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# A critical functional framework for the inhomogeneous Navier–Stokes equations in the half-space

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**Abstract**

This paper is devoted to solving globally the boundary value problem for the incompressible inhomogeneous Navier–Stokes equations in the half-space in the case of small data with critical regularity. In dimension $n \geqslant 3$, we state that if the initial density $\rho_0$ is close to a positive constant in $L_\infty \cap \dot{W}^1_n(\mathbb{R}^n_+)$ and the initial velocity $u_0$ is small with respect to the viscosity in the homogeneous Besov space $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$ then the equations have a unique global solution. The proof strongly relies on new maximal regularity estimates for the Stokes system in the half-space in $L_1(0, T; \dot{B}^0_{p,1}(\mathbb{R}^n_+))$, interesting for their own sake.


*Keywords:* Critical regularity; Inhomogeneous viscous fluids; Stokes system; Homogeneous Besov spaces; Half-space

## 1. Introduction

We want to investigate the global well-posedness for the incompressible inhomogeneous Navier–Stokes equations in the half-space $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n\colon\ x_n > 0\}$. The corresponding system reads

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$$\begin{aligned} &\partial_t\rho + u\cdot\nabla\rho = 0 \quad \text{in } (0,T)\times\mathbb{R}^n_+,\\ &\rho(\partial_t u + u\cdot\nabla u) - \mu\Delta u + \nabla\Pi = \rho f \quad \text{in } (0,T)\times\mathbb{R}^n_+,\\ &\operatorname{div} u = 0 \quad \text{in } (0,T)\times\mathbb{R}^n_+,\\ &u|_{x_n=0} = 0 \quad \text{on } (0,T)\times\mathbb{R}^{n-1},\\ &u_0|_{t=0} = u_0, \quad \rho|_{t=0} = \rho_0 \quad \text{on } \mathbb{R}^n_+, \end{aligned} \tag{INS}$$

where $\rho$, $u = (u_1,\dots,u_n)$ and $\Pi$ stand respectively for the unknown density, velocity and pressure of the fluid. The given positive real number $\mu$ is the viscosity coefficient and $f$ represents external body forces. Due to the compatibility conditions, we assume that the initial velocity $u_0$ is divergence free and that its normal component $u_{0n}$ is zero on $\partial\mathbb{R}^n_+$ in the distributional meaning; the initial density $\rho_0$ is required to be strictly positive in the half-space and we restrict our attention to solutions such that the density tends (weakly) to a positive constant (say 1), and the velocity tends to 0 at infinity.

The homogeneous case $\rho_0 \equiv 1$ — the classical incompressible Navier–Stokes equations — has been extensively studied from a mathematical viewpoint. It is well established that as far as one is interested by *global* existence results with uniqueness, it is important to work with *critical* norms for the initial data $u_0$ and for the solution $u$, that is with norms invariant for all $\lambda > 0$ by the rescaling[1]

$$(u,\Pi)(t,x) \mapsto \big(\lambda u, \lambda^2\Pi\big)\big(\lambda^2 t, \lambda x\big), \qquad u_0(x) \mapsto \lambda u_0(\lambda x). \tag{1.1}$$

This is due to the fact that the homogeneous Navier–Stokes equations are invariant by (1.1) so that any proof based on contracting mapping arguments in a Banach space requires norms with the above scaling invariance.

Solving the (homogeneous) Navier–Stokes equations in critical spaces goes back to the pioneering work by H. Fujita and T. Kato in [17,18]. There, in the case of a bounded domain of $\mathbb{R}^n$, it is stated that any small enough initial velocity with $n/2-1$ derivative(s) in $L^2$ generates a global (unique) solution. In the whole space case, Fujita and Kato's approach has been adapted to a plethora of critical functional frameworks (see e.g. [8,25]). Let us mention in particular that, in the whole space case, the classical incompressible Navier–Stokes equations are globally well-posed if taking $u_0$ small with respect to the viscosity in the Lebesgue space $L_n(\mathbb{R}^n)$ (see [19,23]). This latter result has been adapted to the case of bounded domains by Y. Giga and T. Miyakawa in [20] and to the case of exterior domains by Y. Giga and H. Sohr in [21], and H. Iwashita in [22]. In the half-space case, the well-posedness issue has been studied by H. Kozono in [24] (see also [9] for results related to critical Besov spaces with negative index of regularity).

Motivated by the fact that in real life, a fluid is hardly homogeneous, we want to study whether the aforementioned approach is relevant for the inhomogeneous incompressible Navier–Stokes equations. Now, the *scaling invariance* for (*INS*) reads

$$(\rho,u,\Pi)(t,x) \mapsto \big(\rho, \lambda u, \lambda^2\Pi\big)\big(\lambda^2 t, \lambda x\big), \qquad (\rho_0,u_0)(x) \mapsto (\rho_0, \lambda u_0)(\lambda x) \tag{1.2}$$

which, roughly, means that the critical spaces for the velocity (and pressure) are the same as in the homogeneous case, and that one has to take one more derivative for the density.

[1] Here we take $f \equiv 0$ for simplicity.

The first example of solving (*INS*) in a critical functional framework has been given by the first author in [11], in the whole space case. There, global well-posedness is shown whenever $(\rho_0 - 1, u_0)$ belongs to the *homogeneous Besov space* $\dot{B}^{\frac{n}{2}}_{2,1} \times \dot{B}^{\frac{n}{2}-1}_{2,1}$ which is critical in the sense of (1.2). Data in more general critical Besov spaces related to $L_p$ spaces have been considered in [1,2]. All those results strongly rely on the use of the Fourier transform on $\mathbb{R}^n$ so that their extension to more general domains is far from being obvious. A first attempt in this direction (for bounded domains) has been done in [13]. There, Fourier analysis has been replaced by standard maximal regularity estimates for the Stokes system (after the pioneering work by O. Ladyzhenskaya and V. Solonnikov in [26]). However, it is not clear that the critical index may be attained by this method.

In the present paper, we aim at proving global in time existence (and uniqueness) for (*INS*) *in the half-space* for small data with critical regularity. In fact, we strive for a statement as close as possible to the one given by [24] in the homogeneous case. Based on (1.2), it is thus natural to take the initial data $(\rho_0, u_0)$ so that $\|\nabla\rho_0\|_{L_n(\mathbb{R}^n_+)}$ and $\|u_0\|_{L_n(\mathbb{R}^n_+)}$ be small. However, in order to get a control on the ellipticity of the velocity equation, assuming in addition that $\rho_0$ is bounded away from zero, and in $L_\infty(\mathbb{R}^n_+)$ (a space which has the desired scaling) seems unavoidable. We shall also rather take $u_0$ in a subspace of $L_n(\mathbb{R}^n_+)$, namely the homogeneous Besov space $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$ which still has the right scaling. This assumption will ensure that $\nabla u \in L_{1,\mathrm{loc}}(\mathbb{R}_+; L_\infty(\mathbb{R}^n_+))$, a property which is needed to propagate the regularity of the initial density. In fact, in the framework of critical Besov spaces, having 1 as a third index is the only possibility to get a control over $\nabla u$ in $L_{1,\mathrm{loc}}(\mathbb{R}_+; L_\infty(\mathbb{R}^n_+))$. More explanations (together with the definition of $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$) will be given in the next section. Here we touch an open question, whether one may investigate general parabolic-type systems in $L_1(0,T;X)$, where $X$ stands for a Banach space determining the regularity of solutions with respect to space directions. Positive answers [14,21], obtained by techniques of the theory of semigroups, are known only for spaces $L_q(0,T;X)$ with $q \in (1,\infty)$, but the case $q = 1$ is beyond this approach. Thus, our paper is devoted to this critical case, however only for our particular system.

Let us now introduce the functional spaces that we shall use in our global existence statement. For $p \in [1,+\infty]$ and $T \in [0,+\infty]$ we define $E_p(T)$ as the set of functions $(\rho, u, \Pi)$ such that[2]

$$(\rho - 1) \in L_\infty\big((0,T)\times\mathbb{R}^n_+\big) \cap \mathcal{C}_b\big([0,T); W^1_p\big(\mathbb{R}^n_+\big)\big),$$
$$u \in \mathcal{C}_b\big([0,T); \dot{B}^0_{p,1}\big(\mathbb{R}^n_+\big)\big) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla \Pi \in \big(L_1\big(0,T; \dot{B}^0_{p,1}\big(\mathbb{R}^n_+\big)\big)\big)^n.$$

If $T = +\infty$ then we simply denote the above space by $E_p$. We shall also use the notation $E_{p,\mathrm{loc}} = \bigcap_{T>0} E_p(T)$.

**Theorem 1.** *Let $n \geqslant 2$. Let $\rho_0$ be a bounded positive function such that $(\rho_0 - 1) \in W^1_n(\mathbb{R}^n_+)$. Let $u_0$ be a divergence free vector field on $\mathbb{R}^n_+$ with $u_{0,n} = 0$ at the boundary and coefficients in $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$. Assume that $f \in L_1(\mathbb{R}_+; \dot{B}^0_{n,1}(\mathbb{R}^n_+))$. There exist two positive constants $c$ and $M$ depending only on $n$, and such that if*

$$\|\rho_0 - 1\|_{L_\infty(\mathbb{R}^n_+)} + \|\nabla\rho_0\|_{L_n(\mathbb{R}^n_+)} \leqslant c \quad \textit{and} \quad \|u_0\|_{\dot{B}^0_{n,1}(\mathbb{R}^n_+)} + \|f\|_{L_1(\mathbb{R}_+;\dot{B}^0_{n,1}(\mathbb{R}^n_+))} \leqslant c\mu \qquad (1.3)$$

[2] It is understood that $W^1_p(\mathbb{R}^n_+)$ stands for the set of $L_p$ functions over $\mathbb{R}^n_+$ with (weak) first-order derivatives in $L_p(\mathbb{R}^n_+)$.

*then system (INS) has a global solution $(\rho, u, \Pi)$ in the space $E_n$ such that for all $t \in \mathbb{R}_+$,*

$$\left\|\rho(t)-1\right\|_{L_\infty(\mathbb{R}^n_+)} = \|\rho_0-1\|_{L_\infty(\mathbb{R}^n_+)}, \qquad \left\|\rho(t)-1\right\|_{W^1_n(\mathbb{R}^n_+)} \leqslant 2\|\rho_0-1\|_{W^1_n(\mathbb{R}^n_+)}, \tag{1.4}$$

$$\begin{aligned}&\left\|\nabla\Pi, \partial_t u, \mu D^2 u\right\|_{L_1(\mathbb{R}_+;\dot{B}^0_{n,1}(\mathbb{R}^n_+))} + \|u\|_{L_\infty(\mathbb{R}_+;\dot{B}^0_{n,1}(\mathbb{R}^n_+))}\\ &\quad \leqslant M\left(\|u_0\|_{\dot{B}^0_{n,1}(\mathbb{R}^n_+)} + \|f\|_{L_1(\mathbb{R}_+;\dot{B}^0_{n,1}(\mathbb{R}^n_+))}\right).\end{aligned} \tag{1.5}$$

*If $n \geqslant 3$ then uniqueness holds true in the space $E_n$.*

A few comments are in order.

- Because the space $L_\infty \cap W^1_n(\mathbb{R}^n_+)$ fails to be embedded in the set of continuous functions over $\mathbb{R}^n_+$, the initial density need not be continuous. The initial velocity need not be continuous either. In particular it may have a jump across a smooth interface (if compactly supported, such data are in $\dot{B}^{\frac{1}{n}}_{n,\infty}(\mathbb{R}^n_+)$, hence also in $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$).
- In the case of a *large* initial velocity in $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$, an easy variation over our method would provide a *local* solution.
- In dimension $n \geqslant 3$, one may weaken slightly the assumptions on the density: it is possible to replace the space $W^1_n(\mathbb{R}^n_+)$ by $B^1_{n,\infty}(\mathbb{R}^n_+)$ as done in [1] in the whole space case. However, this improvement requires estimates for the transport equation in spaces $B^1_{n,\infty}(\mathbb{R}^n_+)$, a study that we decided to omit in the present paper, for simplicity.
- Uniqueness in dimension two may be obtained if assuming that $\rho_0 - 1$ is small in $\dot{B}^0_{2,1}(\mathbb{R}^2_+)$ (see the whole space case in [11]). However proving this result also requires estimates for the transport equation in Besov spaces.
- We expect to have global well-posedness for large data in dimension two (this fact is well known for smooth data in the case of a bounded domain, see [26,3,28] and has been extended to the $\mathbb{R}^2$ case with critical data in [12]). This study would require a rather different approach in the treatment of the nonlinear terms, though.

**Remark 1.** Even though taking the initial velocity in $L_n(\mathbb{R}^n_+)$ (as in the homogeneous case) may seem more natural, it is very unlikely that one may prove a global well-posedness result under this assumption. It is not clear either that the space $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$ may be replaced by the larger space $\dot{B}^0_{p,q}(\mathbb{R}^n_+)$ with $q > 1$. The reason why is that at the level of the linearized equations, having $q = 1$ is the only possibility to get a control of $\nabla u$ in $L_{1,\text{loc}}(\mathbb{R}_+; L_\infty(\mathbb{R}^n_+))$, a property which is needed to propagate the $W^1_n(\mathbb{R}^n_+)$ regularity of the density for all time.

Proving Theorem 1 requires three main ingredients:

- time-independent maximal estimates for the linearized velocity equation, namely the *evolutionary Stokes system*:

$$\begin{aligned}\partial_t v - \mu\Delta v + \nabla P = F \quad &\text{in } (0,T)\times\mathbb{R}^n_+,\\ \operatorname{div} v = 0 \quad &\text{in } (0,T)\times\mathbb{R}^n_+,\\ v|_{x_n=0} = 0 \quad &\text{on } (0,T)\times\mathbb{R}^{n-1},\\ v|_{t=0} = v_0 \quad &\text{on } \mathbb{R}^n_+;\end{aligned} \tag{1.6}$$

- bilinear estimates in functional spaces related to homogeneous Besov spaces;
- a compactness argument.

