\Big|; \tag{117}$$

however, the last term is bounded as follows

$$\begin{aligned}\int_{S^1}\int_\Omega\varrho^2\mathrm{div}_x\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t&\leqq\|\nabla_x\mathbf{u}\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^9))}\|\varrho\|^2_{L^\infty(S^1;L^4(\Omega))}\\ &\leqq C\delta^{-1/2}(1+\mathcal{E}^\eta)\Big(\sup_{t\in S^1}\int_\Omega\varrho^4\mathrm{d}x\Big)^{1/2}\leqq C\delta^{-1/2}(1+\mathcal{E}^{2/\Gamma+\eta}).\end{aligned} \tag{118}$$

We find

$$\begin{aligned}\varepsilon\|\nabla_x\varrho\|^2_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))}&\leqq C\big(1+\delta^{-1/2}E^{2/\Gamma+\eta}\big),\quad\text{so}\\ \varepsilon\|\nabla_x\varrho\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))}&\leqq C\varepsilon^{1/2}\big(1+\delta^{-1/4}E^{1/\Gamma+\eta}\big).\end{aligned} \tag{119}$$

Next we write

$$\mathrm{div}_x\partial_t\mathbf{\Phi}=\partial_t\varrho_t=\varepsilon\Delta_x\varrho-\mathrm{div}_x(\varrho\mathbf{u})+\varepsilon(h-\varrho);$$

we set $\partial_t\mathbf{\Phi}=\partial_t\mathbf{\Phi}^1+\partial_t\mathbf{\Phi}^2$ with

$$\mathrm{div}_x\partial_t\mathbf{\Phi}^1=\varepsilon\Delta_x\varrho\quad\text{in }\Omega,\quad\partial_t\mathbf{\Phi}^1=\mathbf{0}\quad\text{at }\partial\Omega. \tag{120}$$

We are able to construct this field since $\frac{\partial\varrho}{\partial n}=0$. Thus

$$\int_{S^1}\int_\Omega(l+\varrho)\mathbf{u}\cdot\partial_t\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t\leqq C(1+\mathcal{E}^{2/\Gamma+\eta})+\int_{S^1}\int_\Omega(l+\varrho)\mathbf{u}\cdot\partial_t\mathbf{\Phi}^1\,\mathrm{d}x\,\mathrm{d}t. \tag{121}$$

The estimate for the constructed field fulfilling (120) gives the following bound

$$\|\partial_t \mathbf{\Phi}^1\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))} \leqq C\varepsilon \|\nabla_x \varrho\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))}. \tag{122}$$

But we know that $\varepsilon^{1/2}\|\nabla_x \varrho\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))} \leqq C(\delta)(1+\mathcal{E}^{1/\Gamma+\eta})$, see (118). Hence

$$\begin{aligned}\int_{S^1}\int_\Omega |\varrho \mathbf{u}\cdot \partial_t \mathbf{\Phi}^1|\,\mathrm{d}x\,\mathrm{d}t &\leqq \varepsilon^{1/2}\|\mathbf{u}\|_{L^2(S^1;L^6(\Omega;\mathbb{R}^3))}\|\varrho\|_{L^\infty(S^1;L^3(\Omega))} \\ &\times \varepsilon^{1/2}\|\nabla_x\varrho\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^3))} \leqq C(\delta)\varepsilon^{1/2}(1+\mathcal{E}^{1/\Gamma+\eta})^2.\end{aligned} \tag{123}$$

At the end, we estimate the last nontrivial integral with ϱ, that is $\varepsilon\int_{S^1}\int_\Omega \nabla_x\varrho\cdot \nabla_x\mathbf{u}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t$. Since $\mathbf{\Phi}\in L^\infty(S^1;L^\infty(\Omega;\mathbb{R}^3))$ and $\nabla_x\mathbf{u}\in L^2(S^1;L^2(\Omega;\mathbb{R}^9))$ so we reduce our task to an estimate of $\varepsilon\|\nabla_x\varrho\|_{L^2(S^1;L^2(\Omega;\mathbb{R}^9))}$ and we proceed as above.

Finally, we consider the term with highest power of ϑ

$$\int_{S^1}\int_\Omega \vartheta^4\,\mathrm{d}x\,\mathrm{d}t \leqq \int_{S^1}\left(\int_\Omega \vartheta^9\,\mathrm{d}x\right)^{1/9}\left(\int_\Omega \vartheta^{3\cdot\frac{9}{8}}\,\mathrm{d}x\right)^{\frac{8}{9}}\mathrm{d}t \tag{124}$$

and from (93) with $N\to\infty$ we find

$$\begin{aligned}\|\vartheta\|_{L^1(S^1;L^9(\Omega))} &\leqq C\Big(1+\|\varrho\|^2_{L^2(S^1;L^{6/5}(\Omega))} \\ &+\varepsilon\int_{S^1}\int_\Omega \varrho s(\varrho,\vartheta)\,\mathrm{d}x\,\mathrm{d}t\Big) \leqq C(\delta)(1+\mathcal{E}^{2/\Gamma}).\end{aligned} \tag{125}$$

So $\int_{S^1}\int_\Omega \vartheta^4\,\mathrm{d}x\,\mathrm{d}t \leqq C(1+\mathcal{E}^{\frac{3}{4}+\frac{2}{\Gamma}})$. Thanks to the structure of E, see (95), we find that

$$\int_{S^1} E(s)\,\mathrm{d}s \leqq C\left(1+\mathcal{E}^{1-a}\right) \tag{126}$$

for some $a>0$, provided Γ is sufficiently large.

