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THE SURFACE DIFFUSION FLOW ON ROUGH PHASE SPACES

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Abstract. The surface diffusion flow is the gradient flow of the surface functional of compact hypersurfaces with respect to the inner product of H^{-1} and leads to a nonlinear evolution equation of fourth order. This is an intrinsically difficult problem, due to the lack of an maximum principle and it is known that this flow may drive smoothly embedded uniformly convex initial surfaces in finite time into non-convex surfaces before developing a singularity [15, 16]. On the other hand it also known that singularities may occur in finite time for solutions emerging from non-convex initial data, cf. [10].

Combining tools from harmonic analysis, such as Besov spaces, multiplier results with abstract results from the theory of maximal regularity we present an analytic framework in which we can investigate weak solutions to the original evolution equation. This approach allows us to prove well-posedness on a large (Besov) space of initial data which is in general larger than C^2 (and which is in the distributional sense almost optimal). Our second main result shows that the set of all compact embedded equilibria, i.e. the set of all spheres, is an invariant manifold in this phase space which attracts all solutions which are close enough (which respect to the norm of the phase space) to this manifold. As a consequence we are able to construct non-convex initial data which generate global solutions, converging finally to a sphere.

1. **Introduction.** In this paper we study the geometric evolution equation

$$V_n = \Delta_{M(t)} H_{M(t)} \quad \text{on} \quad M(t),\ t > 0, \tag{1}$$

subject to the initial condition

$$M|_{t=0} = M_0. \tag{2}$$

Here $M(t)$ denotes for any $t > 0$ an unknown compact closed hypersurface in $\mathbb{R}^{n+1}$, V_n is the normal velocity of the family $\{M(t)\,;\, t > 0\}$, and $\Delta_{M(t)}$ and $H_{M(t)}$ are the Laplace-Beltrami operator and the mean curvature, respectively, of the surface $M(t)$. The orientation of the normal vector and the mean curvature depend on the choice of the orientation of $M(t)$, however we are free to choose whichever we like.

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Equation (1) was first proposed by Mullins [25] to describe thermal grooving in material sciences. It arises also in modelling viscous sintering [20] and J. W. Cahn et al. derived it as a singular limit of the Cahn-Hilliard equation with a concentration dependent mobility [4]. From the point of view of geometrical analysis it appears naturally as the gradient flow of the surface functional with respect to the inner product of H^{-1}, cf. [27]. Local well-posedness in the classical sense, as well as interesting dynamical properties of this flow such as loss of convexity, loss of embeddedness, and formation of singularities have been discussed in [5, 8, 9, 10, 11, 14, 15]. In this paper we offer a further insight into the dynamic picture of (1) by constructing an as large as possible phase space on which weak solutions in the sense of distributions still make sense. This effort will be rewarded by the fact that we have to control a priori less than two derivatives of solutions to (1) in order to guarantee global existence. In order to give a precise statement of our results, let $B^s_{p,2}$, with $p > 1$ and $s > 0$, denote the Besov spaces as introduced in Section 2 below. Using Marcinkiewicz's multiplier theorem [7], we first prove a priori estimates in

$$C([0,T]; B^{5/2-4/p}_{p,2}(S)) \quad \text{and} \quad B^{5/2,5/8}_{p,2}(S \times (0,T)) \quad \text{with} \quad p > 8/3, \tag{3}$$

of solutions to a linear problem associated with (1) against natural norms of the data, cf. Theorem 5.1. Here S denotes a reference surface over which the initial datum M_0 is parameterized as a graph in normal direction. Restrictions on p and the number of derivatives are forced by features of the nonlinear system. Using the above indicated estimates, a suitable choice of function spaces (cf. Section 6), and Banach's fixed point theorem, we prove the following result.

Theorem 1.1. *Let $p > \frac{2n+8}{3}$ and assume that M_0 is a closed compact surface of class $B^{5/2-4/p}_{p,2}$ in the sense of (17). Then there exists $T > 0$ such that there is a unique weak solution $\{M(t)\,;\, t \in [0,T]\}$ to system (1) on the time interval $[0,T]$. Given $t \in [0,T]$, the set $M(t)$ is an immersed hypersurface of class $B^{5/2-4/p}_{p,2}$ and $\mathcal{M} := \bigcup_{0<t<T}(M(t) \times \{t\})$ is a topological manifold of class $B^{5/2,5/8}_{p,2}$.*

The particular choice of the scale of Besov spaces is required by our analysis of the nonlinear terms which appear by parametrising the unknown surfaces in (1). In fact these spaces allow us to control products of functions in spaces of negative order – see the Appendix. The condition $p > (2n + 8)/3$ implies that the field of normal vectors is Hölder continuous in space and time, hence the solutions constructed in Theorem 1.1 consists of $C^{1+\alpha}$-surfaces. Observe furthermore that Theorem 1.1 guarantees the existence of local in time solutions for arbitrary initial surfaces. The description of the regularity of surfaces as submanifolds means that they can be locally viewed as graphs of functions belonging to appropriate Besov spaces.

Using Alexandroff's characterization of embedded surfaces of constant mean curvature [1], it is not difficult to see that spheres are the only embedded steady states of (1). Note that this implies that steady states are not isolated, so that the principle of linearized stability cannot be used to analyse the stability properties of spheres. Instead we shall use an abstract approach which is based on the theory of maximal regularity and spectral properties of an elliptic operator induced by (1). Our analysis will lead us finally to a construction of a centre manifold for (1), which consists only in equilibria. Moreover, this manifold attracts all solutions which are suffciently close nearby. It is worthwhile to emphasize that our techniques are not based on the abstract methods from [26], but also use maximal regularity results, however in anisotropic Besov spaces. This approach is more direct and allows us to essentially improve the results of [11]. More precisely, we have

Theorem 1.2. *If the initial surface M_0 is sufficiently close to a sphere in the $B^{5/2-4/p}_{p,2}$-topology, then the solution to system (1) exists globally in time and stays in a neighbourhood of the sphere. Additionally it converges to a sphere with a radius determined by the volume of the set enclosed by M_0.*

Recall that Theorem 1.1 holds true provided $p > (2n+8)/3$. Hence if $n < 8$ there is a $p > (2n+8)/3$ such that $5/2 - 4/p < 2$, which means that $C^2(\mathbb{S}^n) \hookrightarrow B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$. This implies that there is a sequence of functions in $B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$, say $(r_m)_{m\in\mathbb{N}}$, such that

$$\|r_m\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \to 0 \quad \text{and} \quad \|r_m\|_{C^2(\mathbb{S}^n)} \to \infty \quad \text{as} \quad m \to \infty. \tag{4}$$

This in turn implies that there exist arbitrarily small non-convex perturbations in $B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$ of a given sphere. If $n \geq 8$ the embedding $C^2(\mathbb{S}^n) \hookrightarrow B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$ can no longer be guaranteed. However, an appropriate scaling argument shows, that (4) nevertheless is true. Combining the above reasoning with Theorem 1.2, we get

Theorem 1.3. *There exists a smooth non-convex surface M_0 generating a global solution to (1), which converges eventually to a sphere.*

The paper is organised as follows. First we recall the definition and basic properties of Besov spaces. In Section 3 a parametrisation of solutions to (1) is given. It enables us to transform (1) into a system on a temporal invariant domain. Next, we prove Theorem 1.3, based on Theorems 1.1 and 1.2. In Section 5 we introduce a linear system related to (1) and prove the main a priori estimate (3). Based on the result for the linear system, Theorem 1.1 is proved in Section 6. Subsequently we examine the stability of the unit sphere by constrcting a local centre manifold and show finally Theorem 1.2. In the Appendix we particularly recall some facts concerning pointwise multiplications in Besov spaces. Throughout this paper, generic constants are denoted by the same letter C.

2. **Besov spaces.**

Our analysis is conducted mostly in Besov spaces of $B^s_{p,2}$-type, see e.g. [3, 28, 29]. We briefly recall the corresponding definitions. First we consider the isotropic spaces defined on $\mathbb{R}^n$. Given $s > 0$ and $p \geq 1$, we introduce the Banach space $B^s_{p,2}(\mathbb{R}^n)$, defined by the norm

$$\|u\|_{B^s_{p,2}(\mathbb{R}^n)} = \|u\|_{L_p(\mathbb{R}^n)} + \sum_{|\alpha|=[s]_-} < \partial^\alpha_x u >_{B^{\{s\}}_{p,2}(\mathbb{R}^n)}, \tag{5}$$

where $[s]_-$ is the largest integer less than s and $\{s\} = s - [s]_-$. Moreover $\alpha \in \mathbb{N}^n$ is a standard multi-index. The principal seminorm of $B^s_{p,2}$ – the last term on the right hand side of (5) – is defined by

$$< u >_{B^s_{p,2}(\mathbb{R}^n)} = \left(\int_{\mathbf{R}^n} \frac{1}{|h|^{n+2s}} \|u(\cdot + h) - u(\cdot)\|^2_{L_p(\mathbb{R}^n)} dh \right)^{1/2}, \tag{6}$$

provided $s \in (0,1)$. For the case $s = 1$ of (6), we refer to [3].

The above definitions can be extended to anisotropic spaces, where the time direction is distinguished. Given $T > 0$, we define $B^{s,s'}_{p,2}(\mathbb{R}^n \times (0,T))$ by the norm

$$\|u\|_{B^{s,s'}_{p,2}(\mathbb{R}^n\times(0,T))} = \|u\|_{L_p(\mathbb{R}^n\times(0,T))} \\ + < \partial_t^{[s']_-} u >_{B^{0,\{s'\}}_{p,2}(\mathbb{R}^n\times(0,T))} + \sum_{|\alpha|=[s]_-} < \partial_x^\alpha u >_{B^{\{s\},0}_{p,2}(\mathbb{R}^n\times(0,T))}, \tag{7}$$

where the sum of the last two terms of (7) is the main seminorm of $B^{s,s'}_{p,2}$, given by

$$< u >_{B^{s,0}_{p,2}(\mathbb{R}^n\times(0,T))} = \left(\int_{\mathbb{R}^n} \frac{1}{|h|^{n+2s}} \|u(\cdot+h,\cdot) - u(\cdot,\cdot)\|^2_{L_p(\mathbb{R}^n\times(0,T))} dh \right)^{1/2}, \\ < u >_{B^{0,s'}_{p,2}(\mathbb{R}^n\times(0,T))} = \left(\int_0^T \frac{1}{|h|^{1+s'2}} \|u(\cdot,\cdot+h) - u(\cdot,\cdot)\|^2_{L_p(\mathbb{R}^n\times(0,T-h))} dh \right)^{1/2}, \tag{8}$$

provided $s, s' \in (0,1)$.