In contrast with the homogeneous Navier–Stokes equations and owing to the hyperbolicity of the equation for the density, our existence theorem *does not* come up as a consequence of a contracting mapping argument in a suitable Banach space. Therefore, we shall rather make use of a compactness method, proving first uniform estimates for a sequence of smooth solutions pertaining to smoothed out data. This may be achieved thanks to standard estimates in Lebesgue spaces for the transport equation satisfied by the density and estimates in homogeneous Besov spaces for (1.6) taking for $F$ all the nonlinear terms of the velocity equations. Bilinear estimates in Besov spaces will be needed for bounding those nonlinear terms. Uniqueness will be obtained afterward, proving a stability estimate in low norm and taking advantage of a logarithmic interpolation argument.

Note that at the formal level, the general method and the main ingredients are the same as in the whole space case treated in [11,1,2]. However, in the half-space case, one has to face several additional difficulties. First, in contrast with the $\mathbb{R}^n$ case, the Stokes system cannot be reduced to the basic heat equation after suitable projection. Second, the adaptation of the bilinear estimates to this new framework requires some care. Third, there is no explicit definition of *homogeneous* Besov spaces $\dot{B}^0_{p,1}(\mathbb{R}^n_+)$. In fact, the only reasonable definition is given by restriction of functions defined on the whole space. Furthermore, because we did not make any additional assumption on the potential part of the source term $f$ (in contrast with what has been done in [1,2,11]), recovering the full $L_1$-in-time regularity for $\nabla^2 u$, $\partial_t u$ and $\nabla \Pi$ in (*INS*) turns out to be not so straightforward and requires a novel approach for this problem. Of course this ultimate difficulty would also occur in the whole space case.

Let us now state the estimates that we have obtained for the Stokes system in the half-space:

**Theorem 2.** *Let $p \in (1,\infty)$, $s \in (\frac{1}{p}-1, \frac{1}{p})$ and $T \in (0,\infty]$. Let $F \in L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and $v_0 \in \dot{B}^s_{p,1}(\mathbb{R}^n_+)$ with $\operatorname{div} v_0 = 0$ and $v_{0n}|_{x_n=0} = 0$ in the meaning of the trace. Then there exists a unique solution to problem* (1.6) *such that*

$$v \in \mathcal{C}_b\big([0,T);\dot{B}^s_{p,1}\big(\mathbb{R}^n_+\big)\big), \qquad \nabla^2 v \in L_1\big(0,T;\dot{B}^s_{p,1}\big(\mathbb{R}^n_+\big)\big) \quad \text{and} \quad \nabla P \in L_1\big(0,T;\dot{B}^s_{p,1}\big(\mathbb{R}^n_+\big)\big)$$

*and the following estimate is valid*:

$$\|v\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \big\|\partial_t v, \mu\nabla^2 v, \nabla P\big\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\big[\|F\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}\big], \tag{1.7}$$

*where $C$ is a constant depending only on $s$, $p$ and $n$.*

**Remark 2.** Proving the existence part of Theorem 1 requires only the case $s = 0$. However, combining the general statement with real interpolation will enable us to get another family of estimates (see Lemma 10 below) which turns out to be the key to the uniqueness for (*INS*). The choice of $s$ is restricted by properties of $\dot{B}^s_{p,1}(\mathbb{R}^n_+)$, see Proposition 3.

**Remark 3.** Choosing homogeneous semi-norms enables us to get *time-independent* estimates for the Stokes system. This is of course the key to proving a global result for (*INS*). Indeed using the more standard inhomogeneous framework would introduce a linear time dependency in the estimates.

Here also, having Besov spaces with third index 1 is fundamental: inequality (1.7) is generically false (even in the whole space or for the heat equation), if $\dot{B}^0_{p,1}(\mathbb{R}^n_+)$ is changed into $L_p(\mathbb{R}^n_+)$, or into $\dot{B}^0_{p,q}(\mathbb{R}^n_+)$ for some $q > 1$.

Difficulties are carried by the $L_1$-regularity with respect to time, precluding us from using the standard Marcinkiewicz theorem for the Fourier multipliers or general Calderon–Zygmund theory for singular operators [16,35]. This follows that even the advanced theory of semigroups [14, 21] cannot deliver us such results. This technique deals with the regularity of type $L_q(0,T;X)$ with $q \in (1,\infty)$ and $q = 1$ cannot be reached. Hence Theorem 2 should not be viewed as an element of the standard theory. It is worthwhile to underline that, in contrast with the heat equation, one cannot adapt directly the results for the Stokes system in the whole space to our case by a suitable method of symmetry. Even though the solution may be explicitly computed in terms of the data (see in particular [36,27,34]) it is not clear that the above theorem may be obtained by a direct use of the corresponding formula.

We also think that any approach based on the characterization of the Besov spaces in terms of the heat flow (such that the one used in [9] for instance) is bound to fail for nonnegative index of regularity.

The main tool for proving Theorem 2 is the Fourier transform. After a suitable preparation (reducing the study to the case where $\operatorname{div} F = 0$ and $F_n|_{x_n=0} = 0$ then extending the Stokes problem to the whole real line with nonhomogeneous boundary data on $x_n = 0$) we may perform a Fourier transform with respect to the time $t$ and tangential variables $x' := (x_1, \dots, x_{n-1})$. We then obtain a system of ordinary differential equations with respect to the normal variable $x_n$, which may be explicitly solved. By taking advantage of harmonic analysis techniques and of the theory of Besov spaces, it is then possible to get the maximal regularity estimate (1.7). At this point, handling the trace of the gradient of the velocity is the main problem. Surprisingly, the explicit representation does not give any exploitable information for standard methods in the half-space with homogeneous equations and inhomogeneous boundary conditions are not allowed here. A way to overcome this ultimate difficulty is to obtain the "explicit" form of the velocity in the half-space with the homogeneous boundary data as in [15]. Then the difficulty related to the traces can be omitted by construction of explicit extensions.

To finish with, let us emphasize that the maximal regularity estimates — called sometimes Schauder's estimates — are irreplaceable in the analysis of quasi-linear systems [21,30–33]. They allow us to treat the nonlinear problem as a perturbation of a linear one, since we loose no regularity. This feature is particularly important here as the functions we work with have critical regularity so that the nonlinearities can be controlled only by the highest/whole norms of sought solutions. We thus expect our study to be an important step to understand more advanced boundary problems in (possibly) more general domains.

The paper unfolds as follows. In the next section we recall definitions of the Besov spaces and some auxiliary results from this field. In Section 3 we analyze the Stokes system and prove Theorem 2. Then we return to the nonlinear system (*INS*) and prove Theorem 1. In Appendix A we give the proofs of some technical results needed in the paper.

## 2. Besov spaces

In this section we introduce the homogeneous Besov spaces required in our analysis, and give a few basic results.

Throughout we fix a smooth function $\phi : \mathbb{R}_+ \to [0,1]$ supported in $\{1/2 \leqslant r \leqslant 2\}$ and such that

$$\sum_{k\in\mathbb{Z}} \phi\big(2^{-k} r\big) = 1 \quad \text{for all } r > 0. \tag{2.1}$$

Then we introduce the homogeneous Littlewood–Paley decomposition $(\Delta_k)_{k\in\mathbb{Z}}$ over $\mathbb{R}^n$ by setting

$$\Delta_k u := \varphi\big(2^{-k} D\big) u = \mathcal{F}^{-1}\big(\varphi\big(2^{-k}\cdot\big)\mathcal{F}u\big) \quad \text{with } \varphi(\xi) := \phi\big(|\xi|\big).$$

Above $\mathcal{F}$ stands for the Fourier transform on $\mathbb{R}^n$.

Let us first define the homogeneous Besov spaces on $\mathbb{R}^n$. For that, we introduce the following homogeneous Besov semi-norms (for all $s \in \mathbb{R}$ and $p, q \in [1,\infty]$):

$$\|u\|_{\dot B^s_{p,q}(\mathbb{R}^n)} := \big\| 2^{sk} \|\Delta_k u\|_{L^p(\mathbb{R}^n)} \big\|_{\ell^q(\mathbb{Z})}. \tag{2.2}$$

Owing to the lack of control of low frequencies (in particular when $s$ is large), there is no consensus for defining *homogeneous Besov spaces* $\dot B^s_{p,q}(\mathbb{R}^n)$. When one deals with nonlinear PDEs, the following definition turns out to be convenient:

$$\dot B^s_{p,q}\big(\mathbb{R}^n\big) = \big\{ u \in \mathcal{S}'_h\big(\mathbb{R}^n\big) \colon \ \|u\|_{\dot B^s_{p,q}(\mathbb{R}^n)} < \infty \big\},$$

where $\mathcal{S}'_h(\mathbb{R}^n)$ stands for the set of tempered distributions $u$ over $\mathbb{R}^n$ such that for all smooth compactly supported function $\theta$ over $\mathbb{R}^n$, we have

$$\lim_{\lambda\to+\infty} \theta(\lambda D) u = 0 \quad \text{in } L_\infty\big(\mathbb{R}^n\big).$$

Note that the above condition implies that any distribution in $\mathcal{S}'_h(\mathbb{R}^n)$ tends weakly to 0 at infinity. In particular, $\mathcal{S}'_h(\mathbb{R}^n)$ contains no nonzero polynomial. Note also that if $u \in \mathcal{S}'_h(\mathbb{R}^n)$ then one may write

$$u = \sum_{k\in\mathbb{Z}} \Delta_k u \quad \text{in } \mathcal{S}'_h\big(\mathbb{R}^n\big), \tag{2.3}$$

and that, conversely, if (2.3) is satisfied and $\|u\|_{\dot B^s_{p,q}(\mathbb{R}^n)} < \infty$ for some index $s$ such that $s < n/p$ (or $s \leqslant n/p$ if $q = 1$) then $u$ is in $\dot B^s_{p,q}(\mathbb{R}^n)$.

One can prove the following two fundamental properties (see [4]):

**Proposition 1.**

1. *The space* $\dot B^s_{p,q}(\mathbb{R}^n)$ *is complete whenever* $s \leqslant n/p$ *if* $q = 1$, *and* $s < n/p$ *if* $q > 1$.
2. *The set* $\mathcal{S}_0(\mathbb{R}^n)$ *of Schwartz functions with Fourier transform supported away from the origin is dense in* $\dot B^s_{p,q}(\mathbb{R}^n)$ *if and only if* $p$ *and* $q$ *are finite.*

**Remark 4.** For the special case $s \in (0,1)$ the Besov semi-norms may be defined in terms of finite differences of order 1 according to [35, Chapter 2.5]. More precisely, the quantity defined in (2.2) is equivalent to

$$\left(\sum_{k=1}^{n}\int_0^\infty \frac{dh}{h^{1+sq}}\big\|u(x_1,\dots,x_k+h,\dots,x_n)-u(x_1,\dots,x_k,\dots,x_n)\big\|^q_{L_p(\mathbb{R}^n)}\right)^{\frac1q} \quad \text{if } q<\infty,$$

$$\sum_{k=1}^{n}\sup_{h>0} h^{-s}\big\|u(x_1,\dots,x_k+h,\dots,x_n)-u(x_1,\dots,x_k,\dots,x_n)\big\|_{L_p(\mathbb{R}^n)} \quad \text{if } q=\infty.$$

Let us now consider the Poisson equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n. \tag{2.4}$$

It is obvious that if $f \in \mathcal{S}_0(\mathbb{R}^n)$ then the function $u$ defined by

$$\mathcal{F}u(\xi) = \frac{\mathcal{F}f(\xi)}{|\xi|^2} \tag{2.5}$$

is the unique solution of (2.4) in $\mathcal{S}_0(\mathbb{R}^n)$ and that, in addition, $\|u\|_{\dot B^{s+2}_{p,q}(\mathbb{R}^n)} \leqslant C\|f\|_{\dot B^s_{p,q}(\mathbb{R}^n)}$. Since $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\dot B^s_{p,q}(\mathbb{R}^n)$ if $p$ and $q$ are finite, one may deduce the following result:

**Proposition 2.** *If* $p$ *and* $q$ *are finite then the map* $f \mapsto u$ *defined by* (2.5) *has a unique continuous extension from* $\dot B^s_{p,q}(\mathbb{R}^n)$ *to* $\dot B^{s+2}_{p,q}(\mathbb{R}^n)$.

Let us now define the homogeneous Besov spaces on the half-space.

**Definition 1.** For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, we define the homogeneous Besov space $\dot B^s_{p,q}(\mathbb{R}^n_+)$ over the half-space as the restriction (in the distributional sense) of $\dot B^s_{p,q}(\mathbb{R}^n)$ on $\mathbb{R}^n_+$, that is

$$\phi \in \dot B^s_{p,q}\big(\mathbb{R}^n_+\big) \quad \Leftrightarrow \quad \phi = \psi|_{\mathbb{R}^n_+} \quad \text{for some } \psi \in \dot B^s_{p,q}\big(\mathbb{R}^n\big).$$

We then set

$$\|\phi\|_{\dot B^s_{p,q}(\mathbb{R}^n_+)} := \inf_{\psi|_{\mathbb{R}^n_+}=\phi} \|\psi\|_{\dot B^s_{p,q}(\mathbb{R}^n)}.$$

The result below characterizes $\dot B^s_{p,q}(\mathbb{R}^n_+)$ for small $|s|$. It will enable us to consider the symmetric and antisymmetric extension of functions from $\dot B^0_{p,q}(\mathbb{R}^n_+)$ on the whole space, preserving the class of regularity.

**Proposition 3.** *Let* $1 \leqslant p, q < \infty$. *Then for* $\frac{1}{p} - 1 < s < \frac{1}{p}$, *we have*

$$\dot{B}^s_{p,q}(\mathbb{R}^n_+) = \overline{\{f \in \dot{B}^s_{p,q}(\mathbb{R}^n)\colon \operatorname{supp} f \subset \mathbb{R}^n_+\}}^{\|\cdot\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}}.$$

In other words, if $u \in \dot{B}^s_{p,q}(\mathbb{R}^n_+)$ then one can find a sequence $(u^l)_{l\in\mathbb{N}}$ of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ functions, supported in $\mathbb{R}^n_+$ and such that, denoting by $\tilde{u}$ the extension of $u$ by 0 on $\mathbb{R}^n_-$, we have

$$\lim_{l\to+\infty} \|u^l - \widetilde{u}\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = 0.$$

The proof is based on a result from [35] for nonhomogeneous spaces and the fact that for $0 < s < \frac{1}{p}$, the space $\dot{B}^s_{p,1}(\mathbb{R})$ is embedded in some Lebesgue space with *finite* index. Since it is fundamental for our analysis, a sketch of it is given in Appendix A.