In all estimations, the right-hand sides (115), (121), (123), (118) and (125) depend on $\mathcal{E}$ in powers less than 1. Summing up, we conclude (112). □

Hence, we have the following estimates independent of ε:

$$\begin{aligned}&\sup_{t\in S^1}\int_\Omega\left(\frac{l}{2}|\nabla_x\mathbf{u}|^2 + l\vartheta + \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\delta\Big(\frac{1}{\Gamma-1}\varrho^\Gamma+\varrho^2\Big) - l\ln\vartheta - \varrho s\right)\mathrm{d}x \\ &+\int_{S^1}\int_\Omega\Big(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1}\Big)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_{S^1}\int_\Omega\left(\frac{1}{\vartheta}\mathsf{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}+\delta\vartheta^{-2}\right)\mathrm{d}x\,\mathrm{d}t \\ &+\int_{S^1}\int_{\partial\Omega} d\Big(\frac{\Theta_0}{\vartheta}+\vartheta\Big)\,\mathrm{d}S_x\,\mathrm{d}t + \int_{S^1}\int_\Omega \varrho^{\frac{5}{3}+1}\,\mathrm{d}x\,\mathrm{d}t \leqq C(\delta).\end{aligned} \tag{127}$$

8. Limit $\varepsilon \to 0$

The uniform bounds established in the previous section are exactly the same as in [4, Chapter 3, Section 3.6], therefore the limit passage for $\varepsilon \to 0$ can be carried over by means of the arguments therein, with only one modification concerning the strong pointwise convergence of the densities. Indeed the proof in [4, Chapter 3, Section 3.6] is based on the hypothesis that the initial densities $\varrho_{0,\varepsilon} = \varrho_\varepsilon(0, \cdot)$ converge *strongly* in $L^\gamma(\Omega)$, while this is not a priori known in the time-periodic case. Still, the argument can be modified in the same way as in [3]. As this step is the same for both $\varepsilon \to 0$ and $\delta \to 0$ limit, we provide the details only in the latter case.

Letting $\varepsilon \to 0$ in (106–110) we obtain a family of approximate solutions satisfying

– equation of continuity in the weak sense and in the sense of renormalized solutions:

$$\int_{S^1}\int_\Omega \Big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + \big(b(\varrho) - b'(\varrho)\varrho\big)\,\mathrm{div}_x\mathbf{u}\,\varphi\Big)\ \mathrm{d}x\,\mathrm{d}t = 0 \tag{128}$$

for any $b \in C^\infty([0,\infty))$, $b' \in C_c^\infty([0,\infty))$, and any test function $\varphi \in C^\infty(S^1 \times \overline{\Omega})$;

– momentum equation in the sense of distributions:

$$\begin{aligned}\int_{S^1}\int_\Omega \Big[l\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\\ + \Big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\Big)\,\mathrm{div}_x\boldsymbol{\varphi}\Big]\,\mathrm{d}x\,\mathrm{d}t\\ = \int_{S^1}\int_\Omega \left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} - \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\right)\ \mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{129}$$

for any test function $\boldsymbol{\varphi} \in C_c^\infty(S^1 \times \Omega; \mathbb{R}^3)$;

– approximate entropy balance in the form:

$$\begin{aligned}&\int_{S^1}\int_\Omega \big[(l\ln\vartheta + \varrho s(\varrho,\vartheta))\,\partial_t\psi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\psi\big]\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_{S^1}\int_\Omega \left((\kappa(\vartheta) + \delta\vartheta^{-1} + \delta\vartheta^B)\frac{\nabla_x\vartheta}{\vartheta}\right)\cdot\nabla_x\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_{S^1}\int_\Omega \frac{1}{\vartheta}\left[\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} + (\kappa(\vartheta) + \delta\vartheta^{-1} + \delta\vartheta^B)\frac{|\nabla_x\vartheta|^2}{\vartheta^2}\right]\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_{S^1}\int_{\partial\Omega}\frac{d}{\vartheta}(\vartheta - \Theta_0)\psi\,\mathrm{d}S_x\,\mathrm{d}t + \int_{S^1}\int_\Omega \delta\vartheta^{-2}\psi\,\mathrm{d}x\,\mathrm{d}t \leqq 0\end{aligned} \tag{130}$$

for any $\psi \in C^\infty(S^1 \times \overline{\Omega})$, $\psi \geqq 0$;

– the (modified) total energy balance:

$$\int_{S^1} \partial_t \psi \int_\Omega \Big[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + l\Big(\frac{1}{2}|\mathbf{u}|^2 + \vartheta\Big) + \delta\Big(\frac{1}{\Gamma - 1}\varrho^\Gamma + \varrho^2\Big)\Big] \mathrm{d}x \, \mathrm{d}t = \int_{S^1} \psi \Big(\int_{\partial\Omega} d(\vartheta - \Theta_0) \, \mathrm{d}S_x - \int_\Omega (\varrho \mathbf{f} \cdot \mathbf{u} + \delta \vartheta^{-1}) \, \mathrm{d}x \Big) \mathrm{d}t \quad (131)$$

for any $\psi \in C^\infty(S^1)$.

We remark that introducing a (positive) Radon measure

$$\langle \sigma_{\delta,l} \, ; \, \psi \rangle = - \int_{S^1} \int_\Omega \Big[(l \ln \vartheta + \varrho s(\varrho, \vartheta)) \, \partial_t \psi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \psi \Big] \mathrm{d}x \, \mathrm{d}t + \int_{S^1} \int_\Omega \Big[\Big(\Big(\kappa(\vartheta) + \delta \vartheta^{-1} + \delta \vartheta^B \Big) \frac{\nabla_x \vartheta}{\vartheta} \Big) \cdot \nabla_x \psi \Big] \mathrm{d}x \, \mathrm{d}t - \int_{S^1} \int_{\partial\Omega} \frac{d}{\vartheta} (\vartheta - \Theta_0) \psi \, \mathrm{d}S_x \, \mathrm{d}t, \quad (132)$$

$\psi \in C^\infty(S^1 \times \overline{\Omega})$, formula (130) rewrites

$$\langle \sigma_{\delta,l} \, ; \, \psi \rangle \geqq \int_{S^1} \int_\Omega \frac{1}{\vartheta} \Big[\mathsf{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \Big(\kappa(\vartheta) + \delta \vartheta^{-1} + \delta \vartheta^B \Big) \frac{|\nabla_x \vartheta|^2}{\vartheta} \Big] \psi \, \mathrm{d}x \, \mathrm{d}t + \int_{S^1} \int_\Omega \delta \vartheta^{-2} \psi \, \mathrm{d}x \, \mathrm{d}t$$

for any $\psi \in C^\infty(S^1 \times \overline{\Omega})$, $\psi \geqq 0$.

9. Limit $\delta \to 0$

We now set $l = \delta$ and $\sigma_{\delta,l} = \sigma_\delta$. Our ultimate goal is to pass to the limit for $\delta \to 0^+$. First we show additional estimates independent of δ which determine the regularity of solutions to the original full Navier–Stokes–Fourier system. Second we show the strong convergence of approximations of the temperature and density which is the most difficult task in our limit passage.

