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Directional approach to spatial structure of solutions to the Navier–Stokes equations in the plane

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Abstract

We investigate a steady flow of incompressible fluid in the plane. The motion is governed by the Navier–Stokes equations with prescribed velocity u_∞ at infinity. The main result shows the existence of unique solutions for arbitrary force, provided sufficient largeness of u_∞ . Furthermore a spatial structure of the solution is obtained in comparison with the Oseen flow. A key element of our new approach is based on a setting which treats the direction of the flow as the *time* direction. The analysis is done in the framework of the Fourier transform taken in one (perpendicular) direction and a special choice of function spaces which take into account the inhomogeneous character of the symbol of the Oseen system. From that point of view our technique can be used as an effective tool in examining spatial asymptotics of solutions to other systems modelled by elliptic equations.

Mathematics Subject Classification: 35Q30, 75D05

1. Introduction

We investigate the issue of existence and asymptotic behaviour of solutions to the stationary Navier–Stokes (N–S) equations in the full space:

$$u \cdot \nabla u - \Delta u + \nabla p = F \quad \text{in } \mathbb{R}^2, \quad (1)$$

$$\operatorname{div} u = 0 \quad \text{in } \mathbb{R}^2, \quad (2)$$

where u is a sought velocity, p a corresponding pressure and F is a given external force. This system is coupled with a condition at infinity:

$$u(x) \rightarrow u_\infty \quad \text{as } |x| \rightarrow \infty. \quad (3)$$

This problem is connected with a question of flow around an obstacle (that is when the considered domain Ω is $\Omega = \mathbb{R}^2 \setminus K$, where K is a compact domain)—a problem which has a long history but is far from being completely solved. So far one can present the following main contributions to the problem. The first one is due to Leray [13] where he proves existence of a solution u_L (Leray solution) with finite Dirichlet integral (also referred to as D -solutions)

$$\int_{\Omega} |\nabla u_L|^2 dx < \infty, \quad (4)$$

under the zero flux condition, i.e.

$$\int_{\partial K} u_L \cdot \vec{n} = 0,$$

(which has been relaxed in the recent paper by Russo [17]). The technique used there is to create a sequence of solutions in domains $\Omega_R = \Omega \cap \{x : |x| < R\}$, for which $\int_{\Omega_R} |\nabla u_R|^2 dx < C$, where C is independent of R . After taking a suitable subsequence as $R \rightarrow \infty$ one obtains a solution u_L to (1) and (2). Unfortunately Leray was not able to prove that his solution satisfies (3) or even that it is not trivial. The reason is that the power 2 coincides with a dimension 2 in (4) giving no information about the behaviour of u_L at infinity. Thus a different approach was needed.

Regarding existence to the N–S system with (3), the result of Finn and Smith [5] should be noted. Under smallness assumptions on u_{∞} the authors solve the problem using perturbation approach and the contraction mapping principle. A similar approach was also followed by Galdi [7], where the author used L_p estimates for the Oseen system. Those estimates were also used in [8] by Galdi and Sohr to show that under a suitable integrability condition the solution of the N–S system is physically reasonable (PR) (see definition 1.1). It is important to mention that these results strongly depend on the assumption that $u_{\infty} \neq 0$, that is, in the case where linearization of the N–S equations results in the Oseen and not the Stokes system. Solutions to the former have better properties at infinity due to the presence of the term $u_{,1}$.

The main problem in the case of an exterior domain is to show convergence of the solution to the prescribed velocity field at infinity. It was addressed by Gilbarg and Weinberger in two classical papers [10, 11]. They investigated the behaviour of Leray solutions u_L and later all solutions with finite Dirichlet integral. Their approach uses the structure of the N–S equations, more precisely the equation for the vorticity $\omega = u_{,2} - u_{,1}$:

$$-\Delta \omega + u \cdot \nabla \omega = 0, \quad (5)$$

for which one has a maximum principle for ω (a similar equation is also investigated for the total-head pressure $\Phi = p + \frac{1}{2}|u|^2$). In particular, the authors show that the Leray solution u_L is bounded and there exists a constant vector field \tilde{u}_{∞} such that

$$\int_0^{2\pi} |u_L(r, \theta) - \tilde{u}_{\infty}| d\theta \rightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (6)$$

In the general case (that is not necessarily via Leray's approach) the same property holds if one assumes first that this solution is bounded.

Even though this result was a considerable step forward, many questions remained open, for example if $\tilde{u}_{\infty} = u_{\infty}$ or if the convergence (6) can be improved.

The next important and classical result was due to Amick [1]. In particular the author generalized the results by Gilbarg and Weinberger by showing that any solution with a bounded Dirichlet integral is in $L^{\infty}(\Omega)$. Also the convergence (6) is improved to the pointwise one but only for symmetric flows. In the same paper a solution due to Leray is investigated, in particular it is shown that in the case of symmetric flow this approach results in a non-trivial solution.

It is worth noting that recently Galdi generalized in [9] the pointwise convergence result of Amick using much simpler arguments.

These mentioned results do not address the asymptotic behaviour of solutions, which itself is also an interesting problem. One of the most classical papers regarding the asymptotic profile of solutions is a paper by Finn [6], where the author introduced a class of solutions (to the three-dimensional N–S equations) called PR. The two-dimensional case was covered by Smith [21]. The definition of PR solutions in 2D is as follows:

Definition 1.1. *A solution $u(x)$ of (1) and (2) is said to be PR if there is a positive constant C , a positive number ϵ , and a vector field $u_0 \neq 0$ such that $|u(x) - u_0| < C|u_0 r|^{-\frac{1}{4}-\epsilon}$ as $|u_0 r| \rightarrow \infty$, where $r = |x|$.*

This class is called PR because these solutions have behaviour expected from the physical point of view, for example existence of a wake region behind the obstacle and indeed a faster decay than assumed. In particular Smith showed that all these solutions satisfy the following estimate $|u(x) - u_0| = O(|x|^{-1/2})$, which is sharp in the sense that only very specific flows satisfy $|u(x) - u_0| = o(|x|^{-1/2})$.

It is important to note here the result by Sazonov [19] where he showed that any D -solution which tends to a constant vector field at infinity is a PR solution. For more information on asymptotic behaviour of D -solutions we refer the reader to [18].

The study of the wake region is very important from the point of view of applications, for example in computations of drag and lift. One of the methods is analysis of the flow far from the obstacle using the Trefftz plane approach. It involves investigating the behaviour of the fluid in the plane perpendicular to the wake. As an example of such an approach we should mention [15], where the author treated the direction of the flow as the time direction and analysed vortex sheet evolution—an approach which is in line with our method.

In this paper we consider the case of a full plane flow with a goal to develop new methods to deal with a problem in an exterior domain. Our techniques require non-standard function spaces due to the fact that our analysis is carried through in a Fourier space only in one direction. Of course, one may interpret these results in Sobolev spaces, but we do not address this question here. A hint to discover our space setting one can find in considerations for the evolutionary N–S in [20]; however, this system is homogeneous which is a different situation than in the present paper. Similar techniques were used by Wittwer [22]; however, for a symmetric flow. In [12] the authors omitted this condition; however, these techniques are still very technical. A symmetry condition was also needed in the recent work of Yamazaki [23] for the same problem as ours.

Our result does not rely on any symmetry and is also much simpler than in [22]—we do not consider the system for the rotation of the fluid, but we operate on the velocity vector field itself. Moreover, we give analysis of the flow not only in the half plane, but in the full plane omitting artificial boundary conditions. In such a case one can obtain not only asymptotic behaviour behind the obstacle, but also in front of it.

The existence of the solution is also shown under assumption on the force F in comparison with u_∞ in this sense that we show existence for arbitrary large force F provided u_∞ is large enough. This is an improvement of the result from [23] since we do not impose any symmetry for external force F . Moreover, our method allows us to obtain an exact asymptotic profile of the solution—the result which is typically shown by referring to work of Smith [21] (see [2]). In particular we obtain directly that our solution behaves like $O(|x_1|^{-1/2})$.

We would like to emphasize that the N–S equations admit well-known scaling properties. It is thus possible to consider the original system with $u_\infty = 1$ together with a smallness assumption on F . However, our approach is based on inhomogeneous estimates for the Oseen

system and the condition on F in theorem 1.3 highly depends on β . This makes our result not scale invariant and thanks to this estimate we may observe a direct relation between the magnitude of admissible F and u_∞ .

Although the Oseen system has better properties than the Stokes one, one difference must be emphasized—its inhomogeneity. The term $u_\infty \partial_{x_1} v$ in (10), which sometimes improves information about the sought solution, causes the Oseen system to not be invariant under any rescaling. This follows that there is no natural regularity structure of the system. Hints to find an appropriate function space can be found in our previous papers [14, 16], where the phenomenon of the wake region as a consequence of a parabolic degeneration has been partly explained.

Throughout the paper we use the following Banach spaces:

Definition 1.2. Let $\beta \in \mathbb{R}$. Space \mathcal{X}_β consists of these functions, for which the following norm is finite:

$$\|a\|_{\mathcal{X}_\beta} = \sup_{(t,\xi) \in \mathbb{R}^2} (u_\infty + |t\xi^2|)^\beta |a(t, \xi)|.$$

Function space \mathcal{Y} consists of these functions, for which the following norm is finite:

$$\|b\|_{\mathcal{Y}} = \sup_{(t,\xi) \in \mathbb{R}^2} |t|^{1/2} |b(t, \xi)|.$$

The technique used in this work is to consider the coordinate parallel to u_∞ as time and then use the Fourier transform to obtain a system of ordinary differential equations, which can be solved and analysed to obtain information about its asymptotic behaviour.

