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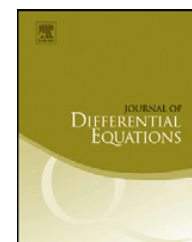


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# Transport equation: Extension of classical results for $\operatorname{div} b \in BMO$

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## ABSTRACT

We investigate the transport equation:  $u_t + b \cdot \nabla u = 0$ . Our result improves the criteria on uniqueness of weak solutions, replacing the classical condition:  $\operatorname{div} b \in L_\infty$  by  $\operatorname{div} b \in BMO$ .

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## 1. Introduction

The goal of this note is to improve classical results concerning the Cauchy problem for the transport equation. The basis of our analysis is the following system

$$\begin{aligned} u_t + b \cdot \nabla u &= 0 \quad \text{in } \mathbb{R}^n \times (0, T), \\ u|_{t=0} &= u_0 \quad \text{on } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where  $u$  is an unknown scalar function,  $b$  – some given vector field and  $u_0$  is an initial datum.

The transport equation is one of the most fundamental examples in the theory of partial differential equations. It describes the motion of matter under influence of the velocity field  $b$ . Classically, for

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smooth data  $b$  and  $u_0$ , (1.1) is solvable elementarily by the method of characteristics. In the language of the fluid mechanics, (1.1) says that  $u$  is constant along streamlines defined by the Lagrangian coordinates. This physical interpretation gives enough reasons for (1.1) to be intensively studied from the mathematical point of view. Here we want to concentrate on the optimal/critical regularity of the vector field  $b$  to control the existence, stability and uniqueness of weak solutions. The last point seems to be the most interesting.

In order to control the uniqueness of weak solutions to (1.1) the classical theory [13] requires that the vector field  $b$  must satisfy

$$\operatorname{div} b \in L_1(0, T, L_\infty(\mathbb{R}^n)), \tag{1.2}$$

provided that the regularity of the field is at least  $b \in L_1(0, T; W_{1(\text{loc})}^1(\mathbb{R}^n))$ . Then thanks to the renormalized meaning of solutions for (1.1), the energy method and Gronwall lemma yield immediately the uniqueness. The main goal of analysis of the present paper is to relax condition (1.2) by

$$\operatorname{div} b \in L_1(0, T; BMO(\mathbb{R}^n)). \tag{1.3}$$

Let us observe that this “slightly” broader class than (1.2) is on the boundary of known counterexamples [13]. For any  $p < \infty$  we are able to construct  $b \in W_p^1(\mathbb{R}^n)$  (time independent) to obtain an example of the loss of uniqueness to (1.1). On the other hand the  $BMO$ -space appears naturally in many considerations, since it is the limit space for the embedding  $W_n^1(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ , where the  $L_\infty$ -space is not reached. We are able to prove existence and uniqueness of weak solutions to (1.1) in the case of bounded solutions and improve the uniqueness criteria for  $L_p$ -solutions. Additionally we show a result concerning stability with respect to initial data. Our approach follows from techniques introduced in [18] to improve the uniqueness criteria for the Euler system in bounded domains. The main tool of our method is a logarithmic type inequality between the Hardy space  $\mathcal{H}^1$  and Lebesgue space  $L_1$ , stated in Theorem D below.

Fundamental results of our issue have been proved by R.J. DiPerna and P.L. Lions in [13], where general questions concerning the well posedness of the problem found positive answer under condition (1.2) with  $b \in L_1(0, T; W_{1(\text{loc})}^1(\mathbb{R}^n))$ . An interesting extension of the theory has been developed by L. Ambrosio [1], for the case of bounded solutions replacing the condition  $b \in W_1^1(\mathbb{R}^n)$  by  $b \in BV(\mathbb{R}^n)$ . In the literature one can find also numerous works on generalizations of the mentioned results on broader classes of function spaces [2,6–9,14–16], but positive answers still require condition (1.2). A step to relax the condition (1.2) has been done recently in [3] and [4]. In [3] the uniqueness is obtained under the assumption that full vector field  $b$  satisfies an Osgood type condition known from ODEs. In [4] the authors consider an abstract transport equation in Banach spaces under a uniqueness criterion:  $\exp[c|\operatorname{div} b|] \in L_1(0, T; L_1(\mathbb{R}^n))$  – for some  $c > 0$ , taking the result for  $\mathbb{R}^n$  with the Lebesgue measure. Note that the condition:  $\exp[c|\operatorname{div} b|]$  describes a slightly larger class than  $\operatorname{div} b \in BMO$ , however [4] requires the  $L_\infty$ -regularity with respect to time.

In the present note we consider weak solutions meant in the following sense:

We say that  $u \in L_\infty(0, T; L_p(\mathbb{R}^n))$  is a weak solution to (1.1) iff the following integral identity holds

$$\int_0^T \int_{\mathbb{R}^n} u \phi_t \, dx dt + \int_0^T \int_{\mathbb{R}^n} \operatorname{div} b u \phi \, dx + \int_0^T \int_{\mathbb{R}^n} b \cdot \nabla \phi u \, dx dt = - \int_{\mathbb{R}^n} u_0 \phi(\cdot, 0) \, dx \tag{1.4}$$

for each  $\phi \in C^\infty([0, T]; C_0^\infty(\mathbb{R}^n))$  such that  $\phi|_{t=T} \equiv 0$ .