9.1. Uniform Bounds

The total mass of the fluid is a constant of motion, in particular,

$$\sup_{\delta > 0} \|\varrho_\delta\|_{L^\infty(S^1; L^1(\Omega))} < \infty. \quad (133)$$

Further, the approximate entropy balances (130), (132) with $\psi \equiv 1$ yield

$$\sup_{\delta > 0} \sigma_\delta[S^1 \times \overline{\Omega}] < \infty$$

$$\sup_{\delta > 0} \|\mathbf{u}_\delta\|_{L^2(S^1; W^{1,2}(\Omega; \mathbb{R}^3))} < \infty$$

$$\sup_{\delta>0} \|\nabla_x \vartheta_\delta^{3/2}\|_{L^2(S^1\times\Omega;\mathbb{R}^3)} < \infty$$

$$\sup_{\delta>0} \|\nabla_x \ln \vartheta_\delta\|_{L^2(S^1\times\Omega;\mathbb{R}^3)} < \infty$$
$$\sup_{\delta>0} \|\vartheta_\delta^{-1}\|_{L^2(S^1;L^1(\Omega))} < \infty, \tag{134}$$

and

$$\delta\left[\int_{S^1}\int_\Omega \left|\nabla_x \vartheta_\delta^{(B+1)/2}\right|^2 \, \mathrm{d}x\mathrm{d}t + \int_{S^1}\int_\Omega \frac{1}{\vartheta_\delta^2}\, \mathrm{d}x\mathrm{d}t\right] \leqq c, \tag{135}$$

where we have used the Korn type inequality [4, Theorem 10.17] to estimate $\mathbf{u}_\delta$.

We observe that

$$\int_{S^1}\int_{\partial\Omega} d(x)\vartheta_\delta \, \mathrm{d}S_x \, \mathrm{d}t = \int_{S^1}\int_\Omega \left(\varrho_\delta \mathbf{f}\cdot\mathbf{u}_\delta + \delta\vartheta_\delta^{-1}\right) \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_{S^1}\int_{\partial\Omega} d(x)\Theta_0 \, \mathrm{d}S_x \, \mathrm{d}t \leqq c\left(1 + \|\varrho_\delta\|_{L^2(S^1;L^{6/5}(\Omega))}\right).$$

Closely following Section 2.4.1, we deduce from the modified total energy balance

$$E_\delta(t) \leqq E_\delta(s) + c\left(1 + \int_{S^1} E_\delta(z)\, \mathrm{d}z\right)$$

for almost all $t \leqq s$, where we have denoted

$$E_\delta(t) = \int_\Omega \Big(\frac{1}{2}\varrho_\delta|\mathbf{u}_\delta|^2 + \varrho_\delta e(\varrho_\delta, \vartheta_\delta)$$
$$+ \delta\big(\frac{1}{2}|\mathbf{u}_\delta|^2 + \frac{1}{2}(\vartheta_\delta + |\ln\vartheta_\delta|) + \varrho_\delta^2 + \frac{1}{\Gamma-1}\varrho_\delta^\Gamma\big)\Big)(t)\, \mathrm{d}x.$$

Again as in Section 2.4.1, we conclude that

$$\sup_{t\in S^1} E_\delta(t) \leqq c\left(1 + \int_{S^1}\int_\Omega \varrho_\delta^{5/3}\, \mathrm{d}x\, \mathrm{d}t + \delta\int_{S^1}\int_\Omega (\varrho_\delta^2 + \varrho_\delta^\Gamma)\, \mathrm{d}x\, \mathrm{d}t\right). \tag{136}$$

For $E_\delta(t)$ we show the following results determining us the final regularity of constructed weak solutions to the original system. Recall $\mathcal{E}$ is defined by (111).

Theorem 6. *Let* $(\varrho, \mathbf{u}, \vartheta)$ *be a solution to* (128, 129, 130, 131)*; then*

$$\mathcal{E} + \int_{S^1}\int_\Omega \left(\varrho^{\frac{5}{3}+\frac{1}{9}} + \delta(\vartheta^{\Gamma+\frac{1}{9}} + \varrho^{2+\frac{1}{9}})\right) \mathrm{d}x\, \mathrm{d}t \leqq C, \tag{137}$$

where C *is independent of the approximation parameters.*

Proof. Within the proof we use the information given by (134). Particularly, recall that the norm $\|\mathbf{u}\|_{L^2(S^1;W^{1,2}(\Omega;\mathbb{R}^3))}$ is uniformly bounded.

Take

$$\mathbf{\Phi} = \mathcal{B}[\varrho^{5/3a} - \{\varrho^{5/3a}\}_\Omega] \tag{138}$$

with $a = \frac{1}{15}$. Since $\mathbf{\Phi} \in L^\infty(S^1; W^{1,9}(\Omega); \mathbb{R}^3)$ we are allowed to use $\mathbf{\Phi}$ as a test function in (129), yielding

$$\begin{aligned}
&\int_{S^1}\int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\big)\varrho^{1/9}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{S^1}\int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\big)\{\varrho^{1/9}\}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{S^1}\int_\Omega (\delta + \varrho)\mathbf{u}\cdot\partial_t\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_\Omega (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{S^1}\int_\Omega \mathsf{S}(\vartheta,\mathbf{u}) : \nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{139}$$

The structure of the pressure implies

$$\begin{aligned}
&\int_{S^1}\int_\Omega \big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2)\big)\varrho^{1/9}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \sim \int_{S^1}\int_\Omega \Big(\varrho^{5/3+1/9} + \varrho^{1+1/9}\vartheta + \vartheta^4\varrho^{1/9} + \delta(\varrho^{\Gamma+\frac{1}{9}} + \varrho^{2+\frac{1}{9}})\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{140}$$

Now we will estimate the terms step by step; we will mainly be concerned with the temperature. From (64) we get

$$\begin{aligned}
\int_{S^1}\int_\Omega \frac{\kappa(\vartheta)|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t + \int_{S^1}\int_{\partial\Omega}\left(\frac{1}{\vartheta} + \vartheta\right)\,\mathrm{d}S_x\,\mathrm{d}t \\
\leqq C\Big(1 + \Big|\int_{S^1}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\Big|\Big).
\end{aligned} \tag{141}$$

