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WEAK SOLUTIONS TO EQUATIONS OF STEADY COMPRESSIBLE HEAT CONDUCTING FLUIDS

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We consider the steady compressible Navier–Stokes–Fourier system in a bounded three-dimensional domain. We prove the existence of a solution for arbitrarily large data under the assumption that the pressure $p(\varrho, \theta) \sim \varrho\theta + \varrho^\gamma$ for $\gamma > \frac{7}{3}$, assuming either the slip or no-slip boundary condition for the velocity and the Newton boundary condition for the temperature. The regularity of solutions is determined by the basic energy estimates, constructed for the system.

Keywords: Steady compressible Navier–Stokes–Fourier equations; slip boundary conditions; Dirichlet boundary conditions; weak solutions; large data.

AMS Subject Classification: 35Q30, 76N10

1. Introduction

The subject of this paper is the issue of the existence of weak solutions to the Navier–Stokes equations for compressible heat conducting fluids. This fundamental system is a background of many models in natural sciences and engineering as motion of meteorological or astrophysical gases and heat transfer in multi-phase flows in engineer models, see e.g. Refs. 21 and 25. From this point of view the careful analysis of such systems seems to be important to ensure physical and thermodynamical properties of studied models. On the other hand, the mathematical difficulties carried by our issue give interesting challenges by themselves. The problems do not fit to the current theory of PDEs, hence the methods are required to be modified or developed to guarantee the positive answer to fundamental questions as existence of a solution, its regularity or uniqueness. Here we want to concentrate our attention on stationary problems in bounded domains, since the theory in this area is almost empty. The choice of the boundary conditions enables us to interpret our system as a model of a heat isolator. Evolutionary systems, from the point of view of weak solutions, and their generalizations have been recently examined by E. Feireisl in Ref. 7. This theory has been further developed in Refs. 3, 9 or 8. However, this technique cannot be directly applied to stationary problems, because of an obstacle which is a lack of *a priori* estimate. Due to this reason there is no general result for this type of problems. Here we fill up this gap, overcoming the obstacle by constructing the basic energy estimate and proving the existence of weak solutions for arbitrarily large data. A key point is the positiveness of the temperature, which is not guaranteed by the structure of the system.

We consider the steady flow of a Newtonian compressible heat conducting fluid in a bounded domain $\Omega \subset \mathbb{R}^3$. It is described by (see e.g. Ref. 1)

$$\begin{aligned} \operatorname{div}(\varrho\mathbf{u}) &= 0, \\ \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbf{S} + \nabla p &= \varrho\mathbf{f}, \\ \operatorname{div}(E\mathbf{u}) &= \varrho\mathbf{f}\cdot\mathbf{u} - \operatorname{div}(p\mathbf{u}) + \operatorname{div}(\mathbf{S}\mathbf{u}) - \operatorname{div}\mathbf{q}, \end{aligned} \tag{1.1}$$

where ϱ is the density of the fluid, $\mathbf{u}$ is the velocity field, $\mathbf{S}$ is the viscous part of the stress−tensor, p is the pressure, $\mathbf{f}$ is the external force, E is the total energy and $\mathbf{q}$ is the heat flux. In addition, the total mass

$$\int_\Omega \varrho dx = M > 0 \tag{1.2}$$

is given. We specify the quantities $\mathbf{S}$, p, E and $\mathbf{q}$. Since the fluid is required to be Newtonian,

$$\mathbf{S} = \mathbf{S}(\mathbf{u}) = 2\mu\mathbf{D}(\mathbf{u}) + \lambda \operatorname{div}\mathbf{u}\mathbf{I}, \tag{1.3}$$

where, for simplicity, the viscosity coefficients μ and λ are assumed to be constant. $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}})$ is the symmetric part of the velocity gradient. Next, the pressure

$$p = p(\varrho,\theta) = c_1\varrho\theta + c_2\varrho^\gamma, \quad c_1, c_2 > 0. \tag{1.4}$$

The heat flux, in accordance with the Fourier law, is considered in the form

$$\mathbf{q} = -\kappa(\theta)\nabla\theta, \tag{1.5}$$

where

$$\kappa(\theta) = c_3(1+\theta^m) \quad \text{with } c_3, m > 0. \tag{1.6}$$

Finally, the total energy

$$E = E(\varrho,\mathbf{u},\theta) = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\theta), \tag{1.7}$$

where the internal energy e depends on the temperature and the density. For the sake of simplicity, we assume

$$e(\varrho, \theta) = c_v \theta + \frac{c_2}{\gamma - 1} \varrho^{\gamma - 1} \tag{1.8}$$

with $c_v > 0$, the specific heat at constant volume. Note that such a choice implies that p and e fulfill the Maxwell relation

$$\frac{1}{\varrho^2}\left(p - \theta \frac{\partial p}{\partial \theta}\right) = \frac{\partial e}{\partial \varrho}. \tag{1.9}$$

Problem (1.1)−(1.6) must be accomplished with boundary conditions. We consider situations when the gas is contained in a fixed domain with impermeable walls, thus

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{at } \partial\Omega. \tag{1.10}$$

Next, we assume that ($\boldsymbol{\tau}$ stands for the tangent vectors to $\partial\Omega$)

$$\beta \mathbf{u}_{\boldsymbol{\tau}} + (1 - \beta)(\mathbf{S}\mathbf{n})_{\boldsymbol{\tau}} = \mathbf{0} \tag{1.11}$$

with $\beta \in [0, 1]$. As a matter of fact, we will treat separately three cases: $\beta = 1$ which corresponds to the homogeneous Dirichlet condition $\mathbf{u} = \mathbf{0}$ at $\partial\Omega$, $\beta = 0$ which corresponds to the total slip and requires additional assumptions on the geometry of Ω, and finally $\beta \in (0, 1)$. For the last two cases, i.e. for the Navier boundary conditions, we denote

$$\alpha = \frac{\beta}{1 - \beta} \geq 0. \tag{1.12}$$

Note that in order to control the ellipticity of the main operator in $(1.1)_2$ for $\beta = 1$ (i.e. the Dirichlet (no-slip) boundary conditions — see Ref. 17) we require

$$\mu > 0, \quad \lambda + \frac{4}{3}\mu > 0 \tag{1.13}$$

and, for $\beta \in [0, 1)$ (i.e. the Navier (slip) boundary conditions — see Ref. 17)

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu > 0. \tag{1.14}$$

The condition for the Dirichlet boundary conditions is less restrictive due to the different representation of the main elliptic term in the weak formulation, cf. Definitions 2.1 and 2.2. Concerning the temperature (internal energy), we assume the condition of the type

$$-\mathbf{q} \cdot \mathbf{n} + L(\theta)(\theta - \theta_0) = 0 \tag{1.15}$$

with $\theta_0(x) \geq c_0 > 0$ a smooth function at the boundary and

$$L(\theta) = c_4(1 + \theta^l) \tag{1.16}$$

with $l > 0$ and $c_4 > 0$. However, to avoid artificial technical computations we restrict ourselves to the case $m = l + 1$. Condition (1.15) allows us to find an interesting physical/thermodynamical interpretation of the system. Possible variability of the given function θ_0, which describes the temperature "outside" the domain, implies that (1.1)–(1.15) is a good model of a thermal isolator.

Problem (1.1)–(1.16) with $\beta \in [0, 1)$ was recently considered by the authors in Ref. 13, where the existence of a weak solution was shown provided $m = l + 1 > \frac{3\gamma-1}{3\gamma-7}$, $\gamma > 3$ and $\beta < 1$. Note that in this situation $\varrho \in L_\infty(\Omega)$ and $\theta, \mathbf{u} \in W^1_p(\Omega)$ for all $1 \leq p < \infty$. It should be emphasized that techniques in Ref. 13 essentially use the fact that the density is bounded. The main difference to the above-mentioned result is that instead of the internal energy balance, we will consider here the total energy balance and both formulations are, unlike the situation in Ref. 13, not equivalent. Except for this existence result, the only large data result for the full system can be found in Ref. 10, however, under a conditional assumption that ϱ is *a priori* bounded in $L_p(\Omega)$ for p sufficiently large, which simplifies the study; such an assumption is evidently physically unrealistic.

The steady compressible Navier–Stokes equations for barotropic gases with arbitrarily large data were for the first time considered in Ref. 10, where the existence of weak solutions was established for $\gamma > 1$ $(N = 2)$ and $\gamma \geq \frac{5}{3}$ $(N = 3)$. The result was improved up to $\gamma > \frac{3}{2}$ in Ref. 14, however only for a potential force with a nonpotential nonvolume force (i.e. $\mathbf{f} = \varrho \nabla F + \mathbf{g}$). Improvements, allowing γ slightly less than $\frac{5}{3}$ even for the nonpotential volume force, can be found in Ref. 2. Further progress, allowing $\gamma > \frac{4}{3}$ $(N = 3)$ and $\gamma = 1$ $(N = 2)$ for the Dirichlet boundary conditions can be found in Refs. 4 and 5. Similar results, also for other boundary conditions, are contained in Refs. 6, 18 and 19.

Our method for the Navier boundary conditions (i.e. $\beta < 1$) will be based on the approach presented for the first time in Ref. 12, in the case of isentropic two-dimensional case, applied to the 3D case in Ref. 20. Note that in all these cases the authors were able to show that $\varrho \in L_\infty(\Omega)$ and thus modifications of the method are needed here. On the other hand, for $\beta = 1$, we will rather follow the method from book, Ref. 17.

In what follows, we will use standard notation for the Lebesgue spaces $L_p(\Omega)$ endowed with the norm $\|\cdot\|_p$ and the Sobolev spaces $W^k_p(\Omega)$ endowed with the norm $\|\cdot\|_{k,p}$. The vector- and tensor-valued functions will be printed in boldface, however, we will not distinguish between function spaces X and X^n, the difference being clear from the context. The generic constants will be denoted by C, its value may vary even in the same line or in the same formula.

Finally, note that, in order to simplify the presentation, we will put $c_1 = c_2 = c_3 = c_4 = 1$, however, keeping the value $c_v > 0$ not fixed. In the next section we introduce main results and describe the structure of the paper.

2. Main Results

The aim of the paper is to prove the following two results.

Theorem 2.1. *Let $\beta \in [0, 1)$, the domain $\Omega \in C^2$ be not axially symmetric if $\beta = 0$ and let $m = l + 1 > \frac{3\gamma-1}{3\gamma-7}$, $\gamma \in (\frac{7}{3}, 3]$. Let $\mathbf{f} \in L_\infty(\Omega)$ and $M > 0$. Then there exists a* *weak solution to problem* (1.1)–(1.16) *such that*

$$\varrho \in L_{3(\gamma-1)}(\Omega), \quad \int_\Omega \varrho dx = M, \quad \mathbf{u} \in W_2^1(\Omega) \quad \textit{and} \quad \theta \in W_2^1(\Omega) \cap L_{3m}(\Omega).$$

Moreover, $\varrho \geq 0$ *and* $\theta > 0$ *a.e. in* Ω.

Remark 2.1. Note that the case $\gamma > 3$ was treated in Ref. 13. Note also that the assumption $\Omega \in C^2$ is not necessary, following the same strategy as in Ref. 15 we could consider less regular domains, but this would produce additional technicalities which we try to avoid. The same also concerns Theorem 2.2 below.

Theorem 2.2. *Let* $\beta = 1$, *the domain* $\Omega \in C^2$ *and let* $m = l + 1 > \frac{3\gamma-1}{3\gamma-7}$, $\gamma > \frac{7}{3}$. *Let* $\mathbf{f} \in L_\infty(\Omega)$ *and* $M > 0$. *Then there exists a weak solution to problem* (1.1)–(1.16) *such that*

$$\varrho \in L_{s(\gamma)}(\Omega), \quad \int_\Omega \varrho dx = M, \quad \mathbf{u} \in W_2^1(\Omega) \quad \textit{and} \quad \theta \in W_r^1(\Omega) \cap L_{3m}(\Omega),$$

where $s(\gamma) = \min\{3(\gamma-1), 2\gamma\}$ *and* $r = \min\{2, \frac{3m}{m+1}\}$. *Moreover,* $\varrho \geq 0$ *and* $\theta > 0$ *a.e. in* Ω.