Definitions (5)-(6) can be reformulated by using the Fourier transform [29]. For this we introduce the Paley-Littlewood decomposition $\{p_k\}_{k\in\mathbb{N}}$ in the following way: Let $p_k : \mathbb{R}^n \to [0,1]$ be a sequence of smooth functions such that

$$\text{supp } \ p_k \subset \{\xi \in \mathbb{R}^n : 2^{k-1} \le |\xi| \le 2^{k+1}\} \quad \text{for} \ \ k \ge 1 \tag{9}$$

and

$$\text{supp } p_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\} \quad \text{and} \quad \sum_{k=0}^\infty p_k \equiv 1. \tag{10}$$

Then for $s \ge 0$, the norm given by (5)-(6) is equivalent to

$$\|u\|_{B^s_{p,2}(\mathbb{R}^n)} = \left(\sum_{k=0}^\infty 2^{ks2} \|u^k\|^2_{L_p(\mathbb{R}^n)} \right)^{1/2}, \tag{11}$$

where $u^k = \mathcal{F}_x^{-1}[p_k(\xi)\mathcal{F}_x[u]]$ and $\mathcal{F}_z$ denotes the Fourier transform with respect to z. Observe that in this definition s may be any real number.

In the case of the spacetime $\mathbb{R}^n \times \mathbb{R}$ the norm (7) reads

$$\|u\|_{B^{s,s'}_{p,2}(\mathbb{R}^n\times\mathbb{R})} = \left(\sum_{k=0}^\infty 2^{ks2} \|u^k_x\|^2_{L_p(\mathbb{R}^n\times\mathbb{R})} \right)^{1/2} + \left(\sum_{k=0}^\infty 2^{ks'2} \|u^k_t\|^2_{L_p(\mathbb{R}^n\times\mathbb{R})} \right)^{1/2}, \tag{12}$$

where $u^k_x = \mathcal{F}_x^{-1}[p^x_k(\xi)\mathcal{F}_x[u]]$ and $u^k_t = \mathcal{F}_t^{-1}[p^t_k(\xi_0)\mathcal{F}_t[u]]$, and where p^x_k, p^t_k denote the Paley-Littlewood decomposition for $\mathbb{R}^n$ and $\mathbb{R}$, respectively.

The above Besov spaces, in particular also Hölder spaces, can be defined in a natural way for functions on smooth manifolds. Using an atlas of the manifold and a suitable partition of unity we define the norm as the sum taken over all maps from the atlas.

A fundamental tool in the analysis of the linear system associated to (1) is Marcinkiewicz's theorem for Fourier multipliers [19]. A simple version with Hörmander's condition of this class reads as follows, cf. [7]:

Theorem 2.1. *Let $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ such that*

$$|\xi^\alpha||\partial^\alpha m| \le C_\alpha M, \ \textit{where}\ 0 \le |\alpha| \le n/2 + 1,$$

$\alpha \in (\mathbb{N})^n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$, $\partial^\alpha = \partial^{\alpha_1}_{\xi_1} \cdot \ldots \cdot \partial^{\alpha_n}_{\xi_n}$ *and* $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

If $g \in L_p(\mathbb{R}^n)$, then $Tg = \mathcal{F}^{-1}[m\mathcal{F}g]$ is a linear bounded operator from $L_p(\mathbb{R}^n)$ into itself and

$$\|Tg\|_{L_p(\mathbb{R}^n)} \le A_{p,n} M \|g\|_{L_p(\mathbb{R}^n)}.$$

In [28, 29] one can find versions of this theorem which work for Besov spaces as well.

3. **Parameterization.** The goal of this section is to transform system (1) into a system on a fixed reference surface S and to parameterize a solution of (1) at any moment t as a graph in the normal direction over S.

More precisely, let S be a compact smooth hypersurface S of $\mathbb{R}^{n+1}$. For convenience we introduce here an atlas of S which is suitable for our later purposes. First we fix two finite coverings $\{s^k\}$ and $\{S^k\}$ such that $s^k \subset S^k \subset S$ and such that $\bigcup_k s^k = \bigcup_k S^k = S$. For each k we define a smooth function $\zeta^k : S \to [0,1]$ such that

$$\zeta^k(x) = \begin{cases} 1 & \text{for} \quad x \in s^k \\ \in [0,1] & \text{for} \quad x \in S^k \setminus s^k \\ 0 & \text{for} \quad x \in S \setminus S^k. \end{cases}$$

Then we set

$$\pi^k(x) = \frac{\zeta^k(x)}{\sum_l (\zeta^l(x))^2} \quad \text{and} \quad \eta^k = \pi^k \zeta^k, \quad \text{so that} \quad \sum_l \eta^l \equiv 1 \text{ on } S. \tag{13}$$

The family $\{\eta^k\}$ is a partition of unity on S. Further we choose an atlas of maps (U^l, Z_l) such that $U^l \subset \mathbb{R}^n$ and $Z_l : U^l \to \mathbb{R}^{n+1}$ with $Z_l(U^l) = S^l$ and

$$\max_l (diam\ U^l + diam\ S^l) \le \lambda, \tag{14}$$

where parameter λ will be specified later. Condition (14) yields a bound for the derivatives of the functions

$$|\partial^\alpha \zeta^k| + |\partial^\alpha \pi^k| + |\partial^\alpha \eta^l| \le \frac{C}{\lambda^{|\alpha|}}. \tag{15}$$

Given a function $\phi : S \times [0,T) \to \mathbb{R}^{n+1}$ we describe a family of surfaces by

$$M(t) = \{x(y,t) = y + \phi(y,t)\vec{n} \ \text{ for all } \ y \in S \text{ and } t \in [0,T)\}, \tag{16}$$

where $\vec{n}$ is the normal vector to S. The initial datum M_0 is assumed to have the following parameterization

$$M_0 = \{x(y) = y + \phi(y,0)\vec{n} \ \text{ such that } \ y \in S\}. \tag{17}$$

Throughout this paper we assume that $\phi(y,0) \in B^{5/2-4/p}_{p,2}(S)$.

If $\phi(\cdot,t)$ is sufficiently small in $C^1(S)$, the function ϕ describes a diffeomorphism between the surfaces $M(t)$ and S. Hence we are able to transform (1) into a problem on the fixed reference domain S in the following way

$$\begin{aligned} v &= -\tilde{\Delta}_{M(t)} \tilde{H}_{M(t)} && \text{on} \quad S \times (0,T), \\ \phi|_{t=0} &= \phi_0 && \text{on} \quad S, \end{aligned} \tag{18}$$

where $v(y,t) = V_n(x(y,t),t)$ and $v = \phi_t \vec{n} \cdot \vec{n}^t$, with $\vec{n}^t$ being the unit outward normal vector to $M(t)$ in the coordinates (y,t). The Laplace-Beltrami and mean curvature operator $\tilde{\Delta}_{M(t)}$ and $\tilde{H}_{M(t)}$ have the following form

$$\tilde{\Delta}_{M(t)} \sim g^{ij} \frac{\partial^2}{\partial y^i \partial y^j} \quad \text{and} \quad \tilde{H}_{M(t)} \sim \frac{1}{n} \tilde{\Delta}_{M(t)} \phi + \text{ lower terms as } H_0(\nabla\phi, \phi), \tag{19}$$

where g_{ij} is the metric induced on S by the coordinates y, and where $g = det\{g_{ij}\}$, locally in each map of the atlas of the manifold S. For the explicit formulas of (19) we refer to [11, 12]. They can be determined in terms of the metric g_{ij}. However, using similar perturbation results as presented in [11], we may focus on the principal parts on a fixed reference surface only.

The construction of the function ϕ allows us to reformulate Theorem 1.1 as follows.

Theorem 3.1. *If $\phi_0 \in B^{5/2-4/p}_{p,2}(S)$, then there exists $T > 0$ such that there exists a unique weak solution to problem (18) on time interval $[0, T)$ such that*

$$\phi \in C([0,T]; B^{5/2-4/p}_{p,2}(S)) \text{ and } \phi \in B^{5/2,5/8}_{p,2}(S \times (0,T)).$$

The proof of Theorem 3.1 will be given in Section 6. The meaning of solutions is given in Definition 6.1.

4. **Proof of Theorem 1.3.** We distinguish three cases: $n = 1$, $1 < n < 8$, and $n \geq 8$. We will construct a smooth surface M_0 close to the unit sphere such that M_0 will be non-convex and it will fulfill the assumptions of Theorem 1.2, namely, M_0 will be sufficiently close to the sphere in the $B^{5/2-4/p}_{p,2}$-topology, generating in this way a global in time solution to (18), converging finally to a sphere.

Proof. In the first case $n = 1$, we introduce

$$M_0 = \{y + \varepsilon \sin my\, \vec{n} : y \in \mathbb{S}^1\}, \tag{20}$$

where $\varepsilon > 0$, $m \in \mathbb{N}$ and $\vec{n}$ is the normal vector to $\mathbb{S}^1$, and where $\mathbb{S}^1$ is the unit circle in the plane. Then we see that the "distance" between M_0 and the circle is measured by the function $\varepsilon \sin my$. A simple calculation leads to

$$< \varepsilon \sin my >_{B^{5/2-4/p}_{p,2}(\mathbb{S}^1)} = \varepsilon\, m^{5/2-4/p} < \sin y >_{B^{5/2-4/p}_{p,2}(\mathbb{S}^1)} . \tag{21}$$

Since $n = 1$, we find $p > \frac{10}{3} (= \frac{2n+8}{3}$ with $n = 1)$ and $\delta > 0$ such that $5/2 - 4/p + \delta < 2$. Taking $\varepsilon = m^{-2+\delta}$ in (21), we conclude that

$$\|m^{-2+\delta} \sin my\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^1)} \to 0 \text{ as } m \to \infty. \tag{22}$$

On the other hand the curvature of M_0 is dominated by the second derivative of this function, for which we have

$$\|m^{-2+\delta} \sin my\|_{C^2(\mathbb{S}^1)} = Cm^{\delta} \to \infty \text{ as } m \to \infty. \tag{23}$$

The simple form of M_0 allows us even to compute its curvature explicitly. Parameterization by angle gives the following formula for the curvature

$$\kappa = 1 + \frac{R'^2 - R''R}{R^2 + R'^2} = 1 + \frac{\varepsilon m^2 \sin mt + \varepsilon^2 m^2}{(1 + \varepsilon \sin mt)^2 + \varepsilon^2 m^2 \cos^2 mt}, \tag{24}$$

with $R(t) = 1 + \varepsilon \sin mt$. Choosing $\varepsilon = m^{-2+\delta}$, we get from (24):

$$\kappa = 1 + \varepsilon m^2 \sin mt + o(1) \text{ for } m \to \infty. \tag{25}$$

Now relation (24) shows that M_0, as given by (20), becomes non-convex provided m is large enough. This proves Theorem 1.3 in the first case.