**Remark 5.** From Proposition 3, it is not difficult to prove that if $1 \leqslant p, q < \infty$ and $1/p - 1 < s < 1/p$ then the space $C^\infty_0(\mathbb{R}^n_+)$ is dense in $\dot{B}^s_{p,q}(\mathbb{R}^n_+)$. This result fails to be true for $\dot{B}^s_{p,\infty}(\mathbb{R}^n_+)$. In other words, the space $\overline{C^\infty_0(\mathbb{R}^n)}^{\|\cdot\|_{\dot{B}^s_{p,\infty}}}$ is a *strict* subspace of $\dot{B}^s_{p,\infty}(\mathbb{R}^n_+)$.

The following embedding results will be of constant use in the study of (*INS*).

**Proposition 4.** *Let* $1 \leqslant p \leqslant \widetilde{p} \leqslant \infty$, $1 \leqslant q \leqslant \infty$ *and* $s \in \mathbb{R}$. *We have*:

- $\dot{B}^0_{p,1}(\mathbb{R}^n_+) \hookrightarrow L_p(\mathbb{R}^n_+) \hookrightarrow \dot{B}^0_{p,\infty}(\mathbb{R}^n_+)$.
- $\dot{B}^s_{p,q}(\mathbb{R}^n_+) \hookrightarrow \dot{B}^{s-n(\frac{1}{p}-\frac{1}{\widetilde{p}})}_{\widetilde{p},q}(\mathbb{R}^n_+)$.

**Proof.** Those properties are well known for the Besov spaces defined on $\mathbb{R}^n$ (see e.g. [4,5]). The result for the spaces on $\mathbb{R}^n_+$ thus follows readily from the definition by restriction. □

Let us now consider the Poisson equation in the half-space.

**Lemma 1.** *For any* $H \in \dot{B}^s_{p,q}(\mathbb{R}^n_+)$ *with* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ *and* $\frac{1}{p} - 1 < s < \frac{1}{p}$, *system*

$$\begin{aligned} &\Delta z = \operatorname{div} H \quad \text{in } \mathbb{R}^n_+, \\ &z|_{x_n=0} = 0 \quad \text{on } \mathbb{R}^{n-1}, \end{aligned} \qquad z \to 0 \quad \text{as } |x| \to \infty, \tag{2.6}$$

*has a unique solution* $z$ *such that* $\nabla z \in \dot{B}^s_{p,q}(\mathbb{R}^n_+)$, *and we have*

$$\|\nabla z\|_{\dot{B}^s_{p,q}(\mathbb{R}^n_+)} \leqslant C\|H\|_{\dot{B}^s_{p,q}(\mathbb{R}^n_+)}. \tag{2.7}$$

**Proof.** The uniqueness stems from the standard theory for the Laplace equation. For proving existence, one may extend $H$ by antisymmetry as in [29]: we define $\widetilde{H}$ on $\mathbb{R}^n$ by $\widetilde{H}|_{\mathbb{R}^n_+} = H$ and, denoting $H_\tau := (H_1, \dots, H_{n-1})$ and $\widetilde{H}_\tau := (\widetilde{H}_1, \dots, \widetilde{H}_{n-1})$,

$$\forall x_n \in (0, +\infty), \quad \widetilde{H}_\tau(x', -x_n) = -H_\tau(x', x_n) \quad \text{and} \quad \widetilde{H}_n(x', -x_n) = H_n(x', x_n). \tag{2.8}$$

Note that as $s \in (\frac{1}{p}-1, \frac{1}{p})$, Proposition 3 guarantees that $\widetilde{H} \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ and

$$\|\widetilde{H}\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \leqslant C\|H\|_{\dot{B}^s_{p,q}(\mathbb{R}^n_+)},$$

so that Proposition 2 ensures that equation $\Delta\widetilde{z} = \operatorname{div}\widetilde{H}$ has a solution $\widetilde{z} \in \dot{B}^{s+1}_{p,q}(\mathbb{R}^n)$ satisfying

$$\|\nabla\widetilde{z}\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \leqslant C\|\widetilde{H}\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}. \tag{2.9}$$

Because the normal component $\widetilde{H}_n$ has no "jump" on $\partial\mathbb{R}^n_+$, we have

$$\forall x_n \in (0,+\infty), \quad (\operatorname{div}\widetilde{H})(x', -x_n) = -\operatorname{div} H(x', x_n) \quad \text{in the sense of distributions}$$

so that $\widetilde{z}$ is antisymmetric with respect to the hyperplane $\partial\mathbb{R}^n_+$. This implies that $\widetilde{z}|_{\partial\mathbb{R}^n_+} = 0$. In addition, as $\Delta\widetilde{z} = \operatorname{div}\widetilde{H}$ on $\mathbb{R}^n$ in the sense of distribution, one may write

$$-\int_{\mathbb{R}^n} \widetilde{z}\Delta\pi\,dx = \int_{\mathbb{R}^n_+} \nabla\widetilde{z}\cdot\nabla\pi\,dx = \int_{\mathbb{R}^n_+} H\cdot\nabla\pi\,dx \quad \text{for all } \pi \in C_0^\infty\big(\mathbb{R}^n_+\big). \tag{2.10}$$

Therefore, $z := \widetilde{z}|_{\mathbb{R}^n_+}$ is a weak solution of (2.6). Inequality (2.7) follows from (2.9). □

Our analysis also requires some estimates for the linear heat equation in the half-space:

$$\begin{aligned} &\partial_t v - \Delta v = F \quad \text{in } (0,T)\times\mathbb{R}^n_+,\\ &v|_{x_n=0} = 0 \quad \text{on } (0,T)\times\mathbb{R}^{n-1}, \qquad v \to 0 \quad \text{as } |x|\to\infty.\\ &v|_{t=0} = v_0 \quad \text{on } \mathbb{R}^n_+, \end{aligned} \tag{2.11}$$

Let us first recall the following result pertaining to the heat equation in the whole space (see the proof in [10]).

**Proposition 5.** *Let $p \in (1,\infty)$ and $s \in \mathbb{R}$. For any $F \in L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))$ and $v_0 \in \dot{B}^s_{p,1}(\mathbb{R}^n)$ there exists a unique solution $v$ to*

$$\begin{aligned} &\partial_t v - \Delta v = F \quad \text{in } (0,T)\times\mathbb{R}^n,\\ &v|_{t=0} = v_0 \quad \text{on } \mathbb{R}^n, \end{aligned} \qquad v \to 0 \quad \text{as } |x|\to\infty,$$

*such that*

$$v \in \mathcal{C}\big([0,T];\dot{B}^s_{p,1}\big(\mathbb{R}^n\big)\big) \quad \text{and} \quad \partial_t v, \nabla^2 v \in L_1\big(0,T;\dot{B}^s_{p,1}\big(\mathbb{R}^n\big)\big).$$

*Besides, the following estimate is valid*

$$\begin{aligned} &\|v\|_{L_\infty(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \big\|\partial_t v, \nabla^2 v\big\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))}\\ &\quad \leqslant C\big(\|F\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)}\big) \end{aligned} \tag{2.12}$$

*where $C$ is a constant depending only on $n$.*

This entails the following result for the heat equation on the half-space:

**Proposition 6.** *Let $p \in (1, \infty)$ and $s \in (-1 + 1/p, 1/p)$. For any $F \in L_1(0, T; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and $v_0 \in \dot{B}^s_{p,1}(\mathbb{R}^n_+)$, system* (2.11) *has a unique solution $v$ such that*

$$v \in \mathcal{C}\big([0, T]; \dot{B}^s_{p,1}\big(\mathbb{R}^n_+\big)\big) \quad \text{and} \quad \partial_t v, \nabla^2 v \in L_1\big(0, T; \dot{B}^s_{p,1}\big(\mathbb{R}^n_+\big)\big)$$

*and inequality* (2.12) *holds true* (*with $\mathbb{R}^n$ replaced by $\mathbb{R}^n_+$*).

**Proof.** The proof is based again on the use of an antisymmetric extension of the data $v_0$ and $F$. Denoting by $\widetilde{v}_0$ and $\widetilde{F}$ the extended data, Proposition 3 ensures that

$$\|\widetilde{v}_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} \leqslant C\|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} \quad \text{and} \quad \|\widetilde{F}\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n))} \leqslant C\|v_0\|_{L_1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}.$$

Now, the previous proposition provides a unique solution

$$\widetilde{v} \in \mathcal{C}\big([0, T]; \dot{B}^s_{p,1}\big(\mathbb{R}^n\big)\big) \cap L_1\big(0, T; \dot{B}^{s+2}_{p,1}\big(\mathbb{R}^n\big)\big)$$

to the heat equation in the whole space with data $\widetilde{v}_0$ and $\widetilde{F}$. Owing to the antisymmetry of $\widetilde{v}_0$ and $\widetilde{F}$, the solution $\widetilde{v}$ is antisymmetric hence vanishes on $x_n = 0$. It is thus clear that the restriction $v$ to $\widetilde{v}$ on $[0, T] \times \mathbb{R}^n_+$ is a solution to (2.11) and satisfies the required properties. □

Let us state another two basic results the proof of which may be found in Appendix A. The first one will enable us to solve the Laplace equation in the half-space for the Dirichlet problem with nonzero boundary conditions.

**Lemma 2.** *Let $s > 0$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Then there exists a constant $C$ such that for all $h \in \dot{B}^{s-1/p}_{p,q}(\mathbb{R}^{n-1})$, we have*

$$\big\|\mathcal{F}^{-1}_{x'}\big[e^{-|\xi|x_n}\mathcal{F}_{x'}[h]\big]\big\|_{\dot{B}^s_{p,q}(\mathbb{R}^n_+)} \leqslant C\|h\|_{\dot{B}^{s-1/p}_{p,q}(\mathbb{R}^{n-1})} \tag{2.13}$$

*where $\mathcal{F}_{x'}$ stands for the Fourier transform with respect to $x' := (x_1, \dots, x_{n-1})$ and $\xi$ denotes the corresponding Fourier variable.*

The second result combined with Proposition 3 will allow us to generalize the standard trace theorem.

**Lemma 3.** *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and $s \in (-1 + 1/p, 1/p)$. For any vector field $F$ with coefficients in $\dot{B}^s_{p,q}(\mathbb{R}^n_+)$ and $\operatorname{div} F = 0$ in $\mathcal{D}'(\mathbb{R}^n_+)$, we have $F_n|_{x_n=0} \in \dot{B}^{s-1/p}_{p,q}(\mathbb{R}^{n-1})$. In addition, there exists a constant $C$ depending only on $n$ and such that*

$$\|F_n|_{x_n=0}\|_{\dot{B}^{s-1/p}_{p,q}(\mathbb{R}^{n-1})} \leqslant C\|F\|_{\dot{B}^s_{p,q}(\mathbb{R}^n_+)}. \tag{2.14}$$

The proof of uniqueness in Theorem 1 requires our introducing *modified* Lebesgue–Besov spaces related to $L_1(I;\dot B^s_{p,q}(\Omega))$ (where $I$ is any interval of $\mathbb{R}$, and $\Omega=\mathbb{R}^n$ or $\mathbb{R}^n_+$). Those spaces, which will be denoted by $\widetilde L_1(I;\dot B^s_{p,q}(\Omega))$, have been introduced in [10]. In the case $\Omega=\mathbb{R}^n$ they may be seen as the set of distributions $u$ in $\mathcal{S}'(I\times\mathbb{R}^n)$ such that

$$\|u\|_{\widetilde L_1(I;\dot B^s_{p,q}(\mathbb{R}^n))}:=\sum_{k\in\mathbb{Z}}2^{ks}\|\Delta_k u\|_{L_1(I;L_p(\mathbb{R}^n))}<\infty\quad\text{and}\quad\lim_{\lambda\to+\infty}\theta(\lambda D)u=0 \tag{2.15}$$

for all smooth compactly supported functions $\theta$ over $\mathbb{R}^n$.

The above definition together with a result by H. Triebel [35, 1.18.2] implies the following important fact.

**Proposition 7.** *Let $p,q\in[1,\infty]$ and $s_1,s_2\in\mathbb{R}$ with $s_1\neq s_2$. Then for any interval $I$ and any $\theta\in(0,1)$ we have*

$$\widetilde L_1\big(I;\dot B^s_{p,q}\big(\mathbb{R}^n\big)\big)=\big(\widetilde L_1\big(I;\dot B^{s_1}_{p,q_1}\big(\mathbb{R}^n\big)\big),\widetilde L_1\big(I;\dot B^{s_2}_{p,q_2}\big(\mathbb{R}^n\big)\big)\big)_{\theta,q}\quad\textit{with } s:=\theta s_2+(1-\theta)s_1.$$

In other words, Proposition 7 gives us the possibility to omit direct analysis on this type of spaces, just by interpolation, taking $q_1=q_2=1$ above. Indeed, by (2.15) we have

$$\widetilde L_1\big(I;\dot B^s_{p,1}\big(\mathbb{R}^n\big)\big)=L_1\big(I;\dot B^s_{p,1}\big(\mathbb{R}^n\big)\big). \tag{2.16}$$

To extend the above definition to the case $\Omega=\mathbb{R}^n_+$, one may follow the same approach as for the standard Besov spaces: the space $\widetilde L_1(I;\dot B^s_{p,q}(\mathbb{R}^n_+))$ is defined as the restriction of $L_1(I;\dot B^s_{p,q}(\mathbb{R}^n))$ on $I\times\mathbb{R}^n_+$, that is

$$\phi\in\widetilde L_1\big(I;\dot B^s_{p,q}\big(\mathbb{R}^n_+\big)\big)\quad\Leftrightarrow\quad\phi=\psi|_{I\times\mathbb{R}^n_+}\quad\text{for some }\psi\in\widetilde L_1\big(I;\dot B^s_{p,q}\big(\mathbb{R}^n\big)\big).$$

We then set

$$\|\phi\|_{\widetilde L_1(I;\dot B^s_{p,q}(\mathbb{R}^n_+))}:=\inf_{\psi|_{I\times\mathbb{R}^n_+}=\phi}\|\psi\|_{\widetilde L_1(I;\dot B^s_{p,q}(\mathbb{R}^n))}.$$

It is easy to prove embeddings similar to those of Proposition 4.