Since $\kappa(\vartheta) \sim \vartheta^3$, we control $\nabla\vartheta \in L^2(S^1; L^2(\Omega;\mathbb{R}^3))$, but the information on the trace gives only $\vartheta \in L^1(S^1 \times \partial\Omega)$. Furthermore $|\int_{S^1}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}| \leqq C\|\varrho\|_{L^2(S^1;L^{6/5}(\Omega))}\|\mathbf{u}\|_{L^2(S^1;L^6(\Omega;\mathbb{R}^3))}$. We keep in mind that the norm $\|\mathbf{u}\|_{L^2(S^1;L^6(\Omega;\mathbb{R}^3))}$ is bounded. We find

$$\|\vartheta\|_{L^1(S^1;L^9(\Omega))} \leqq C(1 + \|\varrho\|_{L^2(S^1;L^{6/5}(\Omega))}). \tag{142}$$

Our analysis will deliver that $\varrho \in L^{16/9}(S^1; L^{16/9}(\Omega))$, so we are able to interpolate the norm in $L^2(S^1; L^{16/9}(\Omega))$ between $L^{16/9}(S^1; L^{16/9}(\Omega))$ and $L^\infty(S^1; L^1(\Omega))$, which is nothing but the balance of mass. Hence we get

$$\|\varrho\|_{L^2(S^1;L^{6/5}(\Omega))} \leqq C(1 + C\|\varrho\|^{8/21}_{L^{16/9}(S^1;L^{16/9}(\Omega))}), \tag{143}$$

since $\frac{5}{6} = (1 - \frac{8}{21}) + \frac{8}{21} \cdot \frac{9}{16}$ and $\frac{1}{2} > \frac{3}{14}$. Thus

$$\|\vartheta\|_{L^1(S^1;L^9(\Omega))} \leqq C\Big(1 + \Big[\int_{S^1}\int_\Omega \varrho^{16/9}\,\mathrm{d}x\,\mathrm{d}t\Big]^{3/14}\Big). \tag{144}$$

As $3/14 < 1/4$, we will get the final estimation. So we start with the estimates. First $\int_{S^1}\int_\Omega \varrho^{5/3}\{\varrho^{1/9}\}_\Omega\,\mathrm{d}x\,\mathrm{d}t$ can be easily controlled since $\{\varrho^{1/9}\}$ is bounded in $L^\infty(S^1; L^\infty(\Omega))$,

$$\begin{aligned}\int_{S^1}\int_\Omega \varrho\vartheta\{\varrho^{1/9}\}_\Omega\,\mathrm{d}x\,\mathrm{d}t &\leqq C\int_{S^1}\int_\Omega \varrho\vartheta\,\mathrm{d}x\,\mathrm{d}t \\ &\leqq C\|\varrho\|_{L^\infty(S^1;L^{5/3}(\Omega))}\|\vartheta\|_{L^\infty(S^1;L^4(\Omega))} \leqq C\mathcal{E}^{3/5}\mathcal{E}^{1/4} = C\mathcal{E}^{17/20},\end{aligned} \tag{145}$$

since $\frac{3}{5} + \frac{1}{4} < 1$. An important element is the estimate on the temperature part

$$\int_{S^1}\int_\Omega \vartheta^4\{\varrho^{1/9}\}_\Omega\,\mathrm{d}x\,\mathrm{d}t \leqq C\int_{S^1}\int_\Omega \vartheta^4\,\mathrm{d}x\,\mathrm{d}t. \tag{146}$$

Let us observe that

$$\int_\Omega \vartheta^4\,\mathrm{d}x \leqq \Big(\int_\Omega \vartheta^9\,\mathrm{d}x\Big)^{1/9}\Big(\int_\Omega \vartheta^{3\cdot\frac{9}{8}}\,\mathrm{d}x\Big)^{8/9} \tag{147}$$

and

$$\Big(\int_\Omega \vartheta^{3\cdot\frac{9}{8}}\,\mathrm{d}x\Big)^{8/9} \leqq C\Big(\int_\Omega \vartheta^4\,\mathrm{d}x\Big)^{\frac{3}{4}\cdot\frac{9}{8}\cdot\frac{8}{9}} \leqq C\mathcal{E}^{3/4}. \tag{148}$$

Using (144) we find that

$$\int_{S^1}\int_\Omega \vartheta^4\,\mathrm{d}x\,\mathrm{d}t \leqq C\Big(1 + \int_{S^1}\int_\Omega \varrho^{16/9}\,\mathrm{d}x\,\mathrm{d}t\Big)^{3/14}\mathcal{E}^{3/4}. \tag{149}$$

So we have

$$\int_{S^1}\int_\Omega \vartheta^4\,\mathrm{d}x\,\mathrm{d}t \leqq \frac{1}{10}\Big[\int_{S^1}\int_\Omega \varrho^{16/9}\,\mathrm{d}x\,\mathrm{d}t\Big] + C\Big(1 + \mathcal{E}^{\frac{3}{4}\cdot\frac{14}{11}}\Big). \tag{150}$$

It is important to note that $\frac{3}{4} \cdot \frac{14}{11} = \frac{21}{22} < 1$. We will see that (146) is the most restrictive term in this consideration. Next we estimate

$$\begin{aligned}\int_{S^1}\int_\Omega (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t &\leqq \|\mathbf{u}\|^2_{L^2(S^1;L^6(\Omega;\mathbb{R}^3))}\|\varrho\nabla\mathbf{\Phi}\|_{L^\infty(S^1;L^{3/2}(\Omega;\mathbb{R}^9))} \\ &\leqq C\|\varrho\|_{L^{5/3}(S^1\times\Omega)}\|\nabla\mathbf{\Phi}\|_{L^{15}(S^1\times\Omega;\mathbb{R}^9)} \leqq C\mathcal{E}^{2/3},\end{aligned} \tag{151}$$

since $\frac{2}{3} = \frac{3}{5} + \frac{1}{15}$, note that this point determines the condition $a = \frac{1}{15}$. Looking at the term with the time derivative we can use the renormalized equation

$$(\varrho^b)_t = -b\mathrm{div}_x(\varrho^b\mathbf{u}) - b\varrho^b\mathrm{div}_x\mathbf{u} \quad \text{in} \quad C_c^\infty(S^1\times\Omega). \tag{152}$$