In the second case, i.e. for $1 < n < 8$, we proceed similarly. First we introduce spherical coordinates on $\mathbb{S}^n$, i.e. we represent $x \in \mathbb{S}^n$ as

$$\begin{aligned} &x_1 = \cos\phi_1,\\ &x_2 = \sin\phi_1\cos\phi_2,\\ &...\\ &x_n = \sin\phi_1 ... \sin\phi_{n-1}\cos\phi_n,\\ &x_{n+1} = \sin\phi_1 ... \sin\phi_{n-1}\sin\phi_n, \end{aligned} \tag{26}$$

where $\phi_1, \phi_2, ..., \phi_{n-1} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\phi_n \in [0, 2\pi)$. Then we define

$$M_0 = \{x = y(\phi_1, ..., \phi_n) + s(\phi_1, ..., \phi_n)\vec{n}\,;\, y(\phi_1, ..., \phi_n) \in \mathbb{S}^n\}, \tag{27}$$

where $s(\cdot)$ depends on two parameters ε and m in the following way:

$$s(\phi_1, ..., \phi_n) = \varepsilon\, l(\phi_1, ..., \phi_{n-1}) \sin m\phi_n, \tag{28}$$

and where l is a smooth function such that supp $l \subset\subset (-\frac{\pi}{2}, \frac{\pi}{2})^{n-1}$. Because of $n < 8$ we can find $\delta > 0$ and $p > \frac{2n+8}{3}$ such that $5/2 - 4/p + \delta < 2$. Next putting $\varepsilon = m^{-2+\delta}$ in (28), we conclude as for (22) that

$$\|s\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \to 0 \text{ as } m \to \infty. \tag{29}$$

Repeating the considerations for (23), we also get that

$$\|s\|_{C^2(\mathbb{S}^n)} \sim Cm^\delta \to +\infty \text{ as } m \to \infty. \tag{30}$$

Thus we may choose $m > 0$ so large that M_0, defined by (27), will be non-convex in the sense that its mean curvature H_0 takes negative values. Indeed, the contributions to the mean curvature H_0 in the direction $\phi_1, ..., \phi_{n-1} = 0$ can be as negative as we wish by increasing m, cf.(24), whereas the other contributions are independent of m.

In the case $n \geq 8$ we proceed in a different way. Here we deform the sphere

$$\Sigma := \{x = (x', x_{n+1}) \in \mathbb{R}^{n+1}\,;\, |x'|^2 + (x_{n+1} - 1)^2 = 1\}$$

near 0 into a surface M_0 such that its mean curvature H_0 will become negative at 0. For this we fix a smooth function $\varphi : \mathbb{R}^n \to [0, 1]$ such that supp $\varphi \subset B(0, 1)$ and such that

$$\varphi(0) = 1, \qquad \varphi'(0) = 0, \qquad \varphi''(0) = 1.$$

Given $\varepsilon > 0$ and $m > 0$, consider

$$g_{\varepsilon,m}(x') := 1 - (1 - |x'|^2)^{1/2} - \varepsilon\varphi(m|x'|) \tag{31}$$

and define

$$\tilde{M}_0^{\varepsilon,m} := \{x = (x', x_{n+1}) \in \mathbb{R}^{n+1}\,;\, x_{n+1} = g_{\varepsilon,m}(x'), \quad |x'| < 1\}.$$

Observe that $\tilde{M}_0^{\varepsilon,m}$ is a perturbation of the lower hemisphere centred at $(0', 1)$. We will choose the parameters in such a way that its mean curvature will be negative in 0 and close $\tilde{M}_0^{\varepsilon,m}$ to get a compact surface M_0 without boundary by glueing it together with the upper hemisphere. Setting $f(x') := \varphi(|x'|)$, we calculate the semi-norm of $\varepsilon f(mx')$ in the space $B^{5/2-4/p}_{p,2}(\mathbb{R}^n)$. Recalling that for $n \geq 8$ we have $5/2 - 4/p > 2$, so that

$$\begin{aligned} &< \varepsilon f(mx) >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}\\ &= \varepsilon m^2 \left(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+(1/2-4/p)2}} \|\nabla^2 f(mx + mh) - \nabla^2 f(mx)\|^2_{L_p(\mathbb{R}^n)} \right)^{1/2}; \end{aligned}$$

and changing coordinates $z = mx$ we get:

$$= \varepsilon\, m^2\, m^{-n/p} \left(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+(1/2-4/p)2}} \|\nabla^2 f(z+mh) - \nabla^2 f(z)\|^2_{L_p(\mathbb{R}^n)} \right)^{1/2}.$$

Taking further $w = mh$, we obtain

$$< \varepsilon f(mx) >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)} = \varepsilon\, m^2\, m^{-n/p}\, m^{1/2-4/p} < f >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}.$$

So we find the following estimate for the semi-norm of the perturbation function

$$< \varepsilon\varphi(m|x|) >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)} \leq C\varepsilon\, m^{5/2-4/p-n/p}. \tag{32}$$

Now observe that we are able to find $p > \frac{2n+8}{3}$ and $\delta > 0$ such that

$$\frac{5}{2} - \frac{4}{p} - \frac{n}{p} + \delta < 2, \text{ since } \frac{5}{2} - \frac{12}{2n+8} - \frac{3n}{2n+8} = 1 < 2. \tag{33}$$

Inserting $\varepsilon = m^{-2+\delta}$ into (31), we see that (29) holds also true for $n \geq 8$ and for m sufficiently large. Thus the assumptions of Theorem 1.2 are fulfilled. Each of the surfaces constructed above will generate a global in time solution, converging eventually to a sphere.

Observe now that the assumptions $\varphi'(0) = 0$ and $\varphi''(0) = 1$ imply that $\nabla f(0) = 0$ and $\Delta f(0) = n$. Hence, in view of the $\mathcal{O}(n)$-invariance of f, a direct calculation gives that the mean curvature of $M^0_{\varepsilon,m}$ (i.e. of M_0) at the point 0 is given by

$$H_0 = 1 - \frac{\varepsilon m^2}{n} \Delta_{\mathbb{R}^n} f(0) = 1 - m^\delta < 0,$$

provided $m > 1$. Hence M_0 cannot be convex. This completes the proof of Theorem 1.3. □

5. **The linear system.** Our next task is to find a suitable linearization of system (18) and to prove an existence result with appropriate Schauder-type estimates. We propose the following linear problem

$$\begin{array}{lll} \phi_t + \overline{\Lambda^*\Lambda}\phi = f & \text{on} & S \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & S, \end{array} \tag{34}$$

where

$$\overline{\Lambda^*\Lambda} = \sum_l Z_l[\Delta^2 Z_l^{-1}(\eta^l \phi)], \tag{35}$$

with Δ being the Laplacian on $\mathbb{R}^n$, and $\{\eta^l\}$, $\{Z_l\}$ are given by (13) – see Section 3. The operator $\overline{\Lambda^*\Lambda}$ is an approximation of Δ_S^2. It should be viewed as an fourth order elliptic operator on S.

The parabolic character of the system of order four implies that (34) should be solved in $B^{4s,s}_{p,2}$-type spaces. From the analysis of the nonlinear system we obtain that the optimal factor is $4s = 5/2$ – see the *Remark* in the proof of Theorem 1.1 in Section 6. In this part we show the key result in proving Theorem 1.1.

Theorem 5.1. *Let $p > \frac{8}{3}$ and $\phi_0 \in B^{5/2-4/p}_{p,2}(S)$ and $f \in B^{-3/2,-3/8}_{p,2}(S \times (0,T))$, i.e. $f \in (B^{3/2,3/8}_{q,2}(S \times (0,T)))^*$, then there exists a unique solution to the system (34) such that*

$$\phi \in C([0,T]; B^{5/2-4/p}_{p,2}(S)) \text{ and } \phi \in B^{5/2,5/8}_{p,2}(S \times (0,T))$$

with the estimate

$$\begin{aligned}&\|\phi\|_{C([0,T];B^{5/2-4/p}_{p,2}(S))}+\|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))}\\&\le C(T,\lambda)\left(\|f\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T))}+\|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)}\right).\end{aligned}\tag{36}$$

First we prove a model version of Theorem 5.1 in the case $S=\mathbb{R}^n$.

Lemma 5.2. *Let $H\in B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n\times(0,T))$ and $\phi_0\in B^{5/2-4/p}_{p,2}(\mathbb{R}^n)$ be given and assume that these data are compactly supported. Then the problem*

$$\begin{aligned}\phi_t+\Delta^2\phi&=H &&\text{in}\quad \mathbb{R}^n\times(0,T),\\ \phi|_{t=0}&=\phi_0 &&\text{on}\quad \mathbb{R}^n\end{aligned}\tag{37}$$

has a unique solution in the class $B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times(0,T))$ and the following estimate

$$\begin{aligned}&\|\phi\|_{C([0,T];B^{5/2-4/p}_{p,2}(\mathbb{R}^n))}+\|\phi\|_{B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times(0,T))}\\&\le C(T)\left(\|H\|_{B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n\times(0,T))}+\|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}\right),\end{aligned}\tag{38}$$

holds true, where C may depend on diameter of supports of the data.

Proof. System (37) can be split into the following two problems

$$\begin{aligned}\phi_{1,t}+\Delta^2\phi_1&=H, &\quad \phi_{2,t}+\Delta^2\phi_2&=0 &&\text{in}\quad \mathbb{R}^n\times(0,T),\\ \phi_1|_{t=0}&=0, & \phi_2|_{t=0}&=\phi_0 &&\text{on}\quad \mathbb{R}^n.\end{aligned}\tag{39}$$

Clearly the solution to (37) is then the sum of ϕ_1 and ϕ_2.