Taking (3) into account we may introduce a new vector field $u = u_\infty + v$, for which we have the following system:

$$u_\infty v_{,1} + v \cdot \nabla v - \Delta v - \nabla p = F \quad \text{in } \mathbb{R}^2, \quad (7)$$

$$\operatorname{div} v = 0 \quad \text{in } \mathbb{R}^2, \quad (8)$$

$$v \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (9)$$

Our main result states the following:

Theorem 1.3. Considering system (7)–(9). Given fixed constant $\beta \in (1/4, 1/2)$ and given F and u_∞ such that

$$\|\hat{F}/|\xi|\|_{\mathcal{Y}} \leq C u_\infty^{2\beta-1/2},$$

where the constant C is independent of the data. Then there exists a unique solution (v, p) to (7)–(9) such that $\hat{v} \in \mathcal{X}_\beta$, $\hat{p} \in \mathcal{Y}$ and the following estimate is valid:

$$\|\hat{v}\|_{\mathcal{X}_\beta} + \|\hat{p}\|_{\mathcal{Y}} \leq C u_\infty^{2\beta-1/2}.$$

The proof of this theorem is given in section 3.1. Our approach to show the existence of such a solution is the following: first we split this problem into two auxiliary ones and define a proper mapping, for which the solution to our problem is a fixed point. Then we give suitable estimates to show that the mapping is a contraction. The last step, together with the Banach fixed point theorem, gives us the existence of the solution.

To obtain the asymptotic behaviour of the fluid we consider the Oseen system:

$$u_\infty \bar{v}_{,1} - \Delta \bar{v} - \nabla \bar{p} = F \quad \text{in } \mathbb{R}^2, \quad (10)$$

$$\operatorname{div} \bar{v} = 0 \quad \text{in } \mathbb{R}^2, \quad (11)$$

$$\bar{v} \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (12)$$

Once we have the result of theorem 1.3 we can show the existence of solution (\bar{v}, \bar{p}) to this system in the same class of functions, that is $\bar{v} \in \mathcal{X}_\beta$ and $\bar{p} \in \mathcal{Y}$. We consider this system to show the following theorem describing an asymptotic profile of solutions to the N–S equations and the Oseen system:

Theorem 1.4. *Considering the solution (v, p) to the system (7)–(9) and the solution (\bar{v}, \bar{p}) to the system (10)–(12), both with the same force F , β and u_∞ . Then the following statement is true:*

$$\hat{v} - \hat{\bar{v}} \in \mathcal{X}_{2\beta}, \tag{13}$$

that is the difference $\hat{v} - \hat{\bar{v}}$ has a better decay at infinity than v and \bar{v} themselves.

The paper is organized as follows: in the next section we introduce two auxiliary problems and derive formulae for their solutions by means of the Fourier transform. Then we provide suitable estimates, which play a fundamental role in the proof of existence of a solution to the main problem (7)–(9). Section 4 is devoted to technical lemmas, which play a fundamental role in the mentioned estimates. We choose this ‘nonlocal’ form of presentation of proofs hoping it is more convenient for the reader. For the same reason we include calculations of all cases even though sometimes they present similarities.

2. Auxiliary systems

This section is devoted to a reformulation of the original N–S equations. The Stokes system cannot be recognized as a purely parabolic system; however, we are able to represent the pressure in terms of the velocity by a nonlocal operator. Then our problem is reduced to two equations.

The first one is for the *pressure*. Taking div from (7) we have

$$\Delta p = \text{div } F - \text{div } \text{div } (v \otimes v), \tag{14}$$

since $\text{div } v = 0$ and $v \cdot \nabla v = \text{div } (v \otimes v)$. We introduce function G as the right-hand side of (14), i.e.

$$G = \text{div } F - \text{div } \text{div } (v \otimes v).$$

The second one is for the *velocity*. We transform (7) into the following system:

$$u_\infty v_{,1} - \Delta v = \nabla p - \text{div } (v \otimes v) \quad \text{in } \mathbb{R}^2, \tag{15}$$

$$v \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \tag{16}$$

As earlier—we introduce function H as the right-hand side of (15), i.e.

$$H = F + \nabla p - \text{div } (v \otimes v).$$

Our mapping used in fixed point theorem will be considered as follows: we start with v in a proper Banach space, then we calculate pressure p from (14). Having v and p we may use (15) to calculate new \tilde{v} . We show that this mapping maps a suitable ball into itself ensuring that there exists v for which $\tilde{v} = v$.

In the following sections we deal with our two problems (14) and (15) using the Fourier transform in x_2 space variable and transforming them into ordinary differential equations. This point of view at the Stokes system via equations (14)–(15) was effectively used in [3, 4] for the issue of existence in the maximal regularity regime. A similar procedure was used in [22]; however, not for the velocity directly, but for the rotation of the fluid.

2.1. Derivation of the solution

Let us focus on the system for the pressure (14) first. Taking Fourier transform in x_2 variable and denoting the new variable as ξ and x_1 as t we obtain

$$\hat{p} - \xi^2 \hat{p} = \hat{G}. \quad (17)$$

For simplicity we omit the hat $\hat{\cdot}$.

Introducing $w = \dot{p}$ we can rewrite (17) as

$$\begin{bmatrix} \dot{p} \\ w \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \xi^2 & 0 \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix}. \quad (18)$$

Eigenvalues and eigenvectors can be easily computed: $\lambda_1 = -|\xi|$, $\lambda_2 = |\xi|$ and $\varphi_1 = [-1/|\xi|, 1]$, $\varphi_2 = [1/|\xi|, 1]$. Introducing matrix $P = [\varphi_1, \varphi_2]$, i.e.

$$P = \begin{bmatrix} -\frac{1}{|\xi|} & \frac{1}{|\xi|} \\ 1 & 1 \end{bmatrix},$$

and new variables $[U_1, U_2] = P^{-1}[p, w]$ we rewrite (18) as

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} -|\xi| & 0 \\ 0 & |\xi| \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + P^{-1} \begin{bmatrix} 0 \\ G \end{bmatrix}.$$

A solution to this system is

$$U_1(t, \xi) = \frac{1}{2} \int_{-\infty}^t e^{|\xi|(s-t)} G(s) ds,$$

$$U_2(t, \xi) = -\frac{1}{2} \int_t^{\infty} e^{-|\xi|(s-t)} G(s) ds$$

which gives us

$$p(t, \xi) = -\frac{1}{2} \frac{1}{|\xi|} \left(\int_{-\infty}^t e^{-|\xi||t-s|} G(s, \xi) ds + \int_t^{\infty} e^{-|\xi||t-s|} G(s, \xi) ds \right) \quad (19)$$

$$= -\frac{1}{2|\xi|} \int_{-\infty}^{+\infty} e^{-|\xi||t-s|} G(s, \xi) ds. \quad (20)$$

For the system for the velocity (15) we may present a similar approach. Using Fourier transform in x_2 variable for the system (15) and taking the right-hand side to be δ we obtain the equation for the fundamental solution:

$$u_\infty \dot{V}(t, \xi) - \ddot{V}(t, \xi) + \xi^2 V(t, \xi) = \delta. \quad (21)$$

Now, assuming without loss of generality, that $u_\infty > 0$ and denoting

$$\Delta = \sqrt{u_\infty^2 + 4\xi^2}, \quad \lambda_1 = \frac{1}{2}(u_\infty - \sqrt{u_\infty^2 + 4\xi^2}), \quad \lambda_2 = \frac{1}{2}(u_\infty + \sqrt{u_\infty^2 + 4\xi^2})$$

we have $\lambda_1 \leq 0$, $\lambda_2 > 0$ and we may write the fundamental solution V as follows

$$V(t, \xi) = \frac{1}{\Delta} \begin{cases} e^{\lambda_2 t} & \text{for } t < 0, \\ e^{\lambda_1 t} & \text{for } t > 0. \end{cases} \quad (22)$$

Now if $v(t, \xi)$ is a solution to (15) with $H(t, \xi)$ as the right-hand side we obtain

$$v(t, \xi) = (V * H)(t, \xi) = \int_{-\infty}^t V(t-s, \xi) H(s) ds + \int_t^{\infty} V(t-s, \xi) H(s) ds$$

$$= -\frac{1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} H(s, \xi) ds + \frac{1}{\Delta} \int_t^{\infty} e^{-\lambda_2(s-t)} H(s, \xi) ds. \quad (23)$$

Let us now focus on detailed information about $G(s)$ and $H(s)$. We start with the former. Since

$$G(s) = \mathcal{F}_{x_2}(\operatorname{div} F - \operatorname{div} \operatorname{div} (v \otimes v))$$

we see that

$$G(s, \xi) = \partial_s \widehat{F}_1(s, \xi) + i\xi \widehat{F}_2(s, \xi) - \partial_s^2 \widehat{(v_1^2)} + 2\partial_s i\xi \widehat{(v_1 v_2)} - \xi^2 \widehat{(v_2^2)}.$$

First we integrate by parts the term from (19) corresponding to $\partial_s^2 \widehat{(v_1^2)}$, namely