The following two lemmas will be used in the proof of the existence part of Theorems 1 and 3. Proving them requires paradifferential calculus (see Appendix A).

**Lemma 4.** *Let $q_0>1$ and $q\in[q_0,\infty]$. There exists a constant $C=C_{q_0,n}$ such that for all $F\in\dot B^0_{n,1}(\mathbb{R}^n_+)\cap\dot B^0_{q,1}(\mathbb{R}^n_+)$ and $G\in L_\infty(\mathbb{R}^n_+)\cap\dot B^1_{q,\infty}(\mathbb{R}^n_+)$ we have*

$$\|FG\|_{\dot B^0_{q,1}(\mathbb{R}^n_+)}\leqslant C\big(\|F\|_{\dot B^0_{n,1}(\mathbb{R}^n_+)}\|\nabla G\|_{\dot B^0_{q,\infty}(\mathbb{R}^n_+)}+\|F\|_{\dot B^0_{q,1}(\mathbb{R}^n_+)}\|G\|_{L_\infty(\mathbb{R}^n_+)}\big).$$

**Lemma 5.** *Let $1<q<\infty$ and $I$ be an interval of $\mathbb{R}$. There exists a constant $C=C_{n,q}$ depending continuously on $q$ such that for all $(r,r_1,r_2)\in[1,+\infty]^3$ such that $1/r=1/r_1+1/r_2$ and all $(F,G)$ in $\widetilde L_{r_1}(I;\dot B^0_{q,\infty}(\mathbb{R}^n))\times(L_{r_2}(I;L^\infty)\cap\widetilde L_{r_2}(I;\dot B^0_{n,\infty}(\mathbb{R}^n_+)))$, we have*

$$\|FG\|_{\widetilde L_r(I;\dot B^0_{q,\infty})}\leqslant C\|F\|_{\widetilde L_{r_1}(I;\dot B^0_{q,\infty})}\big(\|G\|_{L_{r_2}(I;L_\infty)}+\|\nabla G\|_{\widetilde L_{r_2}(I;\dot B^0_{n,\infty})}\big).$$

## 3. The Stokes system

This section is devoted to the study of the Stokes system (1.6) in the half-space. In the first three subsections, we shall focus on the proof of Theorem 2, while the last subsection is devoted to estimates in the (larger) spaces $\widetilde{L}_1(0,T;\dot{B}^0_{p,\infty}(\mathbb{R}^n_+))$, which will be needed for proving the uniqueness for (*INS*). Our technique follows from standard approaches to the subject [15,30,31,33].

For notational simplicity, we shall assume that $\mu=1$. Of course a convenient change of variables gives the general case.

### *3.1. Reduction to a model problem on* $\mathbb{R}\times\mathbb{R}^n_+$.

The first step is to extend the problem on the whole real line for the time direction. Without loss of generality, one may assume that $T=+\infty$, extending the source term $f$ by zero if need be.

Next, we want to "eliminate" the initial datum $v_0$. As $1/p-1<s<1/p$, Proposition 3 guarantees that the function $\widetilde{v}_0$ defined on $\mathbb{R}^n$ by $\widetilde{v}_0|_{\mathbb{R}^n_+}=v_0$ and

$$\widetilde{v}_{0\tau}(x',x_n)=-\widetilde{v}_{0\tau}(x',-x_n)\quad\text{and}\quad\widetilde{v}_{0n}(x',x_n)=\widetilde{v}_{0n}(x',-x_n)\quad\text{for } x_n<0, \tag{3.1}$$

belongs to $\dot{B}^s_{p,1}(\mathbb{R}^n)$, is divergence free and satisfies $\|\widetilde{v}_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)}\leqslant C\|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}$.

Now, according to Proposition 5, the heat equation

$$\begin{aligned}\partial_t v-\Delta v=0\quad&\text{in } \mathbb{R}^+\times\mathbb{R}^n,\\ v|_{t=0}=\widetilde{v}_0\quad&\text{on } \mathbb{R}^n,\end{aligned} \tag{3.2}$$

has a unique solution $E\widetilde{v}_0$ in $\mathcal{C}_b(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n))\cap L_1(\mathbb{R}_+;\dot{B}^{s+2}_{p,1}(\mathbb{R}^n))$ and we have

$$\|E\widetilde{v}_0\|_{L_\infty(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n))}+\|E\widetilde{v}_0\|_{L_1(\mathbb{R}_+;\dot{B}^{s+2}_{p,1}(\mathbb{R}^n))}\leqslant C\|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}. \tag{3.3}$$

Because $\operatorname{div}\widetilde{v}_0=0$, uniqueness for the heat equation guarantees that $\operatorname{div}E\widetilde{v}_0=0$ and, since the symmetry of $\widetilde{v}_0$ is preserved during the evolution, we have

$$(E\widetilde{v}_0)_\tau|_{x_n=0}=0,\quad\text{where } (E\widetilde{v}_0)_\tau:=\big((E\widetilde{v}_0)_1,\dots,(E\widetilde{v}_0)_{n-1}\big). \tag{3.4}$$

Note however that the normal ($n$th) component $(E\widetilde{v}_0)_n$ may be nonzero at $x_n=0$. Now, introducing the new unknown function

$$v_{\text{new}}=v_{\text{old}}-E\widetilde{v}_0\quad\text{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}^n_+, \tag{3.5}$$

reduces our study of (1.6) to the case where the initial data is zero and the boundary data is $v|_{x_n=0}=v_b:=-E\widetilde{v}_0|_{x_n=0}$.

Extending the problem on the whole time line by setting $v=0$ for $t<0$ is the next step. The properties of (3.5) allow us to do it. However, we also have to get a suitable control over $v_b$ for proving Lemma 8 below. In our case, as $v_{b\tau}=0$, we only have to worry about the $n$th component. So let us consider the solution $w$ to the auxiliary problem

$$\begin{aligned}
&\partial_t w - \Delta w = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+,\\
&w|_{x_n=0} = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1},\\
&w|_{t=0} = v_{0n} \quad \text{on } \mathbb{R}^n_+.
\end{aligned} \tag{3.6}$$

From Proposition 6, we get $w \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)) \cap L_1(\mathbb{R}_+; \dot{B}^{s+2}_{p,1}(\mathbb{R}^n_+))$ and

$$\|w\|_{L_\infty(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_t w, \nabla^2 w\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\|v_{0n}\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}.$$

Then we set

$$Ev_{bn} := \begin{cases} w - (E\widetilde{v}_0)_n & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n_+,\\ 0 & \text{for } (t,x) \in \mathbb{R}_- \times \mathbb{R}^n_+. \end{cases} \tag{3.7}$$

As its trace on $t = 0$ is zero, function $Ev_{bn}$ satisfies

$$\begin{aligned}
&\partial_t Ev_{bn} - \Delta Ev_{bn} = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n_+,\\
&Ev_{bn}|_{x_n=0} = \begin{cases} -E\widetilde{v}_0|_{x_n=0} & \text{for } t > 0,\\ 0 & \text{for } t \leqslant 0, \end{cases} \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1}.
\end{aligned} \tag{3.8}$$

In addition, as $Ev_{bn} = 0$ for negative times, Proposition 6 and inequality (3.3) guarantee that $Ev_{bn} \in \mathcal{C}_b(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$, $\partial_t Ev_{bn}, \nabla^2 Ev_{bn} \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and

$$\|Ev_{bn}\|_{L_\infty(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_t Ev_{bn}, \nabla^2 Ev_{bn}\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\|v_{0n}\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}. \tag{3.9}$$

Note that the above inequality provides us with an information on the regularity of $Ev_{bn}$ at the boundary, thus also on $v_b$. Therefore one can now consider the following boundary value problem

$$\begin{aligned}
&\partial_t v - \Delta v + \nabla P = F \quad \text{in } \mathbb{R} \times \mathbb{R}^n_+,\\
&\operatorname{div} v = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n_+,\\
&v|_{x_n=0} = v_b \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1},
\end{aligned} \tag{3.10}$$

with boundary data $v_b$ given by

$$v_{b\tau} \equiv 0, \qquad v_{bn} = Ev_{bn}|_{x_n=0} \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1}, \tag{3.11}$$

and $Ev_{bn} : \mathbb{R} \times \mathbb{R}^n_+ \to \mathbb{R}$ satisfying (3.9).

Removing the potential part of $F$ and the trace of its normal component at the boundary will be our next task. For that, formally, it suffices to solve the elliptic equation

$$\begin{aligned}
&\Delta P = \operatorname{div} F \quad \text{in } \mathbb{R} \times \mathbb{R}^n_+,\\
&(\partial_{x_n} P - F_n)|_{x_n=0} = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1}.
\end{aligned} \tag{3.12}$$

Note that, by virtue of Lemma 3, the second line makes sense as soon as $\nabla P - F$ is in $L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$, since the first line ensures that $\operatorname{div}(\nabla P - F) = 0$. However, because $\partial_{x_n} P$ and $F_n$ need not have a trace on $\partial\mathbb{R}^n_+$, two steps are required for solving system (3.12). So let us first consider the following Poisson equation:

$$\begin{aligned} \Delta P_1 &= \operatorname{div} F \quad \text{in } \mathbb{R} \times \mathbb{R}^n_+, \\ P_1|_{x_n=0} &= 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1}. \end{aligned} \tag{3.13}$$

Applying Lemma 1 for all fixed $t \in \mathbb{R}$ then integrating over $\mathbb{R}$ yields

$$\|\nabla P_1\|_{L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\|F\|_{L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))}. \tag{3.14}$$

By construction, $\operatorname{div}(F - \nabla P_1) = 0$ and $(F - \nabla P_1) \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ to that the $n$th component $\widetilde{F}_n := F_n - \partial_{x_n} P_1$ has a trace on the boundary $\partial\mathbb{R}^n_+$, which, according to Lemma 3 and (3.14) satisfies

$$\|\widetilde{F}_n|_{x_n=0}\|_{L_1(\mathbb{R}; \dot{B}^{s-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \leqslant C\|F - \nabla P_1\|_{L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\|F\|_{L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))}. \tag{3.15}$$

As a second step for solving (3.12), we thus consider the following *Neumann* problem:

$$\begin{aligned} \Delta P_2 &= 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n_+, \\ \partial_{x_n} P_2|_{x_n=0} &= \widetilde{F}_n|_{x_n=0} \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1}. \end{aligned} \tag{3.16}$$

System (3.16) can be solved explicitly by applying the Fourier transform $\mathcal{F}_{x'}$ with respect to the tangential space variables $x' := (x_1, \ldots, x_{n-1})$. Denoting $\xi := (\xi_1, \ldots, \xi_{n-1})$ the corresponding Fourier variables, we get

$$P_2 = \mathcal{F}^{-1}_{x'}\left[-e^{-|\xi|x_n}\frac{1}{|\xi|}\mathcal{F}_{x'}[\widetilde{F}_n|_{x_n=0}]\right]. \tag{3.17}$$

Taking advantage of Lemma 2 and of inequality (3.15), one can conclude that $\nabla P_2$ belongs to $L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and that

$$\|\nabla P_2\|_{L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\|\widetilde{F}_n|_{x_n=0}\|_{L_1(\mathbb{R}; \dot{B}^{s-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \leqslant C\|F\|_{L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))}. \tag{3.18}$$

So finally, changing the divergence free source term $F$ into $F - \nabla P_1 - \nabla P_2$ reduces the study of system (3.10) to the case where

$$\operatorname{div} F = 0 \quad \text{in } \mathbb{R}^n_+ \quad \text{and} \quad F_n = 0 \quad \text{on } \partial\mathbb{R}^n_+. \tag{3.19}$$

*3.2. The boundary problem*

This step, which is the cornerstone of the proof of Theorem 2, is devoted to the study of system (3.10) under hypothesis (3.19). Our main result reads:

**Lemma 6.** *Let $1 < p < \infty$ and $1/p - 1 < s < 1/p$. Let $F \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ satisfy* $\operatorname{div} F = 0$ *and $F_n|_{x_n=0} = 0$ in the meaning of the trace, and let $v_b$ be given by* (3.11)*. Then system* (3.10) *has a solution $(v, P)$ verifying*

$$v \in \mathcal{C}_b\big(\mathbb{R}; \dot{B}^s_{p,1}\big(\mathbb{R}^n_+\big)\big) \quad \text{and} \quad \partial_t v, \nabla^2 v, \nabla P \in L_1\big(\mathbb{R}; \dot{B}^s_{p,1}\big(\mathbb{R}^n_+\big)\big). \tag{3.20}$$

*In addition, we have for some constant $C$ depending only on $n$, $p$ and $s$,*

$$\begin{aligned}&\|v\|_{L_\infty(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \big\|\partial_t v, \nabla^2 v, \nabla P\big\|_{L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\ &\quad \leqslant C\big(\|F\|_{L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}\big).\end{aligned} \tag{3.21}$$