We will prove both results simultaneously which will be possible except for the last step, the strong convergence of the density, where different arguments will be used. First, we recall

Definition 2.1. Let $\beta \in [0,1)$, $\varrho \in L_q(\Omega)$, $q \geq \max\{2, \frac{6}{5}\gamma\}$, $\varrho \geq 0$ a.e. and $\int_\Omega \varrho dx = M$; $\mathbf{u} \in W_2^1(\Omega)$ with $\mathbf{u} \cdot \mathbf{n} = 0$ at $\partial\Omega$ in the sense of traces; $\theta \in W_{\frac{3}{2}}^1(\Omega)$, $\theta^m \nabla\theta \in L_1(\Omega)$, $\theta^{l+1} \in L_1(\partial\Omega)$ and $\theta > 0$ a.e. Then the triple $(\varrho, \mathbf{u}, \theta)$ is a weak solution to problem (1.1)–(1.16), if

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla\eta = 0 \quad \forall\, \eta \in C^\infty(\bar{\Omega}), \tag{2.1}$$

$$\begin{aligned} &-\int_\Omega \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla\boldsymbol{\varphi} + \alpha \int_{\partial\Omega} \mathbf{u} \cdot \boldsymbol{\varphi} d\sigma - \int_\Omega (\varrho^\gamma + \varrho\theta) \operatorname{div} \boldsymbol{\varphi} dx \\ &\quad + \int_\Omega (2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{\varphi}) + \lambda \operatorname{div} \mathbf{u} \operatorname{div} \boldsymbol{\varphi}) dx \\ &\quad = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} dx \quad \forall\, \boldsymbol{\varphi} \in C^2(\bar{\Omega}); \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \quad \text{at } \partial\Omega. \end{aligned} \tag{2.2}$$

$$\begin{aligned} &-\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho\theta + \frac{1}{\gamma - 1} \varrho^\gamma \right) \mathbf{u} \cdot \nabla\psi dx \\ &\quad = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \psi dx + \int_\Omega (\varrho^\gamma + \varrho\theta) \mathbf{u} \cdot \nabla\psi dx - \int_\Omega (2\mu \mathbf{D}(\mathbf{u})\mathbf{u} + \lambda \operatorname{div} \mathbf{u}\mathbf{u}) \cdot \nabla\psi dx \\ &\qquad - \int_\Omega \kappa(\theta) \nabla\theta \cdot \nabla\psi dx - \int_{\partial\Omega} (L(\theta)(\theta - \theta_0) + \alpha |\mathbf{u}_{\boldsymbol{\tau}}|^2) \psi d\sigma \quad \forall\, \psi \in C^\infty(\bar{\Omega}). \end{aligned} \tag{2.3}$$

Definition 2.2. Let $\beta = 1$, $\varrho \in L_q(\Omega)$, $q \geq \max\{2, \frac{6}{5}\gamma\}$, $\varrho \geq 0$ a.e. and $\int_\Omega \varrho dx = M$; $\mathbf{u} \in W^1_2(\Omega)$, $\mathbf{u} = \mathbf{0}$ at $\partial\Omega$ in the sense of traces; $\theta \in W^1_{\frac{3}{2}}(\Omega)$, $\theta^m \nabla\theta \in L_1(\Omega)$, $\theta^{l+1} \in L_1(\partial\Omega)$ and $\theta > 0$ a.e. Then the triple $(\varrho, \mathbf{u}, \theta)$ is a weak solution to the problem (1.1)–(1.16), if

$$\int_\Omega \varrho\mathbf{u} \cdot \nabla\eta = 0 \quad \forall\, \eta \in C^\infty(\bar{\Omega}), \tag{2.4}$$

$$\begin{aligned} -\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - \int_\Omega (\varrho^\gamma + \varrho\theta)\operatorname{div}\boldsymbol{\varphi} dx \\ + \int_\Omega (\mu\nabla\mathbf{u} : \nabla\boldsymbol{\varphi} + (\mu+\lambda)\operatorname{div}\mathbf{u}\operatorname{div}\boldsymbol{\varphi})dx \\ = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}dx \quad \forall\, \boldsymbol{\varphi} \in C^\infty_0(\Omega), \end{aligned} \tag{2.5}$$

$$\begin{aligned} -\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v\varrho\theta + \frac{1}{\gamma-1}\varrho^\gamma\right)\mathbf{u}\cdot\nabla\psi dx \\ = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi dx + \int_\Omega (\varrho^\gamma + \varrho\theta)\mathbf{u}\cdot\nabla\psi dx \\ - \int_\Omega (2\mu\mathbf{D}(\mathbf{u})\mathbf{u} + \lambda\operatorname{div}\mathbf{u}\mathbf{u})\cdot\nabla\psi dx - \int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla\psi dx \\ - \int_{\partial\Omega} L(\theta)(\theta-\theta_0)\psi d\sigma \quad \forall\, \psi \in C^\infty(\bar{\Omega}) \end{aligned} \tag{2.6}$$

and for any $b \in C[0,\infty) \cap C^1(0,\infty)$ such that

$$\begin{aligned} |b'(t)| \leq Ct^{-\lambda_0}, \quad t \in [0,1], \quad \lambda_0 < 1, \\ |b'(t)| \leq Ct^{\lambda_1}, \quad t > 1, \quad -1 < \lambda_1 \leq \frac{q}{2} - 1 \end{aligned} \tag{2.7}$$

we have

$$-\int_{\mathbb{R}^3} b(\varrho)\mathbf{u}\cdot\nabla\psi dx + \int_{\mathbb{R}^3} (\varrho b'(\varrho) - b(\varrho))\operatorname{div}\mathbf{u}\psi dx = 0 \quad \forall\, \psi \in C^\infty_0(\mathbb{R}^3), \tag{2.8}$$

where $\mathbf{u}$ and ϱ are extended by zero outside of Ω.

The proof of Theorems 2.1 and 2.2 will be split in the following steps. First, we introduce an approximation of system (1.1)–(1.16), next we show the existence of smooth solutions for these equations. In Sec. 4 we show *a priori* estimates independent of the approximation parameter. Subsequently, we pass to the limit and show the strong convergence of the density, separately for the Dirichlet and Navier boundary conditions.

In order to explain the conditions on γ and m, we now construct the *a priori* energy estimate. Note that in this part our investigations are formal, i.e. we assume that our solutions are smooth. This point is crucial for our technique since it determines the final regularity of sought solutions as well as it explains some steps in the considerations for the approximation.

Thanks to assumed regularity and Maxwell's law (1.9) we obtain the following version of the energy equation $(1.1)_3$

$$\operatorname{div}(c_v \varrho\theta\mathbf{u}) - \operatorname{div}((1+\theta^m)\nabla\theta) = \mathbf{S}(\mathbf{u}) : \nabla\mathbf{u} - \varrho\theta \operatorname{div}\mathbf{u} \quad \text{in } \Omega, \tag{2.9}$$

being the result of subtraction between $(1.1)_3$ and $(1.1)_2$ multiplied by $\mathbf{u}$.

Lemma 2.1. *Let $(\varrho, \mathbf{u}, \theta)$ be a sufficiently smooth solution to system (1.1)–(1.16) such that $\varrho \geq 0$ and $\theta > 0$ in Ω. Then*

$$\|\mathbf{u}\|_{1,2} + \|\varrho\|_{s(\gamma)} + \|\theta\|_{3m} + \|\nabla\theta\|_r \leq DATA, \tag{2.10}$$

provided

$$\gamma > \frac{7}{3}, \quad m > \frac{3\gamma-1}{3\gamma-7}, \tag{2.11}$$

where $s(\gamma) = \min\{2\gamma, 3(\gamma-1)\}$ and $r = \min\{2, \frac{3m}{m+1}\}$.

Proof. Testing $(1.1)_2$ by $\mathbf{u}$ we obtain the following identity:

$$\int_\Omega \mathbf{S}(\mathbf{u}) : \nabla\mathbf{u}dx + \alpha\int_{\partial\Omega} |\mathbf{u}|^2 d\sigma - \int_\Omega \varrho\theta \operatorname{div}\mathbf{u}dx = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}dx. \tag{2.12}$$

Note that we take formally $\alpha = 0$ for the Dirichlet boundary condition $\beta = 1$.

The integration over Ω of (2.9) reads

$$\int_{\partial\Omega} L(\theta)(\theta-\theta_0)d\sigma = \int_\Omega \mathbf{S}(\mathbf{u}) : \nabla\mathbf{u}dx - \int_\Omega \varrho\theta\operatorname{div}\mathbf{u}dx. \tag{2.13}$$

Adding the above identities we get

$$\int_{\partial\Omega} \big(L(\theta)(\theta-\theta_0) + \alpha|\mathbf{u}|^2\big)d\sigma = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}dx. \tag{2.14}$$

Next, we integrate the entropy equation which is the consequence of the division of (2.9) by θ. Then we obtain the equation on the entropy type quantity $s := \ln\theta$. Let us recall that the thermodynamical entropy is defined as

$$s = \ln\left(\frac{\theta^{c_v}}{\varrho}\right) + \text{const.} \tag{2.15}$$

However, to simplify the notation, we will call our quantity $s = \ln\theta$ the entropy.

We have

$$\begin{aligned}\int_\Omega [c_v\varrho\mathbf{u}\cdot\nabla s - \operatorname{div}((1+\theta^m)\nabla s) - (1+\theta^m)|\nabla s|^2]dx \\ = \int_\Omega \left[\frac{\mathbf{S}(\mathbf{u}):\nabla\mathbf{u}}{\theta} - \varrho\operatorname{div}\mathbf{u}\right]dx\end{aligned} \tag{2.16}$$

and

$$\int_\Omega \left(\frac{\mathbf{S}(\mathbf{u}):\nabla\mathbf{u}}{\theta} + (1+\theta^m)|\nabla s|^2\right)dx + \int_{\partial\Omega}\frac{L(\theta)\theta_0}{\theta}d\sigma = \int_{\partial\Omega} L(\theta)d\sigma. \tag{2.17}$$

Combining identities (2.14) and (2.17), taking into account the form of L given by (1.16), we conclude the following inequality

$$\int_\Omega \frac{1+\theta^m}{\theta^2}|\nabla\theta|^2 dx + \int_{\partial\Omega}\left[\frac{L(\theta)}{\theta} + L(\theta)\theta\right] d\sigma \leq C\left(1 + \int_\Omega |\varrho\mathbf{f}\cdot\mathbf{u}|dx\right). \tag{2.18}$$

The form of the L.H.S. of (2.18) implies that

$$\|\nabla\theta^{m/2}\|^2_{L_2(\Omega)} + \|\theta^{m/2}\|^2_{L_2(\partial\Omega)} \leq \text{L.H.S. of (2.18)}. \tag{2.19}$$

Hence from the standard theory of the Sobolev spaces ($W^1_2(\Omega) \subset L_6(\Omega)$ for $\Omega \subset \mathbb{R}^3$) we deduce

$$\|\theta\|_{3m} \leq C\left(1 + \int_\Omega |\varrho\mathbf{f}\cdot\mathbf{u}|dx\right)^{1/m}. \tag{2.20}$$

Next, we want to examine the integrability of the density. We follow the standard approach known for the barotropic case. We use as a test function for the momentum equation $(1.1)_2$ the solution to

$$\begin{aligned} \operatorname{div}\mathbf{\Phi} &= \varrho^{s(\gamma)-\gamma} - \frac{1}{|\Omega|}\int_\Omega \varrho^{s(\gamma)-\gamma}dx \quad \text{in } \Omega, \\ \mathbf{\Phi} &= \mathbf{0} \quad \text{at } \partial\Omega. \end{aligned} \tag{2.21}$$

The construction of the field $\mathbf{\Phi}$ guarantees that

$$\|\mathbf{\Phi}\|_{1,q} \leq C\|\varrho^{s(\gamma)-\gamma}\|_q. \tag{2.22}$$

The solvability of (2.21)–(2.22) belongs to the standard theory of the barotropic Navier–Stokes equations, thus we omit all details of the construction of $\mathbf{\Phi}$ — see e.g. Ref. 17. Note that the case $\gamma > 3$ was studied in Ref. 13 (the fact which boundary condition for $\mathbf{u}$ is used does not play any role here) and thus we consider only the case $\frac{7}{3} < \gamma \leq 3$, i.e. $s(\gamma) = 3(\gamma-1)$. It yields

$$\begin{aligned} \int_\Omega (\varrho^{2\gamma-2}\theta + \varrho^{3(\gamma-1)})dx &\leq \frac{1}{|\Omega|}\int_\Omega \varrho^{2\gamma-3}dx\int_\Omega \varrho^\gamma dx + \frac{1}{|\Omega|}\int_\Omega \varrho^{2\gamma-3}dx\int_\Omega \varrho\theta dx \\ &+ \frac{1}{2}\int_\Omega (\varrho|\mathbf{u}|^2|\nabla\mathbf{\Phi}| + |\varrho\mathbf{u}\cdot\nabla\mathbf{u}||\mathbf{\Phi}|)dx + \int_\Omega (|\mathbf{S}(\mathbf{u})||\nabla\mathbf{\Phi}| + |\varrho\mathbf{f}\cdot\mathbf{\Phi}|)dx. \end{aligned} \tag{2.23}$$

Keeping in mind that $\int_\Omega \varrho dx = M$ we conclude

$$\|\varrho\|^{3(\gamma-1)}_{3(\gamma-1)} \leq C\left(1 + \int_\Omega (\varrho|\mathbf{u}|^2|\nabla\mathbf{\Phi}| + |\mathbf{S}(\mathbf{u})||\nabla\mathbf{\Phi}|)dx + \int_\Omega \varrho^{2\gamma-3}dx\int_\Omega \varrho\theta dx\right), \tag{2.24}$$

as the other terms are of the same or lower order. Then

$$\begin{aligned} \int_\Omega \varrho|\mathbf{u}|^2|\nabla\mathbf{\Phi}|dx &\leq C\|\nabla\mathbf{\Phi}\|_{\frac{3(\gamma-1)}{2\gamma-3}}\|\mathbf{u}\|^2_{1,2}\|\varrho\|_{3(\gamma-1)} \\ &\leq C\|\varrho\|^{2(\gamma-1)}_{3(\gamma-1)}\|\mathbf{u}\|^2_{1,2} \\ &\leq \frac{1}{4}\|\varrho\|^{3(\gamma-1)}_{3(\gamma-1)} + C\|\mathbf{u}\|^6_{1,2}. \end{aligned} \tag{2.25}$$