Take the first system of (39). We extend H on the whole $\mathbb{R}^{n+1}$ simply by zero. Let $\tilde H$ denote this extension. Hence the first system of (39) can be extended in the whole $\mathbb{R}^{n+1}$ to

$$\tilde\phi_{1,t}+\Delta^2\tilde\phi_1=\tilde H\quad \text{in}\ \ \mathbb{R}^n\times\mathbb{R}.\tag{40}$$

with $\tilde H\in B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n\times\mathbb{R})$, which follows from (121). Solving (40), we get

$$\tilde\phi_1=\mathcal{F}^{-1}_{x,t}\left[\frac{1}{i\xi_0+|\xi|^4}\mathcal{F}_{x,t}[\tilde H]\right].\tag{41}$$

Applying now Marcinkiewicz's theorem to both of the multipliers $i\xi_0/(i\xi_0+|\xi|^4)$ and $|\xi|^4/(i\xi_0+|\xi|^4)$, we get the following estimate for the solution of (40).

$$<\tilde\phi_1>_{B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R})}\le C\|\tilde H\|_{B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n\times\mathbb{R})}.\tag{42}$$

Thus we conclude that the solution to $(39)_1$ satisfies $\phi_1\in B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times(0,T))$.

Next, we consider the second system of (39). To simplify the notation we write ϕ instead of ϕ_2. The Fourier transform applied to space directions yields the explicit solution

$$\phi(x,t)=\phi_2(x,t)=\mathcal{F}_x[\hat\phi_0e^{-|\xi|^4t}].\tag{43}$$

First, we study the regularity with respect to time. By the definition (8) we have

$$<\phi>_{B^{0,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)}=\left(\int_0^\infty\frac{dh}{h^{1+\frac{5}{8}2}}\|\phi(\cdot,\cdot)-\phi(\cdot,\cdot+h)\|^2_{L_p(\mathbb{R}^n\times\mathbb{R}_+)}\right)^{1/2}.\tag{44}$$

Since $L_p(\mathbb{R}^n)\supset B^0_{p,2}(\mathbb{R}^n)$, for $p\ge2$ we have

$$\|\phi(\cdot,\cdot)-\phi(\cdot,\cdot+h)\|^2_{L_p(\mathbb{R}^n\times\mathbb{R}_+)}\le C\left(\sum_{k=0}^\infty\|\phi_h^k\|^2_{L_p(\mathbb{R}^n\times\mathbb{R}_+)}\right)^{1/2},\tag{45}$$

where $\hat{\phi}_h^k = p_k(\xi)\hat{\phi}_0\,(e^{-|\xi|^4 t} - e^{-|\xi|^4(t+h)})$, where p_k are the partition of unity from the Paley-Littlewood decomposition (9)-(11). To apply Marcinkiewicz's theorem it is enough to check the condition from Theorem 2.1. It reduces to the analysis of the following expressions for $l \in N$

$$\left||\xi|^{4l}\left(t^l e^{-|\xi|^4 t} - (t+h)^l e^{-|\xi|^4(t+h)}\right)\right| = \left||\xi|^{4l}\int_t^{t+h}\frac{d}{ds}\left(s^l e^{-|\xi|^4 s}\right)ds\right|$$

$$= \left|\int_{|\xi|^4 t}^{|\xi|^4(t+h)}\frac{d}{d\tau}\left(\tau^l e^{-\tau}\right)d\tau\right| \le C_l\int_{|\xi|^4 t}^{|\xi|^4(t+h)} e^{-\tau/2}d\tau \le C_l(1-e^{-|\xi|^4 h/2})e^{-|\xi|^4 t/2}.$$

Then we obtain

$$\|\phi_h^k(\cdot,t)\|_{L_p(\mathbb{R}^n)} \le C(1-e^{-2^{k4+4}h})\,e^{-2^{k4-4}t}\,\|\phi_0^k\|_{L_p(\mathbb{R}^n)}.$$

Taking the norm with respect to time, we get

$$\|\phi_h^k\|_{L_p(\mathbb{R}^n\times\mathbb{R}_+)} \le C_p(1-e^{-2^{k4+4}h})\,2^{-\frac{k4}{p}}\,\|\phi_0^k\|_{L_p(\mathbb{R}^n)}. \tag{46}$$

Inserting (46) into (44) yields

$$< \phi >_{B^{0,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \le$$

$$C_p\left(\sum_{k=0}^{\infty}\int_0^\infty\frac{dh}{h^{1+5/4}}(1-e^{-2^{k4+4}h})^2\,2^{-k\frac{5}{2}2}\,2^{k(\frac{5}{2}-\frac{4}{p})2}\,\|\phi_0^k\|^2_{L_p(\mathbb{R}^n)}\right)^{1/2} = I_1. \tag{47}$$

To estimate the last integral we note that, taking $l = k+1$, we have

$$\int_0^\infty\frac{dh}{h^{1+5/4}}(1-e^{-2^{k4+4}h})^2\,2^{-k\frac{5}{2}2} = C\int_0^\infty\frac{dh}{h^{1+5/4}}(1-e^{-2^{4l}h})^2\,2^{-5l} = I_2.$$

So putting $w = 2^{4l}h$, we obtain:

$$I_2 \le C\int_0^\infty\frac{2^{-4l}\,dw}{2^{-4l(1+5/4)}\,w^{1+5/4}}(1-e^{-w})^2\,2^{-5l} = C\int_0^\infty\frac{dw(1-e^{-w})^2}{w^{1+5/4}} = M < \infty. \tag{48}$$

Hence inserting this estimate into (47) (I_2 to I_1), recalling definition (11), we conclude

$$< \phi >_{B^{0,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \le C\|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}. \tag{49}$$

Next, we consider regularity with respect to spatial variables. We have

$$< \phi >_{B^{5/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \le \left(\sum_{k=0}^\infty 2^{k\frac{5}{2}2}(\int_0^\infty\|\phi^k\|^p_{L_p(\mathbb{R}^n)}dt)^{2/p}\right)^{1/2} = I_3, \tag{50}$$

where here $\hat{\phi}^k = p_k\hat{\phi}$ – see (9)–(11). Applying Marcinkiewicz's theorem, remembering the form (43), we conclude

$$\|\phi^k(\cdot,t)\|_{L_p(\mathbb{R}^n)} \le Ce^{-2^{k4-4}t}\|\phi_0^k\|_{L_p(\mathbb{R}^n)}.$$

Hence

$$\left(\int_0^\infty\|\phi^k\|^p_{L_p(\mathbb{R}^n)}dt\right)^{1/p} \le C\frac{1}{(p2^{k4})^{1/p}}\|\phi_0^k\|_{L_p(\mathbb{R}^n)}. \tag{51}$$

Returning to (50) we have (see (11))

$$I_3 \le C_p\left(\sum_{k=0}^\infty 2^{k(5/2-4/p)2}\,\|\phi_0^k\|^2_{L_p(\mathbb{R}^n)}\right)^{1/2} = C_p\|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}. \tag{52}$$

Therefore it follows that

$$< \phi >_{B^{5/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \leq C_p ||\phi_0||_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}. \tag{53}$$

Summing up the estimates (42), (49) and (53), we obtain

$$\begin{aligned} &< \phi >_{L_\infty(\mathbb{R}_+;B^{5/2-4/p}_{p,2}(\mathbb{R}^n))} + < \phi >_{B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} \\ &\leq C(\|H\|_{B^{-3/2,-3/8}_{p,2}(\mathbb{R}^n\times\mathbb{R}_+)} + \|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)}), \end{aligned} \tag{54}$$

with a time independent constant C. Estimate (54) controls only the seminorms. To find the information about the whole norm extra conditions on the support of H and for ϕ_0, and finiteness of T are needed. (54) holds for any $p > 1$, so particularity it holds for p_* such that

$$B^{5/2,5/8}_{p_*,2} \subset L_p \text{ with } \|\phi\|_{L_p(\mathbb{R}^n\times\mathbb{R})} \leq C < \phi >_{B^{5/2,5/8}_{p_*,2}(\mathbb{R}^n\times\mathbb{R})}.$$

The above relations hold as $\frac{1}{p_*} - \frac{1}{p} = \frac{5/2}{n+2}$ with $p_* > 1$, which is true for $p > \frac{8}{3}$. Then we easily conclude (38) and the proof of Lemma 5.2 is completed. □

Now we are prepared to consider the problem in on the manifold S. Let

$$\begin{aligned} &\mathcal{L}\phi := \phi_t + \overline{\Lambda^*\Lambda}\phi = H && \text{in} && S\times(0,T), \\ &\phi|_{t=0} = 0 && \text{on} && S. \end{aligned} \tag{55}$$

The method of proving Theorem 5.1 via Lemma 5.2 is adapted from the classical approach to parabolic systems introduced in [17]. It is based on the regularizer operator, see the definition below, which somehow gives the solution by a procedure related to the Banach fixed point theorem. This technique has been effectively applied to similar problems in [21, 22].

By Lemma 5.2 and the properties of the atlas (U^l, Z_l) of S we are able to construct an extension $\tilde{\phi}_0$ of the initial data such that it belongs to $B^{5/2,5/8}_{p,2}(S\times(0,T))$, i.e.

$$\|\tilde{\phi}_0\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} \leq C||\phi_0||_{B^{5/2-4/p}_{p,2}(S)} \quad \text{and} \quad \tilde{\phi}_0|_{t=0} = \phi_0, \tag{56}$$

cf. formula (4) on page 3 in [13], or Chap. 24 in [3] and Chap. 5 in [29], respectively.

Thus we may consider a simplified version of (55) with the initial condition $\phi_0 \equiv 0$ by subtraction of $\tilde{\phi}$ and setting $\phi_{old} = \tilde{\phi}_0 + \phi_{new}$ and $H_{old} = \mathcal{L}\tilde{\phi}_0 + H_{new}$ into (34). By (56) the regularity of the data in (55) is preserved.

Definition 5.3. Introduce a function $\phi^k = R^k(\zeta^k H)$, where $\phi^k = Z_k^{-1*}[\bar{\phi}^k]$ is the solution of the following problem

$$\begin{aligned} &\bar{\phi}^k_t + \Delta^2\bar{\phi} = Z^*_k[\zeta^k H] && \text{in} && \mathbb{R}^n\times(0,T), \\ &\bar{\phi}|_{t=0} = 0 && \text{on} && \mathbb{R}^n. \end{aligned} \tag{57}$$

Then we define an operator $R : B^{-3/2,-3/8}_{p,2}(S\times(0,T)) \to B^{5/2,5/8}_{p,2}(S\times(0,T))$, called the regularizer, by

$$RH = \sum_k \pi^k\phi_k. \tag{58}$$

Clearly, R is linear and by Lemma 5.2 we know that it is bounded. The introduction of the regularizer allows to solve system (55) on short time intervals. The fundamental properties of R are explained in following two lemmas.