**Proof.** Denoting $(\xi_0, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1}$ the Fourier variables pertaining to $(t, x')$ and

$$u(\xi_0, \xi, x_n) = \mathcal{F}_{t,x'}[v] = \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{n-1}} e^{-it\xi_0 - ix'\cdot\xi} v(t, x', x_n)\, dx'\, dt,$$

$$q(\xi_0, \xi, x_n) = \mathcal{F}_{t,x'}[P] = \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{n-1}} e^{-it\xi_0 - ix'\cdot\xi} P(t, x', x_n)\, dx'\, dt,$$

$$f(\xi_0, \xi, x_n) = \mathcal{F}_{t,x'}[F] = \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{n-1}} e^{-it\xi_0 - ix'\cdot\xi} F(t, x', x_n)\, dx'\, dt,$$

system (3.10) reduces to the following ordinary differential system:

$$\begin{gathered} r^2 u_\tau - \partial^2_{x_n} u_\tau = f_\tau - i\xi q \quad \text{in } \mathbb{R} \times \mathbb{R}^{n-1} \times (0, \infty), \\ r^2 u_n - \partial^2_{x_n} u_n = f_n - \partial_{x_n} q \quad \text{in } \mathbb{R} \times \mathbb{R}^{n-1} \times (0, \infty), \\ i\xi \cdot u_\tau + \partial_{x_n} u_n = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^{n-1} \times (0, \infty), \\ u|_{x_n=0} = u_b \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1}, \end{gathered} \tag{3.22}$$

with $r^2 = i\xi_0 + |\xi|^2$ chosen so that[3] $\arg r \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ and

$$u = (u_\tau, u_n), \qquad f = (f_\tau, f_n), \quad \text{where } u_\tau = (u_1, \ldots, u_{n-1}) \text{ and } f_\tau = (f_1, \ldots, f_{n-1}).$$

System (3.22) may be solved explicitly. Indeed, taking the divergence of $(3.10)_1$ and knowing that $\operatorname{div} F = 0$, we discover that

[3] In other words, $r := \big(\frac{\sqrt{\xi_0^2 + |\xi|^4} + |\xi|^2}{2}\big)^{\frac{1}{2}} + i \operatorname{sgn}(\xi_0)\big(\frac{\sqrt{\xi_0^2 + |\xi|^4} - |\xi|^2}{2}\big)^{\frac{1}{2}}$.

$$|\xi|^2 q - \partial_{x_n}^2 q = 0, \tag{3.23}$$

so that, because we want $q$ to tend to 0 for $x_n \to \infty$, we get

$$q(\xi_0, \xi, x_n) = q_0(\xi_0, \xi) e^{-|\xi| x_n} \quad \text{for an unknown function } q_0. \tag{3.24}$$

The first step is to determine $q_0$ and its properties. The theory of ordinary differential equations gives us the following explicit formulae for the solutions to (3.22) — see [15]:

$$u_\tau(\xi_0, \xi, x_n) = u_{b\tau} e^{-r x_n} + \frac{1}{2r} \int_0^\infty \left[e^{-r|x_n - s_n|} - e^{-r(x_n + s_n)}\right]\left[f_\tau(\xi_0, \xi, s_n) - i\xi e^{-|\xi| s_n} q_0\right] ds_n,$$

$$u_n(\xi_0, \xi, x_n) = u_{bn} e^{-r x_n} + \frac{1}{2r} \int_0^\infty \left[e^{-r|x_n - s_n|} - e^{-r(x_n + s_n)}\right]\left[f_n(\xi_0, \xi, s_n) + |\xi| e^{-|\xi| s_n} q_0\right] ds_n.$$

In our case $u_{b\tau}$ is zero and only $u_{bn}$ may be nontrivial — see (3.11). Therefore, taking the derivative of the second equality with respect to $x_n$, we get

$$\partial_{x_n} u_n = -r u_{bn} e^{-r x_n} + \frac{1}{2} \int_0^\infty \left[e^{-r|x_n - s_n|} \operatorname{sgn}(s_n - x_n) + e^{-r(x_n + s_n)}\right]\left[f_n + |\xi| e^{-|\xi| s_n} q_0\right] ds_n.$$

Letting $x_n \to 0$, we find that

$$\partial_{x_n} u_n|_{x_n = 0} = -r u_{bn} + \int_0^\infty e^{-r s_n}\left[f_n(s_n) + |\xi| e^{-|\xi| s_n} q_0\right] ds_n,$$

whence

$$\partial_{x_n} u_n|_{x_n = 0} = -r u_{bn} + \int_0^\infty e^{-r s_n} f_n(s_n)\, ds_n + \frac{|\xi|}{r + |\xi|} q_0. \tag{3.25}$$

Note that our assumption on $F_n$ implies that $f_n|_{x_n = 0} = 0$ in the meaning of the trace. Therefore, owing to $\operatorname{div} F = 0$, one may write

$$-r \int_0^\infty e^{-r s_n} f_n\, ds_n = \int_0^\infty \partial_{s_n}\left(e^{-r s_n}\right) f_n\, ds_n = -\int_0^\infty e^{-r s_n} \partial_{s_n} f_n\, ds_n = \int_0^\infty e^{-r s_n} i\xi \cdot f_\tau\, ds_n.$$

As $\partial_{x_n} u_n = -i\xi \cdot u_\tau$ is zero at the boundary, we thus get

$$q_0 = \left(\frac{i\xi_0}{|\xi|} + r + |\xi|\right) u_{bn} + \int_0^\infty e^{-r s_n}\left[\frac{i\xi}{|\xi|} \cdot f_\tau - f_n\right] ds_n. \tag{3.26}$$

Now we want to investigate the properties of regularity of $q_0$. Together with (3.24) this will enable us to compute the pressure, and to prove that its gradient satisfies (3.21). For this purpose we need two results. The first one will enable us to handle the second term in (3.26):

**Lemma 7.** *Let $h \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and $\widehat{h} := \mathcal{F}_{t,x'} h$. Then*

$$H := \mathcal{F}^{-1}_{t,x'}\left[\int_0^\infty e^{-rs_n}\widehat{h}(\xi_0, \xi, s_n)\, ds_n\right] \tag{3.27}$$

*admits an extension $\widetilde{H} \in L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$ on $\mathbb{R} \times \mathbb{R}^n_+$ such that*

$$\widetilde{H}|_{x_n=0} = H \quad \text{and} \quad \|\widetilde{H}\|_{L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))} \leqslant C\|h\|_{L_1(\mathbb{R}; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+))}. \tag{3.28}$$

**Proof.** In order to construct $\widetilde{H}$, let us consider the following heat equation:

$$\begin{aligned} &\partial_t v - \Delta v = h \quad \text{in } \mathbb{R} \times \mathbb{R}^n_+, \\ &v|_{x_n=0} = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1}. \end{aligned} \tag{3.29}$$

Arguing as for solving (3.10), we obtain the explicit formula:

$$v = \mathcal{F}^{-1}_{t,x'}\left[\frac{1}{2r}\int_0^\infty \left[e^{-r|x_n - s_n|} - e^{-r(x_n+s_n)}\right]\widehat{h}(\xi_0, \xi, s_n)\, ds_n\right].$$

Differentiating the above formulation with respect to $x_n$, and taking $x_n = 0$, we get

$$\partial_{x_n} v|_{x_n=0} = H := \mathcal{F}^{-1}_{t,x'}\left[\int_0^\infty e^{-rs_n}\widehat{h}\, ds_n\right],$$

so that Proposition 6 ensures that $\widetilde{H} := \partial_{x_n} v$ has the required properties.[4] □

The second result concerns the terms of (3.26) related to the boundary data:

**Lemma 8.** *Let $v_b$ be given by* (3.11). *Then $\mathcal{F}^{-1}_{t,x'}[\frac{i\xi_0}{|\xi|}u_{bn}] \in L_1(\mathbb{R}; \dot{B}^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1}))$ and*

$$\left\|\mathcal{F}^{-1}_{t,x'}\left[\frac{i\xi_0}{|\xi|}u_{bn}\right]\right\|_{L_1(\mathbb{R}; \dot{B}^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1}))} \leqslant C\|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}. \tag{3.30}$$

[4] In fact, we have to consider first the case where $h \equiv 0$ on $(-\infty, T)$, and next have $T$ tend to $-\infty$.

*Furthermore, if* $g = \mathcal{F}^{-1}_{t,x'}[(r + |\xi|)u_{bn}]$ *then there exists an extension* $G \in L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$ *such that* $G|_{x_n=0} = g$ *and*

$$\|G\|_{L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))} \leqslant C\|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}. \tag{3.31}$$

**Proof.** Let us first prove (3.31). Remind that $v := Ev_{bn}$ (see (3.8)) satisfies

$$\begin{aligned} \partial_t v - \Delta v &= 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n_+, \\ v|_{x_n=0} &= v_{bn} \quad \text{in } \mathbb{R} \times \mathbb{R}^{n-1}. \end{aligned} \tag{3.32}$$

Therefore we have $\mathcal{F}_{t,x'}[v] = u_{bn}e^{-rx_n}$ and, according to (3.9), $\nabla v \in L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$. Hence it follows that $\mathcal{F}^{-1}_{t,x'}[|\xi|u_{bn}e^{-rx_n}] \in L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$ and that

$$\big\|\mathcal{F}^{-1}_{t,x'}\big[|\xi|u_{bn}e^{-rx_n}\big]\big\|_{L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))} \leqslant C\|v_{0n}\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}. \tag{3.33}$$

Next, we notice that $\partial_{x_n} v = -\mathcal{F}^{-1}_{t,x'}[re^{-rx_n}u_{bn}]$. Hence $\mathcal{F}^{-1}_{t,x'}[re^{-rx_n}u_{bn}] \in L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$ and satisfies (3.33) too. Therefore, one may construct an extension $G$ satisfying (3.31).

Formally, the most difficult term, $\mathcal{F}^{-1}_{t,x'}[\frac{i\xi_0}{|\xi|}u_{bn}]$, may be viewed as the trace of $\partial_t v$ at $x_n = 0$ divided by $|\xi|$. Since (3.32) is satisfied, we already know that $\partial_t v \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ together with a suitable estimate. In order to show that the trace at $x_n = 0$ of $\partial_t v$ is well defined, we plan to express $\partial_t v$ as the $n$th component of a convenient divergence free vector field with coefficients in $L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$. Then applying Lemma 3 will enable us to get (3.31).

So, for $k = 1, \ldots, n-1$, let us consider the following system:

$$\begin{aligned} \partial_t w - \Delta w &= 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ \partial_{x_n} w|_{x_n=0} &= 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ w|_{t=0} &= v_{0k} \quad \text{on } \mathbb{R}^n_+. \end{aligned}$$

By combining symmetric extension and Proposition 5, it is not difficult to see that the above system has a solution $w_k$ with the usual properties of regularity and satisfying in particular

$$\big\|\nabla^2 w_k\big\|_{L_1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}.$$

Now, if we denote by $w_n$ the solution to (3.6), the vector field $V := (w_1, \ldots, w_n)$ is divergence free — because $V|_{t=0} = v_0$ on $\mathbb{R}^n_+$ — and

$$\big\|\partial_t V, \nabla^2 V\big\|_{L_1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}.$$

Of course, we also have $\operatorname{div} \partial_t V = 0$. Hence, as $\partial_t V$ is in $L_1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$, Lemma 3 guarantees that $\partial_t w_n|_{x_n=0} \in L_1(\mathbb{R}_+; \dot{B}^{s-1/p}_{p,1}(\mathbb{R}^n_+))$ with a suitable inequality. Finally, we notice that $\partial_t v = \partial_t w_n - \partial_t(E\widetilde{v}_0)_n$, that $\operatorname{div}(\partial_t E\widetilde{v}_0) = 0$ and that $\partial_t E\widetilde{v}_0 \in L_1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$. So applying again Lemma 3 yields the desired bound for $((\partial_t E\widetilde{v}_0)_n)|_{x_n=0}$ and thus for $(\partial_t v)|_{x_n=0}$. Now, it is easy to conclude that $\mathcal{F}^{-1}_{t,x'}[\frac{i\xi_0}{|\xi|}u_b] \in L_1(\mathbb{R}; \dot{B}^{s+1-1/p}_{p,1}(\mathbb{R}^n_+))$ and satisfies (3.30). Lemma 8 is proved. $\square$

*3.3. The estimate for the pressure*

Here we complete the proof of Lemma 6 and Theorem 2.

**Lemma 9.** *Under the assumptions of Lemma* 6 *we have* $P \in L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$ *and*

$$\|\nabla P\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\big(\|F\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}\big). \tag{3.34}$$

**Proof.** According to (3.24), we have $\mathcal{F}_{t,x'}[P] = q_0 e^{-|\xi|x_n}$, where $q_0$ is given by (3.26). Split $q_0$ into $q_0 = q_1 + q_2$ with

$$q_1 = \frac{i\xi_0}{|\xi|}u_{bn}, \qquad q_2 = \big(r + |\xi|\big)u_{bn} + \int_0^\infty e^{-rs}\left[\frac{i\xi}{|\xi|}\cdot f_\tau - f_n\right]ds. \tag{3.35}$$

By virtue of Lemma 7 and of the first part of Lemma 8, one can find some function $Q_2$ in $L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$ so that $Q_2|_{x_n=0} = \mathcal{F}^{-1}_{t,x'}[q_2]$ and

$$\|Q_2\|_{L_1(\mathbb{R};\dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))} \leqslant C\big(\|F\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}\big). \tag{3.36}$$

Next, combining Lemmas 2 and 8, we gather that the function $Q_1 := \mathcal{F}^{-1}_{t,x}[e^{-|\xi|x_n}q_1]$ is in $L_1(\mathbb{R}; \dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))$, satisfies $Q_1|_{x_n=0} = \mathcal{F}^{-1}_{t,x'}[q_1]$ and $\Delta Q_1 = 0$, and

$$\|Q_1\|_{L_1(\mathbb{R};\dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+))} \leqslant C\|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}. \tag{3.37}$$

So finally, because we want to have $\Delta P = 0$ (remind that $\operatorname{div} F = 0$) the pressure can be sought in the form $P = P_1 + P_2$ where $P_1 := Q_1 + Q_2$ (so that $\nabla P_1$ is in $L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and satisfies the desired estimate), and $P_2$ fulfills the system

$$\begin{aligned} &\Delta P_2 = -\Delta Q_2 \quad \text{in } \mathbb{R}^n_+,\\ &P_2|_{x_n=0} = 0 \quad \text{on } \mathbb{R}^{n-1}. \end{aligned} \tag{3.38}$$

According to (3.36) and (3.37), we have $\nabla Q_2 \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$, hence the assumptions of Lemma 1 are fulfilled. Therefore Eq. (3.38) has a solution $P_2$ such that $\nabla P_2 \in L_1(\mathbb{R}; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and

$$\|\nabla P_2\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\|\nabla Q_2\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leqslant C\big(\|F\|_{L_1(\mathbb{R};\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)}\big).$$

Combining this with (3.36) completes the proof of estimate (3.34). $\square$

Armed with Lemma 9, one may now look at the original system (1.6) as a *heat equation*, namely

$$\begin{aligned} \partial_t v - \Delta v &= F - \nabla P \quad \text{in } (0,T)\times\mathbb{R}^n_+,\\ v|_{x_n=0} &= 0 \quad \text{on } (0,T)\times\mathbb{R}^{n-1},\\ v|_{t=0} &= v_0 \quad \text{on } \mathbb{R}^n_+. \end{aligned} \tag{3.39}$$

Let us emphasize that the incompressibility condition $\operatorname{div} v = 0$ is hidden in the construction of the pressure. In addition, Proposition 6 ensures that the solution $v$ to the above system satisfies

$$\begin{aligned} &\|v\|_{L_\infty(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \big\|\partial_t v, \nabla^2 v\big\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ &\quad\leqslant C\big(\|F-\nabla P\|_{L_1(0,T;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|v_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\big). \end{aligned}$$

Since $\nabla P$ is bounded according to Lemma 9, Theorem 2 is proved.