Observe that the part with $\mathrm{div}_x(\varrho^a \mathbf{u})$ can be bounded as in (151). So we concentrate our attention on the second term, namely on

$$\begin{aligned}&\int_{S^1}\int_\Omega \varrho\mathbf{u}\mathcal{B}[\varrho^{1/9}\mathrm{div}_x\mathbf{u} - \{\varrho^{1/9}\mathrm{div}_x\mathbf{u}\}_\Omega]\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqq \|\varrho\mathbf{u}\|_{L^2(S^1;L^{30/23}(\Omega;\mathbb{R}^3))}\|\mathcal{B}[\varrho^{1/9}\mathrm{div}_x\mathbf{u} - \{\varrho^{1/9}\mathrm{div}_x\mathbf{u}\}_\Omega]\|_{L^2(S^1;L^{30/7}(\Omega;\mathbb{R}^3))},\end{aligned} \tag{153}$$

but we see that

$$\begin{aligned}&\|\mathcal{B}[\varrho^{1/9}\mathrm{div}_x\mathbf{u} - \{\varrho^{1/9}\mathrm{div}_x\mathbf{u}\}_\Omega]\|_{L^2(S^1;L^{30/7}(\Omega;\mathbb{R}^3))}\\ &\quad\leqq C\|\varrho^{1/9}\mathrm{div}_x\mathbf{u}\|_{L^2(S^1;L^{30/17}(\Omega))} \leqq C\|\varrho^{1/9}\|_{L^\infty(S^1;L^{15}(\Omega))} \leqq C\mathcal{E}^{1/15},\end{aligned} \tag{154}$$

since $\frac{3p}{3-p} = \frac{30}{7}$ implies that $p = \frac{30}{17}$ and $\frac{1}{15} + \frac{1}{2} = \frac{17}{30}$.

Keeping in mind that $\mathbf{\Phi} \in L^\infty(S^1; W^{1,9}(\Omega;\mathbb{R}^3))$, we get that $\mathbf{\Phi} \in L^\infty(S^1; L^\infty(\Omega;\mathbb{R}^3))$;

$$\|\varrho\mathbf{u}\|_{L^2(S^1;L^{30/23}(\Omega;\mathbb{R}^3))} \leqq \|\varrho\|_{L^\infty(S^1;L^{5/3}(\Omega))}\|\mathbf{u}\|_{L^2(S^1;L^6(\Omega;\mathbb{R}^3))} \leqq C\mathcal{E}^{3/5} \tag{155}$$

and $\frac{3}{5} + \frac{1}{15} < 1$.

Altogether we find

$$\int_{S^1}\int_\Omega \left[\varrho^{\frac{16}{9}} + \vartheta^4\varrho^{1+\frac{1}{9}} + \delta(\varrho^{\Gamma+\frac{1}{9}} + \varrho^{2+\frac{1}{9}}) + \vartheta^4\right]\mathrm{d}x\,\mathrm{d}t \leqq C\left(1 + \mathcal{E}^{\frac{21}{22}}\right). \tag{156}$$

To get (137) we repeat considerations from the beginning of Section 7 to show

$$\mathcal{E} \leqq C\left(1 + \int_{S^1} E(s)\,\mathrm{d}s + \left|\int_{S^1}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\right|\right) \leqq C\left(1 + \mathcal{E}^{\frac{21}{22}}\right). \tag{157}$$

But

$$\left|\int_{S^1}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\right| \leqq C\|\sqrt{\varrho}\mathbf{u}\|_{L^\infty(S^1;L^2(\Omega;\mathbb{R}^3))}\|\sqrt{\varrho}\|_{L^\infty(S^1;L^2(\Omega))} \leqq C\mathcal{E}^{\frac{1}{2}}; \tag{158}$$

whence (137) is proved. □

We end up with the following bounds for the solutions

$$\begin{aligned}&\sup_{t\in S^1}\int_\Omega \left(\frac{\delta}{2}|\nabla\mathbf{u}|^2 + l\vartheta + \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e + \delta\left(\frac{1}{\Gamma-1}\varrho^\Gamma + \varrho^2\right) - \delta\ln\vartheta - \varrho s\right)\mathrm{d}x\\ &\quad+ \int_{S^1}\int_\Omega \left(\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1}\right)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+ \int_{S^1}\int_\Omega \left(\frac{1}{\vartheta}\mathsf{S}(\vartheta,\mathbf{u}_N) : \nabla\mathbf{u}_N + \delta\vartheta^{-2}\right)\mathrm{d}x\,\mathrm{d}t\\ &\quad+ \int_{S^1}\int_{\partial\Omega} d\left(\frac{\Theta_0}{\vartheta} + \vartheta\right)\mathrm{d}S_x\,\mathrm{d}t + \int_{S^1}\int_\Omega \varrho^{\frac{5}{3}+\frac{1}{9}}\,\mathrm{d}x\,\mathrm{d}t \leqq C.\end{aligned} \tag{159}$$

Returning to (134), we conclude by the Poincaré inequality that

$$\sup_{\delta>0}\left(\|\log\vartheta_\delta\|_{L^2(S^1;W^{1,2}(\Omega))}+\|\vartheta_\delta^\alpha\|_{L^2(S^1;W^{1,2}(\Omega))}\right)<\infty,\ \text{ where } 0\leqq\alpha\leqq 3/2. \tag{160}$$

From bounds (134), (137), (160) we deduce the existence of $(\varrho,\mathbf{u},\vartheta,\sigma)$ such that

$$\begin{aligned}&\varrho_\delta \rightharpoonup^* \varrho \text{ in } L^\infty(S^1,L^{5/3}(\Omega)) \text{ and in } L^p(S^1\times\Omega) \text{ for some } p>1\\ &\mathbf{u}_\delta \rightharpoonup \mathbf{u} \text{ in } L^2(S^1;W^{1,2}(\Omega;\mathbb{R}^3))\\ &\vartheta_\delta \rightharpoonup^* \vartheta \text{ in } L^\infty(S^1,L^4(\Omega)) \text{ and in } L^2(S^1;W^{1,2}(\Omega)),\\ &\sigma_\delta \rightharpoonup^* \sigma \text{ in } [C(S^1\times\overline{\Omega})]^*=\mathcal{M}(S^1\times\overline{\Omega}).\end{aligned} \tag{161}$$