The other terms are less restrictive

$$\begin{aligned}\int_\Omega |\mathbf{S}(\mathbf{u})||\nabla\mathbf{\Phi}|dx &\le C\|\mathbf{u}\|_{1,2}\|\nabla\mathbf{\Phi}\|_2\\ &\le C\|\mathbf{u}\|_{1,2}\|\varrho\|_{4\gamma-6}^{2\gamma-3}\\ &\le C\|\mathbf{u}\|_{1,2}(1+\|\varrho\|_{3(\gamma-1)}^{2(\gamma-1)})\\ &\le \frac{1}{4}\|\varrho\|_{3(\gamma-1)}^{3(\gamma-1)} + C(1+\|\mathbf{u}\|_{1,2}^3)\end{aligned} \tag{2.26}$$

as $\gamma \le 3$. In order to estimate the last term of the R.H.S. of (2.24) we use (2.20)

$$\begin{aligned}\int_\Omega \varrho^{2\gamma-3}dx\int_\Omega \varrho\theta dx &\le \|\varrho\|_{2\gamma-3}^{2\gamma-3}\|\varrho\|_{\frac{3m}{3m-1}}\|\theta\|_{3m}\\ &\le C(\|\varrho\|_{2\gamma-3}^{2\gamma-3}\|\varrho\|_{\frac{3m}{3m-1}}\|\nabla\mathbf{u}\|_2^{\frac{1}{m}}\|\varrho\|_{\frac{6}{5}}^{\frac{1}{m}}+1)\\ &\le C(\|\varrho\|_{3(\gamma-1)}^{2\gamma-2+1/m}\|\mathbf{u}\|_{1,2}^{1/m}+1)\\ &\le \frac{1}{4}\|\varrho\|_{3(\gamma-1)}^{3(\gamma-1)} + C(1+\|\mathbf{u}\|_{1,2}^6).\end{aligned} \tag{2.27}$$

Thus

$$\int_\Omega \varrho^{2\gamma-2}\theta dx + \|\varrho\|_{3(\gamma-1)}^{3(\gamma-1)} \le C(1+\|\mathbf{u}\|_{1,2}^6). \tag{2.28}$$

Let us observe that the first term of the L.H.S. of (2.28) does not play any important role. It can be omitted in the estimation, because (2.20) gives us stronger information. So

$$\begin{aligned}\left|\int_\Omega \varrho\mathbf{u}\cdot\mathbf{f}dx\right| &\le C(1+\|\mathbf{u}\|_{1,2}\|\varrho\|_{\frac{6}{5}}) \le C(1+\|\mathbf{u}\|_{1,2}\|\varrho\|_{3(\gamma-1)}^{\frac{\gamma-1}{2(3\gamma-4)}})\\ &\le C(1+\|\mathbf{u}\|_{1,2}^{\frac{3(\gamma-1)}{3\gamma-4}}).\end{aligned} \tag{2.29}$$

Inserting this estimate into the estimate for the temperature yields

$$\|\theta\|_{3m} \le C(1+\|\mathbf{u}\|_{1,2}^{\frac{1}{m}\frac{3(\gamma-1)}{3\gamma-4}}). \tag{2.30}$$

Returning to (2.12)[a] we also get

$$\begin{aligned}\|\mathbf{u}\|_{1,2}^2 &\le C\left(1+\int_\Omega \varrho|\mathbf{f}\cdot\mathbf{u}|dx + \int_\Omega |\varrho\theta\,\mathrm{div}\,\mathbf{u}|dx\right)\\ &\le C(1+\|\varrho\|_{\frac{6}{5}}\|\mathbf{u}\|_{1,2} + \|\nabla\mathbf{u}\|_2\|\theta\|_{3m}\|\varrho\|_{\frac{6m}{3m-2}})\\ &\le C(1+\|\mathbf{u}\|_{1,2}^{\frac{3(\gamma-3)}{3\gamma-4}} + \|\mathbf{u}\|_{1,2}^{1+\frac{1}{m}\frac{3\gamma-3}{3\gamma-4}+\frac{1}{m}\frac{3m+2}{3\gamma-4}}).\end{aligned} \tag{2.31}$$

[a]Note that for the Dirichlet boundary conditions, i.e. $\beta = 1$, the term $\int_\Omega S(\mathbf{u}) : \nabla\mathbf{u}dx$ can be replaced by $\int_\Omega(\mu|\nabla\mathbf{u}|^2 + (\mu+\lambda)(\mathrm{div}\,\mathbf{u})^2)dx$.

From this inequality we get the desired *a priori* bound $\|\mathbf{u}\|_{1,2} \leq DATA$, provided

$$\frac{1}{m}\left(\frac{3\gamma-3}{3\gamma-4}+\frac{3m+2}{3\gamma-4}\right) < 1, \quad \text{i.e.} \quad m > \frac{3\gamma-1}{3\gamma-7}, \quad \gamma > \frac{7}{3}. \tag{2.32}$$

Eventually, we estimate $\nabla\theta$. Our analysis shows that

$$\int_\Omega \frac{1+\theta^m}{\theta^2}|\nabla\theta|^2 dx + \|\theta\|_{3m} \leq DATA. \tag{2.33}$$

If $m \geq 2$, we immediately get $\nabla\theta \in L_2(\Omega)$ and if $m < 2$ then by (2.30) and (2.33) we have

$$\int_\Omega |\nabla\theta|^{\frac{3m}{m+1}} dx \leq \left(\int_\Omega \theta^{3m} dx\right)^{\frac{2-m}{2(m+1)}} \left(\int_\Omega |\nabla\theta|^2\theta^{m-2} dx\right)^{\frac{3m}{2(m+1)}}. \tag{2.34}$$

Estimate (2.34) completes the proof of (2.10). □

3. The Approximative System

In order to prove Theorems 2.1 and 2.2, we will follow the standard approach — we introduce an approximation of the original problem and prove the existence for the approximative problem. This is the goal of the section below.

For $\epsilon > 0$ consider the following problem in Ω

$$\begin{aligned}
&\epsilon\varrho + \operatorname{div}(\varrho\mathbf{u}) - \epsilon\Delta\varrho = \epsilon h, \\
&\frac{1}{2}\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{2}\varrho\mathbf{u}\cdot\nabla\mathbf{u} - \operatorname{div}(2\mu\mathbf{D}(\mathbf{u}) + \lambda\operatorname{div}\mathbf{u}\mathbf{I}) + \nabla(\varrho^\gamma + \varrho\theta) = \varrho\mathbf{f}, \\
&-\operatorname{div}\left((1+\theta^m)\frac{\epsilon+\theta}{\theta}\nabla\theta\right) + (1+c_v)\operatorname{div}(\varrho\mathbf{u})\theta + c_v\varrho\mathbf{u}\cdot\nabla\theta - \theta\mathbf{u}\cdot\nabla\varrho \\
&\quad = 2\mu|\mathbf{D}(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2 + \epsilon\gamma|\nabla\varrho|^2\varrho^{\gamma-2}
\end{aligned} \tag{3.1}$$

together with the boundary conditions at $\partial\Omega$

$$\begin{aligned}
&\frac{\partial\varrho}{\partial\mathbf{n}} = 0, \\
&\mathbf{u}\cdot\mathbf{n} = 0, \quad \beta\mathbf{u}_{\boldsymbol{\tau}} + (1-\beta)(\mathbf{S}\mathbf{n})_{\boldsymbol{\tau}} = \mathbf{0}, \\
&(1+\theta^m)\frac{\epsilon+\theta}{\theta}\frac{\partial\theta}{\partial\mathbf{n}} + L(\theta)(\theta-\theta_0) + \epsilon\ln\theta = 0.
\end{aligned} \tag{3.2}$$

In what follows, we will show the existence of a solution to problem (3.1)–(3.2) for fixed $\epsilon > 0$. Note that Eq. $(3.1)_3$ corresponds to the balance of the internal energy (2.9). We will use the well-known fact that the balance of the internal energy and the balance of the total energy are equivalent provided the solution is sufficiently smooth (we will use it on the approximative level) and in the limit we get $(1.1)_3$ in the weak form.

Next, note that (provided $\theta > 0$) we may define the "entropy"

$$s = \ln\theta \tag{3.3}$$

— see the remark for (2.15) — and replace $(3.1)_3$ by

$$-\mathrm{div}\Big((1+\mathrm{e}^{sm})\frac{\epsilon+\mathrm{e}^s}{\mathrm{e}^s}\nabla s\Big) + (1+c_v)\,\mathrm{div}(\varrho\mathbf{u}) + c_v\varrho\mathbf{u}\cdot\nabla s - \mathbf{u}\cdot\nabla\varrho$$
$$= \frac{\mathbf{S}(\mathbf{u}):\nabla\mathbf{u}}{\mathrm{e}^s} + (1+\mathrm{e}^{sm})\frac{\epsilon+\mathrm{e}^s}{\mathrm{e}^s}|\nabla s|^2 + \epsilon\gamma|\nabla\varrho|^2\frac{\varrho^{\gamma-2}}{\mathrm{e}^s} \tag{3.4}$$

or

$$-\,\mathrm{div}((1+\mathrm{e}^{sm})(\epsilon+\mathrm{e}^s)\nabla s) + (1+c_v)\,\mathrm{div}(\varrho\mathbf{u})\mathrm{e}^s + c_v\mathrm{e}^s\varrho\mathbf{u}\cdot\nabla s - \mathrm{e}^s\mathbf{u}\cdot\nabla\varrho$$
$$= \mathbf{S}(\mathbf{u}):\nabla\mathbf{u} + \epsilon\gamma|\nabla\varrho|^2\varrho^{\gamma-2}. \tag{3.5}$$

The strategy to prove the existence of a strong solution (i.e. $(\varrho,\mathbf{u},s)\in W^2_q(\Omega)$ for any $q<\infty$) to the approximative problem will be the following. We consider system $(3.1)_{1,2}$ and (3.5) and prove the existence of solutions. We may define the temperature $\theta = \mathrm{e}^s$ and thus we have the warranty that the temperature being a solution to (3.1) is strictly positive. This is one of the most difficult obstacles in proving the final result.

Note that in Ref. 13 the authors used a similar approximative system, however, without the term $\epsilon\gamma\varrho^{\gamma-2}|\nabla\varrho|^2$. This was connected with the fact that for $\varrho\in L_{2\gamma}(\Omega)$, $\mathbf{u}\in W^1_2(\Omega)$, satisfying $\mathrm{div}(\varrho\mathbf{u})=0$ in the sense of distributions, it holds

$$\mathrm{div}(\varrho^\gamma\mathbf{u}) = -(\gamma-1)\varrho^\gamma\,\mathrm{div}\,\mathbf{u}$$

in the weak sense, provided $\gamma\geq 3$. Indeed, we cannot use this equality here and we have to proceed differently.

The main result concerning system (3.1)–(3.2) reads

Theorem 3.1. *Let $\beta\in[0,1]$, $\Omega\in C^2$, $\mathbf{f}\in L_\infty(\Omega)$, $\epsilon>0$. Let Ω be not axially symmetric if $\beta=0$. Let $m=l+1>\frac{3\gamma-1}{3\gamma-7}$, $\gamma>\frac{7}{3}$, $h=\frac{M}{|\Omega|}$. Then there exists a strong solution $(\varrho,\mathbf{u},\theta)$ to problem (3.1)–(3.2) such that*

- $\varrho\in W^2_q(\Omega)$ $\forall\, 1\leq q<\infty$, $\varrho\geq 0$ *a.e. in* Ω, $\int_\Omega \varrho dx = M$;
- $\mathbf{u}\in W^2_q(\Omega)$ $\forall\, 1\leq q<\infty$, $\mathbf{u}\cdot\mathbf{n}=0$ *at* $\partial\Omega$ *for* $\beta\in[0,1)$, $\mathbf{u}=\mathbf{0}$ *at* $\partial\Omega$ *for* $\beta=1$;
- $\theta\in W^2_q(\Omega)$ $\forall\, 1\leq q<\infty$, $\theta\geq C(\epsilon)>0$ *in* Ω.

We first consider the approximative continuity equation and define

$$M^\beta_p = \begin{cases}\{\mathbf{w}\in W^2_p(\Omega);\mathbf{w}=0\,\text{at}\ \partial\Omega\} & \text{if } \beta=1,\\ \{\mathbf{w}\in W^2_p(\Omega);\mathbf{w}\cdot\mathbf{n}=0\,\text{at}\ \partial\Omega\} & \text{if } \beta\in[0,1).\end{cases}$$

We have

Lemma 3.1. *Let $q>3$. Let all assumptions of Theorem 3.1 be satisfied. Then the operator*

$$S: M^\beta_q \to W^2_p(\Omega)$$

such that $S(\mathbf{v}) = \varrho$ *with* ϱ *being the solution to*

$$\begin{aligned} \epsilon\varrho - \epsilon\Delta\varrho &= \epsilon h - \operatorname{div}(\varrho\mathbf{v}) \quad \text{in } \Omega, \\ \frac{\partial\varrho}{\partial\mathbf{n}} &= 0 \qquad\qquad\qquad \text{at } \partial\Omega \end{aligned}$$

is a well-defined continuous compact operator from M_q^β *to* $W_p^2(\Omega)$, $1 \le p < \infty$. *In particular, the solution is unique, non-negative in* Ω, $\int_\Omega \varrho dx = M$ *and*

$$\|\varrho\|_{1,p} \le \begin{cases} C(\epsilon, \|\mathbf{v}\|_p) & \text{if } p > 3, \\ C(\epsilon, \delta, \|\mathbf{v}\|_{3+\delta}) & \text{if } p \le 3, \end{cases} \quad \|\varrho\|_{2,p} \le \begin{cases} C(\epsilon, \|\mathbf{v}\|_{1,p}) & \text{if } p > \frac{3}{2}, \\ C(\epsilon, \delta, \|\mathbf{v}\|_{1,\frac{3}{2}+\delta}) & \text{if } p \le \frac{3}{2}. \end{cases} \tag{3.6}$$

Proof. The L_1-bound of the density follows simply by integration of the approximative continuity equation over Ω. Similarly, integrating over the set $\{x \in \Omega; \varrho(x) < 0\}$ we get $\varrho \ge 0$ a.e. for any solution to $(3.1)_1$. To prove the existence of the solution, we may refer to Proposition 4.29 in Ref. 17, different boundary conditions for $\mathbf{v}$ do not play any role. Finally, using the standard regularity results for the scalar elliptic equation, the L_1-bound of the density and Sobolev imbedding theorem, we get estimates (3.6). □