Let us state the main results of this paper. First we start with the case of pointwise bounded solutions, in that case our technique delivers the most complete result.

**Theorem A.** Let  $T > 0$ ,  $b \in L_1(0, T; W^1_{1(loc)}(\mathbb{R}^n))$ ,  $u_0 \in L_\infty(\mathbb{R}^n)$ , additionally we assume

$$\operatorname{div} b \in L_1(0, T; BMO(\mathbb{R}^n)), \quad \frac{b}{1 + |x|} \in L_1(0, T; L_1(\mathbb{R}^n)) \quad \text{and} \quad (1.5)$$

$$\operatorname{supp} \operatorname{div} b(\cdot, t) \subset B(0, R) \quad \text{for a fixed } R > 0, \quad (1.6)$$

where  $B(0, R)$  denotes the ball centered at the origin with radius  $R$ .

Then there exists a unique weak solution to the system (1.1) such that

$$u \in L_\infty(0, T; L_\infty(\mathbb{R}^n)). \quad (1.7)$$

The above result shows the existence and uniqueness of solutions. It is a consequence of a maximum principle, which is valid for the  $L_\infty$ -solutions. The main difference to the classical results [13] is that having (1.2) we are able to construct  $L_p$ -estimates of the solutions for finite  $p$ . In our case the condition (1.3) is too weak to obtain such information. Additionally we are required to add an extra condition (1.6), which is the price of our improvement of this classical criteria. The technique of the proof of Theorem A allows us to relax this strong restriction to the class of fields  $b$  prescribed by the following conditions

$$\operatorname{div} b = H_\infty + \sum_{k=1}^{\infty} H_k \quad \text{such that}$$

$$H_\infty \in L_1(0, T; L_\infty(\mathbb{R}^n)), \quad H_k \in L_1(0, T; BMO(\mathbb{R}^n)) \quad \text{and}$$

$$\sum_{k=1}^{\infty} \|H_k\|_{L_1(0, T; BMO(\mathbb{R}^n))} < \infty \quad \text{with} \quad \sup_{k \in \mathbb{N}} \operatorname{diam} \operatorname{supp} H_k < \infty \quad (1.8)$$

– see the Remark at the end of this section.

The next result concerns stability of solutions obtained in Theorem A with respect to perturbations of initial data in lower spaces.

**Theorem B.** Let  $1 \leq p < \infty$  and  $b$  fulfill assumptions of Theorem A. Let  $u_0, u_0^k \in L_\infty(\mathbb{R}^n)$  and  $(u_0^k - u_0) \in L_p(\mathbb{R}^n)$  such that  $\sup_{k \in \mathbb{N}} \|u_0^k\|_{L_\infty(\mathbb{R}^n)} + \|u_0\|_{L_\infty(\mathbb{R}^n)} = m < \infty$  and  $(u_0^k - u_0) \rightarrow 0$  in  $L_p(\mathbb{R}^n)$  as  $k \rightarrow \infty$ . Then

$$(u^k - u) \rightarrow 0 \quad \text{in } L_\infty(0, T; L_p(\mathbb{R}^n)) \quad \text{as } k \rightarrow \infty. \quad (1.9)$$

The last result concerns the uniqueness criteria for  $L_p$ -solutions to (1.1).

**Theorem C.** Let  $1 < p < \infty$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $b \in L_1(0, T; W^1_{p'(loc)}(\mathbb{R}^n))$  and conditions (1.5), (1.6) be fulfilled. Let  $u^1, u^2$  be two weak solutions to (1.1) with the same initial datum and  $u^1, u^2 \in L_\infty(0, T; L_p(\mathbb{R}^n))$ ; then  $u^1 \equiv u^2$ .

The above three theorems are proved by a reduction of considerations to an ordinary differential equation of the form

$$\dot{x} = x \ln x, \quad x|_{t=0} = 0. \quad (1.10)$$

The Osgood lemma yields the uniqueness to (1.10). This observation forms our chain of estimations in proofs of the theorems. Due to low regularity of solutions, our analysis requires a special approach. The main tool, which enables us to show the main inequality in the form of (1.10), is the following result.

**Theorem D.** Let  $f \in BMO(\mathbb{R}^n)$ , the support of  $f$  be bounded in  $\mathbb{R}^n$  and  $g \in L_1(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ , then

$$\left| \int_{\mathbb{R}^n} fg \, dx \right| \leq C_0 \|f\|_{BMO(\mathbb{R}^n)} \|g\|_{L_1(\mathbb{R}^n)} \left[ |\ln \|g\|_{L_1(\mathbb{R}^n)}| + \ln(e + \|g\|_{L_\infty(\mathbb{R}^n)}) \right], \quad (1.11)$$

where  $C_0$  depends on the diameter of support of  $f$ .