In addition, due to (128) and (129) we have

$$\begin{aligned}&\varrho_\delta\to\varrho \ \text{ in } C_{\text{weak}}(S^1;L^{5/3}(\Omega))\\ &\varrho_\delta\mathbf{u}_\delta\to\varrho\mathbf{u} \ \text{ in } C_{\text{weak}}(S^1;L^{5/4}(\Omega;\mathbb{R}^3))\\ &b(\varrho_\delta)\to\overline{b(\varrho)} \ \text{ in } C_{\text{weak}}(S^1;L^p(\Omega)),\end{aligned} \tag{162}$$

provided $(\varrho_\delta,\mathbf{u}_\delta)$ satisfy the renormalized continuity equation with $b\in C^1(0,\infty)$ and the sequence $b(\varrho_\delta)$ is bounded in $L^\infty(S^1;L^p(\Omega))$. Here and hereafter, we denote by $\overline{b(\varrho,\vartheta,\mathbf{u})}$ a weak limit in $L^1(S^1\times\Omega)$ (provided it exists) of the sequence $b(\varrho_\delta,\vartheta_\delta,\mathbf{u}_\delta)$. Recall that $f_\delta\to f$ in $C_{\text{weak}}(S^1;L^p(\Omega))$ means that $\lim_{\delta\to0}\int_\Omega f_\delta(t,x)\varphi(x)\,\mathrm{d}x=\int_\Omega f(t,x)\varphi(x)\,\mathrm{d}x$ in $C(S^1)$ for all $\varphi\in L^{p'}(\Omega)$.

9.2. Convergence of Temperatures

The proof closely follows [4, Section 3.7.2]. To begin, we will need a version of the div-curl lemma due to TARTAR [20], MURAT [17] (in the form for example [4, Theorem 10.21]) that reads

Lemma 6. (Div-curl lemma) *Let* $\mathbf{U}_\delta\rightharpoonup\mathbf{U}$ *in* $L^p(\mathbb{R}^4;\mathbb{R}^4)$, $\mathbf{V}_\delta\rightharpoonup\mathbf{V}$ *in* $L^q(\mathbb{R}^4;\mathbb{R}^4)$, *where*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{s}<1.$$

Let $\operatorname{div}\mathbf{V}_\delta$ *be precompact in* $W^{-1,r}(\mathbb{R}^4;\mathbb{R})$ *and* $\operatorname{curl}\mathbf{U}_\delta$ *be precompact in* $W^{-1,r}(\mathbb{R}^{16};\mathbb{R}^4)$ *for* $r\in(1,\infty)$. *Then*

$$\mathbf{V}_\delta\cdot\mathbf{U}_\delta\rightharpoonup\mathbf{V}\cdot\mathbf{U}$$

in $L^s(\mathbb{R}^4)$.

We will apply this lemma to the 4-dimensional vector fields

$$\begin{aligned}&\mathbf{V}_\delta\equiv\left[\delta\log\vartheta_\delta+\varrho_\delta s(\varrho_\delta,\vartheta_\delta),\varrho_\delta s(\varrho_\delta,\vartheta_\delta)\mathbf{u}_\delta-\left(\kappa(\vartheta_\delta)+\frac{\delta}{\vartheta_\delta}+\delta\vartheta_\delta^B\right)\frac{\nabla_x\vartheta_\delta}{\vartheta_\delta}\right],\\ &\mathbf{U}_\delta\equiv[T_k(\vartheta_\delta),0,0,0],\end{aligned}$$

where

$$T_k(z) = kT\left(\frac{z}{k}\right), \quad T(z) = \begin{cases} z \text{ for } 0 \leqq z \leqq 1 \\ \text{concave on } (0, \infty) \\ 2 \text{ for } z \geqq 3. \end{cases}$$

In view of (23), estimates (134), (135), (137) and (160) yield uniform bounds of $\varrho_\delta s(\varrho_\delta, \vartheta_\delta)$, $\varrho_\delta s(\varrho_\delta, \vartheta_\delta)\mathbf{u}_\delta$, $\kappa(\vartheta_\delta)\frac{\nabla_x \vartheta_\delta}{\vartheta_\delta}$ in $L^p(S^1 \times \Omega)$ and in $L^p(S^1 \times \Omega, \mathbb{R}^3)$, respectively, with some $p > 1$. Moreover, all terms in the entropy balance (130) which are proportional to δ, disappear as $\delta \to 0$ in weak convergence $L^p(\Omega)$, $p > 1$. In view of (132) and estimates (133–135), (137) and (160), we easily verify that vector fields $\mathbf{V}_\delta$, $\mathbf{U}_\delta$ satisfy the requested hypotheses of Lemma 6. We thus obtain

$$\overline{T_k(\varrho)\, s_0(\varrho, \vartheta)} + a\overline{T_k(\vartheta)\, \vartheta^3} = \overline{T_k(\vartheta)}\; \overline{s_0(\varrho, \vartheta)} + a\overline{T_k(\vartheta)}\; \overline{\vartheta^3}.$$

Essentially due to monotonicity of $\vartheta \mapsto s_0(\varrho, \vartheta)$,

$$\overline{T_k(\vartheta)\, s_0(\varrho, \vartheta)} \geqq \overline{T_k(\vartheta)}\; \overline{s_0(\varrho, \vartheta)},$$

whence

$$\overline{T_k(\vartheta)}\; \overline{\vartheta^3} \geqq \overline{T_k(\vartheta)\, \vartheta^3}.$$

Letting $k \to \infty$ in the last identity, and using monotonicity of the map $\vartheta \mapsto \vartheta^3$ (see for example [4, Theorem 10.19]) we conclude that

$$\vartheta^3 = \overline{\vartheta^3},$$

which means that

$$\vartheta_\delta \to \vartheta \quad \text{a.e. in } S^1 \times \Omega. \tag{163}$$

Consequently, due to (160), in particular

$$\log \vartheta \in L^2(S^1; W^{1,2}(\Omega)).$$

9.3. Convergence of Densities

9.3.1. Effective Viscous Flux. Using the same arguments as in [4, Chapter 3, Section 3.7] we can show that

$$\overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} = \frac{1}{\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)} \left(\overline{p_0(\varrho, \vartheta) T_k(\varrho)} - \overline{p_0(\varrho, \vartheta)}\; \overline{T_k(\varrho)} \right). \tag{164}$$