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|\xi||t-s|} \partial_s^2 \widehat{(v_1^2)} \, ds &= - \int_{-\infty}^t |\xi| e^{-|\xi|(t-s)} \partial_s \widehat{(v_1^2)} \, ds + \partial_s \widehat{(v_1^2)}(t) \\ &\quad - \int_t^{\infty} (-|\xi|) e^{-|\xi|(s-t)} \partial_s \widehat{(v_1^2)} \, ds - \partial_s \widehat{(v_1^2)}(t) \\ &= \int_{-\infty}^t (|\xi|^2) e^{-|\xi|(t-s)} \widehat{(v_1^2)} \, ds - |\xi| \widehat{(v_1^2)}(t) \\ &\quad - \int_t^{\infty} (-|\xi|^2) e^{-|\xi|(s-t)} \widehat{(v_1^2)} \, ds - |\xi| \widehat{(v_1^2)}(t) \\ &= \int_{-\infty}^{\infty} |\xi|^2 e^{-|\xi||t-s|} \widehat{(v_1^2)} \, ds - 2|\xi| \widehat{(v_1^2)}(t), \end{aligned}$$

hence

$$\frac{1}{2|\xi|} \int_{-\infty}^{\infty} e^{-|\xi||t-s|} \partial_s^2 \widehat{(v_1^2)} \, ds = \frac{1}{2} \int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|} \widehat{(v_1^2)} \, ds - \widehat{(v_1^2)}(t). \tag{24}$$

In the same manner, terms from (19) corresponding to $2i\xi \partial_s \widehat{(v_1 v_2)}$, $\xi^2 \widehat{(v_2^2)}$ are of the same structure as the first term on the right-hand side of (24). Similarly $\partial_s \widehat{F}_1$ and $i\xi \widehat{F}_2$ can be considered as one term. Summarizing, p can be presented as

$$\begin{aligned} p(t, \xi) &= -\frac{1}{2} \int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|} \left(\sum_{ij} c_{ij} \widehat{(v_i v_j)} \right) \, ds \\ &\quad - \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\xi||t-s|} \left(\sum_i b_i \widehat{F}_i \right) \, ds + \widehat{(v_1)^2}(t), \end{aligned} \tag{25}$$

for some constants c_{ij} and b_i such that $|c_{ij}| = |b_i| = 1$ (which is irrelevant for our purposes).

The same calculations can be repeated for v and H , i.e. for (23) to obtain that

$$\begin{aligned} v_1(t, \xi) &= \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} (p(\xi, s) - v_1^2) \, ds - \frac{\lambda_2}{\Delta} \int_t^{\infty} e^{-\lambda_2(s-t)} (p(\xi, s) + v_1^2) \, ds \\ &\quad + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \left(\widehat{(v_1 v_2)} - \frac{\widehat{F}_1}{i\xi} \right) \, ds + \frac{i\xi}{\Delta} \int_t^{\infty} e^{-\lambda_2(s-t)} \left(\widehat{(v_1 v_2)} + \frac{\widehat{F}_1}{i\xi} \right) \, ds, \end{aligned} \tag{26}$$

and

$$\begin{aligned} v_2(t, \xi) &= \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \widehat{(v_1 v_2)} \, ds - \frac{\lambda_2}{\Delta} \int_t^{\infty} e^{-\lambda_2(s-t)} \widehat{(v_1 v_2)} \, ds \\ &\quad + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \left((p - v_2^2) - \frac{\widehat{F}_2}{i\xi} \right) \, ds + \frac{i\xi}{\Delta} \int_t^{\infty} e^{-\lambda_2(s-t)} \left((p - v_2^2) + \frac{\widehat{F}_2}{i\xi} \right) \, ds. \end{aligned}$$

2.2. Main estimates

We are now in a position to formulate lemmas which play a fundamental role in showing that our mapping is a contraction.

Lemma 2.1. *Let $\widehat{v} \in \mathcal{X}_\beta$, such that $\|\widehat{v}\|_{\mathcal{X}_\beta} \leq M$, and $\widehat{F}/\xi \in \mathcal{Y}$, such that $\|\widehat{F}/\xi\|_{\mathcal{Y}} \leq N_F$. Given p in the form*

$$\begin{aligned} \widehat{p}(t, \xi) = & -\frac{1}{2} \int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|} \left(\sum_{ij} c_{ij} \widehat{(v_i v_j)} \right) ds \\ & -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|\xi||t-s|} \left(\sum_i b_i \widehat{F}_i \right) ds + \widehat{(v_1)^2}(t). \end{aligned} \quad (27)$$

Then $\widehat{p} \in \mathcal{Y}$ and

$$\|\widehat{p}\|_{\mathcal{Y}} \leq C u_\infty^{-2\beta+1/2} M^2 + N_F,$$

for some constant C independent of u_∞ and F .

Proof. To prove this lemma we use results from section 4, that is, recalling that $|c_{ij}| = |b_i| = 1$ we estimate \widehat{p} as follows:

$$\begin{aligned} |\widehat{p}(t, \xi)| \leq & \frac{1}{2} \int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|} \left(\sum_{ij} |\widehat{(v_i v_j)}| \right) ds \\ & + \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\xi||t-s|} \left(\sum_i |\widehat{F}_i| \right) ds + |\widehat{(v_1)^2}|(t). \end{aligned} \quad (28)$$

Since $\widehat{v} \in \mathcal{X}_\beta$ we use estimate (43) from lemma 4.3 to obtain that

$$\|\widehat{(v_i v_j)}\|_{\mathcal{Y}} = \|\widehat{v}_i * \widehat{v}_j\|_{\mathcal{Y}} \leq C u_\infty^{-2\beta+1/2} M^2.$$

It estimates the last term on the right-hand side of (28), but also, with a help of estimate (48) from lemma 4.4, gives us the following estimate:

$$\left\| \frac{1}{2} \int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|} \left(\sum_{ij} |\widehat{(v_i v_j)}| \right) ds \right\|_{\mathcal{Y}} \leq 4C u_\infty^{-2\beta+1/2} M^2.$$

To finish the estimate of \widehat{p} we present the remaining term and estimate it as follows

$$\begin{aligned} & \left\| \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\xi||t-s|} \left(\sum_i |\widehat{F}_i| \right) ds \right\|_{\mathcal{Y}} \\ & \leq \left\| \frac{1}{2} \int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|} \left(\sum_i |\widehat{F}_i|/|\xi| \right) ds \right\|_{\mathcal{Y}} \leq \|\widehat{F}_i/|\xi|\|_{\mathcal{Y}} = N_F, \end{aligned}$$

where again we used (48). □

The second lemma we need to prove is the following:

Lemma 2.2. *Let $\widehat{w} \in \mathcal{X}_\beta$, such that $\|\widehat{w}\|_{\mathcal{X}_\beta} \leq M$, and $\widehat{p} \in \mathcal{Y}$, such that $\|\widehat{p}\|_{\mathcal{Y}} \leq N$. Then, for the following terms \widehat{v}_1 and \widehat{v}_2*

$$\begin{aligned} \widehat{v}_1(t, \xi) = & \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} (p(\xi, s) - w_1^2) ds - \frac{\lambda_2}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} (p(\xi, s) + w_1^2) ds \\ & + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \left(\widehat{(w_1 w_2)} - \frac{\widehat{F}_1}{i\xi} \right) ds + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_2(s-t)} \left(\widehat{(w_1 w_2)} + \frac{\widehat{F}_1}{i\xi} \right) ds, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \widehat{v}_2(t, \xi) = & \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \widehat{(w_1 w_2)} ds - \frac{\lambda_2}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} \widehat{(w_1 w_2)} ds \\ & + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \left(\widehat{(p - w_2^2)} - \frac{\widehat{F}_2}{i\xi} \right) ds \\ & + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_2(s-t)} \left(\widehat{(p - w_2^2)} + \frac{\widehat{F}_2}{i\xi} \right) ds. \end{aligned} \tag{30}$$

the following estimate is valid:

$$\|\widehat{v}_i\|_{\mathcal{X}_\beta} \leq C(\|\widehat{p}\|_{\mathcal{Y}} + u_\infty^{-2\beta+1/2} \|\widehat{w}\|_{\mathcal{X}_\beta}^2 + \|\widehat{F}/\xi\|_{\mathcal{Y}}) = C(N + u_\infty^{-2\beta+1/2} M^2 + \|\widehat{F}/\xi\|_{\mathcal{Y}}). \tag{31}$$

Proof. The proof is analogous to the proof of the previous lemma. First we note that

$$\|\widehat{(w_i w_j)}\|_{\mathcal{Y}} = \|\widehat{w}_i * \widehat{w}_j\|_{\mathcal{Y}} \leq C u_\infty^{-2\beta+1/2} M^2,$$

like also $\|p\|_{\mathcal{Y}} \leq N$.

Then, we find that in the form of v_i , i.e. in (29) and (30), these integrals in a sequence are, respectively, in the form of integrals \widehat{B} , \widehat{D} , \widehat{A} , \widehat{C} from lemma 4.5. Thus, applying this lemma we obtain

$$\|v_i\|_{\mathcal{X}_\beta} \leq C(\|p\|_{\mathcal{Y}} + \|\widehat{(w_i w_j)}\|_{\mathcal{Y}}) \leq C(N + u_\infty^{-2\beta+1/2} M^2).$$

3. Main results

In this section we gather our main result on the existence of the solution to our problem.

3.1. The proof of theorem 1.3.

In this section we would like to prove the existence of a solution to our problem. We use the standard approach, namely Banach’s fixed point theorem for a contraction mapping.