### *3.4. The estimate in* $\widetilde L_1(\mathbb{R}; \dot B^0_{p,\infty}(\mathbb{R}^n_+))$

As a consequence of the above analysis we readily get the following estimates which turn out to be the key to the proof of uniqueness in Theorem 1.

**Lemma 10.** *Let $1<p<\infty$ and $s\in(-1+1/p, 1/p)$. Assume that the time-dependent vector field $F$ has coefficients in $\widetilde L_1(\mathbb{R}; \dot B^s_{p,\infty}(\mathbb{R}^n_+))$. Then system* (3.10) *with null boundary data has a unique solution $(v,P)$ such that*

$$v\in L_\infty\big(\mathbb{R}; \dot B^s_{p,\infty}\big(\mathbb{R}^n_+\big)\big) \quad \textit{and} \quad \partial_t v, \nabla^2 v, \nabla P \in \widetilde L_1\big(\mathbb{R}; \dot B^s_{p,\infty}\big(\mathbb{R}^n_+\big)\big). \tag{3.40}$$

*In addition,*

$$\|v\|_{L_\infty(\mathbb{R};\dot B^s_{p,\infty}(\mathbb{R}^n_+))} + \big\|\partial_t v, \nabla^2 v, \nabla P\big\|_{\widetilde L_1(\mathbb{R};\dot B^s_{p,\infty}(\mathbb{R}^n_+))} \leqslant C\|F\|_{\widetilde L_1(\mathbb{R};\dot B^s_{p,\infty}(\mathbb{R}^n_+))}. \tag{3.41}$$

**Proof.** The proof follows from a direct application of the interpolation theory (see Proposition 7) to Theorem 2. □

## 4. The nonlinear problem

The previous section will enable us to solve the nonlinear system (*INS*) with the initial velocity in a critical Besov space of index 0. In order to prove our main existence result, Theorem 1, we shall proceed as follows:

- first, we show the existence of solutions for data with more integrability;
- second, we prove the uniqueness part of Theorem 1;
- last, we tackle the proof of existence in the case of data with critical regularity. For that, we shall use the first step to construct smoother solutions pertaining to smoothed out data, then resort to compactness arguments.

**Notation.** In this section, we shall only consider functions or distributions defined on the half-space $\mathbb{R}^n_+$ so that, for notational simplicity, we shall write $\dot B^s_{p,r}$ instead of $\dot B^s_{p,r}(\mathbb{R}^n_+)$.

### 4.1. Smoother solutions

This subsection is dedicated to the proof of existence of "smooth" global solutions. Our main result reads:

**Theorem 3.** *Let $(\rho_0, u_0, f)$ satisfy the assumptions of Theorem 1 for a small enough constant $c$. Assume in addition that $(\rho_0 - 1) \in W^1_p$, $u_0 \in \dot{B}^0_{p,1}$ and $f \in L_{1,\mathrm{loc}}(\mathbb{R}_+; \dot{B}^0_{p,1})$ for some $p \in (n, \infty)$. Then system (INS) has a unique global solution*

$$(\rho, u, \nabla\Pi) \in E_n \cap E_{p,\mathrm{loc}}$$

*satisfying the inequalities of Theorem 1 and, for all $t \in \mathbb{R}_+$,*

$$\left\|\nabla\rho(t)\right\|_{L_q} \leqslant 2\|\nabla\rho_0\|_{L_q} \quad \text{for } q = n, p, \tag{4.1}$$

$$\|\nabla u\|_{L_1(0,t;L_\infty)} \leqslant \log 2, \tag{4.2}$$

$$\begin{aligned}&\|u\|_{L_\infty(0,t;\dot{B}^0_{p,1})} + \left\|\partial_t u, \mu\nabla^2 u, \nabla\Pi\right\|_{L_1(0,t;\dot{B}^0_{p,1})} \\ &\quad \leqslant C\big(\|u_0\|_{\dot{B}^0_{p,1}} + \|f\|_{L_1(0,t;\dot{B}^0_{p,1})} + \mu\|\nabla\rho_0\|_{L_p}\big)\end{aligned} \tag{4.3}$$

*with $C$ depending only on $n$.*

**Proof.** As a first step, let us state a priori estimates in $E_n \cap E_{p,\mathrm{loc}}$ for system (*INS*).

So we assume that we are given a solution $(\rho, u) \in E_n(T) \cap E_p(T)$. We claim that if condition (1.3) is satisfied for some small enough constant $c$ then estimates (1.4), (1.5), (4.1), (4.2) and (4.3) are true for all $t \in [0, T]$.

Let us first consider the density. Owing to the incompressibility condition and to the fact that $u \cdot n = 0$ on $\partial\mathbb{R}^n_+$, all the $L_p$ norms of $\rho$ are time independent and we obtain by a Gronwall type argument the following inequalities:

$$\begin{aligned}\left\|\nabla\rho(t)\right\|_{L_q} &\leqslant e^{\int_0^t \|\nabla u\|_{L_\infty} d\tau}\|\nabla\rho_0\|_{L_q} \quad \text{for } q \in \{n, p\}, \\ \left\|1 - \rho(t)\right\|_{L_\infty} &= \|1 - \rho_0\|_{L_\infty}.\end{aligned} \tag{4.4}$$

Therefore, in particular,

$$\left\|1 - \rho(t)\right\|_{L_\infty\cap\dot{W}^1_n} \leqslant e^{\int_0^t \|\nabla u\|_{L_\infty} d\tau}\|1 - \rho_0\|_{L_\infty\cap\dot{W}^1_n}. \tag{4.5}$$

In order to bound the velocity, we may apply Theorem 2 to the system

$$\begin{aligned}&\partial_t u - \mu\Delta u + \nabla\Pi = (1-\rho)\partial_t u + \rho(f - u\cdot\nabla u), \qquad \operatorname{div} u = 0, \\ &u|_{x_n=0} = 0, \qquad u|_{t=0} = u_0.\end{aligned} \tag{4.6}$$

We get for $q \in \{n, p\}$,

$$U_q(t) \leqslant C\big(U_q(0) + \left\|(1-\rho)\partial_t u\right\|_{L_1(0,t;\dot{B}^0_{q,1})} + \left\|\rho(f - u\cdot\nabla u)\right\|_{L_1(0,t;\dot{B}^0_{q,1})}\big)$$

with $U_q(0) := \|u_0\|_{\dot B^0_{q,1}}$ and

$$U_q(t) := \|u\|_{L_\infty(0,t;\dot B^0_{q,1})} + \|\partial_t u\|_{L_1(0,t;\dot B^0_{q,1})} + \mu\|\nabla^2 u\|_{L_1(0,t;\dot B^0_{q,1})} + \|\nabla\Pi\|_{L_1(0,t;\dot B^0_{q,1})}.$$

For bounding the right-hand side of $U_n(t)$ one may use Proposition 4 from Section 2, which gives

$$\dot B^1_{n,1} \hookrightarrow L_\infty \cap \dot W^1_n \hookrightarrow L_\infty \cap \dot B^1_{n,\infty},$$

and Lemma 4. We find that

$$\big\|(1-\rho)\partial_t u\big\|_{L_1(0,t;\dot B^0_{n,1})} \leqslant C\|\rho-1\|_{L_\infty(0,t;L_\infty\cap\dot W^1_n)}\|\partial_t u\|_{L_1(0,t;\dot B^0_{n,1})},$$

$$\big\|\rho(f-u\cdot\nabla u)\big\|_{L_1(0,t;\dot B^0_{n,1})} \leqslant C\big(1+\|\rho-1\|_{L_\infty(0,t;L_\infty\cap\dot W^1_n)}\big)\|f-u\cdot\nabla u\|_{L_1(0,t;\dot B^0_{n,1})}.$$

Since Lemma 4 also ensures that

$$\|u\cdot\nabla u\|_{L_1(0,t;\dot B^0_{n,1})} \leqslant C\|u\|_{L_\infty(0,t;\dot B^0_{n,1})}\|\nabla u\|_{L_1(0,t;\dot B^1_{n,1})} \tag{4.7}$$

we end up with

$$\begin{aligned} U_n(t) \leqslant C\big(&U_n(0) + \|\rho-1\|_{L_\infty(0,t;L_\infty\cap\dot W^1_n)}U_n(t) \\ &+ \big(1+\|\rho-1\|_{L_\infty(0,t;L_\infty\cap\dot W^1_n)}\big)\big(\|f\|_{L_1(0,t;\dot B^0_{n,1})} + \mu^{-1}U_n^2(t)\big)\big). \end{aligned}$$

Therefore, there exist two positive constants $c$ and $M$ depending only on $n$ such that if

$$\|\rho-1\|_{L_\infty(0,T;L_\infty\cap\dot W^1_n)} \leqslant c \quad \text{and} \quad U_n(T) \leqslant c\mu$$

then the above inequality implies that

$$U_n(t) \leqslant M\big(U_n(0) + \|f\|_{L_1(0,t;\dot B^0_{n,1})}\big) \quad \text{for all } t\in[0,T]. \tag{4.8}$$

Obviously, inequality (4.5) implies that the smallness condition for $\rho-1$ is satisfied on $[0,T]$ provided

$$\|\rho_0-1\|_{L_\infty\cap\dot W^1_n} \leqslant c/2 \quad \text{and} \quad \|\nabla u\|_{L_1(0,T;L^\infty)} \leqslant \log 2. \tag{4.9}$$

Because $\dot B^1_{n,1}(\mathbb{R}^n_+) \hookrightarrow L_\infty(\mathbb{R}^n_+)$ (see Proposition 4), the latter condition is satisfied provided $U_n(T) \leqslant c\mu$ with $c$ small enough. As the function $t \mapsto U_n(t)$ is continuous, combining inequality (4.8) with a standard bootstrap argument enables us to conclude that (4.8) and the second part of (4.9) are true for all $t\in[0,T]$ provided $\|u_0\|_{\dot B^0_{n,1}} + \|f\|_{L_1(\mathbb{R}_+;\dot B^0_{n,1})} < c\mu$ for a small enough positive constant $c$.

Let us now bound $U_p(t)$. By virtue of Lemma 4 and inequalities (4.4), (4.9), one can write

$$\big\|(1-\rho)\partial_t u\big\|_{L_1(0,t;\dot B^0_{p,1})} \leqslant C\big(\|\nabla\rho_0\|_{L_p}\|\partial_t u\|_{L_1(0,t;\dot B^0_{n,1})} + \|1-\rho_0\|_{L_\infty}\|\partial_t u\|_{L_1(0,t;\dot B^0_{p,1})}\big),$$
$$\big\|\rho(f-u\cdot\nabla u)\big\|_{L_1(0,t;\dot B^0_{p,1})} \leqslant C\big(\big(1+\|\rho_0-1\|_{L_\infty}\big)\|f-u\cdot\nabla u\|_{L_1(0,t;\dot B^0_{p,1})} + \|\nabla\rho_0\|_{L_p}\|f-u\cdot\nabla u\|_{L_1(0,t;\dot B^0_{n,1})}\big).$$

Now, since $\dot B^1_{n,1}(\mathbb{R}^n_+)\hookrightarrow L_\infty\cap\dot W^1_n(\mathbb{R}^n_+)$ and $\dot B^1_{p,1}(\mathbb{R}^n_+)\hookrightarrow\dot W^1_p(\mathbb{R}^n_+)$, Lemma 4 also yields

$$\|u\cdot\nabla u\|_{L_1(0,t;\dot B^0_{p,1})} \leqslant C\big(\|u\|_{L_\infty(0,t;\dot B^0_{n,1})}\|\nabla u\|_{L_1(0,t;\dot B^1_{p,1})} + \|u\|_{L_\infty(0,t;\dot B^0_{p,1})}\|\nabla u\|_{L_1(0,t;\dot B^1_{n,1})}\big) \leqslant C\mu^{-1}U_n(t)U_p(t).$$

Using also (4.7), we finally find that

$$U_p(t) \leqslant C\big(U_p(0) + U_n(t)\|\nabla\rho_0\|_{L_p} + \|\rho_0-1\|_{L_\infty}U_p(t) + \mu^{-1}U_n(t)U_p(t)\big(1+\|1-\rho_0\|_{L_\infty}\big) + \mu^{-1}U_n^2(t)\|\nabla\rho_0\|_{L_p} + \big(1+\|1-\rho_0\|_{L_\infty}\big)\|f\|_{L_1(0,t;\dot B^0_{p,1})} + \|\nabla\rho_0\|_{L_p}\|f\|_{L_1(0,t;\dot B^0_{n,1})}\big).$$

Because $\|1-\rho_0\|_{L_\infty}+\mu^{-1}U_n(t)$ is *small*, one can deduce (use (4.8)) that

$$U_p(t) \leqslant C\big(U_p(0) + \|f\|_{L_1(0,t;\dot B^0_{p,1})} + \|\nabla\rho_0\|_{L_p}\big(\|u_0\|_{\dot B^0_{n,1}} + \|f\|_{L_1(0,t;\dot B^0_{n,1})}\big)\big), \tag{4.10}$$

which implies inequality (4.3).