Recall that this *effective viscous flux identity* was discovered and exploited by Lions [11] in the case of constant viscosity coefficients. To get it, one needs to subtract the limit $\delta \to 0$ of the momentum equation (129) tested by $\boldsymbol{\varphi} = \zeta \nabla \Delta^{-1}[T_k(\varrho_\delta) 1_\Omega]$, $\zeta \in C_c^\infty(S^1 \times \Omega)$ from the limit $\delta \to 0$ of (129) tested by $\boldsymbol{\varphi} = \zeta \nabla \Delta^{-1}[\overline{T_k(\varrho)} 1_\Omega]$;

in what follows $\mathcal{R}$ denotes the Riesz operator, $(\mathcal{R}[v])_{ij} = (\nabla \otimes \nabla \Delta^{-1})_{ij} v = \mathcal{F}^{-1}\left[\frac{\xi_i \xi_j}{|\xi|^2} \mathcal{F}(v)(\xi)\right]$ with $\mathcal{F}$ the Fourier transform. This procedure yields

$$
\begin{aligned}
&\lim_{\delta \to 0^+} \int_{S^1} \int_\Omega \zeta(t,x) \Big(p(\varrho_\delta, \vartheta_\delta) T_k(\varrho_\delta) - \mathsf{S}(\vartheta_\delta, \mathbf{u}_\delta) : \mathcal{R}[1_\Omega T_k(\varrho_\delta)] \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_{S^1} \int_\Omega \zeta(t,x) \Big(\overline{p(\varrho, \vartheta)} \; \overline{T_k(\varrho)} - \mathsf{S}(\vartheta, \mathbf{u}) : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \lim_{\delta \to 0^+} \int_{S^1} \int_\Omega \zeta(t,x) \Big(T_k(\varrho_\delta) \mathbf{u}_\delta \cdot \mathcal{R}[1_\Omega \varrho_\delta \mathbf{u}_\delta] \\
&\quad - \varrho_\delta (\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \mathcal{R}[1_\Omega T_k(\varrho_\delta)] \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_{S^1} \int_\Omega \zeta(t,x) \Big(\overline{T_k(\varrho)} \mathbf{u} \cdot \mathcal{R}[1_\Omega \varrho \mathbf{u}] - \varrho (\mathbf{u} \otimes \mathbf{u}) : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] \Big) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{165}
$$

At this stage, we will need the following lemma, see [4, Theorem 10.27].

Lemma 7. (Commutators I) *Let* $\mathbf{U}_\delta \rightharpoonup \mathbf{U}$ *in* $L^p(\mathbb{R}^3; \mathbb{R}^3)$*,* $v_\delta \rightharpoonup v$ *in* $L^q(\mathbb{R}^3)$*, where*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.$$

Then

$$v_\delta \mathcal{R}[\mathbf{U}_\delta] - \mathcal{R}[v_\delta] \mathbf{U}_\delta \rightharpoonup v \mathcal{R}[\mathbf{U}] - \mathcal{R}[v] \mathbf{U}$$

in $L^s(\mathbb{R}^3; \mathbb{R}^3)$.

Combining this lemma with the convergence established in (162), we get

$$
\begin{aligned}
&\int_{S^1} \int_\Omega \zeta(t,x) \mathbf{u}_\delta \cdot \Big(T_k(\varrho_\delta) \mathcal{R}[1_\Omega \varrho_\delta \mathbf{u}_\delta] - \varrho_\delta \mathcal{R}[1_\Omega T_k(\varrho_\delta)] \mathbf{u}_\delta \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \to \int_{S^1} \int_\Omega \zeta(t,x) \mathbf{u} \cdot \Big(\overline{T_k(\varrho)} \mathcal{R}[1_\Omega \varrho \mathbf{u}] - \varrho \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] \mathbf{u} \Big) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

Thus (165) yields

$$
\begin{aligned}
&\int_{S^1} \int_\Omega \zeta(t,x) \Big(\overline{p(\varrho, \vartheta) T_k(\varrho)} - \overline{p(\varrho, \vartheta)} \; \overline{T_k(\varrho)} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_{S^1} \int_\Omega \zeta(t,x) \Big(\overline{\mathsf{S}(\vartheta, \mathbf{u}) : \mathcal{R}[1_\Omega T_k(\varrho)]} - \mathsf{S}(\vartheta, \mathbf{u}) : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] \Big) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{166}
$$

In the sequel we will need another another commutator lemma in the spirit of COIFMAN and MEYER [2], see [4, Theorem 10.28].

Lemma 8. (Commutators II) *Let* $w \in W^{1,r}(\mathbb{R}^3)$*,* $\mathbf{z} \in L^p(\mathbb{R}^3; \mathbb{R}^3)$*,* $1 < r < 3$*,* $1 < p < \infty$*,* $\frac{1}{r} + \frac{1}{p} - \frac{1}{3} < \frac{1}{s} < 1$*. Then for all such s we have*

$$\| \mathcal{R}[w\mathbf{z}] - w \mathcal{R}[\mathbf{z}] \|_{a,s,\mathbb{R}^3} \leqq C \|w\|_{1,r,\mathbb{R}^3} \|\mathbf{z}\|_{p,\mathbb{R}^3},$$

where $\frac{a}{3} = \frac{1}{s} + \frac{1}{3} - \frac{1}{p} - \frac{1}{r}$*. Here,* $\|\cdot\|_{a,s,\mathbb{R}^3}$ *denotes the norm in the Sobolev–Slobodetskii space* $W^{a,s}(\mathbb{R}^3)$*.*

We can write

$$\int_{S^1}\int_\Omega \zeta(t,x)\overline{\mathbb{S}(\vartheta,\mathbf{u}):\mathcal{R}[1_\Omega T_k(\varrho)]}\,\mathrm{d}x\,\mathrm{d}t = \lim_{\delta\to 0^+}\int_{S^1}\int_\Omega \omega(\vartheta_\delta,\mathbf{u}_\delta)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\lim_{\delta\to 0^+}\int_{S^1}\int_\Omega \zeta(t,x)\Big(\frac{4}{3}\mu(\vartheta_\delta)+\xi(\vartheta_\delta)\Big)\mathrm{div}_x\mathbf{u}_\delta\, T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t, \tag{167}$$

where

$$\omega(\vartheta_\delta,\mathbf{u}_\delta) = T_k(\vartheta_\delta)\Big(\mathcal{R}\Big[\zeta(t,x)\mu(\vartheta_\delta)\big(\nabla\mathbf{u}_\delta+(\nabla\mathbf{u}_\delta)^T\big)\Big] - \zeta(t,x)\mu(\vartheta_\delta)\mathcal{R}:\big[\nabla\mathbf{u}_\delta+(\nabla\mathbf{u}_\delta)^T\big]\Big).$$