We recall that our mapping is defined as follows: having a vector field w and force F we calculate the pressure p using formula (27), then we calculate a vector field v using p , w and formulae (29)–(30). Let us denote this mapping as $G : \mathcal{X}_\beta \rightarrow \mathcal{X}_\beta$. First we would like to show that there exists a constant ϵ such that for sufficiently small $\|F/|\xi|\|_{\mathcal{Y}}$ the mapping G maps a ball of radius ϵ in the space \mathcal{X}_β into itself, namely

$$G(\mathcal{B}_\alpha(\epsilon)) \subset \mathcal{B}_\alpha(\epsilon). \tag{32}$$

We take $w \in \mathcal{X}_\beta$ such that $\|w\|_{\mathcal{X}_\beta} \leq M$ and F such that $\|\widehat{F}/|\xi|\|_{\mathcal{Y}} \leq N_F$. From lemma 2.1 we have

$$\|p\|_{\mathcal{Y}} \leq C u_\infty^{-2\beta+1/2} M^2 + N_F.$$

We may now use lemma 2.2 to obtain

$$\|v\|_{\mathcal{X}_\beta} \leq C(\|p\|_{\mathcal{Y}} + \|\hat{F}/\xi\|_{\mathcal{Y}} + u_\infty^{-2\beta+1/2}M^2) \leq C(u_\infty^{-2\beta+1/2}M^2 + N_F).$$

To find ϵ in (32) we have to solve an inequality

$$C(u_\infty^{-2\beta+1/2}M^2 + N_F) \leq M.$$

From this inequality we can read the following constraints:

$$N_F \leq \frac{1}{4C^2}u_\infty^{2\beta-1/2}, \quad \epsilon = \frac{1}{2C}u_\infty^{2\beta-1/2}. \quad (33)$$

For this choice of N_F and ϵ one gets G to map a ball $B_\alpha(\epsilon)$ into itself.

In a similar way we show, that on a smaller ball the mapping G is a contraction. For $w_1, w_2 \in \mathcal{B}_\beta(\epsilon/2)$ and corresponding v_1, v_2 we have

$$\|v_1 - v_2\|_{\mathcal{X}_\beta} \leq C u_\infty^{-2\beta+1/2} \|w_1 - w_2\|_{\mathcal{X}_\beta}^2 \leq \gamma \|w_1 - w_2\|_{\mathcal{X}_\beta},$$

where $\gamma < 1$. Using the Banach fixed point theorem we obtain, that there exists a vector field $v \in \mathcal{B}_\beta(\epsilon/2)$, such that $v = G(v)$.

3.2. The proof of the theorem 1.4

In order to investigate the asymptotic profile of solutions to the N–S equations we observe that the same existence argument as was presented in the preceding section can be adapted for the Oseen system (10)–(12) to obtain the solution (\bar{v}, \bar{p}) such that $\bar{v} \in \mathcal{X}_\beta$ and $\bar{p} \in \mathcal{Y}$.

This solution satisfies the following identities (we focus only on \bar{v}_1 and \bar{p}):

$$\begin{aligned} \bar{v}_1(t, \xi) &= \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \widehat{\bar{p}(\xi, s)} ds - \frac{\lambda_2}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} \widehat{\bar{p}(\xi, s)} ds \\ &\quad + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \left(-\frac{\hat{F}_1}{i\xi} \right) ds + \frac{i\xi}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} \frac{\hat{F}_1}{i\xi} ds, \\ \widehat{\bar{p}}(t, \xi) &= -\frac{1}{2} \int_{-\infty}^\infty e^{-|\xi||t-s|} \left(\sum_i b_i \hat{F}_i \right) ds. \end{aligned}$$

For the readability of the paper we recall identities for the solution (v, p) of the N–S equations:

$$\begin{aligned} v_1(t, \xi) &= \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} (p(\xi, s) - v_1^2) ds - \frac{\lambda_2}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} (p(\xi, s) + v_1^2) ds \\ &\quad + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} \left(\widehat{(v_1 v_2)} - \frac{\hat{F}_1}{i\xi} \right) ds + \frac{i\xi}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} \left(\widehat{(v_1 v_2)} + \frac{\hat{F}_1}{i\xi} \right) ds, \end{aligned} \quad (34)$$

$$\begin{aligned} \widehat{p}(t, \xi) &= -\frac{1}{2} \int_{-\infty}^\infty |\xi| e^{-|\xi||t-s|} \left(\sum_{ij} c_{ij} \widehat{(v_i v_j)} \right) ds \\ &\quad - \frac{1}{2} \int_{-\infty}^\infty e^{-|\xi||t-s|} \left(\sum_i b_i \hat{F}_i \right) ds + \widehat{(v_1)^2}(t). \end{aligned} \quad (35)$$

We are interested in an estimate of the difference $v_1 - \bar{v}_1$. For that we first find:

$$p(t, \xi) - \bar{p}(t, \xi) = -\frac{1}{2} \int_{-\infty}^\infty |\xi| e^{-|\xi||t-s|} \left(\sum_{ij} c_{ij} \widehat{(v_i v_j)} \right) ds + \widehat{(v_1)^2}(t). \quad (36)$$

Using our assumptions on v and using lemma 4.3 we see that all convolutions appearing in (36) can be estimated in a pointwise manner by the term $|t|^{-1/2}(u_\infty + t\xi^2)^{-2\beta+1/2}$. Then one may use lemma 4.4 to obtain the following pointwise estimate for the difference of pressures:

$$|p(t, \xi) - \bar{p}(t, \xi)| \leq C|t|^{-1/2}(u_\infty + |t\xi^2|)^{-2\beta+1/2}.$$

To estimate the difference of velocities first we denote $\pi(t, \xi) = p(t, \xi) - \bar{p}(t, \xi)$ and write this difference as follows:

$$\begin{aligned} v_1(t, \xi) - \bar{v}_1(t, \xi) &= \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} (\widehat{\pi(\xi, s) - v_1^2}) \, ds - \frac{\lambda_2}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} (\widehat{\pi(\xi, s) + v_1^2}) \, ds \\ &\quad + \frac{i\xi}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} (\widehat{v_1 v_2}) \, ds + \frac{i\xi}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} (\widehat{v_1 v_2}) \, ds. \end{aligned}$$

Again, using results of lemma 4.3 to estimate convolutions in the above equations we combine them with the previous estimate of $\pi(t, \xi)$ and we write

$$|\widehat{\pi}(t, \xi)| + |\widehat{v_1^2}(t, \xi)| + |\widehat{v_1 v_2}(t, \xi)| \leq C|t|^{-1/2}(u_\infty + |t\xi|^2)^{-2\beta+\frac{1}{2}}.$$

Now we are in a position to use lemma 4.6 and complete the proof of theorem 1.4.

3.3. Note on the exterior problem

Results presented in this paper address the flow in the full plane. A natural question is how they can be adapted to the exterior flow problem. It does not seem to be an easy task but one of the main ideas which it is natural to consider is to try to ‘represent’ the obstacle as a force F . This idea has been followed in other problems and in our case the difficulty might be if such a representation would result in an admissible F , e.g. whether $\widehat{F}/|\xi|$ would be finite or not.

It does seem to be possible, however, to adapt those methods to obtain a representation formula for the solution in front and behind the obstacle, which itself could be useful. It would require considering the problem in the halfspace $x_1 \geq 1$ (or $x_1 \leq -1$, respectively) with suitable boundary conditions on the line $x_1 = 1$ ($x_1 = -1$). A similar approach was considered in [22].

4. Main lemmas

The main auxiliary lemma, which will be used many times, is the following:

Lemma 4.1. *Given $\theta > 0$. Then the following estimate is valid:*

$$\int_0^\infty e^{-u}|u - \theta|^{-1/2} \, du \leq (1 + \theta)^{-1/2}. \tag{37}$$

Proof. Let $I = \int_0^\infty e^{-u}|u - \theta|^{-1/2} \, du$. We split this integral into three parts:

$$I = I_1 + I_2 + I_3 = \int_0^{\theta/2} + \int_{\theta/2}^{2\theta} + \int_{2\theta}^\infty,$$

and estimate them separately. Let us focus on I_1 :

$$\begin{aligned} I_1 &= \int_0^{\theta/2} e^{-u}|u - \theta|^{-1/2} \, du \leq 2 \int_0^{\theta/2} e^{-u}|\theta|^{-1/2} \, du \\ &= 2|\theta|^{-1/2}(1 - e^{-\theta}) \leq (1 + \theta)^{-1/2}, \end{aligned}$$

since $(1 - e^{-\theta}) \sim \theta/(1 + \theta)$.

For I_2 we proceed as follows:

$$I_2 = \int_{\theta/2}^{2\theta} e^{-u} |u - \theta|^{-1/2} du \leq C \int_0^{2\theta} e^{-\theta} |u|^{-1/2} du \leq C e^{-\theta} |\theta|^{1/2} \leq \frac{C}{(1+\theta)^{1/2}},$$

which is a desired estimate.

Integral I_3 we estimate for small and large θ ($\theta < 1$ and $\theta > 1$, respectively). In the first case

$$I_3 = \int_{\theta}^1 + \int_1^{\infty} = (1 - \theta^{1/2}) + C \leq C \leq (1 + \theta)^{-1/2}$$

while in the second one we have

$$I_3 \leq \int_{\theta}^{\infty} e^{-u} u^{-1/2} \leq \int_{\theta}^{\infty} e^{-u} |\theta|^{-1/2} \leq (1 + \theta)^{-1/2}. \quad \square$$

We also consider the following modification of the previous lemma:

Lemma 4.2. *Given $\theta > 0$. Then the following estimate is valid:*

$$\int_0^{\infty} e^{-u} |u - \theta|^{-1/2} (1 + |u - \theta|)^{-2\beta+1/2} du \leq C(1 + |\theta|)^{-2\beta}. \quad (38)$$

Proof. Let $I = \int_0^{\infty} e^{-u} |u - \theta|^{-1/2} (1 + |u - \theta|)^{-2\beta+1/2} du$. We split this integral into three parts:

$$I = I_1 + I_2 + I_3 = \int_0^{\theta/2} + \int_{\theta/2}^{2\theta} + \int_{2\theta}^{\infty},$$

and estimate them separately. Let us focus on I_1 :

$$\begin{aligned} I_1 &= \int_0^{\theta/2} e^{-u} |u - \theta|^{-1/2} (1 + |u - \theta|)^{-2\beta+1/2} du \leq 2 \int_0^{\theta/2} e^{-u} |\theta|^{-1/2} (1 + |\theta|)^{-2\beta+1/2} du \\ &= 2|\theta|^{-1/2} (1 + |\theta|)^{-2\beta+1/2} (1 - e^{-\theta}) \leq (1 + |\theta|)^{-2\beta}, \end{aligned}$$

since $(1 - e^{-\theta}) \sim \theta/(1 + \theta)$.