Lemma 5.4. *Assume that $T > 0$ is small enough. Then $\mathcal{L}RH = H + \mathcal{T}H$ and $\|\mathcal{T}\| \leq \frac{1}{2}$.*

Proof. From the definition of the operator R we obtain

$$\mathcal{L}RH = \sum_k \pi^k \mathcal{L}\phi^k + \sum_k \left(\mathcal{L}(\pi^k \phi^k) - \pi^k \mathcal{L}\phi^k \right). \tag{59}$$

Assume that supp $\eta^l \cap$ supp $\pi^k \neq \emptyset$. Then we conclude from (57) that

$$\bar{\eta}^l \bar{\phi}^k_t + \Delta^2(\bar{\eta}^l \bar{\phi}^k) = Z_k[\eta^l \pi^k H] + r_{lk}, \tag{60}$$

where the term r_{lk} has the following structure (in the highest order of derivatives): $r_{lk} \sim \lambda^{-1} \nabla \bar{\phi}^k$ supported on supp $\eta^l \pi^k$. Hence taking into account (60) and recalling the properties of η^l, we get

$$\sum_k \pi^k \mathcal{L}\phi^k = \sum_{k,l} \pi^k Z_k[\bar{\phi}^k_t + \Delta^2(\bar{\eta}^l \bar{\phi}^k)] = H + \sum_{k,l} Z_k[r_{lk}]. \tag{61}$$

Moreover the second term on the right hand side of (59) has a structure similar to r_{kl}, namely $\sum_k \mathcal{L}(\pi^k \phi^k) - \pi^k \mathcal{L}\phi^k \sim \lambda^{-1} \sum_k \nabla^3 \phi^k$.

To estimate these remainders, we apply the imbedding theorem (120)

$$\begin{aligned} \|\nabla^3 \phi^k\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T))} &\leq C\|\phi^k\|_{B^{3/2,3/8}_{p,2}(S\times(0,T))} \\ &\leq \varepsilon \|\phi^k\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} + c(\varepsilon)\|\phi^k\|_{L_p(S\times(0,T))} \\ &\leq (\varepsilon + c(\varepsilon)T^{1/p})\|H\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T))}, \end{aligned} \tag{62}$$

where we used the fact that

$$\sup_{t\leq T} < \phi^k(\cdot,t) >_{B^{5/2-4/p}_{p,2}(\mathbb{R}^n)} \leq C < \phi^k >_{B^{5/2,5/8}_{p,2}(\mathbb{R}^n\times(0,T))}, \tag{63}$$

cf. again formula (4) on page 3 in [13], or [3] Chap. 24, [29] Chap. 5. Observe that C is independent of T, since $\phi^k|_{t=0} \equiv 0$. Thus by (62) and the form of the remainder there exists $T_* > 0$ such that the norm of the operator $\mathcal{T}$ is less than $1/2$ on the time interval $[0,T]$ for $T \leq T_*$. This completes the proof of Lemma 5.4. □

Note that the constant in (62) must be small enough in order to absorb the terms of order λ^{-1}. This is possible by choosing T sufficiently small.

Lemma 5.5. *Assume that $T > 0$ is small enough. Then $R\mathcal{L}\phi = \phi + \mathcal{W}\phi$ and $\|\mathcal{W}\| \leq \frac{1}{2}$.*

Proof. Repeating the considerations from the proof of Lemma 5.4, we get

$$R\mathcal{L}\phi = \sum_k \pi^k R^k[\zeta^k(\sum_l Z_k[\bar{\eta}^l \bar{\phi}_l + \Delta^2(\bar{\eta}^l \bar{\phi}^l)])] = \phi + \mathcal{W}\phi,$$

with a suitable estimate for the norm of operator $\mathcal{W}$ for small $T \leq T*$. Thus we claim that Lemma 5.5 is true. Similar results with detailed proofs can be found in [22]. □

To prove Theorem 5.1 we first prove the existence of solutions to system (55) on a time interval $[0,T_*]$, where T_* is chosen such that Lemmas 5.4 and 5.5 hold true. From Lemmas 5.4 and 5.5 we conclude that

$$R\mathcal{L} = Id + \mathcal{W} \quad \text{and} \quad \mathcal{L}R = Id + \mathcal{T}. \tag{64}$$

By the smallness of the norms of $\mathcal{T}$ and $\mathcal{W}$ we get that the right hand side from (64) are invertible on a time interval $[0,T_*]$, so

$$(Id + \mathcal{W})^{-1}R\mathcal{L} = Id \qquad \text{and} \qquad \mathcal{L}R(Id + \mathcal{T})^{-1} = Id.$$

Thus, the solution to (55) is given by the following formula

$$\phi = (Id + \mathcal{W})^{-1} RH \quad \text{ on } S \times (0,T) \quad \text{for} \quad T \leq T_*. \tag{65}$$

The definition of R and Lemma 5.2 imply that the solution to (55) fulfills

$$\|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} \leq C\|H\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T))} \qquad \text{for} \quad T \leq T_*. \tag{66}$$

Thus applying remark (56), we obtain the solvability of the problem (34) on the time interval $[0, T_*]$. Moreover we have

$$\|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T_*))} \leq C\left(\|H\|_{B^{-3/2,-3/8}_{p,2}(S\times(0,T_*))} + \|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)}\right). \tag{67}$$

Next, by the trace theorem we extend the existence of the solutions to $[T_*, 2T_*]$, $[2T_*, 3T_*]$, etc. So we finally obtain the estimate (36) with a constant depending on T. This completes the proof of Theorem 5.1.

Remark 1. It is worthwhile to underline that the magnitude of the constant in (67) is independent of λ and T_*. It depends only on the geometry of S. However if we choose an atlas with possible smaller λ, then T_* will be smaller too. On the other hand (67) holds for any $T \in (0, T_*]$ for the same constant in (67). This remark is crucial in the treating of the nonlinear problem in the next section – see in particular (83). The simplest solution to avoid the mentioned difficulty is just to use this 'universal' constant from (67), and then prescribe first λ and then T.

6. **Local in time existence.** In this section we prove Theorem 3.1 which implies the unique solvability of system (1) in the sense of Theorem 1.1. The system (1) will be considered in the following form

$$\begin{array}{rcll} \phi_t &=& -L^\phi \tilde{\Delta}_{M(t)} \tilde{H}_{M(t)} & \text{in} \quad S \times (0,T), \\ \phi|_{t=0} &=& \phi_0 & \text{on} \quad S, \end{array} \tag{68}$$

where L^ϕ is defined by

$$L^\phi = (\vec{n} \cdot \vec{n}^t)^{-1}, \tag{69}$$

$\vec{n}$ and $\vec{n}^t$ are normal vectors to S and $M(t)$, respectively. Additionally

$$\|\vec{n}^t\|_{L_\infty(S\times(0,T))} \leq C(1 + \|\phi\|_{L_\infty((0,T),W^1_\infty(S))}), \tag{70}$$

where the constant C depends only on S.

The goal of this section is to prove existence and uniqueness of weak solutions to (68) in the following sense:

Definition 6.1. We say that a function $\phi : S \times (0,T) \to \mathbf{R}$ is a weak solution to problem (18) iff $\phi \in B^{5/2,5/8}_{p,2}(S \times (0,T))$ with $p > \frac{2n+8}{3}$ and $\phi|_{t=0} = \phi_0$ and the following identity is valid

$$(\phi_t, \pi)_{L_2(S)} + (\tilde{H}_{M(t)}\phi, \tilde{\Delta}_{M(t)}[L^\phi \pi])_{L_2(S)} = 0 \tag{71}$$

in the distributional sense on $[0,T)$ for each $\pi \in B^{3/2,3/8}_{q,2}(S\times(0,T))$ with $1/p + 1/q = 1$.

Proof. The assertion of Theorem 3.1 will follow from Banach's fixed point theorem and an application of Theorem 5.1 to the weak formulation (71). Since we are looking for local in time solutions we shall always underline the dependence of T.

The solution to (18) will be constructed in the set

$$\begin{aligned}\Xi = \{u \in B^{5/2,5/8}_{p,2}(S\times(0,T)) \quad \text{with} \quad p > \tfrac{2n+8}{3} \quad \text{and} \quad u|_{t=0} = \phi_0 \\ \text{such that} \quad \|u\|_\Xi = < u >_{B^{5/2,5/8}_{p,2}(S\times(0,T))} + \|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)} < \delta\}.\end{aligned} \tag{72}$$

Let us emphasize that the norm in (72) is equivalent to the standard norm in $B^{5/2,5/8}_{p,2}(S\times(0,T))$, but this form gives us directly the trace theorem (in time) and an inequality with the constant independent of T – see (63). In other words we control all constants in following considerations uniformly in time. This is of particular importance, since T has to be small. In order to underline the domain or the time interval we shall write $\Xi(S\times(0,T))$ or just $\Xi(0,T)$.

Let now

$$\varepsilon := \|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)}. \tag{73}$$

Our goal is to show if $\varepsilon > 0$ is small enough the there exists $\delta > \varepsilon$ such that the mapping

$$K : \Xi \to \Xi \text{ is a contraction, where } K(\psi) = \phi, \tag{74}$$

is the solution to the following problem

$$\begin{aligned}(\phi_t,\pi)_{L_2(S)} + (\Lambda\phi,\Lambda\pi)_{L_2(S)} \\ = (\Lambda\psi - \tilde H_{N(t)}, \tilde\Delta_{N(t)}[L^\psi\pi])_{L_2(S)} + (\Lambda\psi, \Lambda\pi - \tilde\Delta_{N(t)}[L^\psi\pi])_{L_2(S)}\end{aligned} \tag{75}$$

in the distributional sense on $[0,T)$ for each $\pi \in B^{3/2,3/8}_{q,2}(S\times(0,T))$ with $1/p+1/q = 1$. In (75) $N(t)$ is the surface generated by ψ via definition (16), and L^ψ is the version of (70) for $N(t)$. Let us explain the notation in (75). Here we use Λ as an approximation of Δ_S – see (35). We put $\Lambda\phi = \sum_l Z^l[\Delta Z_l^{-1}(\eta^l\phi)]$. We note that $\Lambda^2 = \overline{\Lambda^*\Lambda}$ up to lower terms.