The above inequality can be viewed as a representative of the family of logarithmic Sobolev inequalities [5,10,11,17], however there is one important difference between this one and others. Here an extra information about derivatives of the function is not required, in contrast to  $L_\infty - BMO$  inequalities. The crucial assumption is the boundedness of the support of the function  $f$ , it is a consequence of results of the classical theory [21,22]. Unfortunately, it is not expected that it could be possible to omit this restriction in Theorem D. Methods of proving (1.11) distinguish this result from others, too. They base on relations between the Zygmund space  $L \ln L$  and Riesz operators. Theorem D has been proved in [18], applied to the evolutionary Euler system. Outlines of the proof of Theorem D one can find in Appendix A.1.

The below remark shows us a possible generalization of stated theorems.

**Remark.** The results stated in Theorems A, B and C can be extended on the following linear system

$$\begin{aligned} u_t + b \cdot \nabla u &= cu + f \quad \text{in } \Omega \times (0, T), \\ u|_{t=0} &= u_0 \quad \text{on } \Omega \end{aligned} \quad (1.12)$$

in an arbitrary domain  $\Omega \subset \mathbb{R}^n$  with a sufficiently smooth boundary  $\partial\Omega$ , enough to allow integration by parts, and with given

$$c, f \in L_1(0, T; L_\infty(\mathbb{R}^n)) \quad \text{and} \quad b \cdot n = 0 \quad \text{on } \partial\Omega \times (0, T),$$

where  $n$  is the normal vector to the boundary  $\partial\Omega$ .

Additionally, we find a natural generalization of (1.5)–(1.6)

$$\begin{aligned} \operatorname{div} b &= H_\infty + \sum_{k=1}^{\infty} H_k \quad \text{such that} \\ H_\infty &\in L_1(0, T; L_\infty(\Omega)), \quad H_k \in L_1(0, T; BMO(\Omega)) \quad \text{and} \\ \sum_{k=1}^{\infty} \|H_k\|_{L_1(0, T; BMO(\Omega))} &< \infty \quad \text{with} \quad \sup_{k \in \mathbb{N}} \operatorname{diam} \operatorname{supp} H_k < \infty. \end{aligned} \quad (1.13)$$

In the case of bounded  $\Omega$  condition (1.13) is trivially fulfilled. We leave the proof of the Remark to the kind reader, it is almost the same as for (1.1), the estimations are just more technical, but the core of the problem is the same.

Throughout the paper we use the standard notation.  $L_p(\mathbb{R}^n)$  denotes the common Lebesgue space, generic constants are denoted by  $C$ . Let us recall only the definition of the BMO-space. We say that  $f \in BMO(\mathbb{R}^n)$ , if  $f$  is locally integrable and the corresponding semi-norm

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - \{f\}_{B(x, r)}| \, dy \quad (1.14)$$

is finite, where  $\{f\}_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy$  and  $B(x, r)$  is a ball with radius  $r$  centered at  $x$  – see [20]. The above definition implies that (1.14) is a semi-norm only, however in our case from assumptions on  $\operatorname{div} b$  follows

$$\|\operatorname{div} b\|_{L_1(\mathbb{R}^n)} \leq |\operatorname{diam} \operatorname{supp} \operatorname{div} b|^n \|\operatorname{div} b\|_{BMO(\mathbb{R}^n)}$$

which is a consequence of the properties of the support restricted by (1.6). In general the norm  $BMO$  can be defined as a sum of (1.14) and the  $L_1$ -norm, then  $BMO$  is a Banach space.

## 2. Proof of Theorem A

The proof of existence of solutions in our case is standard, we present a sketch in Appendix A.2. Thus, we claim that there exists a weak solution fulfilling the definition (1.4) with (1.7). The high regularity of test functions required in (1.4) does not allow us to obtain any information concerning the uniqueness of solutions to (1.4) in a direct way. To solve this issue we start with an application of the standard procedure. We introduce

$$S_\epsilon(f) = m_\epsilon * f = \int_{\mathbb{R}^n} m_\epsilon(\cdot - y) f(y) dy, \tag{2.1}$$

where  $m_\epsilon$  is a smooth function with suitable properties tending weakly to the Dirac delta – see (A.12) in Appendix A. Applying the above operator to (1.1) we get

$$\partial_t S_\epsilon(u) + S_\epsilon(b \cdot \nabla u) = 0, \tag{2.2}$$

where  $b \cdot \nabla u = \operatorname{div}(bu) - u \operatorname{div} b$  and the r.h.s. is well defined as a distribution. In fact (2.2) implies that  $\partial_t S_\epsilon(u)$  is well defined as a Lebesgue function, too.