For I_2 we proceed as follows:

$$\begin{aligned} I_2 &= \int_{\theta/2}^{2\theta} e^{-u} |u - \theta|^{-1/2} (1 + |u - \theta|)^{-2\beta+1/2} du \\ &\leq C e^{-\theta} \int_0^{2\theta} |u|^{-1/2} (1 + |u|)^{-2\beta+1/2} du =: \tilde{I}_2. \end{aligned}$$

Now we distinguish two cases: first $\theta \leq 1$. Then we have:

$$\tilde{I}_2 \leq C e^{-\theta} |\theta|^{1/2} \leq C(1 + |\theta|)^{-2\beta}. \quad (39)$$

In the second case $\theta > 1$ we simply have

$$\tilde{I}_2 \leq e^{-\theta} \int_0^{\theta} |u|^{-1/2} (1 + |u|)^{-2\beta+1/2} du \leq C(\beta) p(\theta) e^{-\theta} \leq C(\beta) (1 + \theta)^{-2\beta}, \quad (40)$$

where $p(\theta)$ is a suitable polynomial with respect to θ . Integral I_3 we first rewrite as $I_3 = e^{-\theta} \int_{\theta}^{\infty} e^{-u} |u|^{-1/2} (1 + |u|)^{-2\beta+1/2} du$. Then since $\beta > 1/4$ we have

$$I_3 \leq e^{-\theta} (1 + \theta)^{-2\beta+1/2} \int_{\theta}^{\infty} e^{-u} |u|^{-1/2} \leq C(1 + \theta)^{-2\beta}, \quad (41)$$

which is the desired estimate. \square

Lemma 4.3. *Let $a, b \in \mathcal{X}_\beta$. Then for $\beta < 1/2$ the following estimate is valid:*

$$(a * b)(t, \xi) \leq C \|a\|_{\mathcal{X}_\beta} \|b\|_{\mathcal{X}_\beta} |t|^{-1/2} (u_\infty + |t\xi^2|)^{-2\beta + \frac{1}{2}}. \tag{42}$$

In particular for $\beta > 1/4$ one easily gets a weaker estimate:

$$\|a * b\|_Y \leq u_\infty^{-2\beta + 1/2} \|a\|_{\mathcal{X}_\beta} \|b\|_{\mathcal{X}_\beta}. \tag{43}$$

Proof. Without any loss we may assume that $\|a\|_{\mathcal{X}_\beta} = \|b\|_{\mathcal{X}_\beta} = 1$. Therefore

$$|a(t, \xi)| \leq (u_\infty + |t\xi^2|)^{-\beta} \quad \text{for all } (t, \xi) \in \mathbb{R}^2,$$

and we may write

$$|(a * b)(t, \xi)| \leq \int_{\mathbb{R}} \frac{1}{(u_\infty + |t||y - \xi|^2)^\beta} \frac{1}{(u_\infty + |ty^2|)^\beta} dy. \tag{44}$$

By I we denote the right-hand side of (44). Using the substitution $u = |t|^{1/2}y$ we have

$$I = |t|^{-1/2} \int_{\mathbb{R}} \frac{1}{(u_\infty + |u - |\xi||t|^{1/2}|^2)^\beta (u_\infty + |u|^2)^\beta} du. \tag{45}$$

Because of the presence of the term $|t|^{-1/2}$ it is sufficient to estimate the integral in (45).

We split domain \mathbb{R} in the integral into three parts:

- $A_1 = \{u : |u| \leq \frac{1}{2}|t|^{1/2}|\xi|\},$
- $A_2 = \{u : \frac{1}{2}|t|^{1/2}|\xi| < |u| \leq 2|t|^{1/2}|\xi|\},$
- $A_3 = \{u : 2|t|^{1/2}|\xi| < |u|\},$

and we denote by J_1, J_2, J_3 corresponding integrals. First we estimate J_1 :

$$\begin{aligned} J_1 &= \int_{A_1} \frac{1}{(u_\infty + |u - |\xi||t|^{1/2}|^2)^\beta (u_\infty + |u|^2)^\beta} du \\ &\leq \int_{A_1} \frac{1}{(u_\infty + |\xi|^2|t|/4)^\beta (u_\infty + |u|^2)^\beta} du \\ &\leq (u_\infty + |t||\xi|^2)^{-\beta} \int_{A_1} \frac{1}{(\sqrt{u_\infty} + |u|)^{2\beta}} du \\ &= (u_\infty + |t||\xi|^2)^{-\beta} (\sqrt{u_\infty} + |u|)^{-2\beta + 1} \Big|_0^{t^{1/2}|\xi|/2}. \end{aligned}$$

We use the assumption that $\beta < 1/2$ to obtain an estimate:

$$J_1 \leq (u_\infty + |t\xi|^2)^{-2\beta + 1/2}.$$

For J_2 we proceed in a similar way:

$$\begin{aligned} J_2 &= \int_{A_2} \frac{1}{(u_\infty + |u - |\xi||t|^{1/2}|^2)^\beta (u_\infty + |u|^2)^\beta} du \\ &\leq C \int_{A_2} \frac{1}{(u_\infty + |u - |\xi||t|^{1/2}|^2)^\beta (\sqrt{u_\infty} + |t|^{1/2}|\xi|)^{2\beta}} du, \end{aligned}$$

using a substitution $t^{1/2}\xi - u = y$ we can write

$$\begin{aligned} J_2 &\leq 2 \int_{-t^{1/2}|\xi|}^{\frac{1}{2}t^{1/2}|\xi|} \frac{1}{(\sqrt{u_\infty} + |y|)^{2\beta} (\sqrt{u_\infty} + |t|^{1/2}|\xi|)^{2\beta}} dy \\ &\leq C (\sqrt{u_\infty} + |t|^{1/2}|\xi|)^{-4\beta + 1}, \end{aligned}$$

which is the same estimate as for J_1 . Finally we estimate J_3 :

$$J_3 = \int_{A_3} \frac{1}{(u_\infty + |u - |\xi||t|^{1/2}|^2)^\beta (u_\infty + |u|^2)^\beta} du$$

$$\leq \int_{|t|^{1/2}|\xi|}^\infty \frac{1}{(\sqrt{u_\infty} + |u|)^{4\beta}} du = (\sqrt{u_\infty} + |t|^{1/2}|\xi|)^{-4\beta+1}.$$

Gathering estimates for J_1, J_2 and J_3 the proof of lemma 4.3 is completed. □

In order to estimate the difference of pressures we also need the following lemma:

Lemma 4.4. *Let f satisfy the following inequality:*

$$|f(t, \xi)| \leq |t|^{-1/2}(u_\infty + |t\xi|^2)^{-2\beta+\frac{1}{2}}. \tag{46}$$

Then for $1/2 > \beta > 1/4$ and \tilde{I} defined as follows

$$\tilde{I} := \xi \int_{\mathbb{R}} e^{-|\xi||t-y|} f(y, \xi) dy$$

the following estimate holds:

$$|\tilde{I}(t, \xi)| \leq C|t|^{-1/2}(u_\infty + |t\xi|^2)^{-2\beta+\frac{1}{2}}, \tag{47}$$

where C does not depend on f .

Moreover if $f \in \mathcal{Y}$ then the following estimate is valid

$$\|\tilde{I}(t, \xi)\|_{\mathcal{Y}} \leq C u_\infty^{-2\beta+\frac{1}{2}} \|f\|_{\mathcal{Y}}. \tag{48}$$

Proof. It is straightforward that we may focus on estimate of the following integral:

$$I = |\xi| \int_{\mathbb{R}} e^{-|\xi||t-y|} |y|^{-1/2} (u_\infty + |y\xi|^2)^{-2\beta+\frac{1}{2}} dy.$$

First we use a substitution $u = \xi y$ to obtain

$$I = |\xi|^{1/2} \int_{\mathbb{R}} e^{-|\xi t|-|u|} u^{-1/2} (u_\infty + |u\xi|)^{-2\beta+1/2} du. \tag{49}$$

We split this integral into three parts: $A_1 = \{u : |u| < |\xi||t|/2\}$, $A_2 = \{u : |\xi||t|/2 \leq u \leq 2|\xi||t|\}$, $A_3 = \{u : 2|\xi||t| \leq u\}$ and we introduce I_1, I_2 and I_3 as the corresponding parts of integral I from (49).

In what follows we denote $-2\beta + 1/2 = -\gamma$. Then condition $\beta < 1/2$ corresponds to $\gamma < 1/2$, while $\beta > 1/4$ implies $\gamma > 0$. For I_1 we distinguish two cases: first for $|t\xi^2| < u_\infty$, and then for $|t\xi^2| \geq u_\infty$. In the former we have

$$I_1 \leq \int_0^{|\xi||t|/2} e^{-|\xi||t|/2} |u|^{-1/2} |\xi|^{1/2} (u_\infty + u|\xi|)^{-\gamma} du \leq \int_0^{|\xi||t|/2} e^{-|\xi||t|/2} |u|^{-1/2} |\xi|^{1/2} (u_\infty)^{-\gamma} du$$

$$= |t|^{-1/2} (u_\infty)^{-\gamma} e^{-|\xi||t|/2} |t\xi| \leq |t|^{-1/2} (u_\infty + |t\xi^2|)^{-\gamma}.$$

which is the desired estimate.