Applying Theorem 5.1 to system (75), we get

$$\begin{aligned}\|\phi\|_{L_\infty(0,T;B^{5/2-4/p}_{p,2}(S))} + \|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} \le C(\|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)} \\ + \sup_\pi \Big|\int_0^T (\Lambda\psi - \tilde H_{N(t)}, \tilde\Delta_{N(t)}[L^\psi\pi])_{L_2(S)}dt\Big| \\ + \sup_\pi \Big|\int_0^T (\Lambda\psi, \Lambda\pi - \tilde\Delta_{N(t)}[L^\psi\pi])_{L_2(S)}dt\Big|),\end{aligned} \tag{76}$$

where the supremum is taken over all $\pi \in B^{3/2,3/8}_{q,2}(S\times(0,T))$ with norm 1.

To find the estimate of the right hand side of (76), we consider the terms in local maps. The highest order expression has the following form

$$\int_0^T \int_{U^l} (\Delta\psi - A(\nabla\psi) : \nabla^2\psi)(A(\nabla\psi) : \nabla^2[L^\psi\pi])\eta^l dxdt, \tag{77}$$

where $|L^\psi| \le C(1+|\nabla\psi|)$. The matrix A fulfills the following estimate

$$\|id - A(\nabla\psi)\|_{L_\infty(U^l\times(0,T))} \le C\lambda^a(1+\|\nabla\psi\|_{C^a(S\times(0,T))}) \tag{78}$$

for all $\psi \in \Xi$, with a constant C which depends only on S. Indeed (78) measure the difference between Λ and $\tilde\Delta_{N(t)}$ in terms of the highest derivatives. But in a single chart Λ may be treated as Δ. Without loss of generality we may choose $x^l \in U^l$ such that at the initial time $t = 0$ we have that

$$\tilde\Delta_{N(t)}\psi|_{(x^l,0)} = \Delta\psi|_{(x^l,0)} + \text{lower terms of } \psi.$$

In fact we may take here ϕ_0. Hence if λ is sufficiently small and diam $U^l \le \lambda$ and $T \le \lambda$, then for $x \in U^l$ and $t \in [0,T]$ we have

$$|\tilde{\Delta}_{N(t)}\psi - \Delta\psi| \le C\lambda^a|\nabla\psi| + \text{lower terms of } \psi,$$

where C depends on $\|S\|_{C^a} + \|\nabla\psi\|_{C^a}$. This implies that (78) holds true. As a consequence we are required to control integrals with the following structure

$$\int_0^T \int_{U^l} (\nabla^2\psi : \nabla^2[(1+\nabla\psi)\pi])\eta^l dxdt \text{ with } \phi \in B^{5/2,5/8}_{p,2}, \pi \in B^{3/2,3/8}_{q,2}. \tag{79}$$

But $\nabla^2\psi \in B^{1/2,1/8}_{p,2}$, hence we have to prove that

$$\nabla^2[(1+\nabla\psi)\pi] \in B^{-1/2,-1/8}_{q,2}(S \times (0,T)), \tag{80}$$

which holds true because we have

$$\nabla^3\psi\,\pi \in B^{-1/2,-1/8}_{q,2}, \qquad \nabla^2\psi\cdot\nabla\pi \in B^{-1/2,-1/8}_{q,2}, \quad \text{and } \nabla\psi\cdot\nabla^2\pi \in B^{-1/2,-1/8}_{q,2} \tag{81}$$

for $\psi \in B^{5/2,5/8}_{p,2}$, $\pi \in B^{3/2,3/8}_{q,2}$. The last inclusion is obvious, the first one follows from (122) and the second one from (123) – see the Appendix.

Remark 2. (79) explains the choice of regularity of our solutions. In order to estimate it we need 5 derivatives acting on ψ and π. Hence the space of solutions requires at least 5/2 derivative. Next, inclusions (81) and restrictions coming from the multiplication results for the Besov spaces imply that $p > \frac{2n+8}{3}$ – see Proposition 8.3. In this way we find our space $B^{5/2,5/8}_{p,2}$ with the trace in the $B^{5/2-4/p}_{p,2}$-space. This reasoning explains why our result is close to optimal.

The regularity given by Ξ implies $\nabla\psi \in L_\infty$ – even the Hölder continuity in time and space. Remembering that $\nabla\psi \in C^{a,a/4}(S \times (0,T))$ which follows from (120), lower terms (in derivatives of functions ψ and π) from (76) can be estimated by

$$C(1/\lambda)T^a(\|\psi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} + \|M_0\|_{B^{5/2-4/p}_{p,2}})\|\pi\|_{B^{3/2,3/8}_{q,2}(S\times(0,T))}. \tag{82}$$

We emphasize that the estimates for the lower terms in local coordinates are connected to estimates of terms with derivatives of the localizers ζ^k, π^k and η^k. From (15) we know that these quantities are bounded only by $\frac{1}{\lambda}$. However thanks to (63) we estimate them with a constant T^a for some $a > 0$. Thus a suitable small T will absorb the influence of $\frac{1}{\lambda}$ – see (82) and the Remark at the end of the previous Section 5.

Our considerations lead to the following bound of the left hand side of (76)

$$\begin{aligned} &\|\phi\|_{L_\infty(0,T;B^{5/2-4/p}_{p,2}(S))} + \|\phi\|_{B^{5/2,5/8}_{p,2}(S\times(0,T))} \le C\Big(\|\phi_0\|_{B^{5/2-4/p}_{p,2}(S)} \\ &+T^a\|M_0\|_{B^{5/2-4/p}_{p,2}} + \Big(\varepsilon + C(1/\lambda)T^a + \lambda^a\|M_0\|_{B^{5/2-4/p}_{p,2}} + \|\psi\|_\Xi\Big)\|\psi\|_\Xi\Big). \end{aligned} \tag{83}$$

Thanks to the smallness of ε and λ, we find $\delta > 0$ such that $K : \Xi \to \Xi$ – see (72).

The proof of the contraction property of K on Ξ is almost the same as for (83). So repeating all steps done for (75), we get

$$\|K(\psi_1) - K(\psi_2)\|_\Xi \le 1/2\|\psi_1 - \psi_2\|_\Xi, \tag{84}$$

provided T sufficiently small.

This shows the unique solvability of system (18) in the meaning of definition (71). Theorems 3.1 and 1.1 are therefore proved. □

7. **Stability of spheres.** In this part we establish stability of spheres in the $B^{5/2-4/p}_{p,2}$-topology. For simplicity we focus our analysis on the behaviour of solutions to (1) in a neighbourhood of the unit sphere. Since the required regularity guarantees that the normal vector is at least Hölder continuous we can choose $\mathbb{S}^n$ as the reference surface. Moreover rescaling the temporal variable by the factor L^ϕ, we may re-express system (68) as

$$\phi_t + G(\phi) = 0, \qquad \phi|_{t=0} = \phi_0, \tag{85}$$

where $G(\phi(t)) := \tilde{\Delta}_{M(t)} \tilde{H}_{M(t)}$. Our analysis is concerned with the stability of the zero solution, so we need the Fréchet derivative of G at 0. To verify that G is Fréchet differentiable from $B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$ to $B^{-3/2-4/p}_{p,2}(\mathbb{S}^n)$ we first observe that G is a quasilinear differential operator of fourth order with coefficients which are rational expressions in ϕ and $\nabla\phi$. Thus it suffices to verify that G maps $B^{5/2-4/p}_{p,2}(\mathbb{S}^n)$ into $B^{-3/2-4/p}_{p,2}(\mathbb{S}^n)$. To do so, fix $t \in [0, T]$. The coefficients of $\tilde{H}_{M(t)}$ are of class $B^{3/2-4/p}_{p,2}(\mathbb{S}^n)$ and thus $\tilde{H}_{M(t)} \in B^{1/2-4/p}_{p,2}(\mathbb{S}^n)$, cf. [2] Theorem 4.1. Writing further $g_{jk}(\phi)$ for the metric on $\mathbb{S}^n$ induced by $\phi(t)$, we have that $g_{jk}(\phi) \in B^{3/2-4/p}_{p,2}(\mathbb{S}^n)$. Thus invoking Theorem 4.3 in [2], we get $g_{jk}(\phi)\partial_k \tilde{H}_{M(t)} \in B^{-1/2-4/p}_{p,2}(\mathbb{S}^n)$. This implies that $\partial_j(g_{jk}(\phi)\partial_k \tilde{H}_{M(t)}) \in B^{-3/2-4/p}_{p,2}(\mathbb{S}^n)$ and therefore $G(\phi(t)) \in B^{-3/2-4/p}_{p,2}(\mathbb{S}^n)$ for $t \in [0, T]$. The Fréchet derivative of G at 0 can be calculated by using Lemmas 3.1 and 3.2 in [11]. Summarizing we have:

Lemma 7.1. *Let*

$$A = \frac{1}{n}\Delta^2_{\mathbb{S}^n} + \Delta_{\mathbb{S}^n}, \tag{86}$$

where $\Delta_{\mathbb{S}^n}$ is the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^n$, then

$$\partial G(0)h = Ah \text{ for } h \in B^{5/2-4/p}_{p,2}(\mathbb{S}^n). \tag{87}$$

Furthermore the spectrum A consists of a sequence of real eigenvalues.

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < ..., \tag{88}$$

where λ_0 is of multiplicity $n + 2$.

The properties of operator A allow us to introduce the following decomposition of the phase space

$$X = X_c \oplus X_s, \text{ where } X = B^{5/2-4/p}_{p,2}(\mathbb{S}^n), \tag{89}$$

$X_c = span\{Y_0, Y_1, ..., Y_{n+1}\}$ and $Y_0, ..., Y_{n+1}$ are eigenvectors for the eigenvalue $\lambda_0 = 0$, $Y_0 = 1$ and $Y_1, ..., Y_{n+1}$ are homogeneous functions of degree one. In the canonical setting they are just $Y_l = x_l$. Furthermore we define projections P_c and P_s on the subspaces X_c and X_s, respectively. The space X_s is called the stable subspace since the spectrum of A here is strictly positive, i.e. spec $A|_{X_s} \subset [\lambda_1, +\infty)$. Since X_c is finite dimensional, the decomposition (89) is well defined.