Next we define the operator

$$\mathcal{T} : M_p^\beta \times W_p^2(\Omega) \to M_p^\beta \times W_p^2(\Omega)$$

such that $\mathcal{T}(\mathbf{v}, s) = (\mathbf{w}, z)$, where $(\mathbf{w}, z)$ is the solution to the system

$$\left.\begin{aligned} -\operatorname{div} S(\mathbf{w}) &= -\frac{1}{2}\operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v}) - \frac{1}{2}\varrho\mathbf{v}\cdot\nabla\mathbf{v} - \nabla(\varrho^\gamma + \varrho \mathrm{e}^s) + \varrho\mathbf{f}, \\ &\quad - \operatorname{div}((1+\mathrm{e}^{ms})(\epsilon + \mathrm{e}^s)\nabla z) \\ &= \mathbf{S}(\mathbf{v}) : \nabla\mathbf{v} - (1+c_v)\operatorname{div}(\varrho\mathbf{v})\mathrm{e}^s - c_v \mathrm{e}^s \varrho\mathbf{v}\cdot\nabla s \\ &\quad + \mathrm{e}^s\mathbf{v}\cdot\nabla\varrho + \epsilon\gamma|\nabla\varrho|^2\varrho^{\gamma-2} \end{aligned}\right\} \text{ in } \Omega,$$

$$\left.\begin{aligned} \mathbf{w}\cdot\mathbf{n} = 0, \quad \beta\mathbf{w}_{\boldsymbol{\tau}} + (1-\beta)(\mathbf{S}(\mathbf{w})\mathbf{n})_{\boldsymbol{\tau}} = \mathbf{0}, \\ (1+\mathrm{e}^{ms})(\epsilon+\mathrm{e}^s)\frac{\partial z}{\partial\mathbf{n}} + \epsilon z = -L(\mathrm{e}^s)(\mathrm{e}^s - \theta_0) \end{aligned}\right\} \text{ at } \partial\Omega, \tag{3.7}$$

where $\varrho = S(\mathbf{v})$ is given by Lemma 3.1.

Note that, except for the term $\epsilon\gamma|\nabla\varrho|^2\varrho^{\gamma-2}$ in $(3.7)_2$ and possibly different boundary conditions, we are in a similar situation as in Ref. 13. Thus we only briefly sketch the proof of the following lemma, giving the main properties of the operator $\mathcal{T}$.

Lemma 3.2. *Let* $p > 3$ *and all assumptions of Theorem 2.1 or 2.2 be satisfied. Then* $\mathcal{T}$ *is a continuous and compact operator from* $M_p^\beta \times W_p^2(\Omega)$ *into itself.*

Proof. As $\epsilon > 0$, the system is strongly elliptic. As $p > 3$, all the terms on the R.H.S. belong to $W^1_p(\Omega)$ ($\gamma > 2$, $\|\varrho\|_{2,p} \le C(\epsilon, \|\mathbf{v}\|_{1,p})$) and all boundary terms belong to $W_p^{1-\frac{1}{p}}(\partial\Omega)$ which gives the compactness. The unique solvability and the continuity of the operator $\mathcal{T}$ is straightforward. □

The next aim is to show that the set of all solutions satisfying

$$t\mathcal{T}(\mathbf{w}, z) = (\mathbf{w}, z), \quad t \in [0, 1] \tag{3.8}$$

is bounded in $M^\beta_p \times W^2_p(\Omega)$. This gives the next lemma:

Lemma 3.3. *Let all assumptions of Theorem 3.1 be satisfied. Then there exists a constant $C > 0$ such that all solutions to problem (3.8) in the class $M^\beta_p \times W^2_p(\Omega)$ satisfy*

$$\|\mathbf{w}\|_{2,p} + \|z\|_{2,p} + \|\theta\|_{2,p} + \|\varrho\|_{2,p} \le C,$$

where $\theta = \mathrm{e}^z$, $\varrho = S(\mathbf{w})$ and the constant C is independent of $t \in [0, 1]$.

Proof. Identity (3.8) implies that we consider the system

$$\left.\begin{aligned} -\operatorname{div} S(\mathbf{w}) &= -\frac{1}{2}t\operatorname{div}(\varrho\mathbf{w}\otimes\mathbf{w}) - \frac{1}{2}t\varrho\mathbf{w}\cdot\nabla\mathbf{w} - t\nabla(\varrho^\gamma + \varrho\theta) + t\varrho\mathbf{f}, \\ -\operatorname{div}\Big((1+\theta^m)\frac{\epsilon+\theta}{\theta}\nabla\theta\Big) &= t\mathbf{S}(\mathbf{w}) : \nabla\mathbf{w} \\ -(1+c_v)t\operatorname{div}(\varrho\mathbf{w})\theta &- c_v t\varrho\mathbf{w}\cdot\nabla\theta + t\theta\mathbf{w}\cdot\nabla\varrho + \epsilon\gamma t|\nabla\varrho|^2\varrho^{\gamma-2}, \\ \epsilon\varrho - \epsilon\Delta\varrho &= \epsilon h - \operatorname{div}(\varrho\mathbf{w}), \end{aligned}\right\} \text{ in } \Omega,$$

$$\left.\begin{aligned} \mathbf{w}\cdot\mathbf{n} = 0, \quad \beta\mathbf{w}_{\boldsymbol{\tau}} &+ (1-\beta)(\mathbf{S}(\mathbf{w})\mathbf{n})_{\boldsymbol{\tau}} = \mathbf{0}, \\ (1+\theta^m)\frac{\epsilon+\theta}{\theta}\frac{\partial\theta}{\partial\mathbf{n}} &+ \epsilon\ln\theta = -tL(\theta)(\theta-\theta_0), \\ \frac{\partial\varrho}{\partial\mathbf{n}} &= 0, \end{aligned}\right\} \text{ at } \partial\Omega, \tag{3.9}$$

where we denoted $\theta = \mathrm{e}^z$. First, we multiply $(3.9)_1$ by $\mathbf{w}$, integrate over Ω and use $(3.9)_3$. We get

$$\begin{aligned} \int_\Omega S(\mathbf{w}) : \nabla\mathbf{w}dx + \alpha\int_{\partial\Omega}|\mathbf{w}|^2 d\sigma + \frac{\epsilon\gamma t}{\gamma-1}\int_\Omega\varrho^\gamma dx + \epsilon\gamma t\int_\Omega|\nabla\varrho|^2\varrho^{\gamma-2}dx \\ -\ \epsilon\frac{\gamma}{\gamma-1}ht\int_\Omega\varrho^{\gamma-1}dx = t\int_\Omega\varrho\theta\operatorname{div}\mathbf{w}dx + t\int_\Omega\varrho\mathbf{f}\cdot\mathbf{w}dx. \end{aligned} \tag{3.10}$$

We put formally $\alpha = 0$ for $\beta = 1$. Next, we integrate $(3.9)_2$ over Ω and get

$$\begin{aligned} t\int_{\partial\Omega} L(\theta)(\theta-\theta_0)d\sigma + \epsilon\int_{\partial\Omega}\ln\theta d\sigma \\ = t\int_\Omega S(\mathbf{w}) : \nabla\mathbf{w}dx - t\int_\Omega\varrho\theta\operatorname{div}\mathbf{w}dx + t\int_\Omega\epsilon\gamma|\nabla\varrho|^2\varrho^{\gamma-2}dx. \end{aligned} \tag{3.11}$$

Thus (3.10) and (3.11) imply

$$
\begin{aligned}
(1-t)\int_\Omega S(\mathbf{w}) : \nabla \mathbf{w} dx + \alpha \int_{\partial\Omega} |\mathbf{w}|^2 d\sigma + \frac{\epsilon\gamma t}{\gamma - 1}\int_\Omega \varrho^\gamma dx \\
+ t\int_{\partial\Omega} L(\theta)(\theta-\theta_0) d\sigma + \epsilon \int_{\partial\Omega} \ln\theta d\sigma \\
= t\int_\Omega \varrho \mathbf{f}\cdot \mathbf{w} dx + \epsilon t \frac{\gamma}{\gamma-1} h \int_\Omega \varrho^{\gamma-1} dx. \qquad (3.12)
\end{aligned}
$$

Using the fact that the solution is smooth and thus $\theta = \mathrm{e}^z \geq c_0(\epsilon) > 0$, we may rewrite $(3.9)_2$ into the "entropy" formulation, i.e.

$$
\begin{aligned}
-\mathrm{div}\bigg((1+\theta^m)\frac{\epsilon+\theta^m}{\theta}\nabla z\bigg) + (1+c_v)t\,\mathrm{div}(\varrho\mathbf{w}) + tc_v\varrho\mathbf{w}\cdot\nabla z - t\mathbf{w}\cdot\nabla\varrho \\
= t\frac{\mathbf{S}(\mathbf{w}) : \nabla\mathbf{w}}{\theta} + (1+\theta^m)\frac{\epsilon+\theta}{\theta}|\nabla z|^2 + t\epsilon\gamma\frac{|\nabla\varrho|^2\varrho^{\gamma-2}}{\theta} \quad \text{in } \Omega, \quad (3.13) \\
(1+\theta^m)\frac{\epsilon+\theta}{\theta}\frac{\partial z}{\partial \mathbf{n}} + \frac{\epsilon z}{\mathrm{e}^z} = -t\frac{L(\theta)}{\theta}(\theta-\theta_0) \quad \text{at } \partial\Omega.
\end{aligned}
$$

Thus, integrating $(3.13)_1$ over Ω and using $(3.13)_2$ reads

$$
\begin{aligned}
t\int_\Omega \frac{\mathbf{S}(\mathbf{w}) : \nabla\mathbf{w}}{\theta} dx + \int_\Omega (1+\theta^m)\frac{(\epsilon+\theta)}{\theta}|\nabla z|^2 dx + t\int_{\partial\Omega}\frac{L(\theta)}{\theta}\theta_0 d\sigma \\
- \epsilon\int_{\partial\Omega} z\mathrm{e}^{-z} d\sigma + t\epsilon\gamma\int_\Omega \frac{|\nabla\varrho|^2\varrho^{\gamma-2}}{\theta} dx - t\int_\Omega (c_v\varrho\mathbf{w}\cdot\nabla z - \mathbf{w}\cdot\nabla\varrho) dx \\
= t\int_{\partial\Omega} L(\theta) d\sigma. \qquad (3.14)
\end{aligned}
$$

Let us now look at the last term on the L.H.S. We have formally, using $(3.9)_3$

$$
\begin{aligned}
\int_\Omega \mathbf{w}\cdot\nabla\varrho dx &= -\int_\Omega \mathrm{div}(\varrho\mathbf{w})\ln\varrho dx \\
&= \int_\Omega (-\epsilon\Delta\varrho + \epsilon\varrho - \epsilon h)\ln\varrho dx \\
&= \int_\Omega \bigg(\epsilon\frac{|\nabla\varrho|^2}{\varrho} + \epsilon\varrho\ln\varrho - \epsilon h\ln\varrho\bigg) dx. \qquad (3.15)
\end{aligned}
$$

Thus, in the last integral, the first term, the second term for $\varrho > 1$ and the last term for $\varrho < 1$ have good signs, while the other terms can be controlled by $C + \frac{1}{4}\frac{\gamma\epsilon}{\gamma-1}\int_\Omega \varrho^\gamma dx$.

To make the calculations rigorous, we realize that

$$
\begin{aligned}
\int_\Omega \mathbf{w}\cdot\nabla\varrho dx &= -\int_\Omega \mathrm{div}((\varrho+\delta)\mathbf{w})\ln(\varrho+\delta) dx \\
&= -\delta\int_\Omega \mathrm{div}\,\mathbf{w}\ln(\varrho+\delta) dx - \int_\Omega \mathrm{div}(\varrho\mathbf{w})\ln(\varrho+\delta) dx \qquad (3.16)
\end{aligned}
$$

and we proceed as above and finally pass with δ to zero.