In the later case, that is for $|t\xi^2| \geq u_\infty$ one estimates integral I_1 as follows:

$$I_1 \leq \int_0^{|\xi||t|/2} e^{-|\xi||t|/2} |u|^{-1/2} |\xi|^{1/2} (u_\infty + u|\xi|)^{-\gamma} du \leq \int_0^{|\xi||t|/2} e^{-|\xi||t|/2} |u|^{-1/2} |\xi|^{1/2} (u|\xi|)^{-\gamma} du$$

$$= e^{-|\xi||t|/2} |t\xi|^{-1/2-\gamma+1} |\xi|^{1/2-\gamma}$$

$$= e^{-|\xi||t|/2} |t\xi^2|^{-\gamma} |t\xi| |t|^{-1/2} \leq |t|^{-1/2} (u_\infty + |t\xi^2|)^{-\gamma},$$

where we used the assumption that $\gamma < 1/2$. This completes the estimate for I_1 for the first part of the lemma.

For integral I_2 we proceed as follows:

$$\begin{aligned} I_2 &\leq |\xi|^{1/2} \int_{|\xi t|/2}^{2|\xi t|} e^{-|\xi t| - u} u^{-1/2} (u_\infty + |u\xi|)^{-2\beta+1/2} du \\ &\leq |\xi|^{1/2} |\xi t| e^{-|\xi t|} |\xi t|^{-1/2} (u_\infty + |t\xi^2|)^{-2\beta+1/2} \\ &\leq |t|^{-1/2} \frac{|\xi t|}{(1 + |\xi t|)} (u_\infty + |t\xi^2|)^{-2\beta+1/2}. \end{aligned}$$

For I_3 we must distinguish two cases: one if $|\xi t| < 1$ and the other one for $|\xi t| \geq 1$. For the first case we have

$$\begin{aligned} I_3 &\leq (u_\infty + |t\xi^2|)^{-2\beta+1/2} |\xi|^{1/2} \int_{|\xi t|}^{\infty} e^{-u} u^{-1/2} du = (u_\infty + |t\xi^2|)^{-2\beta+1/2} \left(\int_{|\xi t|}^1 + \int_1^{\infty} \right) \\ &\leq (u_\infty + |t\xi^2|)^{-2\beta+1/2} \left(|\xi|^{1/2} \int_{|\xi t|}^1 u^{-1/2} du + |\xi|^{1/2} \int_1^{\infty} e^{-u} u^{-1/2} du \right) \\ &\leq (u_\infty + |t\xi^2|)^{-2\beta+1/2} |\xi|^{1/2} ((1 - |\xi t|) + |\xi|^{1/2}) \\ &\leq (u_\infty + |t\xi^2|)^{-2\beta+1/2} |\xi|^{1/2} \leq (u_\infty + |t\xi^2|)^{-2\beta+1/2} |t|^{-1/2}, \end{aligned}$$

since $|\xi t| < 1$. For $|\xi t| \geq 1$ we have

$$I_3 \leq (u_\infty + |t\xi^2|)^{-2\beta+1/2} |\xi|^{1/2} \int_{|\xi t|}^{\infty} e^{-u} du = (u_\infty + |t\xi^2|)^{-2\beta+1/2} |\xi|^{1/2} e^{-|\xi t|},$$

and again using (51) we obtain

$$I_3 \leq C |t|^{-1/2} (u_\infty + |t\xi^2|)^{-2\beta+1/2}.$$

This completes the first part of the lemma.

To show (48) we proceed in a similar way. Since $f \in \mathcal{Y}$ we have

$$|f(t, \xi)| \leq |t|^{-1/2} \|f\|_{\mathcal{Y}}$$

for all t and ξ . Hence we can assume without any loss that $\|f\|_{\mathcal{Y}} = 1$ and we can focus on the estimate of the following integral:

$$I = |\xi| \int_{\mathbb{R}} e^{-|\xi||t-y|} |y|^{-1/2} dy.$$

We use again a substitution $u = \xi y$ to obtain

$$I = |\xi|^{1/2} \int_{\mathbb{R}} e^{-|\xi t| - |u|} u^{-1/2} du. \tag{50}$$

We split this integral into three parts: $A_1 = \{u : |u| < |\xi||t|/2\}$, $A_2 = \{u : |\xi||t|/2 \leq u \leq 2|\xi||t|\}$, $A_3 = \{u : 2|\xi||t| \leq u\}$ and we introduce I_1 , I_2 and I_3 as the corresponding parts of integral I .

For I_1 we have

$$I_1 \leq \frac{1}{2} |\xi|^{1/2} \int_{A_1} e^{-|\xi||t|/2} u^{-1/2} = |\xi|^{1/2} e^{-|\xi||t|/2} |\xi t|^{1/2}.$$

Now since

$$|\xi t| e^{-|\xi t|} \leq C, \tag{51}$$

where C is independent of ξt , we have

$$I_1 \leq |\xi|^{1/2} \frac{|\xi t|^{1/2}}{(1 + |\xi t|)} = |t|^{-1/2} \frac{|\xi t|}{(1 + |\xi t|)} \leq |t|^{-1/2},$$

which is the desired estimate.

To estimate integrals I_2 and I_3 one can repeat the reasoning from the first part of this proof. It is sufficient to note that previously for integrals I_2 and I_3 in the first step we were extracting the term $(u_\infty + t\xi^2)^{-2\beta+1/2}$ in front of these integrals. Here we do not have this term so all other estimates are the same.

The proof of the lemma is completed. \square

Lemma 4.5. *Let $f \in \mathcal{Y}$. Given the following terms:*

$$\begin{aligned}\tilde{A} &:= \frac{|\xi|}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} f(s, \xi) \, ds, & \tilde{B} &:= \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} f(s, \xi) \, ds, \\ \tilde{C} &:= \frac{|\xi|}{\Delta} \int_t^{\infty} e^{-\lambda_2(s-t)} f(s, \xi) \, ds, & \tilde{D} &:= \frac{\lambda_2}{\Delta} \int_t^{\infty} e^{-\lambda_2(s-t)} f(s, \xi) \, ds.\end{aligned}$$

Then $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathcal{X}_\beta$ provided $2\beta \leq 1$ and the following estimate is valid:

$$\|\tilde{A}\|_{\mathcal{X}_\beta} + \|\tilde{B}\|_{\mathcal{X}_\beta} + \|\tilde{C}\|_{\mathcal{X}_\beta} + \|\tilde{D}\|_{\mathcal{X}_\beta} \leq c \|f\|_{\mathcal{Y}}.$$

$$\tilde{I} := \frac{|\xi|}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} f(s, \xi) \, ds$$

belongs to the function space \mathcal{X}_β provided $2\beta \leq 1$ and the following estimate is valid:

$$\|\tilde{I}\|_{\mathcal{X}_\beta} \leq \|f\|_{\mathcal{Y}}.$$

Remark. The same estimate is valid for

$$\tilde{I} := \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} f(s, \xi) \, ds.$$

Proof. Let us start with integral \tilde{A} . Without loss of generality we may assume that $t > 0$. As in previous lemmas we may also assume that $\|f\|_{\mathcal{Y}} = 1$ and consider the integral

$$A := \frac{|\xi|}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} |s|^{-1/2} \, ds.$$

Since $\lambda_1 = (u_\infty - \Delta)/2$ and $\Delta = \sqrt{u_\infty^2 + 4\xi^2}$ we see that the behaviour of λ_1 is different for small ξ and large ξ .

Let us assume that $|\xi| < u_\infty$. In this case we have $\Delta \sim u_\infty$ and $\lambda_1 \sim -|\xi|^2/u_\infty$, thus

$$A \leq C \frac{|\xi|}{u_\infty} \int_{-\infty}^t e^{|\xi|^2/u_\infty(s-t)} |s|^{-1/2} \, ds.$$

Using a substitution $-u = |\xi|^2(s-t)$ we obtain

$$A \leq C \frac{1}{u_\infty} \int_0^\infty e^{-u/u_\infty} |u - t\xi^2|^{-1/2} \, du = C u_\infty^{-1/2} \int_0^\infty e^{-y} |y - t\xi^2|^{-1/2} \, dy.$$

We use inequality (37) to obtain

$$A \leq C u_\infty^{-1/2} (1 + t\xi^2/u_\infty)^{-1/2} = C (u_\infty + t\xi^2)^{-1/2}, \quad (52)$$

which is the desired estimate, since for $2\beta \leq 1$ we have

$$(u_\infty + t\xi^2)^\beta A \leq (u_\infty + t\xi^2)^{\beta-1/2} \leq c,$$

where constant c does not depend on t , ξ and u_∞ , thus $A \in \mathcal{X}_\beta$.

Let us now assume that $|\xi| \geq u_\infty$. In this case we have $\lambda_1 \sim -|\xi|$ and $\Delta \sim |\xi|$, thus

$$A \leq C \int_{-\infty}^t e^{|\xi|(s-t)} |s|^{-1/2} ds. \tag{53}$$

Using a substitution $|\xi|s = -u + t|\xi|$ we end up with

$$\begin{aligned} A &\leq |\xi|^{-1/2} \int_0^\infty e^{-u} |t|\xi| - u|^{-1/2} du \leq |\xi|^{-1/2} (1 + t|\xi|)^{-1/2} \\ &= (|\xi| + t|\xi|^2)^{-1/2} \leq (u_\infty + t|\xi|^2)^{-1/2}, \end{aligned}$$

where we again used inequality (37) with $\theta = t|\xi|$ and the fact that $|\xi| > u_\infty$. This is the same inequality as (52), thus we have proved the results of lemma 4.5 for \tilde{A} .