We rewrite Eq. (2.2) as follows

$$\partial_t S_\epsilon(u) + b \cdot \nabla S_\epsilon(u) = R_\epsilon, \quad \text{where } R_\epsilon = b \cdot \nabla S_\epsilon(u) - S_\epsilon(b \cdot \nabla u). \tag{2.3}$$

Standard facts, known from the DiPerna–Lions theory [13], implies (see (A.13) in Appendix A.3) that the remainder is controlled in the limit:  $R_\epsilon \rightarrow 0$  in  $L_1(0, T; L_1(\operatorname{loc})(\mathbb{R}^n))$ . Since  $R_\epsilon$  convergences locally in space, only, we introduce a smooth function  $\pi_r : \mathbb{R}^n \rightarrow [0, 1]$  such that  $\pi_r(x) = \pi_1(\frac{x}{r})$  and

$$\pi_1(x) = \begin{cases} 1, & |x| < 1, \\ \in [0, 1], & 1 \leq |x| \leq 2, \\ 0, & |x| > 2 \end{cases} \quad \text{with } |\nabla \pi_r| \leq \frac{C}{r}. \tag{2.4}$$

In order to prove the uniqueness for our system it is enough to consider (2.3) with zero initial data (due to its linearity). Since we are forced to localize the problem, we multiply (2.3) by  $S_\epsilon(u)\pi_r$  and integrate over the space, getting

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} (S_\epsilon(u))^2 \pi_r dx - \frac{1}{2} \int_{\mathbb{R}^n} \operatorname{div} b (S_\epsilon(u))^2 \pi_r dx - \frac{1}{2} \int_{\mathbb{R}^n} b \cdot \nabla \pi_r (S_\epsilon(u))^2 dx \\ & = \int_{\mathbb{R}^n} R_\epsilon S_\epsilon(u) \pi_r dx. \end{aligned} \tag{2.5}$$

Then integrating over time, using properties of  $S_\epsilon$  and letting  $\epsilon \rightarrow 0$ , next differentiating with respect  $t$  we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} u^2 \pi_r dx - \frac{1}{2} \int_{\mathbb{R}^n} \operatorname{div} bu^2 \pi_r dx = \frac{1}{2} \int_{\mathbb{R}^n} b \cdot \nabla \pi_r u^2 dx \quad \text{for } r > 0. \quad (2.6)$$

The r.h.s. of (2.6) is estimated as follows

$$\left| \int_{\mathbb{R}^n} b \cdot \nabla \pi_r u^2 dx \right| \leq C \|u\|_{L^\infty(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n \setminus B(0,r)} \frac{|b|}{1+|x|} (1+|x|) |\nabla \pi_r| dx \rightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (2.7)$$

By definition  $(1+|x|)|\nabla \pi_r| \leq C$ , because the support of  $\nabla \pi_k$  is a subset of the set:  $\{r \leq |x| \leq 2r\}$ . By (A.9) the norm  $\|u\|_{L^\infty}$  is controlled, too.

Eqs. (2.5)–(2.6) give the following form after integration over time

$$\int_{\mathbb{R}^n} u^2(x, t) \pi_r dx = \int_0^t \int_{\mathbb{R}^n} \operatorname{div} bu^2 \pi_r dx ds + \int_0^t \int_{\mathbb{R}^n} b \cdot \nabla \pi_r u^2 dx ds. \quad (2.8)$$

Let us observe that for sufficiently large  $r$

$$\operatorname{div} bu^2 \pi_r = \operatorname{div} bu^2 \pi_{r'} = \operatorname{div} bu^2$$

for any  $r' > r$ , which is just an elementary consequence of the boundedness of the support of  $\operatorname{div} b$ . To avoid questions about integrability of  $u^2$  and possible pathologies as in [12] we fix  $r_0$  so large as in the last remark and introduce

$$g(x, t) := u^2(x, t) \pi_{r_0}(x). \quad (2.9)$$

Then from (2.8) we find

$$\int_{\mathbb{R}^n} g(x, t) dx \leq \int_0^t \int_{\mathbb{R}^n} |\operatorname{div} b| g dx ds + R_r, \quad (2.10)$$

where by (2.7)

$$R_r = \left| \int_0^t \int_{\mathbb{R}^n} |b \nabla \pi_r| u^2 dx ds \right| \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

So from (2.10), letting  $r \rightarrow \infty$  ( $r_0$  is fixed) we get

$$\int_{\mathbb{R}^n} g(x, t) dx \leq \int_0^t \int_{\mathbb{R}^n} |\operatorname{div} b| g dx ds. \quad (2.11)$$

The form of (2.11) guarantees us that we are able to find  $T_1 > 0$  such that

$$\int_{\mathbb{R}^n} g(x, t) dx \leq e^{-1} \quad \text{for } t \in [0, T_1]. \tag{2.12}$$

So taking

$$\beta(t) = \|g(t)\|_{L_1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u^2(x, t) \pi_{r_0} dx, \quad \gamma(t) = \|\operatorname{div} b(t)\|_{BMO}, \tag{2.13}$$

by Theorem D, we get from (2.11) the following inequality

$$\beta(t) \leq \int_0^t C_0 \gamma(s) \beta(s) [|\ln \beta(s)| + \ln(m^2 + e)], \tag{2.14}$$

where  $m = \|u\|_{L_\infty(0, T; L_\infty(\mathbb{R}^n))}$ . Note that on the interval  $[0, e^{-1}]$  the function  $t \rightarrow t|\ln t|$  is increasing, hence for a fixed  $\epsilon > 0$  we may introduce a function  $\beta^*(\cdot)$  by the identity

$$\beta^*(t) = \epsilon + \int_0^t C_0 \gamma(s) \beta^*(s) [|\ln \beta^*(s)| + \ln(m^2 + e)]. \tag{2.15}$$