The other term in the last integral of the L.H.S. in (3.14) can be treated as follows:

$$-\int_\Omega \varrho \mathbf{w}\cdot\nabla z dx = \int_\Omega (\epsilon\Delta\varrho - \epsilon\varrho + \epsilon h) z dx = \epsilon\int_\Omega (-\nabla\varrho\cdot\nabla z - \varrho\ln\theta + h\ln\theta)dx. \tag{3.17}$$

Now, the first term can be estimated by

$$\frac{1}{4}\epsilon\int_\Omega \frac{|\nabla\varrho|^2}{\varrho}dx + \frac{1}{4}\epsilon\gamma\int_\Omega \frac{|\nabla\varrho|^2\varrho^{\gamma-2}}{\theta}dx + \frac{1}{4}\int_\Omega |\nabla z|^2\theta^{\frac{1}{\gamma-1}}dx \tag{3.18}$$

and as $m > \frac{1}{\gamma-1}$, all the terms can be controlled by the terms on the L.H.S. Next, for $\theta \le 1$ the term $\varrho\ln\theta$ has a good sign, while for $\theta > 1$

$$\begin{aligned}\epsilon\int_\Omega \varrho(\ln\theta)^+ dx &\le \epsilon\|\varrho\|_2\|(\ln\theta)^+\|_2\\ &\le \frac{\epsilon}{4}(\|(\ln\theta)^+\|_{L_1(\partial\Omega)} + \|\nabla z\|_{L_2(\Omega)})^2 + \frac{\epsilon}{4}\|\varrho^\gamma\|_1 + C\\ &\le \frac{\epsilon}{4}\int_{\partial\Omega} L(\theta)\theta d\sigma + \|\nabla z\|^2_{L_2(\Omega)} + \frac{\epsilon}{4}\|\varrho^\gamma\|_1 + C.\end{aligned} \tag{3.19}$$

Finally, the last term must be controlled for $\theta \le 1$. But

$$\epsilon\int_\Omega h(\ln\theta)^- dx \le C\epsilon\int_\Omega |z^-|dx \le \frac{1}{2}\epsilon\int_{\partial\Omega}|z^-|\mathrm{e}^{|z^-|}d\sigma + \frac{1}{4}\|\nabla z\|_2. \tag{3.20}$$

Thus, summing up all estimates in (3.12)–(3.20) we end up with

$$\begin{aligned}&\int_\Omega \frac{1+\theta^m}{\theta^2}|\nabla\theta|^2 dx + t\int_{\partial\Omega}\left(\frac{L(\theta)}{\theta} + L(\theta)\theta\right)d\sigma + \epsilon t\int_\Omega |\ln\theta|dx\\ &\quad \le Ct\left(1 + \left|\int_\Omega \varrho\mathbf{f}\cdot\mathbf{w}dx\right|\right).\end{aligned} \tag{3.21}$$

Note that, as t is on the R.H.S., we have (recall that $m = l+1$)

$$\begin{aligned}\left(\int_\Omega \theta^{3m}dx\right)^{\frac{1}{3m}} &\le C\left(\left(\int_\Omega |\nabla\theta^{\frac{m}{2}}|^2 dx\right)^{\frac{1}{m}} + \left(\int_{\partial\Omega}\theta^{l+1}d\sigma\right)^{\frac{1}{l+1}}\right)\\ &\le C\left(1 + \int_\Omega |\varrho\mathbf{f}\cdot\mathbf{w}|dx\right)^{\frac{1}{m}}.\end{aligned} \tag{3.22}$$

Next, returning to (3.10),[b] we have

$$\begin{aligned}\|\varrho\|^\gamma_{3\gamma} + \|\mathbf{w}\|^2_{1,2} &\le C(\epsilon)\left(\int_\Omega (S(\mathbf{w}):\nabla\mathbf{w} + \epsilon\gamma|\nabla\varrho|^2\varrho^{\gamma-2})dx + \alpha\int_{\partial\Omega}|\mathbf{w}|^2 d\sigma\right)\\ &\le C(\epsilon)\left(\int_\Omega |\varrho\theta\,\mathrm{div}\,\mathbf{w}|dx + \int_\Omega |\varrho\mathbf{f}\cdot\mathbf{w}|dx + 1\right).\end{aligned} \tag{3.23}$$

[b] See the footnote below (2.30) in the case $\beta = 1$.

It is worthwhile to underline that in (3.23) the constant $C(\epsilon)$ blows up as $\epsilon \to 0$. We are not able to control uniformly the norm $\|\varrho\|_{3\gamma}$ — see Lemma 4.1. However, in this part ϵ is fixed and we are allowed to use this point of view.

In both (3.22) and (3.23), the constants C are independent of t, moreover, in (3.22), it is also independent of ϵ. We will use this fact later on.

First, we have

$$\left|\int_\Omega \varrho \mathbf{f}\cdot\mathbf{w}dx\right| \le \|\mathbf{f}\|_\infty \|\varrho\|_{\frac{6}{5}}\|\mathbf{w}\|_6 \le C\|\varrho\|_{3\gamma}^{\frac{\gamma}{6\gamma-2}}\|\nabla\mathbf{w}\|_2 \le \delta\|\nabla\mathbf{w}\|_2^2 + C(\delta)\|\varrho\|_{3\gamma}^{\frac{\gamma}{3\gamma-1}}$$

and thus, for any $\gamma > 1$ ($\frac{\gamma}{3\gamma-1} < \gamma$) we can control this term by the L.H.S. On the other hand, the term involving the temperature is more complicated. We have, using also (3.22)

$$\begin{aligned}\int_\Omega |\varrho\theta \operatorname{div}\mathbf{w}|dx &\le C\|\operatorname{div}\mathbf{w}\|_2\|\theta\|_{3m}\|\varrho\|_{\frac{6m}{3m-2}}\\ &\le C\|\varrho\|_{3\gamma}^{\gamma\frac{3m+2}{2m(3\gamma-1)}+\frac{1}{m}\frac{\gamma}{6\gamma-2}}\|\nabla\mathbf{w}\|_2^{\frac{m+1}{m}}\\ &\le \delta\|\nabla\mathbf{w}\|_2^2 + C(\delta)\|\varrho\|_{3\gamma}^{\frac{3\gamma(m+1)}{(m-1)(3\gamma-1)}}\end{aligned}$$

and thus, this term can be estimated provided $\frac{3(m+1)}{(m-1)(3\gamma-1)} < 1$, i.e. $m > \frac{3\gamma+2}{3\gamma-4}$. Note that for $\gamma > \frac{7}{3}$ we have $m > \frac{3\gamma+2}{3\gamma-4}$. This gives the bound

$$\|\nabla\mathbf{w}\|_2 + \|\varrho\|_{3\gamma} + \|\theta\|_{3m} \le C(\epsilon)$$

with the R.H.S. independent of t. Note that from (3.21) we also have $\|\nabla\theta\|_2 \le C(\epsilon)$ for $m \ge 2$ while for $m < 2$ we only get $\|\nabla\theta\|_{\frac{3m}{m+1}} \le C(\epsilon)$, see the proof of Lemma 2.1. To conclude, we need to estimate $(\mathbf{w},\theta,\varrho)$ in $W^2_p(\Omega)\times W^2_p(\Omega)\times W^2_p(\Omega)$ for any $p < \infty$, independently of t.

From Lemma 3.1 we immediately see that $\|\varrho\|_{2,2} \le C(\epsilon)$ with the constant independent of t. Thus ϱ is bounded in $L_\infty(\Omega)$ and from $(3.9)_2$ we immediately get $\|\mathbf{w}\|_{1,3} \le C(\epsilon)$, thus $\|\mathbf{w}\|_q \le C(\epsilon)$ for any $q < \infty$ and once again from $(3.9)_2$, $\|\mathbf{w}\|_{1,3m} \le C(\epsilon)$; thus from Lemma 3.1 $\|\varrho\|_{2,3m} \le C(\epsilon)$. Next we need to improve the regularity of the temperature/entropy. To this aim, we rewrite $(3.9)_2$ into the form

$$\begin{aligned}-\Delta\Phi(z) &= t[\mathbf{S}(\mathbf{w}):\nabla\mathbf{w} - (1+c_v)\operatorname{div}(\varrho\mathbf{w})\mathrm{e}^z - c_v\mathrm{e}^z\varrho\mathbf{w}\cdot\nabla z\\ &\quad + \mathrm{e}^z\mathbf{w}\cdot\nabla\varrho + \epsilon\gamma|\nabla\varrho|^2\varrho^{\gamma-2}],\\ \frac{\partial\Phi(z)}{\partial\mathbf{n}} &= -\epsilon z - tL(\mathrm{e}^z)(\mathrm{e}^z-\theta_0)\end{aligned} \tag{3.24}$$

with

$$\Phi(z) = \int_0^z (1+\mathrm{e}^{m\tau})(\epsilon+\mathrm{e}^\tau)d\tau.$$

We multiply $(3.24)_1$ by Φ and integrate over Ω. As the R.H.S. of (3.23) belongs to $L_{\frac{3m}{m+1}}(\Omega)$ and the boundary terms ($\Phi(z) \sim \epsilon z$ for $z \to -\infty$, $\Phi(z) \sim \mathrm{e}^{(m+1)z}$ for

$z \to +\infty$) do not cause any troubles, we get $\|\Phi(z(\cdot))\|_{1,2} \le C(\epsilon)$ with the R.H.S. independent of t which implies

$$\|\theta^{m+1}\|_6 = \|\mathrm{e}^{(m+1)z}\|_6 \le C(\epsilon) \quad \text{and} \quad \|\nabla\theta\|_2 \le \|\mathrm{e}^z \nabla z\|_2 \le C(\epsilon).$$

Thus, again from (3.24) we get $\|\Phi\|_{2,p^*} \le C$ with $p^* = \min\{\frac{3m}{2}, 2\}$. As $m > 1$, we immediately see that

$$\|z\|_\infty + \|\theta\|_\infty \le C(\epsilon), \quad \|\nabla z\|_q + \|\nabla\theta\|_q \le C(\epsilon)$$

with $1 \le q \le q^* = \frac{3p^*}{3-p^*} > 3$. We now return to the balance of momentum $(3.14)_1$ and see that $\|\mathbf{w}\|_{2,q^*} \le C(\epsilon)$, hence $\|\varrho\|_{2,r} \le C(\epsilon)$ for all $r < \infty$. Therefore $(3.9)_2$ implies

$$\|z\|_{2,q^*} + \|\theta\|_{2,q^*} \le C(\epsilon), \quad \|\nabla z\|_\infty + \|\nabla\theta\|_\infty \le C(\epsilon).$$

Using this information once more, we conclude

$$\|\mathbf{w}\|_{2,r} + \|z\|_{2,r} + \|\theta\|_{2,r} \le C(\epsilon) \quad \text{for any } r < \infty \text{ and } \epsilon \text{ fixed.}$$

This ends the proof of Lemma 3.3 and subsequently the proof of Theorem 3.1. □

4. *A Priori* Estimates, Limit Passage

Having proved solvability of the approximative problem, the next aim is to pass to the limit $\epsilon \to 0^+$. In order to be able to do so, we need *a priori* estimates of solutions to (3.1)–(3.2) independent of the parameter ϵ. We have

Lemma 4.1. *Let assumptions of Theorems 2.1 and 2.2 be satisfied. Let $\epsilon > 0$. Then there exists $C > 0$, independent of ϵ, such that for $(\varrho, \mathbf{u}, \theta)$, solutions to (3.1)–(3.2)*

$$\|\mathbf{u}\|_{1,2} + \sqrt{\epsilon}\|\nabla\varrho\|_2 + \|\varrho\|_{s(\gamma)} + \|\theta\|_{3m} + \|\theta\|_{1,r} \le C, \tag{4.1}$$

where $s(\gamma) = \min\{2\gamma, 3(\gamma-1)\}$ and $r = \min\{2, \frac{3m}{m+1}\}$.

Proof. First of all, taking $t = 1$, we use (3.10)–(3.22). However, in order to get the estimate for the density, we cannot use the term $\epsilon\int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2 dx$. Therefore we repeat the estimates from the proof of Lemma 2.1 (2.12)–(2.34), getting (4.1), but without the term $\sqrt{\epsilon}\|\nabla\varrho\|_2$.

Testing the approximative continuity equation $(3.1)_1$ by ϱ yields

$$\begin{aligned}\epsilon(\|\varrho\|_2^2 + \|\nabla\varrho\|_2^2) &\le \int_\Omega (\epsilon h\varrho - \operatorname{div}(\varrho\mathbf{u})\varrho)dx \\ &\le C + \frac{\epsilon}{2}\|\varrho\|_2^2 + \frac{1}{2}\int_\Omega \varrho^2 \operatorname{div}\mathbf{u}dx,\end{aligned}$$

which gives $\sqrt{\epsilon}\|\nabla\varrho\|_2 \le C$ as $3(\gamma-1) > 4$ for $\gamma > \frac{7}{3}$. Lemma 4.1 is proved. □

In the next section, in the case of the Navier boundary conditions, we also need an estimate of $\operatorname{curl}\mathbf{u}$. As $\gamma > 3$ is studied in Ref. 13, we consider only $\frac{7}{3} < \gamma \le 3$ here.

Lemma 4.2. *Let assumptions of Theorem 2.2 be satisfied, $\beta \in [0,1)$ and $\epsilon > 0$. Then there exists C, independent of ϵ, such that for $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ we have*

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$$

with

$$\|\boldsymbol{\omega}_0\|_{1,2} \le C, \quad \|\boldsymbol{\omega}_1\|_{1,\frac{3(\gamma-1)}{2\gamma-1}} \le C, \quad \|\boldsymbol{\omega}_2\|_r \le C\sqrt{\epsilon}$$

with $r < \frac{3}{2}$. In particular, for $\epsilon \to 0^+$ there exists a subsequence $\boldsymbol{\omega}_{\epsilon_n}$ such that $\boldsymbol{\omega}_{\epsilon_n}$ is strongly convergent in $L_r(\Omega)$ for $1 \le r < \frac{3}{2}$.