It is convenient to we restate (85) in the following way:

$$\phi_t + A\phi = g(\phi), \text{ where } g(\phi) = A\phi - G(\phi). \tag{90}$$

In order to analyse (90), we have to have a closer look at g. Thanks to (87) we have

$$G(\phi) = A\phi + \frac{1}{2}\partial^2 G(0)(\phi, \phi) + o(\|\phi\|^2) \qquad \text{as} \quad \phi \to 0, \tag{91}$$

so

$$g(\phi) = \frac{1}{2}\partial^2 G(0)(\phi, \phi) + o(\|\phi\|^2) \qquad \text{as} \quad \phi \to 0. \tag{92}$$

This point of view on the right hand side of (90) is helpful in the following considerations. In particular, we have (see definition (72))

Proposition 7.2. *Let $T > 0$, then there exists $M > 0$, $\delta > 0$, such that there exists a unique solution to (90) on time interval $[0, T]$ fulfilling the following a priori estimate*

$$\|\phi\|_{\Xi(\mathbb{S}^n \times (0,T))} \leq M\varepsilon, \tag{93}$$

provided $\|\phi_0\|_{B^{5/2-4/p}_{p,2}(\mathbb{S}^n)} \leq \varepsilon$ and $0 < \varepsilon \leq \delta$.

In fact, Proposition 7.2 follows from the results and methods presented in Sections 5 and 6 as well as from Lemma 7.1 and the Taylor expansion (91). In order to avoid too much technicalities we omit the proof here. Results of this type are called "almost global in time existence" and reflect the continuity of the solution with respect to the initial datum, remembering that 0 is an equilibrium. The proof is standard for regular parabolic systems and follows from the application of Banach's fixed point theorem in the proof of the local in time existence theorem. Details in our setting can be found in [23].

Our next goal is to parametrise the set of all small static solutions to (90). Eventually it will appear that they build a local unique centre manifold for (85). Let S_z be a sphere close to $\mathbb{S}^n$ and let $(z_1, ..., z_{n+1})$ be the coordinates of its centre. Then z_0 is chosen such that $1 + z_0$ is the radius of S_z. By the properties of Y_l we find that

$$(1 + z_0)^2 = \sum_{l=1}^{n+1} ((1 + \psi(z))Y_l - z_l)^2,$$

where ψ is the distance to $\mathbb{S}^n$. Straightforward calculations lead us to the following representation of the distance to the original sphere

$$\psi(z) = \sum_{l=1}^{n+1} z_l Y_l - 1 + \sqrt{(\sum_{l=1}^{n+1} z_l Y_l)^2 + (1 + z_0)^2 - \sum_{l=1}^{n+1} z_l^2}. \tag{94}$$

We restrict our attention to a neighbourhood of zero in $\mathbb{R}^{n+2}$. Then we get

$$\partial\psi(0)h = \sum_{l=0}^{n+1} h_l Y_l \text{ for } h \in \mathbb{R}^{n+2}, \tag{95}$$

i.e. $P_c \partial\psi(0) = Id$. The identity (95) implies that $P_c\psi : \mathcal{O} \to \mathcal{O}'$, where $\mathcal{O}, \mathcal{O}'$ are suitable small neighbourhoods of zero in $X_c \simeq \mathbb{R}^{n+2}$. The inverse function theorem implies that there exists a smooth diffeomorphism $m : \mathcal{O}' \to \mathcal{O}$ such that

$$P_c\psi(m(h)) = h \text{ for } h \in \mathcal{O}' \subset X_c \simeq \mathbb{R}^{n+2}. \tag{96}$$

So the set of all S_z sufficiently close to $\mathbb{S}^n$ is represented in X by

$$\mathcal{M} = \left\{ \sum_{l=0}^{n+1} h_l Y_l + P_s\psi(m(h)) : h \in \mathcal{O}' \right\}. \tag{97}$$

The set $\mathcal{M}$ is a smooth (n+2)-dimensional manifold in X, tangent to X_c at zero, so by the properties of m, P_c and (94) we find

$$\|P_s\psi(m(h))\|_X \leq \frac{1}{2}\|h\|_{\mathbb{R}^{n+2}} \text{ for } h \in \mathcal{O}'. \tag{98}$$

Summarizing the above analysis, we get that the set of all small static solutions to (90) is described in the following way

$$\mathcal{M} = \{z + \sigma(z) : z \in \mathcal{O}'\} \text{ with } \sigma := P_s\psi \circ m \text{ and } \nabla\sigma(0) = 0. \tag{99}$$

Now we want to apply (99) for our investigations of (85). Using (89) the system (90) can be decomposed in the following way:

$$\begin{aligned} (P_c\phi)_t + P_cA(P_c\phi) &= P_cg(\phi),\\ (P_s\phi)_t + P_sA(P_s\phi) &= P_sg(\phi). \end{aligned} \tag{100}$$

Additionally from Lemma 7.1 we get

$$A\phi - g(\phi)|_{\mathcal{M}} \equiv 0, \text{ hence } P_cg(\phi)|_{\mathcal{M}} \equiv 0. \tag{101}$$

Now we want to show that all small solutions close to the unit sphere (or rather to $\mathcal{M}$) are global in time and stay in a neighbourhood of $\mathbb{S}^n$. In order to prove this we reduce equations (100). Note that the set of all small equilibria (99) is the graph of the function $\sigma : P_c\mathcal{M} \to X_s$ in the sense that $z + \sigma(z) \in \mathcal{M}$ with $z \in P_c\mathcal{M}$. Consequently, for any $z \in P_c\mathcal{M}$, by (101) there holds $P_sg(z + \sigma(z)) = P_sA\sigma(z)$, too. Hence for fixed $z \in P_c\mathcal{M}$ we can reformulate the second equation of (100) as follows

$$(P_s\phi - \sigma(z))_t + P_sA[P_s\phi - \sigma(z)] = P_s[g(\phi) - g(z + \sigma(z))], \tag{102}$$

The mild solution to (7.18) is given by the variation-of-constant formula

$$P_s\phi(t) - \sigma(z) = e^{-P_sAt}(P_s\phi_0 - \sigma(z)) + \int_0^t e^{-P_sA(t-s)}P_s[g(\phi) - g(z+\sigma(z)]ds, \tag{103}$$

cf. Section 4.2 in [18]. The quantity $P_s\phi - \sigma(z)$ measures the distance to $\mathcal{M}$ in X_s.

Next we remark that in view of (92) we have

$$\|g(\phi) - g(z + \sigma(z))\|_Y \le (P_s\phi - \sigma(z))O(\|\phi\| + \|z\|)\|\phi - z - \sigma(z)\|_X, \tag{104}$$

where $Y := B^{-3/2-4/p}_{p,2}(\mathbb{S}^n)$.

Now we start with the proof of existence of global in time solutions. Let ϕ_0 be so small that for a given $\varepsilon_0 > 0$ there holds

$$\|\phi_0\|_X + \|P_c\phi_0\|_X + \|P_s\phi_0 - \sigma(P_c\phi_0)\|_X \le \varepsilon_0. \tag{105}$$

The size of ε_0 will be determined later. However it is clear that $P_c\phi_0 \in \mathcal{O}' \subset P_c\mathcal{M}$, in other words ϕ_0 has to be sufficiently close to $\mathcal{M}$.

The idea of the proof of global in time solutions is to show that our solution will stay near $\mathcal{M}$. First we choose a point at $\mathcal{M}$ determined by the initial datum. Then we will consider the solution/perturbation on a time interval $[0, T]$ with sufficiently large T. This is possible by Proposition 7.2 and the smallness required by (105). Next, we will choose again a point at $\mathcal{M}$ and start over our consideration with initial time T. The obtained information should guarantee that the solution is always near $\mathcal{M}$. By induction we then get a globally defined solution. Let us underline that T is fixed within the below analysis.

In order to realize the above idea we define the following sequence of systems of equations on $[0,T]$:

$$\begin{aligned}(P_c\phi^k - z_k)_t + P_cA(P_c\phi^k - z_k) &= P_c[g(\phi^k) - g(z_k + \sigma(z_k))],\\ (P_s\phi^k - \sigma(z_k))_t + P_sA(P_s\phi^k - \sigma(z_k)) &= P_s[g(\phi^k) - g(z_k + \sigma(z_k))],\\ P_c\phi^k(0) - z_k = 0, \qquad P_s\phi^k(0) - \sigma(z_k) &= P_s\phi^{k-1}(T) - \sigma(z_k),\\ z_k = P_c\phi^{k-1}(T) \text{ and } \|P_s\phi_k(0) - \sigma(z_k)\|_X &\leq \varepsilon_k,\end{aligned} \tag{106}$$

for $k \geq 1$ and with $\phi^0 = \phi_0$ — the initial datum from (85). The above system describes the evolution of the distance between (90) and equilibrium state described by z_k, where z_k is the projection of the solution in the previous step, i.e. for ϕ^{k-1}. Our method will work as long as the following condition will hold

$$\|\phi(t)\|_{\Xi(t-T,t)} \leq \Delta \text{ for } T \leq t < T_{max} \tag{107}$$

and our goal is to show that $T_{max} = \infty$. The size of Δ will be determined a few steps later, however it should be viewed as a constant fixed for the whole procedure.

Thanks to (106) and Lemma 7.1 we find that

$$P_cA(P_c\phi^k - z_k) \equiv 0 \text{ and } \|e^{-P_sAt}\| \leq Be^{-\lambda t}, \tag{108}$$

where $B > 0$ and $\lambda \in (0, \lambda_1)$, cf. Proposition 2.3.1. in [18].

Then the structure of the nonlinear problem – the right hand side being controlled by (104) – and the analysis for the local in time result of Section 6 imply that

$$\begin{aligned}\|P_c\phi^k(T) - z_k\|_X \leq B_2(T)\|\phi\|_\Xi\|\phi^k - z_k - \sigma(z_k)\|_\Xi,\\ \|P_s\phi^k(T) - \sigma(z_k)\|_X\\ \leq Be^{-\lambda T}\|P_s\phi^k(0) - \sigma(z_k)\|_X + B_2(T)\|\phi\|_\Xi\|\phi^k - z_k - \sigma(z_k)\|_\Xi.\end{aligned} \tag{109}$$

In order to control the right hand side of (109) we analyse the system (106) in the form:

$$\begin{aligned}(\phi^k - z_k - \sigma(z_k))_t + A(\phi^k - z_k - \sigma(z_k)) &= g(\phi^k) - g(z_k + \sigma(z_k)),\\ \phi^k(0) - z_k - \sigma(z_k) &= P_s\phi^{k-1}(T) - \sigma(P_c\phi^{k-1}(T)).\end{aligned} \tag{110}$$

Then using the results of Section 6 similarly as to derive Proposition 7.2, we are able to prove the following almost global in time result for (110).