The monotonicity of the r.h.s. yields immediately

$$0 \leq \beta(t) \leq \beta^*(t) \quad \text{for } t \in [0, T_1]. \tag{2.16}$$

The form of (2.15) gives us the following implicate formula on  $\beta^*$

$$\beta^*(t) = \epsilon \exp \left\{ \int_0^t C_0 \gamma(s) [|\beta^*(s)| + \ln(m^2 + e)] ds \right\}. \tag{2.17}$$

Additionally it is clear that

$$\beta^*(t) \geq \epsilon \quad \text{and} \quad |\ln \beta^*(t)| \leq |\ln \epsilon| \quad \text{for } \beta^*(t) < 1 \tag{2.18}$$

and the above conditions hold for  $t \in [0, T_2]$  for some  $T_2 > 0$ .

Applying these facts we obtain

$$\beta^*(t) \leq \epsilon \exp \left\{ \int_0^t C_0 \gamma(s) \ln \epsilon^{-1} ds \right\} \exp \left\{ \int_0^t C_0 \gamma(s) \ln [m^2 + e] ds \right\} \leq C \epsilon^{1 - \int_0^t C_0 \gamma(s) ds}. \tag{2.19}$$

Taking  $T_3 > 0$  so small that

$$\int_0^{T_3} C_0 \gamma(t) dt \leq \frac{1}{2}, \quad (2.19) \text{ yields } \beta^*(t) \leq C \epsilon^{1/2} \text{ for } t \in [0, T_3].$$



Note that none of  $T_1, T_2, T_3$  is depending on  $\epsilon$ , so it is allowed to pass  $\epsilon \rightarrow 0$  getting finally

$$0 \leq \beta(t) \leq \beta^*(t) = 0 \quad \text{for all } t \in [0, T_3],$$

which implies immediately that

$$u^2(x, t)\pi_{r_0}(x) = 0 \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T_3) \text{ a.e.}$$

However the choice of  $r_0$  was arbitrary, thus the only admitted solution is

$$u(x, t) \equiv 0 \quad \text{a.e. in } \mathbb{R}^n \times (0, T_3).$$

From that we conclude the end of the proof of Theorem A.

### 3. Proof of Theorem B

The next result concerns the stability of solutions from Theorem A. We start with the mollified equation (2.2) for reasons same as previously, testing it now by  $|S_\epsilon(u)|^{p-2}S_\epsilon(u)\pi_r$  with  $p$  as in Theorem B. Repeating the considerations from (2.3)–(2.7) we deduce

$$\frac{d}{dt} \int_{\mathbb{R}^n} |u^k - u|^p dx \leq \int_{\mathbb{R}^n} |\operatorname{div} b| |u^k - u|^p dx. \tag{3.1}$$

The r.h.s. of (3.1) is controlled due to the boundedness of the support of  $\operatorname{div} b$ .

For a given  $1 \geq \epsilon > 0$ , we fix  $K_\epsilon \in \mathbb{N}$  such that for all  $k > K_\epsilon$

$$\|u_0^k - u_0\|_{L^p} \leq \epsilon. \tag{3.2}$$

Let  $X = |u^k - u|^p$ , then by Theorem D (3.1) reads

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} X dx &\leq \int_{\mathbb{R}^n} |\operatorname{div} b| X dx \\ &\leq C_0 \|\operatorname{div} b\|_{BMO(\mathbb{R}^n)} \|X\|_{L_1(\mathbb{R}^n)} \left[ |\ln \|X\|_{L_1(\mathbb{R}^n)}| + \ln(e + 2m) \right], \end{aligned} \tag{3.3}$$

with  $\int_{\mathbb{R}^n} X(x, 0) dx \leq \epsilon$ .

By our assumptions the r.h.s. of (3.3) is at least locally integrable, hence there exists a positive time  $T_0$  so small that

$$\sup_{t \in [0, T_0]} \int_{\mathbb{R}^n} X(x, t) dx \leq e^{-1}. \tag{3.4}$$

It follows that the function  $w|\ln w|$  is increasing, since  $\int_{\mathbb{R}^n} X(x, \cdot) dx$  on chosen time interval takes the values only from the interval  $[0, e^{-1}]$ . Monotonicity allows us to introduce a function  $B : [0, T_0] \rightarrow [0, \infty)$  such that

$$\frac{d}{dt} B = C_0 \|\operatorname{div} b\|_{BMO(\mathbb{R}^n)} B \left[ |\ln B| + \ln(e + 2m) \right] \quad \text{and} \quad B|_{t=0} = \epsilon, \tag{3.5}$$

where  $m$  is defined in the statement of Theorem 3. The definition of  $B$  guarantees that it is an increasing and continuous function, thus there exists  $T_1 > 0$  such that  $0 < T_1 \leq T_0$  and

$$B(t) \leq e^{-1} < 1 \quad \text{for } t \in [0, T_1]. \tag{3.6}$$

Taking the difference between (3.5) and (3.3) we get

$$\begin{aligned} \frac{d}{dt} \left[ B - \int_{\mathbb{R}^n} X dx \right] &\geq C_0 \|\operatorname{div} b\|_{BMO(\mathbb{R}^n)} \\ &\cdot \left[ B |\ln B| - \int_{\mathbb{R}^n} X dx \left| \ln \int_{\mathbb{R}^n} X dx \right| + \ln(e + 2m) \left( B - \int_{\mathbb{R}^n} X dx \right) \right] \end{aligned} \tag{3.7}$$

with  $B(0) - \int_{\mathbb{R}^n} X(x, 0) dx \geq 0$ .