To prove the estimate for \tilde{B} we note that λ_1 can be estimated by $|\xi|$, since for small $|\xi|$, i.e. $|\xi| \leq u_\infty$ one has $\lambda_1 \sim |\xi|^2/u_\infty \leq |\xi|$, and for large $|\xi|$ one has $\lambda_1 \sim |\xi|$.

To estimate \tilde{C} we first note, that without loss of generality we may assume $t < 0$, i.e. $t = -|t|$. It is easy to see that we may show this inequality for $t < 0$, i.e. as earlier, the behaviour of $\lambda_2 = (u_\infty + \Delta)/2$ is different for small $|\xi|$ and large $|\xi|$, in particular $\lambda_2 \sim u_\infty$ for small $|\xi|$ and $\lambda_2 \sim |\xi|$ for large $|\xi|$. Term Δ behaves exactly the same.

Let us first consider the case $|\xi| \leq u_\infty$. We have

$$\begin{aligned} C &= \frac{|\xi|}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} |s|^{-1/2} ds \leq \int_t^\infty e^{-u_\infty(s-t)} |s|^{-1/2} ds \\ &= u_\infty^{1/2} \int_0^\infty e^{-u_\infty u} |u - |t||^{-1/2} du = u_\infty^{-1/2} \int_0^\infty e^{-y} |y - u_\infty|t||^{-1/2} dy. \end{aligned}$$

We may now use (37) to obtain

$$C \leq u_\infty^{-1/2} (1 + u_\infty|t|)^{-1/2} = (u_\infty + |t|u_\infty^2)^{-1/2},$$

but since $|\xi| < u_\infty$ the last term can be estimated by $(u_\infty + |t||\xi|^2)^{-1/2}$, which, as we have seen earlier, is the desired estimate.

Let us now assume that $|\xi| > u_\infty$. In this case we have

$$C = \frac{|\xi|}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} |s|^{-1/2} ds \leq \int_t^\infty e^{-|\xi|(s-t)} |s|^{-1/2} ds,$$

and the last term estimates exactly in the same way as in (53), thus

$$C \leq C(u_\infty + t|\xi|^2)^{-1/2}.$$

Estimate of \tilde{D} is analogous to the previous one, since for large $|\xi|$ both integrals behave exactly the same, and for small $|\xi|$ one has $\lambda_2 \sim u_\infty$, hence we must estimate the integral:

$$D = \frac{\lambda_2}{\Delta} \int_t^\infty e^{-u_\infty(s-t)} |s|^{-1/2} ds \leq \int_t^\infty e^{-u_\infty(s-t)} |s|^{-1/2} ds,$$

but exactly the same integral has been estimated during the proof of the estimate for C . \square

The previous theorem needs to be modified in order to be used for investigation of the asymptotic profile:

Lemma 4.6. *Let f satisfy the following pointwise inequality:*

$$|f(t, \xi)| \leq |t|^{-1/2} (u_\infty + |t|\xi|^2)^{-2\beta + \frac{1}{2}}. \tag{54}$$

Then for $1/2 \geq \beta > 1/4$ and for the given terms:

$$\begin{aligned} \tilde{A} &:= \frac{|\xi|}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} f(s, \xi) ds, & \tilde{B} &:= \frac{\lambda_1}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} f(s, \xi) ds, \\ \tilde{C} &:= \frac{|\xi|}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} f(s, \xi) ds, & \tilde{D} &:= \frac{\lambda_2}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} f(s, \xi) ds. \end{aligned}$$

Then $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathcal{X}_{2\beta}$ and the following estimate is valid:

$$\|\tilde{A}\|_{\mathcal{X}_{2\beta}} + \|\tilde{B}\|_{\mathcal{X}_{2\beta}} + \|\tilde{C}\|_{\mathcal{X}_{2\beta}} + \|\tilde{D}\|_{\mathcal{X}_{2\beta}} \leq C.$$

Proof. Let us start with integral \tilde{A} . Without loss of generality we may assume that $t > 0$ and consider the following integral:

$$A := \frac{|\xi|}{\Delta} \int_{-\infty}^t e^{-\lambda_1(s-t)} |s|^{-1/2} (u_\infty + |s\xi^2|)^{-2\beta+1/2} ds.$$

Since $\lambda_1 = (u_\infty - \Delta)/2$ and $\Delta = \sqrt{u_\infty^2 + 4\xi^2}$ we see that the behaviour of λ_1 is different for small ξ and large ξ .

Let us assume that $|\xi| < u_\infty$. In this case we have $\Delta \sim u_\infty$ and $\lambda_1 \sim -|\xi|^2/u_\infty$, thus

$$A \leq C \frac{|\xi|}{u_\infty} \int_{-\infty}^t e^{|\xi|^2/u_\infty(s-t)} |s|^{-1/2} (u_\infty + |s\xi^2|)^{-2\beta+1/2} ds.$$

Using a substitution $-u = |\xi|^2(s-t)$ we obtain

$$\begin{aligned} A &\leq C \frac{1}{u_\infty} \int_0^\infty e^{-u/u_\infty} |u - t\xi^2|^{-1/2} (u_\infty + |u - t\xi^2|)^{-2\beta+1/2} du \\ &= C u_\infty^{-2\beta+1/2-1/2} \int_0^\infty e^{-y} |y - t\xi^2/u_\infty|^{-1/2} (1 + |y - t\xi^2/u_\infty|)^{-2\beta+1/2} dy. \end{aligned}$$

We use inequality (38) to obtain

$$A \leq C u_\infty^{-2\beta} (1 + t\xi^2/u_\infty)^{-2\beta} = C (u_\infty + t\xi^2)^{-2\beta}, \quad (55)$$

which is the desired estimate.

Let us now assume that $|\xi| \geq u_\infty$. In this case we have $\lambda_1 \sim -|\xi|$ and $\Delta \sim |\xi|$, thus

$$A \leq C \int_{-\infty}^t e^{|\xi|(s-t)} |s|^{-1/2} (u_\infty + s\xi^2)^{-2\beta+1/2} ds. \quad (56)$$

Using a substitution $|\xi|s = -u + t|\xi|$ we end up with

$$\begin{aligned} A &\leq \int_0^\infty |\xi|^{-2\beta} e^{-u} |t|\xi| - u|^{-1/2} \left(\frac{u_\infty}{|\xi|} + |t|\xi| - u \right)^{-2\beta+1/2} du \\ &= \int_0^{t\xi/2} + \int_{t\xi/2}^{2t\xi} + \int_{2t\xi}^\infty =: A_1 + A_2 + A_3. \end{aligned}$$

Now we estimate integrals A_1, A_2 and A_3 separately.

Integral A_1 can be estimated as follows:

$$\begin{aligned} A_1 &= |\xi|^{-2\beta} \int_0^{t\xi/2} e^{-u} |t\xi|^{-1/2} \left(\frac{u_\infty}{|\xi|} + |t\xi| \right)^{-2\beta+1/2} du \\ &= |\xi|^{-2\beta} (1 - e^{-t\xi/2}) |t\xi|^{-1/2} \left(\frac{u_\infty}{|\xi|} + |t\xi| \right)^{-2\beta+1/2}. \end{aligned}$$

We now consider two cases: the first one is for $|t\xi| < u_\infty/|\xi|$. Remembering that $u_\infty/|\xi| \leq 1$ and that $(1 - e^{-t\xi/2})|t\xi|^{-1/2} \sim |t\xi|^{1/2}$ we estimate A_1 :

$$\begin{aligned} A_1 &\leq |\xi|^{-2\beta} |t\xi|^{1/2} \left(\frac{u_\infty}{|\xi|} + |t\xi| \right)^{-2\beta+1/2} \leq |\xi|^{-2\beta} \left(\frac{u_\infty}{|\xi|} + |t\xi| \right)^{-2\beta+1/2+1/2} \\ &\leq (u_\infty + t\xi^2)^{-2\beta}, \end{aligned}$$

where we used the same assumption $|t\xi| < u_\infty/|\xi|$.

In case $t|\xi| > u_\infty/|\xi|$ we may write $|t\xi|^{-1/2} \leq (|t\xi| + u_\infty/|\xi|)^{-1/2}$ which gives us the following easy estimate:

$$A_1 \leq |\xi|^{-2\beta} \left(\frac{u_\infty}{|\xi|} + |t\xi| \right)^{-2\beta+1/2-1/2} = (u_\infty + t\xi^2)^{-2\beta}.$$

Let us now estimate integral A_2 . We first estimate it in the following way:

$$A_2 \leq \int_0^{|t\xi|} |\xi|^{-2\beta} e^{-|t\xi|u} |u|^{-1/2} \left(\frac{u_\infty}{|\xi|} + |u| \right)^{-2\beta+1/2} du.$$

Now for the first case $|t\xi| < u_\infty/|\xi|$ one can write

$$A_2 \leq |\xi|^{-2\beta} e^{-|t\xi|} |t\xi|^{1/2} \left(\frac{u_\infty}{|\xi|} \right)^{-2\beta+1/2} \leq |\xi|^{-2\beta} \left(\frac{u_\infty}{|\xi|} \right)^{-2\beta+1} \leq u_\infty^{-2\beta},$$

which under our assumption is the desired estimate since $u_\infty^{-2\beta} \leq (u_\infty + t\xi^2)^{-2\beta}$.