Proposition 7.3. *Let $T > 0$, $\phi^k(0) - z_k - \sigma(z_k) \in X$, and assume that $\|\phi^k\|_{\Xi(0,T)} \leq \Delta$. Then there exist M and δ, both independent of Δ and T, such that*

$$\|\phi^k - z_k - \sigma(z_k)\|_{\Xi(0,T)} \leq M\varepsilon, \tag{111}$$

provided $\|\phi^k(0) - z_k - \sigma(z_k)\|_X \leq \varepsilon$ and $0 < \varepsilon \leq \delta$.

An application of Proposition 7.3 to (110) leads us to the following bound

$$\|P_c\phi^k(T) - z_k\|_X + \|P_s\phi^k(T) - \sigma(z_k)\|_X \leq Be^{-\lambda T}\varepsilon_k + B_2(T)\Delta M\varepsilon_k \leq \frac{1}{2}\varepsilon_k, \tag{112}$$

provided

$$Be^{-\lambda T} + B_2(T)\Delta M \leq \frac{1}{2}. \tag{113}$$

The above condition is crucial in our setting. It determines the important constants in the whole procedure. First, we choose T so large that $Be^{-\lambda T} \leq \frac{1}{4}$, then $B_2(T)$ is defined and give the condition on Δ, i.e. $\Delta \leq \frac{1}{4B_2(T)M}$.

Let us now consider ε_{k+1}. We have to find a good bound on this quantity being a bound of the initial datum for the next step. By (106) - in order to control $P_s\phi^{k+1}(0) - \sigma(z_{k+1})$ - we first note that

$$\|P_s\phi^k(T) - \sigma(P_c\phi^k(T))\|_X \leq \|P_s\phi^k(T) - \sigma(z_k)\|_X + \|\sigma(P_c\phi^k(T)) - \sigma(z_k)\|_X. \quad (114)$$

Remembering that $\mathcal{M}$ is tangent to X_c at zero, so in the neighbourhood of zero the function σ is continuous with the Lipschitz constant less than one – see (98) and (99), hence (112) implies that

$$\|P_s\phi^k(T) - \sigma(P_c\phi^k(T))\|_X \leq \|P_s\phi^k(T) - \sigma(z_k)\| + \|P_c\phi^k(T) - z_k\| \leq \frac{1}{2}\varepsilon_k. \quad (115)$$

Thus we can put $\varepsilon_{k+1} := \varepsilon_k/2$, i.e.

$$\|P_s\phi^k(T) - \sigma(P_c\phi^k(T))\|_X \leq \varepsilon_{k+1} = \frac{1}{2}\varepsilon_k. \quad (116)$$

Particularly (112) implies that

$$\|z_{k+1} - z_k\|_X \leq \frac{1}{2^k}\varepsilon_0 \quad (117)$$

which by definition of ϕ^k implies

$$\|\phi(kT)\| \leq \|\phi_0\| + 2\varepsilon_0 \leq 3\varepsilon_0 \quad (118)$$

and the right hand side is independent of k and so small that Proposition 7.2 implies the existence of a solution on the time interval $[0,T]$ with estimate (107). This gives the relation $\varepsilon_0 = \varepsilon_0(\Delta)$, and closes the chain of relations between the constants T, Δ, ε_0. Hence we conclude that our procedure holds for all $k \in \mathbb{N}$ and ϕ is the globally defined solution.

Additionally, thanks to (117) there exists z^* such that $\lim z_k = z^*$ in X. By definition $z^* \in X_c$, so it prescribes $\phi^* = z^* + \sigma(z^*) \in \mathcal{M}$. Finally (112) shows that

$$\|\phi(t) - \phi^*\|_X \to 0 \text{ as } t \to \infty. \quad (119)$$

The proof of Theorem 1.2 is thus completed.

8. Appendix.

Let us first we recall some basic facts concerning Besov spaces. For the corresponding proofs we refer to e.g. [3],[29],[28]. A crucial role is played by the following imbedding theorems.

Proposition 8.1. *We have:*

$$\begin{aligned} B^s_{p,2}(\mathbb{R}^n) \subset L_m(\mathbb{R}^n), \quad &if \quad \frac{n}{s}(\frac{1}{p} - \frac{1}{m}) \leq 1, \\ B^{s,s'}_{p,2}(\mathbb{R}^n \times (0,T)) \subset L_m(\mathbb{R}^n \times (0,T)), \quad &if \quad (\frac{n}{s} + \frac{1}{s'})(\frac{1}{p} - \frac{1}{m}) \leq 1. \end{aligned} \quad (120)$$

The next result concerns $B^{3/2,3/8}_{q,2}(\mathbb{R}^n \times (0,T))$ for $q < 8/5$. One can check elementary that these spaces can be obtained as the closure of compactly supported functions:

Proposition 8.2. *We have*

$$B^{3/2,3/8}_{q,2}(\mathbb{R}^n \times (0,T)) = \overline{C^\infty_0(\mathbb{R}^n \times (0,T))}^{\|\cdot\|_{B^{3/2,3/8}_{q,2}}}. \quad (121)$$

The above fact allows to extend any function from this class trivially by zero. It follows that the constant in the imbedding inequality from (120) does not depend on smallness of T which will be important in our considerations.

Next we prove a result which controls the regularity of the product of two functions from Besov spaces. It is an important tool for analysis of the nonlinear terms appeared in Section 6.

Proposition 8.3. *Let* $p > \frac{2n+8}{3}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) Let $f \in B^{-1/2,-1/8}_{p,2}(S \times (0,T))$ *and* $g \in B^{3/2,3/8}_{q,2}(S \times (0,T))$.
Then $fg \in B^{-1/2,-1/8}_{q,2}(S \times (0,T))$ *and*

$$\|fg\|_{B^{-1/2,-1/8}_{q,2}(S\times(0,T))} \leq C\|f\|_{B^{-1/2,-1/8}_{p,2}(S\times(0,T))}\|g\|_{B^{3/2,3/8}_{q,2}(S\times(0,T))}, \tag{122}$$

(ii) Let $f \in B^{1/2,1/8}_{p,2}(S \times (0,T))$ *and* $g \in B^{1/2,1/8}_{q,2}(S \times (0,T))$.
Then $fg \in B^{-1/2,-1/8}_{q,2}(S \times (0,T))$ *and*

$$\|fg\|_{B^{-1/2,-1/8}_{q,2}(S\times(0,T))} \leq C\|f\|_{B^{1/2,1/8}_{p,2}(S\times(0,T))}\|g\|_{B^{1/2,1/8}_{q,2}(S\times(0,T))}. \tag{123}$$

Proof. All calculations are done in local coordinates of the surface S. By (121) we are able to assume that the function g is defined on the whole $S \times \mathbb{R}$. This property implies that constants in (122) and (123) are independent of T.

In order to prove (i) it is enough to show that $gk \in B^{1/2,1/8}_{q,2}$ for $k \in B^{1/2,1/8}_{p,2}$. Then $fgk \in L_1$ is controlled. Recalling the definition (8), we find

$$\begin{aligned} < gk >_{B^{1/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R})} \leq & (\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\frac{1}{2}2}}\|[k(\cdot+h,\cdot)-k(\cdot,\cdot)]g(\cdot,\cdot)\|^2_{L_q(\mathbb{R}^n\times\mathbb{R})})^{1/2} \\ & +(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\frac{1}{2}2}}\|k(\cdot,\cdot)[g(\cdot+h,\cdot)-g(\cdot,\cdot)\|^2_{L_q(\mathbb{R}^n\times\mathbb{R})})^{1/2} = I_1 + I_2. \end{aligned} \tag{124}$$

The last two expressions can be estimated as follows

$$\begin{aligned} I_1 \leq & (\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\frac{1}{2}2}}\|k(\cdot+h,\cdot)-k(\cdot,\cdot)\|^2_{L_p(\mathbb{R}^n\times\mathbb{R})}\|g(\cdot,\cdot)\|^2_{L_m(\mathbb{R}^n\times\mathbb{R})})^{1/2} \\ \leq & \|k\|_{B^{1/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R})}\|g\|_{L_m(\mathbb{R}^n\times\mathbb{R})} \leq C\|k\|_{B^{1/2,0}_{p,2}(\mathbb{R}^n\times\mathbb{R})}\|g\|_{B^{3/2,3/8}_{q,2}(\mathbb{R}^n\times\mathbb{R})} \end{aligned} \tag{125}$$

with $\frac{1}{p} + \frac{1}{m} = \frac{1}{q}$, and by (120) $B^{3/2,3/8}_{q,2} \subset L_m$ as $p > \frac{2n+8}{3}$. Next,

$$I_2 \leq (\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\frac{1}{2}2}}\|k(\cdot,\cdot)\|^2_{L_m(\mathbb{R}^n\times\mathbb{R})}\|g(\cdot+h,\cdot)-g(\cdot,\cdot)\|^2_{L_{q_1}(\mathbb{R}^n\times\mathbb{R})})^{1/2} \tag{126}$$

with q_1 such that $B^{3/2,3/8}_{p,2} \subset B^{1/2,1/8}_{q_1,2}$ with $\frac{1}{q} + \frac{1}{q_1} = \frac{1}{n+4}$; and $\frac{1}{m} + \frac{1}{q_1} = \frac{1}{q}$. On the other hand m is described by the imbedding $B^{1/2,1/8}_{p,2} \subset L_m$, so $\frac{1}{p} - \frac{1}{m} = \frac{1}{2n+8}$. Remembering that $\frac{1}{q} + \frac{1}{p} = 1$, we find the condition $p > \frac{2n+8}{3}$. The considerations for the time semi-norm are almost the same and the proofs are omitted therefore. Similar considerations for B^s_{pp}-spaces can be found in [24]. In more general case, we have to apply the paraproduct formula as it was done in [6] for estimates in $B^0_{p,1}$.

In order to prove (ii) we show that $fgk \in L_1$ for $k \in B^{1/2,1/8}_{p,2}$. Then a direct application of (120) implies the above inclusion. the proof of Proposition 8.3 is therefore completed. ∎

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