**Remark 6.** Let us stress the fact that since the constant $c$ may be computed from the constant involved in Lemma 4, it is *independent* of $p\in(n,\infty)$.

The proof of the existence part of Theorem 3 unfolds as follows:

- first, we construct a sequence of approximate solutions by solving a linear system;
- second, we prove uniform bounds for those approximate solutions;
- third, we prove convergence in low norm;
- last, we check that the limit is indeed a solution to (*INS*), and satisfies the required properties of regularity.

Throughout, we assume that condition (1.3) is satisfied and that, in addition, $\rho_0-1$ belongs to $W^1_p$, $u_0$ is in $\dot B^0_{p,1}$ and $f\in L_{1,\mathrm{loc}}(\mathbb{R}_+;\dot B^0_{p,1})$ for some $p\in(n,\infty)$.

1. *Construction of a sequence of approximate solutions*. Starting from $(\rho^0,u^0):=(1,0)$, one can solve inductively the following system of linear PDEs:

$$\begin{aligned}
&\partial_t \rho^{l+1} + u^l \cdot \nabla \rho^{l+1} = 0 \quad \text{in } (0,T) \times \mathbb{R}^n_+,\\
&\partial_t u^{l+1} - \mu \Delta u^{l+1} + \nabla \Pi^{l+1} = \rho^{l+1}\big(f - u^l \cdot \nabla u^l\big) + \big(1-\rho^{l+1}\big)\partial_t u^l \quad \text{in } (0,T) \times \mathbb{R}^n_+,\\
&\operatorname{div} u^{l+1} = 0 \quad \text{in } (0,T) \times \mathbb{R}^n_+,\\
&u^{l+1}|_{x_n=0} = 0 \quad \text{on } (0,T) \times \mathbb{R}^{n-1},\\
&u^{l+1}|_{t=0} = u_0, \qquad \rho^{l+1}|_{t=0} = \rho_0 \quad \text{on } \mathbb{R}^n_+.
\end{aligned} \tag{4.11}$$

Having $u^l$ in $E_n$ implies that $\nabla u^l \in L_1(\mathbb{R}_+; L_\infty)$ (see Proposition 4). Therefore the classical theory for transport equations provides a solution $\rho^{l+1}$ to $(4.11)_1$. Next, Theorem 2 enables us to solve the Stokes system $(4.11)_{2,3,4,5}$. Then an easy induction ensures that for all $l \in \mathbb{N}$, the above system has a global solution $(\rho^{l+1}, u^{l+1}, \nabla\Pi^{l+1})$ with

$$\big(\rho^{l+1} - 1\big) \in \mathcal{C}\big(\mathbb{R}_+; W^1_n \cap W^1_p\big),$$
$$u^{l+1} \in \mathcal{C}\big(\mathbb{R}_+; \dot B^0_{n,1} \cap \dot B^0_{p,1}\big) \quad \text{and} \quad \partial_t u^{l+1}, \nabla^2 u^{l+1}, \nabla\Pi^{l+1} \in L_{1,\mathrm{loc}}\big(\mathbb{R}_+; \dot B^0_{n,1} \cap \dot B^0_{p,1}\big).$$

2. *Uniform bounds in* $E_n \cap E_{p,\mathrm{loc}}$. Arguing exactly as in the first part of the proof, it is easy to see that if the smallness condition (1.3) is satisfied then $(\rho^l, u^l, \nabla\Pi^l)_{l\in\mathbb{N}}$ is bounded in $E_n \cap E_{p,\mathrm{loc}}$. Besides inequality (4.4) is satisfied and

$$\big\|u^l\big\|_{E_n} \leqslant C\big(\|u_0\|_{\dot B^0_{n,1}} + \|f\|_{L_1(\mathbb{R}_+; \dot B^0_{n,1})}\big), \tag{4.12}$$

$$\big\|u^l\big\|_{E_p(T)} \leqslant C\big(\|u_0\|_{\dot B^0_{p,1}} + \|f\|_{L_1(0,T; \dot B^0_{p,1})} + \mu\|\nabla\rho_0\|_{L_p}\big) \quad \text{for all } T>0. \tag{4.13}$$

3. *Convergence in low norm.* In order to complete the proof of existence, one has to show that $(\rho^l, u^l, \nabla\Pi^l)_{l\in\mathbb{N}}$ converges to some function $(\rho, u, \nabla\Pi)$ with the required properties of regularity. Owing to the hyperbolic nature of the equation for the density, it is not clear that convergence may be proved in the space $E_n \cap E_{p,\mathrm{loc}}$. Therefore, we shall show that convergence holds true in a *larger* space. More precisely, we claim that for all $T_0 > 0$,

$$\big(\rho^l - 1\big)_{l\geqslant 1} \quad \text{is a Cauchy sequence in } \mathcal{C}\big([0,T_0]; L_p\big),$$
$$\big(u^l - u^1\big)_{l\geqslant 1} \quad \text{is a Cauchy sequence in } L_\infty\big(0,T_0; \dot B^0_{\frac p2,\infty}\big) \cap \widetilde L_1\big(0,T_0; \dot B^2_{\frac p2,\infty}\big), \quad \text{and}$$
$$\big(\Pi^l - \Pi^1\big)_{l\geqslant 1} \quad \text{is a Cauchy sequence in } \widetilde L_1\big(0,T_0; \dot B^1_{\frac p2,\infty}\big).$$

In all that follows, we fix some positive time $T_0$. Let us first consider the density. Denoting $(\delta\rho^l_m, \delta u^l_m) := (\rho^{l+m} - \rho^l, u^{l+m} - u^l)$, we see that $\delta\rho^l_m$ satisfies

$$\partial_t \delta\rho^{l+1}_m + u^{l+m} \cdot \nabla \delta\rho^{l+1}_m = -\delta u^l_m \cdot \nabla\rho^l,$$

whence, because $\operatorname{div} u^{l+m} = 0$,

$$\big\|\delta\rho^{l+1}_m(t)\big\|_{L_p} \leqslant \int_0^t \big\|\delta u^l_m\big\|_{L_\infty} \big\|\nabla\rho^l\big\|_{L_p}\, d\tau.$$

Because $(\nabla\rho^l)_{l\in\mathbb{N}}$ is bounded according to (4.1), we thus have for all $t\in\mathbb{R}_+$,

$$\left\|\delta\rho_m^{l+1}(t)\right\|_{L_p}\leqslant 2\|\nabla\rho_0\|_{L_p}\int_0^t\left\|\delta u_m^l(\tau)\right\|_{L_\infty}d\tau. \tag{4.14}$$

Next, we notice that

$$\partial_t\delta u_m^{l+1}-\mu\Delta\delta u_m^{l+1}+\nabla\delta\Pi_m^{l+1}=R_m^l\quad\text{with }\delta\Pi_m^{l+1}:=\Pi^{l+1+m}-\Pi^{l+1}$$

and

$$\begin{aligned}R_m^l:=&-\delta\rho_m^{l+1}\left(f-\partial_t u^l-u^l\cdot\nabla u^l\right)-\rho^{l+1+m}\left(\delta u_m^l\cdot\nabla u^l+u^{l+m}\cdot\nabla\delta u_m^l\right)\\&+\left(1-\rho^{l+1+m}\right)\partial_t\delta u_m^l.\end{aligned}$$

By virtue of Lemma 10 from Section 3, for proving that, for all $l\geqslant 1$, we have

$$\begin{cases}\left(u^l-u^1\right)\in L_\infty\left(0,T_0;\dot B^0_{\frac p2,\infty}\left(\mathbb{R}^n_+\right)\right)\cap\widetilde L_1\left(0,T_0;\dot B^2_{\frac p2,\infty}\left(\mathbb{R}^n_+\right)\right),\\ \nabla\left(\Pi^l-\Pi^1\right)\in\widetilde L_1\left(0,T_0;\dot B^0_{\frac p2,\infty}\left(\mathbb{R}^n_+\right)\right),\end{cases} \tag{4.15}$$

it suffices to show that $R^1_{l-1}$ belongs to $\widetilde L_1(0,T_0;\dot B^0_{\frac p2,\infty})$. Actually, combining inequalities (4.12) and (4.13), Proposition 4 and interpolation, we discover that

- $(1-\rho^l)_{l\in\mathbb{N}}$ is bounded in $L_\infty(0,T_0;L_p)$,
- $(u^l)_{l\in\mathbb{N}}$ is bounded in $L_\infty(0,T_0;L_p)\cap L_2(0,T_0;L_\infty)$,
- $(\nabla u^l)_{l\in\mathbb{N}}$ is bounded in $L_2(0,T_0;L_p)$,
- $(\partial_t u^l)_{l\in\mathbb{N}}$ is bounded in $L_1(0,T_0;L_p)$.

Hence $R^1_{l-1}$ belongs to $L_1(0,T_0;L_{\frac p2})$ which is a subspace of $\widetilde L_1(0,T_0;\dot B^0_{\frac p2,\infty})$ by virtue of the following chain of embedding:

$$L_1(0,t;L_{\frac p2})\hookrightarrow L_1\left(0,t;\dot B^0_{\frac p2,\infty}\right)\hookrightarrow\widetilde L_1\left(0,t;\dot B^0_{\frac p2,\infty}\right). \tag{4.16}$$

Let $\delta U_m^l(t):=\|\partial_t\delta u_m^l,\mu\nabla^2\delta u_m^l,\nabla\delta\Pi_m^l\|_{\widetilde L_1(0,t;\dot B^0_{\frac p2,\infty})}+\|\delta u_m^l\|_{L_\infty(0,t;\dot B^0_{\frac p2,\infty})}$. Applying Proposition 10 to the equation satisfied by $\delta u_m^{l+1}$, we see that

$$\delta U_m^{l+1}(t)\leqslant C\left(A_m^l(t)+B_m^l(t)+C_m^l(t)+D_m^l(t)\right)$$

with

$$A^l_m(t) := \big\|\delta\rho^{l+1}_m\big(f-\partial_t u^l - u^l\cdot\nabla u^l\big)\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{p}{2},\infty})},$$
$$B^l_m(t) := \big\|\rho^{l+1+m}\delta u^l_m\cdot\nabla u^l\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{p}{2},\infty})},$$
$$C^l_m(t) := \big\|\rho^{l+1+m}u^{l+m}\cdot\nabla\delta u^l_m\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{p}{2},\infty})},$$
$$D^l_m(t) := \big\|\big(1-\rho^{l+1+m}\big)\partial_t\delta u^l_m\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{p}{2},\infty})}.$$

Bounding $A^l_m$ is easy. Indeed, thanks to (4.16) and Hölder's inequality, we have

$$A^l_m(t)\leqslant C\int_0^t \big\|\delta\rho^{l+1}_m\big\|_{L_p}\big\|f-\partial_t u^l-u^l\cdot\nabla u^l\big\|_{L_p}\,d\tau. \tag{4.17}$$

Next, by virtue of Lemma 5 and Proposition 4, there exists a constant $C$ depending continuously on $(p,n)$ such that

$$B^l_m(t)\leqslant C\int_0^t \big\|\rho^{l+1+m}\nabla u^l\big\|_{\dot{W}^1_n\cap L_\infty}\big\|\delta u^l_m\big\|_{\dot{B}^0_{\frac{p}{2},\infty}}\,d\tau.$$

The uniform bounds of the previous step and the smallness condition (1.3) guarantee that

$$B^l_m(t)\leqslant Cc\big\|\delta u^l_m\big\|_{L_\infty(0,t;\dot{B}^0_{\frac{p}{2},\infty})}. \tag{4.18}$$

Similarly, Lemma 5 and Proposition 4 yield

$$C^l_m(t)\leqslant C\big\|\rho^{l+1+m}u^{l+m}\big\|_{L_2(0,t;\dot{W}^1_n\cap L_\infty)}\big\|\nabla\delta u^l_m\big\|_{\widetilde{L}_2(0,t;\dot{B}^0_{\frac{p}{2},\infty})}.$$

Lemma 4 combined with an obvious embedding yields

$$\big\|\rho^{l+1+m}u^{l+m}\big\|_{L_2(0,t;\dot{W}^1_n\cap L_\infty)}\leqslant C\big\|\rho^{l+1+m}\big\|_{L_\infty(0,t;L_\infty\cap\dot{W}^1_n)}\big\|u^{l+m}\big\|_{L_2(0,t;\dot{B}^1_{n,1})}.$$

Hence, using condition (1.3), inequality (4.12) and interpolation, we gather that

$$C^l_m(t)\leqslant Cc\mu^{\frac12}\big\|\delta u^l_m\big\|_{\widetilde{L}_2(0,t;\dot{B}^1_{\frac{p}{2},\infty})}\leqslant Cc\,\delta U^l_m(t). \tag{4.19}$$

Similar arguments lead to

$$D^t_l(t)\leqslant\big\|1-\rho^{l+1+m}\big\|_{L_\infty(0,t;L_\infty\cap\dot{W}^1_n)}\big\|\partial_t\delta u^l_m\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{p}{2},\infty})}\leqslant Cc\big\|\partial_t\delta u^l_m\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{p}{2},\infty})}. \tag{4.20}$$

So finally, putting together inequalities (4.17)–(4.20) and taking $c$ smaller if needed, we get

$$\delta U_m^{l+1}(t) \leqslant \frac{1}{2}\delta U_m^l(t) + C\int_0^t \big\|\big(f-\partial_t u^l - u^l\cdot\nabla u^l\big)(\tau)\big\|_{L_p}\big\|\delta\rho_m^{l+1}(\tau)\big\|_{L_p}\,d\tau$$

for all $t\in[0,T_0]$.