Proof. Denote $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$. The momentum equation $(3.1)_2$ immediately yields

$$\begin{aligned}
-\mu\Delta\boldsymbol{\omega} &= \operatorname{curl}\left(\varrho\mathbf{f} - \varrho\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{2}\epsilon h\mathbf{u} + \frac{1}{2}\epsilon\varrho\mathbf{u}\right) - \operatorname{curl}\left(\frac{1}{2}\epsilon\Delta\varrho\mathbf{u}\right) \\
&= \operatorname{curl}\mathbf{H}_1 + \operatorname{curl}\mathbf{H}_2 \quad \text{in } \Omega, \\
\boldsymbol{\omega}\cdot\boldsymbol{\tau}_1 &= -\left(2\chi_2 - \frac{\alpha}{\mu}\right)\mathbf{u}\cdot\boldsymbol{\tau}_2 \quad \text{at } \partial\Omega, \\
\boldsymbol{\omega}\cdot\boldsymbol{\tau}_2 &= \left(2\chi_1 - \frac{\alpha}{\mu}\right)\mathbf{u}\cdot\boldsymbol{\tau}_1 \quad \text{at } \partial\Omega, \\
\operatorname{div}\boldsymbol{\omega} &= 0 \quad \text{at } \partial\Omega,
\end{aligned} \tag{4.2}$$

where χ_i are the curvatures related with directions $\boldsymbol{\tau}_i$. Relations $(4.2)_{2,3}$ are a consequence of the slip boundary conditions, see Refs. 11 and 24. We write

$$\begin{aligned}
&-\mu\Delta\boldsymbol{\omega}_0 = \mathbf{0} \quad -\mu\Delta\boldsymbol{\omega}_1 = \operatorname{curl}\mathbf{H}_1 \quad -\mu\Delta\boldsymbol{\omega}_2 = \operatorname{curl}\mathbf{H}_2 \quad \text{in } \Omega, \\
&\boldsymbol{\omega}_0\cdot\boldsymbol{\tau}_1 = -\left(2\chi_2 - \frac{\alpha}{\mu}\right)\mathbf{u}\cdot\boldsymbol{\tau}_2 \quad \boldsymbol{\omega}_1\cdot\boldsymbol{\tau}_1 = 0 \quad \boldsymbol{\omega}_2\cdot\boldsymbol{\tau}_1 = 0 \quad \text{at } \partial\Omega, \\
&\boldsymbol{\omega}_0\cdot\boldsymbol{\tau}_2 = \left(2\chi_1 - \frac{\alpha}{\mu}\right)\mathbf{u}\cdot\boldsymbol{\tau}_1 \quad \boldsymbol{\omega}_1\cdot\boldsymbol{\tau}_2 = 0 \quad \boldsymbol{\omega}_2\cdot\boldsymbol{\tau}_2 = 0 \quad \text{at } \partial\Omega, \\
&\operatorname{div}\boldsymbol{\omega}_0 = 0 \quad \operatorname{div}\boldsymbol{\omega}_1 = 0 \quad \operatorname{div}\boldsymbol{\omega}_2 = 0 \quad \text{at } \partial\Omega.
\end{aligned} \tag{4.3}$$

We study the three problems separately, based on results from Refs. 22 and 23. First, let us consider $\boldsymbol{\alpha}_0$, the solution to the Stokes problem

$$\begin{aligned}
-\mu\Delta\boldsymbol{\alpha}_0 + \nabla p_0 &= \mathbf{0} && \text{in } \Omega, \\
\operatorname{div}\boldsymbol{\alpha}_0 &= 0 && \text{in } \Omega, \\
\boldsymbol{\alpha}_0\cdot\boldsymbol{\tau}_1 &= -\left(2\chi_2 - \frac{\alpha}{\mu}\right)\mathbf{u}\cdot\boldsymbol{\tau}_2 && \text{at } \partial\Omega, \\
\boldsymbol{\alpha}_0\cdot\boldsymbol{\tau}_2 &= \left(2\chi_1 - \frac{\alpha}{\mu}\right)\mathbf{u}\cdot\boldsymbol{\tau}_1 && \text{at } \partial\Omega, \\
\boldsymbol{\alpha}_0\cdot\mathbf{n} &= 0 && \text{at } \partial\Omega,
\end{aligned}$$

i.e. we have

$$\|\boldsymbol{\alpha}_0\|_{W_2^1(\Omega)} \le C\|\mathbf{u}\|_{W_2^{\frac{1}{2}}(\partial\Omega)} \le C\|\mathbf{u}\|_{W_2^1(\Omega)}.$$

Moreover, as $\boldsymbol{\alpha}_0$ has zero divergence and $\Omega \in C^2$, we have that

$$\frac{\partial \boldsymbol{\alpha}_0}{\partial \mathbf{n}} \cdot \mathbf{n} = ((\mathbf{n} \cdot \nabla)\boldsymbol{\alpha}_0) \cdot \mathbf{n} \in W_2^{-\frac{1}{2}}(\partial\Omega) \quad \text{and} \quad \left\| \frac{\partial \boldsymbol{\alpha}_0}{\partial \mathbf{n}} \cdot \mathbf{n} \right\|_{W_2^{-\frac{1}{2}}(\partial\Omega)} \leq C\|\mathbf{u}\|_{W_2^1(\Omega)}.$$

Thus, denoting

$$V_0^{-1,p'} = (V_0^{1,p})^* = \{\mathbf{w} \in W_p^1(\Omega); \mathbf{w} - (\mathbf{w} \cdot \mathbf{n})\mathbf{n} = \mathbf{0} \text{ at } \partial\Omega\}^*,$$

we have for $\mathbf{w} = \boldsymbol{\omega}_0 - \boldsymbol{\alpha}_0$

$$\begin{aligned} -\mu\Delta\mathbf{w} &= \mu\Delta\boldsymbol{\alpha}_0 \quad \text{in } \Omega, \\ \left.\begin{aligned} \mathbf{w} \cdot \boldsymbol{\tau}_1 &= 0 \\ \mathbf{w} \cdot \boldsymbol{\tau}_2 &= 0 \\ \operatorname{div} \mathbf{w} &= 0 \end{aligned}\right\} & \qquad \text{at } \partial\Omega; \end{aligned}$$

hence

$$\begin{aligned} \|\mathbf{w}\|_{W_2^1(\Omega)} &\leq C\|\boldsymbol{\alpha}_0\|_{V_0^{-1,2}} \\ &= C \sup_{\|\boldsymbol{\varphi}_0\|_{V_0^{1,2}} \leq 1} \langle \Delta\boldsymbol{\alpha}_0, \boldsymbol{\varphi}_0 \rangle \\ &= C \sup_{\|\boldsymbol{\varphi}_0\|_{V_0^{1,2}} \leq 1} \int_\Omega \Delta\boldsymbol{\alpha}_0 \cdot \boldsymbol{\varphi}_0 dx \\ &= C \sup_{\|\boldsymbol{\varphi}_0\|_{V_0^{1,2}} \leq 1} \left(-\int_\Omega \nabla\boldsymbol{\alpha}_0 : \nabla\boldsymbol{\varphi}_0 dx + \int_{\partial\Omega} \frac{\partial \boldsymbol{\alpha}_0}{\partial \mathbf{n}} \cdot \boldsymbol{\varphi}_0 d\sigma \right) \\ &\leq C \left(\|\nabla\boldsymbol{\alpha}_0\|_{L_2(\Omega)} + \left\| \frac{\partial \boldsymbol{\alpha}_0}{\partial \mathbf{n}} \cdot \mathbf{n} \right\|_{W_2^{-\frac{1}{2}}(\partial\Omega)} \right) \\ &\leq C\|\mathbf{u}\|_{W_2^1(\Omega)}. \end{aligned}$$

Thus $\|\boldsymbol{\omega}_0\|_{1,2} \leq \|\mathbf{w}\|_{1,2} + \|\boldsymbol{\alpha}_0\|_{1,2} \leq C\|\mathbf{u}\|_{1,2}$.

Next, for $\boldsymbol{\omega}_1$, we have

$$\|\boldsymbol{\omega}_1\|_{1,p} \leq C\|\operatorname{curl} \mathbf{H}_1\|_{V_0^{-1,p}} \leq C(\|\varrho\mathbf{f}\|_p + \|\varrho\mathbf{u} \cdot \nabla\mathbf{u}\|_p + \epsilon\|\mathbf{u}\|_p + \epsilon\|\varrho\mathbf{u}\|_p).$$

Evidently, the most restrictive term is the convective one and for $p = \frac{3(\gamma-1)}{2\gamma-1}$ $(\in (\frac{12}{11}, \frac{6}{5})$ if $\gamma \in (\frac{7}{3}, 3])$ we have

$$\|\varrho\mathbf{u} \cdot \nabla\mathbf{u}\|_p \leq \|\nabla\mathbf{u}\|_2\|\mathbf{u}\|_6\|\varrho\|_{3(\gamma-1)} \leq C.$$

Finally, for $\boldsymbol{\omega}_2$ and $p < \frac{3}{2}$

$$\begin{aligned} \|\boldsymbol{\omega}_2\|_p &\leq C\epsilon\|\Delta\varrho\mathbf{u}\|_{-1,p} \\ &= C\epsilon \sup_{\|\boldsymbol{\varphi}\|_{1,p'} \leq 1} \int_\Omega \Delta\varrho\mathbf{u} \cdot \boldsymbol{\varphi} dx \\ &\leq C\epsilon \sup_{\|\boldsymbol{\varphi}\|_{1,p'} \leq 1} \int_\Omega (\nabla\varrho \cdot (\nabla\mathbf{u}\boldsymbol{\varphi}) + \nabla\varrho \cdot (\nabla\boldsymbol{\varphi}\mathbf{u}))dx \\ &\leq C\sqrt{\epsilon}\sqrt{\epsilon}\|\nabla\varrho\|_2\|\mathbf{u}\|_{1,2} \leq C\sqrt{\epsilon}. \end{aligned}$$

Lemma 4.2 is proved. □

Using the bounds obtained in Lemma 4.1, we can take a sequence $\epsilon_n \to 0^+$ such that for $(\mathbf{u}_{\epsilon_n}, \varrho_{\epsilon_n}, \theta_{\epsilon_n}) = (\mathbf{u}_n, \varrho_n, \theta_n)$

$$
\begin{aligned}
&\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } W^1_2(\Omega), && \mathbf{u}_n \to \mathbf{u} \quad \text{in } L_q(\Omega), \quad 1 \le q < 6,\\
&\varrho_n \rightharpoonup \varrho \quad \text{in } L_{\min\{3(\gamma-1),2\gamma\}}(\Omega), &&\\
&\theta_n \rightharpoonup \theta \quad \text{in } W^1_{\min\{\frac{3m}{m+1},2\}}(\Omega), && \theta_n \to \theta \quad \text{in } L_q(\Omega), \quad 1 \le q < 3m.
\end{aligned}
$$

The above convergences of approximative solutions preserve the boundary conditions for the velocity, if $\beta = 1$ then $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ and if $\beta < 1$ we have only $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

We can now pass to the limit in the weak formulation for our approximative system. The passage for the continuity equation is rather standard. In the momentum equation, the only difficulty is the pressure. It requires the strong convergence of the density and this analysis will be done in Secs. 5 and 6. A more delicate situation is for the energy equation. There is no chance to pass to the limit in the internal energy balance, due to the presence of the term $\mathbf{S}(\mathbf{u}_n) : \nabla \mathbf{u}_n$, which is only bounded in $L_1(\Omega)$. Therefore we replace the internal energy balance by the total energy balance, see below. It is not necessary only in the case $\gamma > 3$ in Ref. 13, where we can prove the strong convergence of the velocity gradient, due to higher integrability of all terms; particularly due to fact that $p \in L_2(\Omega)$, which, in the presented case, we are not able to guarantee.

Thus, passing to the limit $\epsilon \to 0^+$ we get from the continuity and momentum equations

$$
\int_\Omega \varrho \mathbf{u} \cdot \nabla \eta dx = 0 \quad \forall \eta \in C^\infty(\overline{\Omega}) \tag{4.4}
$$

(recall that $\epsilon \|\nabla \varrho_n\|_2 \to 0$)

$$
\begin{aligned}
&\int_\Omega ((-\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} + 2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{\varphi}) + \lambda \operatorname{div} \mathbf{u} \operatorname{div} \boldsymbol{\varphi} - (\overline{\varrho^\gamma} + \varrho\theta) \operatorname{div} \boldsymbol{\varphi}) dx\\
&\qquad + \alpha \int_{\partial\Omega} \mathbf{u} \cdot \boldsymbol{\varphi} d\sigma = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} dx,
\end{aligned} \tag{4.5}
$$

where the elliptic term can be replaced by $\int_\Omega (\mu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} + (\mu + \lambda) \operatorname{div} \mathbf{u} \operatorname{div} \boldsymbol{\varphi}) dx$ if $\beta = 1$; in this case we set $\alpha = 0$. Equality (4.5) is satisfied for all $\boldsymbol{\varphi} \in C^\infty_0(\Omega)$ if $\beta = 1$, i.e. for the Dirichlet boundary condition, and for all $\boldsymbol{\varphi} \in C^2(\overline{\Omega})$ with $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$ at $\partial\Omega$ if $\beta < 1$, i.e. for the Navier boundary conditions. We also introduced the known notation $\overline{\varrho^\gamma}$ to be the weak limit of ϱ_n^γ, generally $\overline{b(a)}$ stands for a weak limit of $b(a_n)$, where a_n converges weakly to a.

In order to get the total energy balance, we first use as test function in the approximative momentum equation $\mathbf{u}_n \psi$ with $\psi \in C^\infty(\overline{\Omega})$. It yields

$$
\begin{aligned}
&\int_\Omega (2\mu \mathbf{D}(\mathbf{u}_n) : \mathbf{D}(\mathbf{u}_n)\psi + \lambda (\operatorname{div} \mathbf{u}_n)^2 \psi + 2\mu \mathbf{D}(\mathbf{u}_n)\mathbf{u}_n \cdot \nabla\psi + \lambda \operatorname{div} \mathbf{u}_n \mathbf{u}_n \cdot \nabla \psi) dx\\
&\qquad + \alpha \int_{\partial\Omega} |\mathbf{u}_n|^2 \psi d\sigma - \int_\Omega \left(\frac{1}{2} \varrho_n |\mathbf{u}_n|^2 \mathbf{u}_n \cdot \nabla\psi + (\varrho_n^\gamma + \varrho_n\theta_n) \operatorname{div} \mathbf{u}_n \psi\right) dx\\
&\qquad - \int_\Omega (\varrho_n^\gamma + \varrho_n \theta_n) \mathbf{u}_n \cdot \nabla\psi dx = \int_\Omega \varrho_n \mathbf{f} \cdot \mathbf{u}_n \psi dx.
\end{aligned} \tag{4.6}
$$

Note that we keep the same structure for both boundary conditions, only the integral over $\partial\Omega$ disappears for the Dirichlet boundary conditions (i.e. formally $\alpha = 0$).