Since the monotonicity of the function  $w|\ln w|$  on  $[0, e^{-1}]$  implies

$$\left( B |\ln B| - \int_{\mathbb{R}^n} X dx \left| \ln \int_{\mathbb{R}^n} X dx \right| \right) \left( B - \int_{\mathbb{R}^n} X dx \right) \geq 0, \tag{3.8}$$

remembering that we consider  $t \in [0, T_1]$ , from (3.7) we get

$$0 \leq \int_{\mathbb{R}^n} X(x, t) dx \leq B(t) \quad \text{for } t \in [0, T_1]. \tag{3.9}$$

The above fact reduces our analysis to the considerations of the function  $B$ . Additionally, by the choice of the time interval it follows that  $B(t) < 1$  for  $t \in [0, T_1]$ , hence we can use the estimate (remember  $0 < \epsilon < 1$ )

$$|\ln B| \leq \ln \epsilon^{-1} \quad \text{for } t \in [0, T_1]. \tag{3.10}$$

Solving (3.5) we get

$$B(t) \leq \epsilon \exp \left\{ C_0 [\ln(e + 2m) + \ln \epsilon^{-1}] \int_0^t f(s) ds \right\} \leq C \epsilon e^{-C_0 \int_0^t f(s) ds}, \tag{3.11}$$

where  $f(t) = \|\operatorname{div} b(\cdot, t)\|_{BMO(\mathbb{R}^n)}$  and  $C$  depends on data given in Theorems A and B.

Next, we choose  $T_2$  so small that  $0 < T_2 \leq T_1$  and  $C_0 \int_0^{T_2} f(s) ds \leq 1/2$ , then (3.11) yields

$$\sup_{t \in [0, T_2]} B(t) \leq C \epsilon^{1/2}. \tag{3.12}$$

Here we shall emphasize that  $T_2$  is independent from the smallness of  $\epsilon$  – see (3.2). Thus we are able to start our analysis over from the very beginning, but for the initial time  $t = T_2$ . Since  $C_0$  in (3.11) is an absolute constant we find the next interval  $[T_2, T_3]$ , where we obtain

$$\sup_{t \in [T_2, T_3]} \|u^k - u\|_{L_p(\mathbb{R}^n)} \leq C \epsilon^{1/4} \tag{3.13}$$

for all  $k > K_\epsilon$  – see (3.2). Since  $T$  is fixed and finite and by the assumptions  $f \in L_1(0, T)$ , we are always able to cover the whole interval  $[0, T]$  in finite steps, so finally we obtain

$$\sup_{t \in [0, T]} B(t) \leq C\epsilon^a \tag{3.14}$$

with  $a > 0$  defined by the properties of  $f$  and again  $C$  depending on all data, but independent from  $\epsilon$ . Letting  $\epsilon \rightarrow 0$  we prove (1.9). Theorem B is proved.

#### 4. Proof of Theorem C

Our last result describes the uniqueness criteria for weak solutions, provided their existence in the  $L_\infty(0, T; L_p(\mathbb{R}^n))$ -class in the meaning of the definition (1.4). The problem reduces to (1.1) with zero initial data and  $u \in L_\infty(0, T; L_p(\mathbb{R}^n))$ . To work in optimal regularity of coefficients we consider (2.3)

$$\partial_t S_\epsilon(u) + b \cdot \nabla S_\epsilon(u) = R_\epsilon \rightarrow 0 \quad \text{in } L_1(0, T; L_1(\text{loc})(\mathbb{R}^n)).$$

Next, we introduce the renormalized solution to (1.1) – we refer here to [13] where this approach has been developed. Take  $\beta \in C^1(\mathbb{R})$ , i.e.  $\|\beta\|_{L_\infty(\mathbb{R})} + \|\beta'\|_{L_\infty(\mathbb{R})} < \infty$ , then

$$\partial_t \beta(S_\epsilon(u)) + b \cdot \nabla \beta(S_\epsilon(u)) = R_\epsilon \beta'(S_\epsilon(u)) \tag{4.1}$$

which implies the limit for  $\epsilon \rightarrow 0$

$$\partial_t \beta(u) + b \cdot \nabla \beta(u) = 0. \tag{4.2}$$

As the function  $\beta$  we choose  $T_m : \mathbb{R} \rightarrow [0, m^p]$  such that

$$T_m(s) = \begin{cases} |s|^p & \text{for } |s| < m, \\ m^p & \text{for } |s| \geq m \end{cases} \tag{4.3}$$

defined for fixed  $m \in \mathbb{R}_+$ .  $T_m$  is not a  $C^1$ -function, but a simple approximation procedure will lead us to (4.2) with  $\beta = T_m$ .