Due to Lemma 8, $\omega(\vartheta_\delta,\mathbf{u}_\delta)$ is bounded in $L^1(S^1;W^{a,s}(\Omega;\mathbb{R}^3))$ with some $a\in(0,1)$, $s>1$; whence we may apply the div-curl lemma to 4-dimensional vectors

$$\mathbf{V}_\delta \equiv [T_k(\varrho_\delta), T_k(\varrho_\delta)\mathbf{u}_\delta],\quad \mathbf{U}_\delta\equiv[\omega(\vartheta_\delta,\mathbf{u}_\delta),0,0,0]$$

to get

$$\omega(\vartheta_\delta,\mathbf{u}_\delta)\,T_k(\varrho_\delta) \rightharpoonup \overline{\omega(\vartheta,\mathbf{u})}\;\overline{T_k(\varrho)},$$

where, due to (163),

$$\overline{\omega(\vartheta,\mathbf{u})} = \omega(\vartheta,\mathbf{u}).$$

This result in combination with (166) and (167) yields the effective viscous flux identity (164).

9.3.2. Oscillations Defect Measure and Renormalized Continuity Equation. We have

$$\begin{aligned} b_0\limsup_{\delta\to 0}&\int_{S^1}\int_\Omega \zeta(t,x)|T_k(\varrho_\delta)-T_k(\varrho)|^{8/3}\,\mathrm{d}x\,\mathrm{d}t\\ &\leqq b_0\limsup_{\delta\to 0}\int_{S^1}\int_\Omega\zeta(t,x)\Big((T_k(\varrho_\delta)-T_k(\varrho)\Big)(\varrho_\delta^{5/3}-\varrho^{5/3})\,\mathrm{d}x\,\mathrm{d}t\\ &\leqq \int_{S^1}\int_\Omega\zeta(t,x)\Big(\overline{\varrho^{5/3}\,T_k(\varrho)}-\overline{\varrho^{5/3}}\;\overline{T_k(\varrho)}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqq \int_{S^1}\int_\Omega\zeta(t,x)\Big(\overline{p_0(\varrho,\vartheta)\,T_k(\varrho)}-\overline{p_0(\varrho,\vartheta)}\;\overline{T_k(\varrho)}\Big)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\zeta\in C_c^\infty(S^1\times\Omega)$, $\zeta\geqq 0$. The first inequality is an algebraic one. To derive the second one, we have used convexity of $\varrho\mapsto\varrho^{5/3}$ and concavity of $\varrho\mapsto T_k(\varrho)$, and finally, to derive the third one, we have employed (20).

The right-hand side of the last inequality can be calculated from (164). Consequently, with help of bounds (134), (160) for sequences $\mathbf{u}_\delta$, ϑ_δ, Hölder inequality and interpolation, one concludes that

$$\mathbf{osc_q}[\varrho_\delta\to\varrho](S^1\times\Omega)\equiv\sup_{k>0}\limsup_{\delta\to 0}\int_{S^1}\int_\Omega|T_k(\varrho_\delta)-T_k(\varrho)|^q\,\mathrm{d}x\,\mathrm{d}t<\infty, \tag{168}$$

with some $q > 2$. The expression at the right-hand side is called *oscillations defect measure*, see [4, Chapter 3, Section 3.7.5].

On the other hand, relation (168) implies that the limit quantities ϱ, $\mathbf{u}$ satisfy the renormalized equation of continuity (128), see [4, Lemma 3.8] that reads

Lemma 9. (Renormalized continuity equation) *Let $\Omega \subset \mathbb{R}^3$ be open and let*

$$\begin{array}{ll} \varrho_\delta \rightharpoonup \varrho & \text{in } L^1(S^1 \times \Omega) \\ \mathbf{u}_\delta \rightharpoonup \mathbf{u} & \text{in } L^r(S^1 \times \Omega; \mathbb{R}^3) \\ \nabla \mathbf{u}_\delta \rightharpoonup \nabla \mathbf{u} & \text{in } L^r(S^1 \times \Omega; \mathbb{R}^9), \quad r > 1. \end{array}$$

Let

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](\Omega) < \infty \tag{169}$$

for $\frac{1}{q} < 1 - \frac{1}{r}$, where $(\varrho_\delta, \mathbf{u}_\delta)$ solve the renormalized continuity equation (128). Then the limit functions ϱ, $\mathbf{u}$ solve (128) for all $b \in C^1([0,\infty)) \cap W^{1,\infty}(0,\infty)$.

Since in our case, the validity of the renormalized continuity equation can be extended via a Lebesgue dominated convergence theorem for example to any $b \in C^1[0,\infty)$ with growth $z|b'(z)| + |b(z)|/z^{\gamma/2} \in L^\infty(1,\infty)$, equations (128) with $(\varrho_\delta, \mathbf{u}_\delta)$ and $(\varrho, \mathbf{u})$, respectively, yield, in particular

$$\lim_{\delta \to 0} \int_{S^1} \int_\Omega T_k(\varrho_\delta) \mathrm{div}_x \mathbf{u}_\delta \, \mathrm{d}x \, \mathrm{d}t = \int_{S^1} \int_\Omega T_k(\varrho) \mathrm{div}_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{170}$$

for any $k > 0$, where we have taken $\varphi = 1$ and

$$b(\varrho) = \varrho \int_1^\varrho \frac{T_k(z)}{z^2} \, \mathrm{d}z.$$

Combining (164), (170) we obtain that

$$\lim_{k \to 0} \int_{S^1} \int_\Omega \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta) \right)^{-1} \left(\overline{p_0(\varrho,\vartheta) T_k(\varrho)} - \overline{p_0(\varrho,\vartheta)}\ \overline{T_k(\varrho)} \right) \mathrm{d}x \, \mathrm{d}t = 0,$$

therefore, by virtue of hypotheses (15–18),

$$\lim_{k \to 0} \int_{S^1} \int_\Omega \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta) \right)^{-1} \left(\overline{\varrho^{5/3} T_k(\varrho)} - \overline{\varrho^{5/3}}\ \overline{T_k(\varrho)} \right) \mathrm{d}x \, \mathrm{d}t = 0. \tag{171}$$

Relation (171) yields the desired conclusion

$$\varrho_\delta \to \varrho \ \text{ a.e. in } S^1 \times \Omega,$$

which completes the proof of Theorem 1.

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