Next, the weak form of the internal energy balance reads

$$\int_\Omega (1+\theta_n^m)\frac{\epsilon_n+\theta_n}{\theta_n}\nabla\theta_n\cdot\nabla\psi dx + \int_{\partial\Omega}(\epsilon_n \ln\theta_n + L(\theta_n)(\theta_n-\theta_0))\psi d\sigma$$
$$= \int_\Omega (-(1+c_v)\operatorname{div}(\varrho_n\mathbf{u}_n)\theta_n - c_v\varrho_n\mathbf{u}_n\cdot\nabla\theta_n + \theta_n\mathbf{u}_n\cdot\nabla\varrho_n)\psi dx$$
$$+ \int_\Omega (2\mu|\mathbf{D}(\mathbf{u}_n)|^2 + \lambda(\operatorname{div}\mathbf{u}_n)^2 + \epsilon_n\gamma|\nabla\varrho_n|^2\varrho_n^{\gamma-2})\psi dx. \tag{4.7}$$

We apply the Green formula to the last term on the R.H.S. and use the approximative continuity equation:

$$\epsilon_n\gamma\int_\Omega |\nabla\varrho_n|^2\varrho_n^{\gamma-2}\psi dx$$
$$= -\frac{\epsilon_n\gamma}{\gamma-1}\int_\Omega (\Delta\varrho_n\varrho_n^{\gamma-1}\psi + \nabla\varrho_n\cdot\nabla\psi\varrho_n^{\gamma-2})dx$$
$$= -\frac{\gamma}{\gamma-1}\int_\Omega (\epsilon_n\varrho_n^\gamma\psi - \epsilon_n h\varrho_n^{\gamma-1}\psi + \operatorname{div}(\varrho_n\mathbf{u}_n)\varrho_n^{\gamma-1}\psi$$
$$+ \epsilon_n\nabla\varrho_n\cdot\nabla\psi\varrho_n^{\gamma-2})dx$$
$$= -\frac{\gamma}{\gamma-1}\int_\Omega (\epsilon_n\varrho_n^\gamma\psi - \epsilon_n h\varrho_n^{\gamma-1}\psi + \epsilon_n\nabla\varrho_n\cdot\nabla\psi\varrho_n^{\gamma-2})dx$$
$$+ \frac{1}{\gamma-1}\int_\Omega \varrho_n^\gamma\mathbf{u}_n\cdot\nabla\psi dx - \int_\Omega \varrho_n^\gamma\operatorname{div}\mathbf{u}_n\psi dx. \tag{4.8}$$

Thus we get from (4.6)–(4.8)

$$-\int_\Omega\left(\frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + c_v\varrho_n\theta_n + \frac{1}{\gamma-1}\varrho_n^\gamma\right)\mathbf{u}_n\cdot\nabla\psi dx$$
$$+ \int_\Omega (1+\theta_n^m)\frac{\epsilon_n+\theta_n}{\theta_n}\nabla\theta_n\cdot\nabla\psi dx$$
$$+ \int_{\partial\Omega}(\epsilon_n\ln\theta_n + L(\theta_n)(\theta_n-\theta_0) + \alpha|\mathbf{u}_n|^2)\psi d\sigma$$
$$= \int_\Omega \varrho_n\mathbf{f}\cdot\mathbf{u}_n\psi dx + \int_\Omega (\varrho_n^\gamma + \varrho_n\theta_n)\mathbf{u}_n\cdot\nabla\psi dx$$
$$- \int_\Omega (2\mu\mathbf{D}(\mathbf{u}_n)\mathbf{u}_n + \lambda\operatorname{div}\mathbf{u}_n\mathbf{u}_n)\cdot\nabla\psi_n dx$$
$$- \frac{\gamma}{\gamma-1}\int_\Omega (\epsilon_n\varrho_n^\gamma\psi - \epsilon_n h\varrho_n^{\gamma-1}\psi + \epsilon_n\nabla\varrho_n\cdot\nabla\psi\varrho_n^{\gamma-2})dx.$$

Passing to the limit reads

$$
\begin{aligned}
&-\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v\varrho\theta + \frac{1}{\gamma-1}\overline{\varrho^\gamma}\right)\mathbf{u}\cdot\nabla\psi dx + \int_\Omega (1+\theta^m)\nabla\theta\cdot\nabla\psi dx \\
&\qquad + \int_{\partial\Omega} (L(\theta)(\theta-\theta_0) + \alpha|\mathbf{u}|^2)\psi d\sigma \\
&\quad = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi dx + \int_\Omega (\overline{\varrho^\gamma} + \varrho\theta)\mathbf{u}\cdot\nabla\psi dx \\
&\qquad - \int_\Omega (2\mu\mathbf{D}(\mathbf{u})\mathbf{u} + \lambda\operatorname{div}\mathbf{u}\mathbf{u})\cdot\nabla\psi dx. \qquad (4.9)
\end{aligned}
$$

Thus, (4.4), (4.5) and (4.9) give almost the weak formulation to the original problem; it only remains to show that $\overline{\varrho^\gamma} = \varrho^\gamma$. This will be studied in the remaining two sections, separately for the Navier and the Dirichlet boundary conditions.

5. Strong Convergence of the Density — the Navier Boundary Conditions

In order to complete the proof of Theorem 2.1, we must show that $\varrho_n \to \varrho$ strongly in $L_p(\Omega)$ for a certain $p \geq 1$. Due to the *a priori* estimates it immediately implies the strong convergence for any $1 \leq p < \min\{2\gamma, 3(\gamma-1)\}$. In the case of the Navier boundary conditions, i.e. for $\beta < 1$, we may use the method from Ref. 13, developed originally in Ref. 12.

To this aim, we use the Helmholtz decomposition of the velocity, i.e.

$$\mathbf{u} = \nabla\Phi + \operatorname{curl}\mathbf{A},$$

where $\mathbf{A}$ solves the elliptic problem

$$
\left.\begin{aligned}
\operatorname{curl}\operatorname{curl}\mathbf{A} &= \operatorname{curl}\mathbf{u} = \boldsymbol{\omega} \\
\operatorname{div}\operatorname{curl}\mathbf{A} &= 0
\end{aligned}\right\} \quad \text{in } \Omega,
$$
$$\operatorname{curl}\mathbf{A}\cdot\mathbf{n} = 0 \quad \text{at } \partial\Omega$$

and the scalar potential Φ solves

$$\Delta\Phi = \operatorname{div}\mathbf{u} \quad \text{in } \Omega, \qquad \frac{\partial\Phi}{\partial\mathbf{n}} = 0 \quad \text{at } \partial\Omega.$$

Analogously we find $\mathbf{A}_n$ and $\boldsymbol{\Phi}_n$ for $\mathbf{u}_n$.

The classical theory for elliptic equations yields the existence of unique solutions to the above-mentioned problems (up to an additive constant) together with the estimates[22]

$$\|\nabla \operatorname{curl}\mathbf{A}\|_q \leq C\|\boldsymbol{\omega}\|_q, \quad \|\nabla^2\operatorname{curl}\mathbf{A}\|_q \leq C\|\boldsymbol{\omega}\|_{1,q}, \quad \|\nabla^2\Phi\|_q \leq C\|\operatorname{div}\mathbf{u}\|_q$$

for any $1 < q < \infty$. We may rewrite the approximative balance of momentum into the form

$$\nabla(-(2\mu+\lambda)\Delta\Phi_n + \varrho_n^\gamma + \varrho_n\theta_n)$$
$$= \mu\Delta \operatorname{curl} \mathbf{A}_n + \varrho_n \mathbf{f} - \varrho_n \mathbf{u}_n \cdot \nabla \mathbf{u}_n - \frac{1}{2}\epsilon_n h \mathbf{u}_n + \frac{1}{2}\epsilon_n \varrho_n \mathbf{u}_n - \frac{1}{2}\epsilon_n \Delta\varrho_n \mathbf{u}_n$$

and we denote

$$G_n = -(2\mu+\lambda)\Delta\Phi_n + \varrho_n^\gamma + \varrho_n\theta_n = -(2\mu+\lambda)\operatorname{div}\mathbf{u}_n + \varrho_n^\gamma + \varrho_n\theta_n$$

and its limit version

$$G = -(2\mu+\lambda)\operatorname{div}\mathbf{u} + \overline{\varrho^\gamma} + \varrho\theta.$$

The quantities G_n and G are in the theory of compressible Navier–Stokes equations known as *effective viscous fluxes* and the structure of the system implies better properties of this quantity than one can deduce from the *a priori* estimates. We have (recall that we deal only with $\frac{7}{3} < \gamma \le 3$).

Lemma 5.1. *We have, up to a subsequence, for* $\epsilon_n \to 0^+$

$$G_n \to G \quad \text{in } L_q(\Omega) \quad \forall\, q < \frac{3}{2}.$$

Proof. Similarly as in the proof of Lemma 4.2, we decompose

$$G_n = G_n^1 + G_n^2,$$

where

$$\int_\Omega G_n^2 dx = 0, \quad \nabla G_n^2 = -\frac{1}{2}\epsilon_n\Delta\varrho_n\mathbf{u}_n - \mu \operatorname{curl} \boldsymbol{\omega}_n^2,$$

with $\boldsymbol{\omega}_n^2$ from Lemma 4.2. Thus

$$\|G_n^2\|_q \le C(\epsilon_n\|\Delta\varrho_n\mathbf{u}_n\|_{-1,q} + \|\operatorname{curl}\boldsymbol{\omega}_n^2\|_{-1,q}) \le C\sqrt{\epsilon_n} \quad \text{for } q < \frac{3}{2}.$$

Next, recalling calculations from Lemma 4.2 we have

$$\|G_n^1\|_{1,r} \le C(\|\mathbf{u}_n\|_{1,2}, \|\varrho_n\|_{3(\gamma-1)}, \|\mathbf{f}\|_\infty), \quad r = \frac{3(\gamma-1)}{2\gamma-1} \in \left(\frac{12}{11}, \frac{6}{5}\right],$$

i.e. the sequence is relatively compact in $L_q(\Omega)$ for $q < \frac{3}{2}$. Lemma 5.1 is proved. □

Next we show

Lemma 5.2. *We have*

$$\int_\Omega \overline{(\varrho^\gamma + \varrho\theta)\varrho}\,dx \le \int_\Omega G\varrho dx, \quad \int_\Omega (\overline{\varrho^\gamma} + \varrho\theta)\varrho = \int_\Omega G\varrho dx,$$

consequently $\overline{\varrho^2} = \varrho^2$ *and up to a subsequence,* $\varrho_n \to \varrho$ *in* $L_q(\Omega)$ $\forall\, q < 3(\gamma-1)$. *Moreover, the limit temperature* $\theta > 0$ *a.e. in* Ω.

Proof. We consider the approximative momentum equation. First note that due to (3.6) we have

$$\|\varrho_n\|_{1,r} \le C(\epsilon_n, \|\mathbf{u}_n\|_r) \le C(\epsilon_n)$$

for $r > 3$, sufficiently close to 3. Thus, denoting $c_n = \|\varrho_n\|_\infty + 1$ and realizing that for $\delta \in (0,1)$

$$\int_\Omega \epsilon_n \Delta\varrho_n(\ln c_n - \ln(\varrho_n+\delta))dx = \epsilon_n \int_\Omega |\nabla\varrho_n|^2 \frac{1}{\varrho_n+\delta}dx \ge 0$$

and thus

$$\int_\Omega (\operatorname{div}(\varrho_n\mathbf{u}_n) + \epsilon_n\varrho_n - \epsilon_n h)(\ln c_n - \ln(\varrho_n+\delta))dx \ge 0,$$

we get

$$\int_\Omega \left(\frac{\varrho_n\mathbf{u}_n\cdot\nabla\varrho_n}{\varrho_n+\delta} + \epsilon_n\varrho_n \ln\frac{c_n}{\varrho_n+\delta}\right)dx \ge \epsilon_n h\int_\Omega \ln\frac{c_n}{\varrho_n+\delta} \ge 0.$$

Passing with $\delta \to 0^+$

$$\int_\Omega \mathbf{u}_n\cdot\nabla\varrho_n dx \ge -\epsilon_n\int_\Omega \varrho_n \ln\frac{c_n}{\varrho_n}dx, \quad \text{i.e.}\ -\int_\Omega \varrho_n \operatorname{div}\mathbf{u}_n dx \ge o(\epsilon_n) \quad \text{for } \epsilon_n \to 0^+.$$

Moreover, the definition of G_n yields

$$\int_\Omega (\varrho_n^\gamma + \varrho_n\theta_n)\varrho_n dx \le \int_\Omega G_n\varrho_n dx + o(\epsilon_n).$$

Passing with ϵ_n to 0 we get

$$\int_\Omega \overline{(\varrho^\gamma + \varrho\theta)\varrho}dx \le \int_\Omega G\varrho dx \quad \text{as } 3(\gamma-1) > 3 \quad \text{for } \gamma > \frac{7}{3}.$$

Next we consider the limit equation

$$\operatorname{div}(\varrho\mathbf{u}) = 0, \quad \text{i.e.} \int_\Omega \varrho\mathbf{u}\cdot\nabla\eta dx = 0, \quad \forall\,\eta \in C^\infty(\bar\Omega).$$