Since we do not control integrability of all terms in (4.2), we use the function  $\pi_r$  from (2.4) to localize the problem, getting

$$\frac{d}{dt} \int_{\mathbb{R}^n} T_m(u) \pi_r dx \leq \int_{\mathbb{R}^n} |\text{div } b| T_m(u) \pi_r dx + \int_{\mathbb{R}^n} |b \cdot \nabla \pi_r| T_m(u) dx. \tag{4.4}$$

Repeating considerations from the proof of Theorem A (2.5)–(2.19) we deduce  $T_m(u) \equiv 0$ . Letting  $m \rightarrow \infty$ , by (4.3) we conclude  $u \equiv 0$ . Thus,  $u_1 \equiv u_2$ . Theorem C is proved.

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### Appendix A

#### A.1. Sketch of the proof of Theorem D

We proceed almost as in [18]. The assumption of the boundedness of  $\text{supp } f$  allows us to consider the integral on the l.h.s. of (1.11) on a torus  $\mathbb{T}^n = \mathbb{R}^n / (d\mathbb{Z}^n)$  ( $= [0, d]^n$ ) with sufficiently large  $d$  guaranteeing that  $\text{supp } f$  can be treated as a subset of  $\mathbb{T}^n$ . Consider the Hardy space on  $\mathbb{T}^n$  with the following norm

$$\|g\|_{\mathcal{H}^1(\mathbb{T}^n)} = \|g\|_{L_1(\mathbb{T}^n)} + \sum_{k=1}^n \|R_k g\|_{L_1(\mathbb{T}^n)}, \tag{A.1}$$

where  $R_k$  are the Riesz operators – [20,21]. Since  $BMO(\mathbb{T}^n) = (\mathcal{H}^1(\mathbb{T}^n))^*$ , we get

$$\left| \int_{\mathbb{T}^n} fg \, dx \right| \leq \|f\|_{BMO(\mathbb{T}^n)} \|g\|_{\mathcal{H}^1(\mathbb{T}^n)}. \tag{A.2}$$

Hence to control the norm (A.1) an estimate of  $\|R_k g\|_{L_1(\mathbb{T}^n)}$  is required. The classical Zygmund’s result [22] (we refer to [21], too) says:

$$\|R_k h\|_{L_1(\mathbb{T}^n)} \leq C + C \int_{\mathbb{T}^n} |h| \ln^+ |h| \, dx, \tag{A.3}$$

where  $\ln^+ a = \max\{\ln a, 0\}$  and constants  $C$  depend on  $d$ , so on the diameter of  $\text{supp } f$ .

Let us observe that  $\ln^+(g/\lambda) = \ln g - \ln \lambda$  for  $g \geq \lambda$  and

$$\begin{aligned} |\ln g|_{g \geq \lambda} &\leq \ln(1 + \|g\|_{L_\infty(\mathbb{T}^n)}) + \left| \ln \frac{g}{1 + \|g\|_{L_\infty(\mathbb{T}^n)}} \right|_{g \geq \lambda} \\ &\leq 2 \ln(1 + \|g\|_{L_\infty(\mathbb{T}^n)}) + |\ln \lambda|. \end{aligned} \tag{A.4}$$

Taking  $h = \frac{g}{\|g\|_{L_1(\mathbb{T}^n)}}$  in (A.3), employing (A.4), we conclude

$$\|R_k g\|_{L_1(\mathbb{T}^n)} \leq C \|g\|_{L_1(\mathbb{T}^n)} + C \int_{\mathbb{T}^n} |g| [\ln(1 + \|g\|_{L_\infty(\mathbb{T}^n)}) + |\ln \|g\|_{L_1(\mathbb{T}^n)}|] \, dx. \tag{A.5}$$

Inequalities (A.2), (A.5) yield (1.11).

#### A.2. The proof of existence (Theorem A)

Here we prove the existence of weak solutions to (1.1). To construct them we find a sequence of approximations of the function  $b$  and initial datum  $u_0$ . We require that

$$\begin{aligned} b^\epsilon &\in C^\infty(\mathbb{R}^n \times (0, T)), \quad \text{supp div } b^\epsilon(\cdot, t) \subset B(0, 2R) \quad \text{and} \\ b^\epsilon &\rightarrow b \quad \text{in } L_1(0, T; W_{1(loc)}^1(\mathbb{R}^n)) \end{aligned}$$

with suitable behavior of norms. For a given initial datum we find  $u_0^\epsilon \in C_0^\infty(\mathbb{R}^n)$  with  $u_0^\epsilon \rightharpoonup^* u_0$  in  $L_\infty(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$  and  $\|u_0^\epsilon\|_{L_\infty(\mathbb{R}^n)} \leq \|u_0\|_{L_\infty(\mathbb{R}^n)}$ . Then we consider the following equation with smooth coefficients  $b^\epsilon$  and initial data  $u_0^\epsilon$ :

$$\begin{aligned} u_t^\epsilon + b^\epsilon \cdot \nabla u^\epsilon &= 0 \quad \text{in } \mathbb{R}^n \times (0, T), \\ u^\epsilon|_{t=0} &= u_0^\epsilon \quad \text{on } \mathbb{R}^n. \end{aligned} \tag{A.6}$$