Let us now consider the case $|t\xi| > u_\infty/|\xi|$. For this we split integral A_2 as $A_2 = \int_0^{u_\infty/|\xi|} + \int_{u_\infty/|\xi|}^{|t\xi|} =: A_{21} + A_{22}$.

Integral A_{21} we estimate as follows:

$$A_{21} \leq \int_0^{u_\infty/|\xi|} |\xi|^{-2\beta} e^{-|t\xi|u} |u|^{-1/2} (u_\infty/|\xi|)^{-2\beta+1/2} du = u_\infty^{-2\beta} e^{-t\xi} |u_\infty/|\xi||^1.$$

We estimate $e^{-|t\xi|} \leq (1 + |t\xi|)^{-1}$ and get

$$A_{21} \leq u_\infty^{-2\beta+1} (|\xi| + t\xi^2)^{-1} \leq u_\infty^{-2\beta+1} (u_\infty + t\xi^2)^{-1},$$

where we again used $|\xi| > u_\infty$. Now if $\beta \leq 1/2$ (i.e. $-1 \leq -2\beta$) one estimates the last term by $u_\infty^{-2\beta+1} (u_\infty + t\xi^2)^{-2\beta}$.

For integral A_3 one changes variables $-y = |t\xi| - u$ to obtain

$$\begin{aligned} A_3 &= \int_{|t\xi|}^\infty |\xi|^{-2\beta} e^{-t\xi} e^{-y} |y|^{-1/2} \left(\frac{u_\infty}{|\xi|} + |y| \right)^{-2\beta+1/2} dy \\ &\leq e^{-|t\xi|} |\xi|^{-2\beta} \left(\frac{u_\infty}{|\xi|} + |t\xi| \right)^{-2\beta+1/2} \int_0^\infty e^{-y} |y|^{-1/2} dy. \end{aligned}$$

Under the assumption that $u_\infty \leq |\xi|$ one easily gets

$$A_3 \leq C(1 + |t\xi|)^{1/2} e^{-|t\xi|} (u_\infty + t\xi^2)^{-2\beta} \leq C(u_\infty + t\xi^2)^{-2\beta}. \tag{57}$$

This finishes the estimate for the term A .

As in the proof of lemma 4.5 to prove estimate for \tilde{B} we note that λ_1 can be estimated by $|\xi|$, since for small $|\xi|$, i.e. $|\xi| \leq u_\infty$ one has $\lambda_1 \sim |\xi|^2/u_\infty \leq |\xi|$, and for large $|\xi|$ one has $\lambda_1 \sim |\xi|$. Thus $B < A$.

To estimate C we first note that without loss of generality we may assume $t < 0$. Using the identity $\lambda_1 + \lambda_2 = u_\infty$ we may estimate integral C as follows:

$$\begin{aligned} C(t, \xi) &\leq \frac{|\xi|}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} |f(s, \xi)| ds \\ &= \frac{|\xi|}{\Delta} \int_t^\infty e^{\lambda_1(s-t)} e^{-u_\infty(s-t)} |f(s, \xi)| ds \leq \frac{|\xi|}{\Delta} \int_t^\infty e^{\lambda_1(s-t)} |f(s, \xi)| ds. \end{aligned} \tag{58}$$

Next using a substitution $s = -u$ and denoting $\tau = -t$ we obtain

$$C(-\tau, \xi) \leq \frac{|\xi|}{\Delta} \int_{-\infty}^\tau e^{-\lambda_1(u-\tau)} |f(-u, \xi)| du, \tag{59}$$

but this integral has been already estimated above (that is in the case of integral A), one just needs to remember that the estimate for $|f|$ does not depend on the sign of u .

To complete the proof of this theorem we need to estimate integral D . First we note that for large $|\xi|$, that is for $|\xi| > u_\infty$ integrals D and C behave exactly the same, thus we estimate D only under condition $|\xi| < u_\infty$. Then $\lambda_2 \sim u_\infty$ and $\Delta \sim u_\infty$.

As in the case of integral C we also assume without loss of generality that $t < 0$. For convenience let us denote $-\gamma = -2\beta + 1/2$. We then have

$$\begin{aligned} D &\leq \int_t^\infty e^{-u_\infty(s-t)} |s|^{-1/2} (u_\infty + s\xi^2)^{-\gamma} ds \\ &= \int_0^\infty e^{-u} |u_\infty|^{-1/2} |u + u_\infty t|^{-1/2} \left(u_\infty + |u + u_\infty t| \frac{\xi^2}{u_\infty} \right)^{-\gamma} du \\ &= \int_0^{tu_\infty/2} + \int_{tu_\infty/2}^{2tu_\infty} + \int_{2tu_\infty}^\infty =: D_1 + D_2 + D_3, \end{aligned}$$

where we use a substitution $u_\infty(s - t) = u$.

We estimate integral D_1 as follows:

$$\begin{aligned} D_1 &\leq \int_0^{tu_\infty/2} e^{-u} |u_\infty|^{-1/2} |tu_\infty|^{-1/2} \left(u_\infty + tu_\infty \frac{\xi^2}{u_\infty} \right)^{-\gamma} \\ &= (1 - e^{-tu_\infty/2}) |u_\infty|^{-1/2} |tu_\infty|^{-1/2} (u_\infty + t\xi^2)^{-\gamma} \\ &\leq \frac{|tu_\infty|^{1/2}}{(1 + |tu_\infty|)} |u_\infty|^{-1/2} (u_\infty + |t\xi^2|)^{-\gamma} \leq (u_\infty + |tu_\infty^2|)^{-1/2} (u_\infty + |t\xi^2|)^{-\gamma}. \end{aligned}$$

But now since $|\xi| < u_\infty$ one has $(u_\infty + |tu_\infty^2|)^{-1/2} \leq (u_\infty + |t\xi^2|)^{-1/2}$ and thus

$$D_1 \leq (u_\infty + |t\xi^2|)^{-\gamma-1/2} = (u_\infty + |t\xi^2|)^{-2\beta},$$

which is the desired estimate.

For integral D_2 we proceed as follows:

$$D_2 = \int_{tu_\infty/2}^{2tu_\infty} e^{-tu_\infty/2} |u_\infty|^{-1/2} \int_0^{tu_\infty} |u|^{-1/2} \left(u_\infty + u \frac{\xi^2}{u_\infty} \right)^{-\gamma} du.$$

Now we distinguish two cases. The first one is for $|t\xi^2| < u_\infty$, that is for $tu_\infty \frac{\xi^2}{u_\infty} < u_\infty$. Then we may estimate D_2 as

$$D_2 \leq e^{-tu_\infty/2} |u_\infty|^{-1/2} |tu_\infty|^{1/2} u_\infty^{-\gamma} \leq u_\infty^{-\gamma-1/2} \leq (u_\infty + |t\xi^2|)^{-\gamma-1/2},$$

where we used the fact that $e^{-tu_\infty/2} |tu_\infty|^{1/2}$ is uniformly bounded.

For $|t\xi^2| \geq u_\infty$ one has

$$\begin{aligned} D_2 &\leq e^{-2tu_\infty/2} |u_\infty|^{-1/2} \int_0^{tu_\infty} |u|^{-1/2} |u|^{-\gamma} |\xi^2|^{-\gamma} u_\infty^\gamma du \\ &= e^{-tu_\infty/2} |u_\infty|^{-1/2} |tu_\infty|^{-1/2-\gamma+1} |\xi^2|^{-\gamma} u_\infty^\gamma \leq |t\xi^2|^{-\gamma-1/2} u_\infty^{1/2}. \end{aligned}$$

where we again used the assumption $|\xi| < u_\infty$ and assumption $\gamma < 1/2$ which corresponds to $\beta < 1/2$. The last inequality under our assumption of $|t\xi^2| \geq u_\infty$ gives us the desired estimate $D_2 \leq (u_\infty + |t\xi^2|)^{-2\beta}$.

To finish the proof of the lemma we need to estimate integral D_3 . First we obtain

$$\begin{aligned} D_3 &= \int_{2tu_\infty}^\infty \leq \int_{2tu_\infty}^\infty e^{-u} |u_\infty|^{-1/2} |u|^{-1/2} \left(u_\infty + |u| \frac{\xi^2}{u_\infty} \right)^{-\gamma} \\ &\leq (u_\infty + |t\xi^2|)^{-\gamma} \int_{2tu_\infty}^\infty e^{-u} |u_\infty|^{-1/2} |u|^{-1/2}. \end{aligned}$$

Now in the case of $t\xi^2 < u_\infty$ one easily gets

$$D_3 \leq (u_\infty + |t\xi^2|)^{-\gamma} u_\infty^{-1/2} \int_0^\infty e^{-u} |u|^{-1/2} \leq (u_\infty + t\xi^2)^{-\gamma-1/2},$$

which is the desired estimate. In case $t\xi^2 \geq u_\infty$ one first notes that this condition implies $tu_\infty^2 > t\xi^2 > u_\infty$, that is $tu_\infty > 1$. This allows us to estimate D_3 as follows:

$$\begin{aligned} D_3 &\leq (u_\infty + |t\xi^2|)^{-\gamma} (2tu_\infty)^{-1/2} u_\infty^{-1/2} \int_0^\infty e^{-u} du \\ &\leq (u_\infty + |t\xi^2|)^{-\gamma} (1 + tu_\infty)^{-1/2} u_\infty^{-1/2} \leq (u_\infty + |t\xi^2|)^{-\gamma} (u_\infty + |tu_\infty^2|)^{-1/2} \\ &\leq (u_\infty + |t\xi^2|)^{-\gamma-1/2}, \end{aligned}$$

where again we used assumption $|\xi| < u_\infty$. This completes the proof of lemma 4.6. \square

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