As $\mathbf{u} \in W_2^1(\Omega)$, $\varrho \in L_{3(\gamma-1)}(\Omega)$, $3 \ge \gamma > \frac{7}{3}$, we have that

$$\mathbf{u}\cdot\nabla\varrho = \operatorname{div}(\varrho\mathbf{u}) - \varrho\operatorname{div}\mathbf{u} \in L_{\frac{6(\gamma-1)}{3\gamma-1}}(\Omega)$$

and there exists $\varrho^n \in C^\infty(\bar\Omega)$, $\varrho^n \ge 0$, $\varrho^n \to \varrho$ in $L_{3(\gamma-1)}(\Omega)$, $\mathbf{u}\cdot\nabla\varrho^n \to \mathbf{u}\cdot\nabla\varrho$ in $L_{\frac{6(\gamma-1)}{3\gamma-1}}(\Omega)$, see Ref. 16. In this consideration the Friedrichs lemma is hidden. Thus

$$\int_\Omega (\varrho\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla\varrho)dx = 0. \tag{5.1}$$

Next, using as test function in the weak formulation of the limit continuity equation $\eta = \ln\delta - \ln(\varrho^n + \delta)$, $\delta > 0$ reads

$$0 = \int_\Omega \varrho\mathbf{u}\cdot\nabla\ln\frac{\delta}{\varrho^n+\delta}dx = \int_\Omega \varrho\frac{\mathbf{u}\cdot\nabla\varrho^n}{\varrho^n+\delta}dx.$$

As $\varrho^n \to \varrho$ in $L_{3(\gamma-1)}(\Omega)$, $\mathbf{u}\cdot\nabla\varrho^n \to \mathbf{u}\cdot\nabla\varrho$ in $L_{\frac{6(\gamma-1)}{3\gamma-1}}(\Omega)$, $\varrho^n + \delta \to \varrho + \delta$ a.e., $0 \le \frac{1}{\varrho^n+\delta} \le \frac{1}{\delta}$,

$$\int_\Omega \frac{\varrho\mathbf{u}\cdot\nabla\varrho}{\varrho+\delta}dx = 0.$$

Passing with $\delta \to 0^+$ we get, due to (5.1), $\int_\Omega \varrho\,\mathrm{div}\,\mathbf{u} = 0$. Hence, the definition of G yields

$$\int_\Omega \overline{(\varrho^\gamma + \varrho\theta)\varrho}dx = \int_\Omega G\varrho dx.$$

Next, as $s_n \to s$ in $L_2(\Omega)$, we get exactly as in Lemma 5.1 in Ref. 13 that $\mathrm{e}^{s_n} \to \mathrm{e}^s$ in $L_r(\Omega)$, $1 \le r < 3m$, $\theta = \mathrm{e}^s$ and thus $\theta > 0$ a.e. as s is finite a.e. This step guarantees us the positiveness of the temperature a.e.

Using the properties of the weak convergence (see e.g. Ref. 17) $\overline{\varrho^{\gamma+1}} \ge \overline{\varrho^\gamma}\varrho$, $\overline{\varrho^2}\theta \ge \varrho^2\theta$ and due to

$$\int_\Omega \overline{(\varrho^\gamma + \varrho\theta)\varrho}dx = \int_\Omega \overline{(\varrho^\gamma + \varrho\theta)}\varrho dx$$

we get

$$\overline{\varrho^2} = \varrho^2 \quad \text{a.e. in } \Omega \quad \text{and} \quad \lim_{n\to\infty}\|\varrho_n - \varrho\|_2^2 = \int_\Omega(\overline{\varrho^2} - \varrho^2)dx = 0.$$

Lemma 5.2 and thus also Theorem 2.1 are proved. □

6. Strong Convergence of the Density — the Dirichlet Boundary Conditions

We cannot apply the method from the previous section as the compactness of the vorticity up to the boundary is no longer true in the case of the Dirichlet boundary conditions. Thus we proceed differently; we follow the proof from Ref. 17 for the case of the barotropic fluid, i.e. $p = p(\varrho)$. First, recall that (for the proof see Proposition 4.26 in Ref. 17).

Lemma 6.1. *Let* $\Omega \subset \mathbb{R}^3$ *be a domain,* $1 < r < \infty$, $\max\{2, r'\} \le q \le \infty$, $\frac{1}{s} + \frac{1}{q^*} < 1$, $q^* = \frac{3q}{3-q}$ *if* $q < 3$, $q^* = \infty$ *otherwise. Let* $\frac{1}{t} + \frac{1}{q} < \frac{5}{6}$. *Assume that for any subdomain* Ω' *such that* $\overline{\Omega'} \subset \Omega$, *there holds*

$$\begin{aligned}
&\mathbf{q}_n \rightharpoonup \mathbf{q} \quad \text{in } L_t(\Omega'), && \mathrm{div}\,\mathbf{q}_n \to \mathrm{div}\,\mathbf{q} \quad \text{in } W_t^{-1}(\Omega'),\\
&\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L_2(\Omega'), && \nabla\mathbf{u}_n \rightharpoonup \nabla\mathbf{u} \quad \text{in } L_2(\Omega'),\\
&p_n \rightharpoonup p \quad \text{in } L_r(\Omega'), && \mathbf{F}_n \rightharpoonup \mathbf{F} \quad \text{in } L_s(\Omega'),\\
&g_n \rightharpoonup g \quad \text{in } L_q(\Omega'),
\end{aligned}$$

and suppose that

$$\frac{\partial}{\partial x_j}(q_n^i u_n^j) - \mu \Delta u_n^i - (\mu+\lambda)\frac{\partial}{\partial x_i}\operatorname{div}\mathbf{u}_n + \frac{\partial}{\partial x_i}p_n = F_n^i \quad \text{in } \mathcal{D}'(\Omega).$$

Then

$$\lim_{n\to\infty}\int_\Omega \eta(p_n g_n - (2\mu+\lambda)g_n \operatorname{div}\mathbf{u}_n)dx = \int_\Omega \eta(pg - (2\mu+\lambda)g\operatorname{div}\mathbf{u})dx,$$

$\eta \in \mathcal{D}(\Omega)$. *Moreover, if* $q > \max\{r', 2\}$, *then there exists a subsequence such that* $g_n \operatorname{div}\mathbf{u}_n \rightharpoonup \overline{g\operatorname{div}\mathbf{u}}$ *in* $L_{\frac{2q}{2+q}}(\Omega)$, $p_n g_n \rightharpoonup \overline{pg}$ *in* $L_{\frac{rq}{r+q}}(\Omega)$ *and*

$$\overline{pg} - (2\mu+\lambda)\overline{g\operatorname{div}\mathbf{u}} = pg - (2\mu+\lambda)g\operatorname{div}\mathbf{u} \quad \text{a.e. in } \Omega.$$

Recall also that we have (4.4), (4.5) and (4.9) and for any $b \in C[0,\infty)\cap C^1(0,\infty)$,

$$\begin{aligned} &|b'(t)| \le Ct^{-\lambda_0}, \quad t\in[0,1], \quad \lambda_0 < 1\\ &|b'(t)| \le Ct^{\lambda_1}, \quad t > 1, \quad -1 < \lambda_1 \le \frac{3}{2}(\gamma-1)-1 \end{aligned}$$

there holds

$$\operatorname{div}(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho))\operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3),$$

(see Ref. 17, Lemma 3.3).

We take Lemma 6.1 in the following quantities

$$\begin{aligned} &\mathbf{q}_n = \frac{1}{2}\varrho_n\mathbf{u}_n, && \mathbf{q} = \frac{1}{2}\varrho\mathbf{u},\\ &\mathbf{u}_n = \mathbf{u}_n, && \mathbf{u} = \mathbf{u},\\ &p_n = \varrho_n^\gamma + \varrho_n\theta_n, && p = \overline{\varrho^\gamma} + \varrho\theta,\\ &\mathbf{F}_n = \varrho_n\mathbf{f} - \frac{1}{2}\varrho_n\mathbf{u}_n\cdot\nabla\mathbf{u}_n, && \mathbf{F} = \varrho\mathbf{f} - \frac{1}{2}\varrho\mathbf{u}\cdot\nabla\mathbf{u},\\ &g_n = b(\varrho_n) \quad \text{with } \lambda_1 = 0,\ \ b\in C^1[0,\infty), && g = \overline{b(\varrho)}, \end{aligned}$$

together with $t = 2$, $r = \frac{3(\gamma-1)}{\gamma}$, $s = \frac{3(\gamma-1)}{2\gamma-1}$, $q > 3$; thus $\frac{1}{s} + \frac{1}{q^*} = \frac{1}{s} < 1$ and $\frac{1}{t} + \frac{1}{q} < \frac{5}{6}$. Therefore all assumptions of Lemma 6.1 are satisfied and

$$\begin{aligned} &\overline{(\varrho^\gamma + \varrho\theta)b(\varrho)} - (2\mu+\lambda)\overline{b(\varrho)(\operatorname{div}\mathbf{u})}\\ &\quad = (\overline{\varrho^\gamma} + \varrho\theta)\overline{b(\varrho)} - (2\mu+\lambda)\overline{b(\varrho)}\operatorname{div}\mathbf{u} \quad \text{a.e. in } \Omega. \end{aligned} \tag{6.1}$$

Our aim is to take $b(t) = t^\vartheta$ for $0 < \vartheta < 1$. The problem is that such b is not continuously differentiable at 0. On the other hand, we know that

$$\varrho_n^\vartheta \rightharpoonup \overline{\varrho^\vartheta} \quad \text{in } L_{\frac{3(\gamma-1)}{\vartheta}}(\Omega), \quad \overline{\varrho^\vartheta} = 0 \quad \text{in } \mathbb{R}^3\backslash\bar{\Omega}.$$

We have

Lemma 6.2. *Let* $0 < \vartheta < 1$. *Then for* $\mathbf{u}$, $\overline{\varrho^\gamma} + \varrho\theta$, $\overline{\varrho^\vartheta}$, $\overline{(\varrho^\gamma + \varrho\theta)\varrho^\vartheta}$ *defined above, it holds*

$$\operatorname{div}((\overline{\varrho^\vartheta})^{\frac{1}{\vartheta}}\mathbf{u}) \geq \frac{1-\vartheta}{\vartheta(2\mu+\lambda)}(\overline{(\varrho^\gamma + \varrho\theta)\varrho^\vartheta} - (\overline{\varrho^\gamma} + \varrho\theta)\,\overline{\varrho^\vartheta})(\overline{\varrho^\vartheta})^{\frac{1}{\vartheta}-1} \quad in\ \mathcal{D}'(\mathbb{R}^3).$$

Proof. We may follow step by step the proof of Lemma 4.39 in Ref. 17, just replacing $p_\delta(\varrho_n)$ by $\varrho_n^\gamma + \varrho_n\theta_n$, 2β by $3(\gamma-1)$ and set $\alpha = 0$. □

Thus we have

$$\operatorname{div}(r\mathbf{u}) \geq f \quad \text{in } \mathcal{D}'(\mathbb{R}^3)$$

with

$$r = \begin{cases} (\overline{\varrho^\vartheta})^{\frac{1}{\vartheta}} - \varrho & \text{in } \Omega \\ 0 & \text{otherwise} \end{cases}$$

$$f = \begin{cases} \dfrac{1-\vartheta}{\vartheta(2\mu+\lambda)}(\overline{(\varrho^\gamma + \varrho\theta)\varrho^\vartheta} - (\overline{\varrho^\gamma} + \varrho\theta)\,\overline{\varrho^\vartheta})(\overline{\varrho^\vartheta})^{\frac{1}{\vartheta}-1} & \text{in } \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Hence, taking as test function above the function Φ, non-negative in $\mathbb{R}^3$, equal 1 in Ω, we have

$$\int_\Omega (\overline{(\varrho^\gamma + \varrho\theta)\varrho^\vartheta} - (\overline{\varrho^\gamma} + \varrho\theta)\,\overline{\varrho^\vartheta})(\overline{\varrho^\vartheta})^{\frac{1}{\vartheta}-1} dx \leq 0.$$

We may easily show that the limit temperature $\theta > 0$ a.e. in Ω, see Sec. 5, proof of Lemma 5.2. The functions $t \mapsto t^\gamma + \theta t$ and $t \mapsto t^\vartheta$ are increasing on $[0,\infty)$ and therefore

$$\overline{(\varrho^\gamma + \varrho\theta)\varrho^\vartheta} \geq (\overline{\varrho^\gamma} + \varrho\theta)\,\overline{\varrho^\vartheta} \quad \text{a.e. in } \Omega,$$

$$\text{i.e. } (\overline{(\varrho^\gamma + \varrho\theta)\varrho^\vartheta} - (\overline{\varrho^\gamma} + \varrho\theta)\,\overline{\varrho^\vartheta})(\overline{\varrho^\vartheta})^{\frac{1}{\vartheta}-1} = 0 \quad \text{a.e. in } \Omega.$$

As

$$\int_\Omega \varrho_\epsilon^\vartheta 1_{\overline{\varrho^\vartheta}=0} dx \to \int_\Omega \overline{\varrho^\vartheta} 1_{\overline{\varrho^\vartheta}=0} dx,$$

we have

$$\varrho_\epsilon \to 0 \quad \text{in } L_p(\{\overline{\varrho^\vartheta} = 0\}), \quad 1 \leq p < 3(\gamma-1).$$

Thus

$$\overline{\varrho^{\gamma+\vartheta}} = \overline{\varrho^\gamma}\,\overline{\varrho^\vartheta}, \quad \overline{\varrho^{1+\vartheta}} = \varrho\,\overline{\varrho^\vartheta}$$

which implies (see Lemma 3.39 in Ref. 17) the strong convergence of the density in $L_p(\Omega)$, $1 \leq p < 3(\gamma-1)$. Theorem 2.2 is proved.

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