The method of characteristics implies the existence of smooth solutions to (A.6) for  $t \in (0, T)$  together with the pointwise bound

$$\|u^\epsilon\|_{L_\infty(0, T; L_\infty(\mathbb{R}^n))} \leq \|u_0^\epsilon\|_{L_\infty(\mathbb{R}^n)} \leq \|u_0\|_{L_\infty(\mathbb{R}^n)}. \tag{A.7}$$

Note that we do not use any uniform bound on  $\text{div} b^\epsilon$ .

Now we pass to the limit with  $\epsilon \rightarrow 0$  in (A.6). The solutions to (A.6) are classical, in particular it implies they fulfill the following integral identity

$$-\int_0^T \int_{\mathbb{R}^n} u^\epsilon \phi_t \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \text{div} b^\epsilon u^\epsilon \phi \, dx - \int_0^T \int_{\mathbb{R}^n} b^\epsilon \cdot \nabla \phi u^\epsilon \, dx \, dt = \int_{\mathbb{R}^n} u_0^\epsilon \phi(\cdot, 0) \, dx \tag{A.8}$$

for any  $\phi \in C^\infty([0, T]; C_0^\infty(\mathbb{R}^n))$  such that  $\phi|_{t=T} \equiv 0$ .

Estimate (A.7) implies that for a subsequence  $\epsilon_k \rightarrow 0$

$$u^{\epsilon_k} \rightharpoonup^* u \quad \text{in } L_\infty(0, T; L_\infty(\mathbb{R}^n)) \quad \text{with } \|u\|_{L_\infty(0, T; L_\infty(\mathbb{R}^n))} \leq \|u_0\|_{L_\infty(\mathbb{R}^n)}. \tag{A.9}$$

Then taking the limit of (A.6) for  $\epsilon_k \rightarrow 0$ , by the properties of sequences  $\{b^\epsilon\}$  and  $\{u_0^\epsilon\}$ , we obtain

$$-\int_0^T \int_{\mathbb{R}^n} u \phi_t \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \text{div} b u \phi \, dx - \int_0^T \int_{\mathbb{R}^n} b \cdot \nabla \phi u \, dx \, dt = \int_{\mathbb{R}^n} u_0 \phi(\cdot, 0) \, dx \tag{A.10}$$

for the same set of test functions as in (A.8).

### A.3. The commutator estimate

Let us recall the well-known facts concerning the mollification of the equation and the behavior of the commutators [13,19]. Introduce  $m_1 : \mathbb{R}^n \rightarrow [0, \infty)$  such that

$$m_1(x) = N_n \begin{cases} \exp\{-\frac{1}{1-|x|^2}\} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

where the number  $N_n$  is determined by the constraint  $\int_{\mathbb{R}^n} m_1 \, dx = 1$ . Then for given  $\epsilon > 0$  we define

$$m_\epsilon(x) := \frac{1}{\epsilon^n} m_1\left(\frac{x}{\epsilon}\right) \quad \text{with } \int_{\mathbb{R}^n} m_\epsilon \, dx = 1. \tag{A.11}$$

It is clear that  $m_\epsilon \rightarrow \delta$  in  $\mathcal{D}'(\mathbb{R}^n)$ , where  $\delta$  is the Dirac mass located at the origin of  $\mathbb{R}^n$ . The function  $m_\epsilon$  introduces an operator  $S_\epsilon : L_1(\text{loc})(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$

$$S_\epsilon(h) = m_\epsilon * h = \int_{\mathbb{R}^n} m_\epsilon(x - y) h(y) \, dy. \tag{A.12}$$

The standard theory [13] guarantees the following estimate for the commutator

$$b \cdot \nabla S_\varepsilon(u) - S_\varepsilon(b \cdot \nabla u) \rightarrow 0 \quad \text{in } L_1(0, T; L_1(\text{loc}) (\mathbb{R}^n)), \quad (\text{A.13})$$

provided that  $u \in L_\infty(0, T; L_{p(\text{loc})}(\mathbb{R}^n))$ ,  $b \in L_1(0, T; W_{p'(\text{loc})}^1(\mathbb{R}^n))$  and  $1 = \frac{1}{p} + \frac{1}{p'}$  for  $p \in [1, \infty]$ . The convergence (A.13), known also as the Friedrichs lemma [19], allows us to test weak solutions by functions with lower regularity than it is required by the weak formulation. In our case it enables application of energy methods to obtain inequalities for  $L_2$  and  $L_p$  norms – considerations: (2.5)–(2.6), (3.1) and (4.1)–(4.2). In other words, thanks to (A.13) we are able to test the equation by the solution, although it does not belong to class required by the definition (1.4).

The proof of (A.13) belongs to the by now classical theory, and since it is quite technical we omit it here and refer again to [13,19].

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