Plugging inequality (4.14) in the above inequality, we end up with

$$\delta U_m^{l+1}(t) \leqslant \frac{1}{2}\delta U_m^l(t) + C\|\nabla\rho_0\|_{L^p}\int_0^t \big\|\big(f-\partial_t u^l - u^l\cdot\nabla u^l\big)(\tau)\big\|_{L_p}\big\|\delta u_m^l\big\|_{L_1(0,\tau;L_\infty)}\,d\tau. \tag{4.21}$$

Let us admit for a while that there exists some constant $C$ such that

$$\big\|\delta u_m^l\big\|_{L_{p/n}(0,T;L_\infty)} \leqslant C\delta U_m^l(T)\quad \text{for all } T\in[0,T_0]. \tag{4.22}$$

Then inserting inequality (4.22) in inequality (4.21), we see that for all $T\in[0,T_0]$,

$$\delta U_m^{l+1}(T) \leqslant \frac{1}{2}\delta U_m^l(T) + CT^{1-\frac{n}{p}}\|\nabla\rho_0\|_{L_p}\big\|f-\partial_t u^l - u^l\cdot\nabla u^l\big\|_{L_1(0,T;L_p)}\delta U_m^l(T).$$

Now, because $f-\partial_t u^l - u^l\cdot\nabla u^l$ is bounded in $L_1(0,T_0;L_p)$, we see that if $T$ has been chosen small enough then

$$\delta U_m^{l+1}(T) \leqslant \frac{3}{4}\delta U_m^l(T),$$

so that $(u^l-u^1)_{l\in\mathbb{N}}$ and $(\Pi^l-\Pi^1)_{l\in\mathbb{N}}$ are Cauchy sequences in the desired space restricted to interval $[0,T]$. Let us emphasize that, according to inequality (4.10), the smallness of $T$ depends only on the magnitude of the data and on $T_0$. Therefore, starting from time $T$, the above arguments can be used again to show that $(u^l-u^1)_{l\in\mathbb{N}}$ and $(\Pi^l-\Pi^1)_{l\in\mathbb{N}}$ are also Cauchy sequences in the desired space restricted to interval $[T,2T]$, and so on, until the whole interval $[0,T_0]$ is exhausted.

In order to justify inequality (4.22), one may use the fact that $z:=\delta u_m^l$ satisfies $z=\sum_{k<0}\Delta_k z+\sum_{k\geqslant 0}\Delta_k z$. Hence

$$\begin{aligned}
\|z\|_{L_{p/n}(0,T;L_\infty)} &\leqslant \sum_{k<0}\|\Delta_k z\|_{L_{p/n}(0,T;L_\infty)} + \sum_{k\geqslant 0}\|\Delta_k z\|_{L_{p/n}(0,T;L_\infty)}\\
&\leqslant \sum_{k<0}2^{-\frac{2kn}{p}}\|\Delta_k z\|_{L_{p/n}(0,T;L_\infty)} + \sum_{k\geqslant 0}\|\Delta_k z\|_{L_1(0,T;L_\infty)}^{\frac{n}{p}}\|\Delta_k z\|_{L_\infty(0,T;L_\infty)}^{1-\frac{n}{p}}\\
&\leqslant T^{\frac{n}{p}}\sum_{k<0}2^{-\frac{2kn}{p}}\|\Delta_k z\|_{L_\infty(0,T;L_\infty)}\\
&\quad + \sum_{k\geqslant 0}\big(2^{k(2-\frac{2n}{p})}\|\Delta_k z\|_{L_1(0,T;L_\infty)}\big)^{\frac{n}{p}}\big(2^{-\frac{2nk}{p}}\|\Delta_k z\|_{L_\infty(0,T;L_\infty)}\big)^{1-\frac{n}{p}}
\end{aligned}$$

$$\leqslant T^{\frac{n}{p}}\|z\|_{L_\infty(0,T;\dot B^{-\frac{2n}{p}}_{\infty,\infty})} + \|z\|^{1-\frac{n}{p}}_{L_\infty(0,T;\dot B^{-\frac{2n}{p}}_{\infty,\infty})}\|z\|^{\frac{n}{p}}_{L_1(0,T;\dot B^{2-\frac{2n}{p}}_{\infty,\infty})}.$$

Now, Proposition 4 ensures that

$$\widetilde L_1\big(0,T;\dot B^2_{\frac{p}{2},\infty}\big)\hookrightarrow \widetilde L_1\big(0,T;\dot B^{2-\frac{2n}{p}}_{\infty,\infty}\big)\quad\text{and}\quad L_\infty\big(0,T;\dot B^0_{\frac{p}{2},\infty}\big)\hookrightarrow L_\infty\big(0,T;\dot B^{-\frac{2n}{p}}_{\infty,\infty}\big).$$

So we get inequality (4.22).

4. *Regularity of the solution.* Let $\rho$ be the limit of $(\rho^l)_{l\in\mathbb N}$, and $u := \bar u + u^1$ (resp. $\Pi := \overline{\Pi} + \Pi^1$) where $\bar u$ (resp. $\overline{\Pi}$) stands for the limit of $(u^l - u^1)_{l\in\mathbb N}$ (resp. $(\Pi^l - \Pi^1)_{l\in\mathbb N}$). Interpolating the bounds of step 2 with the results of convergence of step 3, it is easy to check that $(\rho, u, \nabla\Pi)$ is a global solution to (*INS*). In addition, the compactness properties for the weak $*$ topology guarantee that $\rho - 1$ (resp. $u$) is in $L_\infty(\mathbb R_+; W^1_n\cap W^1_p)$ (resp. $L_\infty(\mathbb R_+;\dot B^0_{n,1}\cap\dot B^0_{p,1})$) and satisfies the desired inequalities.

In order to show that $\nabla^2 u\in L_1(\mathbb R_+;\dot B^0_{n,1}(\mathbb R^n_+))$, one can proceed as follows. First, we extend the terms $\nabla^2 u^l$ (resp. $\nabla^2 u$) by 0 on the whole space. Denoting this extension by $\widetilde v^l$ (resp. $\widetilde v$), we thus have according to Propositions 3 and 7,

$$\widetilde v^l - \widetilde v^1 \to \widetilde v - \widetilde v^1\quad\text{in }\widetilde L_1\big(\mathbb R_+;\dot B^0_{\frac{p}{2},\infty}(\mathbb R^n)\big). \tag{4.23}$$

Introduce the spectral cut-off operator $E_k := \sum_{|j|\leqslant k}\Delta_j$. It is easy to check that $E_k\widetilde v^l$ satisfies exactly the same inequalities as $\nabla^2 u^l$ (up to an irrelevant constant). In addition, owing to the spectral localization of $E_k\widetilde v^l$, the following Bernstein inequality[5]

$$\big\|\Delta_j\big(\widetilde v^l-\widetilde v\big)\big\|_{L_n}\leqslant C2^{j(\frac{2n}{p}-1)}\big\|\Delta_j\big(\widetilde v^l-\widetilde v\big)\big\|_{L_{\frac{p}{2}}}$$

associated with (4.23) implies that for fixed $k\in\mathbb N$,

$$E_k\widetilde v^l\to E_k\widetilde v\quad\text{in }L_1\big(\mathbb R_+;\dot B^0_{n,1}\big(\mathbb R^n\big)\big). \tag{4.24}$$

On the other hand for all $k,l\in\mathbb N$

$$\mu\big\|E_k\widetilde v^l\big\|_{L_1(\mathbb R_+;\dot B^0_{n,1}(\mathbb R^n))}\leqslant C\big(\|u_0\|_{\dot B^0_{n,1}(\mathbb R^n_+)}+\|f\|_{L_1(\mathbb R_+;\dot B^0_{n,1}(\mathbb R^n_+))}\big)$$

with $C$ independent of $k$ and $l$.

Therefore by (4.24) we have for all $k\in\mathbb N$,

$$\mu\|E_k\widetilde v\|_{L_1(\mathbb R_+;\dot B^0_{n,1}(\mathbb R^n))}\leqslant C\big(\|u_0\|_{\dot B^0_{n,1}(\mathbb R^n_+)}+\|f\|_{L_1(\mathbb R_+;\dot B^0_{n,1}(\mathbb R^n_+))}\big). \tag{4.25}$$

So finally, using the definition of the norm in $\dot B^0_{n,1}(\mathbb R^n)$, and (4.25), one may write

[5] Note that one can assume here that $p/2\leqslant n$ taking $p$ smaller in step 3 if need be.

$$\|\widetilde{v}\|_{L_1(\mathbb{R}_+;\dot{B}^0_{n,1}(\mathbb{R}^n))} = \lim_{k\to+\infty}\sum_{|l|\leqslant k}\|\Delta_l\widetilde{v}\|_{L_1(\mathbb{R}_+;L_p)} \leqslant C\big(\|u_0\|_{\dot{B}^0_{n,1}(\mathbb{R}^n_+)} + \|f\|_{L_1(\mathbb{R}_+;\dot{B}^0_{n,1}(\mathbb{R}^n_+))}\big).$$

Hence, one can conclude that $\widetilde{v}\in L_1(\mathbb{R}_+;\dot{B}^0_{n,1}(\mathbb{R}^n))$, and thus $\nabla^2 u\in L_1(\mathbb{R}_+;\dot{B}^0_{n,1}(\mathbb{R}^n_+))$.

Of course, the same method works for proving that $\nabla^2 u\in L_{1,\mathrm{loc}}(\mathbb{R}_+;\dot{B}^0_{p,1})$ and that $\partial_t u$ and $\nabla\Pi$ have the desired regularity. Finally, because we have $\nabla u\in L_1(\mathbb{R}_+;L_\infty)$, continuity with respect to time for the density stems from standard properties for the transport equation. Continuity for the velocity follows from Theorem 2.

Uniqueness may be justified from arguments similar to those which have been used for showing that $(\rho^l,u^l,\nabla\Pi^l)$ is a Cauchy sequence. We do not give any details since a more general result will be proved below in Proposition 8. □

### 4.2. *The proof of uniqueness*

The following statement implies uniqueness in Theorem 1.

**Proposition 8.** *Let $(\rho_1,u_1,\nabla\Pi_1)$ and $(\rho_2,u_2,\nabla\Pi_2)$ solve system (INS) on $[0,T]\times\mathbb{R}^n_+$ with the same data. Assume that $n\geqslant 3$ and that for $i=1,2$ we have*

$$(\rho_i-1)\in L_\infty\big(0,T;L_\infty\cap W^1_n\big)\quad \textit{and}\quad u_i\in L_1\big(0,T;\dot{B}^2_{n,1}\big)\cap\mathcal{C}\big([0,T];\dot{B}^0_{n,1}\big).$$

*There exists a constant $c=c(n)$ such that if in addition*

$$\|\rho_1-1\|_{L_\infty(0,T\times\mathbb{R}^n_+)} + \|\nabla\rho_1\|_{L_\infty(0,T;L_n)} \leqslant c, \tag{4.26}$$

*then $(\rho_2,u_2,\nabla\Pi_2)\equiv(\rho_1,u_1,\nabla\Pi_1)$ on $[0,T]\times\mathbb{R}^n_+$.*

**Proof.** The system satisfied by the difference

$$(\delta\rho,\delta u,\delta\Pi) := (\rho_2-\rho_1,u_2-u_1,\Pi_2-\Pi_1)$$

between the two solutions reads

$$\begin{aligned}
\partial_t\delta\rho + u_1\cdot\nabla\delta\rho &= -\delta u\cdot\nabla\rho_2,\\
\partial_t\delta u - \mu\Delta\delta u + \nabla\delta\Pi &= \big(f-(\partial_t+u_2\cdot\nabla)u_2\big)\delta\rho - \rho_1 u_1\cdot\nabla\delta u\\
&\quad - \delta u\cdot(\rho_1\nabla u_2) + (1-\rho_1)\partial_t\delta u,\\
\operatorname{div}\delta u &= 0.
\end{aligned}\tag{4.27}$$

Note that the right-hand side of $(4.27)_1$ is at most in $L_\infty(0,T;L_n)$ no matter how smooth $\delta u$ is. Therefore, we shall estimate $\delta\rho$ in $L_\infty(0,T;L_n)$. Owing to the coupling between the two equations, this will induce also a loss in the stability estimates for the velocity. For instance, by using the fact that $\partial_t u_2\in L_1(0,T;\dot{B}^0_{n,1})$, $u_2\in L_\infty(0,T;\dot{B}^0_{n,1})$, $\nabla u_2\in L_1(0,T;\dot{B}^1_{n,1})$ and that

$$\dot{B}^0_{n,1}\hookrightarrow L_n \quad\text{and}\quad \dot{B}^1_{n,1}\hookrightarrow L_\infty,$$

we readily see that $(f-(\partial_t+u_2\cdot\nabla)u_2)\in L_1(0,T;L_n)$[6] so that the first term of $(4.27)_2$ may be estimated in $L_1(0,T;L_{\frac{n}{2}})$ if a bound on $\|\delta\rho\|_{L_1(0,t;L_n)}$ is available. However, the solution to the evolutionary Stokes equations with the right-hand side in $L_1(0,T;L_{\frac{n}{2}})$ fails to have its first-order time derivative in $L_1(0,T;L_{\frac{n}{2}})$. Actually, according to Lemma 10, it belongs to the slightly larger space $\widetilde{L}_1(0,T;\dot{B}^0_{\frac{n}{2},\infty})$, that we shall use for bounding $\delta u$.

Let us now tackle the proof of uniqueness. Bounding $\delta\rho$ in $L_\infty(0,T;L_n)$ is straightforward. Indeed, since $\operatorname{div} u_1=0$, one can write for all $t\in[0,T]$:

$$\|\delta\rho(t)\|_{L_n}\leqslant\int_0^t\|\nabla\rho_2\|_{L_n}\|\delta u\|_{L^\infty}\,d\tau\leqslant\|\nabla\rho_2\|_{L_\infty(0,t;L_n)}\|\delta u\|_{L_1(0,t;L_\infty)}. \tag{4.28}$$

Let us now check that $\delta u$ is in $L_\infty(0,T;\dot{B}^0_{\frac{n}{2},\infty})$ and that $\partial_t\delta u,\nabla^2\delta u,\nabla\delta\Pi\in\widetilde{L}_1(0,T;\dot{B}^0_{\frac{n}{2},\infty})$. We claim that the right-hand side of $(4.27)_2$ belongs to $L_1(0,T;L_{\frac{n}{2}})$ which is a subspace of $\widetilde{L}_1(0,T;\dot{B}^0_{\frac{n}{2},\infty})$ according to the following chain of embeddings:

$$L_1(0,t;L_{\frac{n}{2}})\hookrightarrow L_1\big(0,t;\dot{B}^0_{\frac{n}{2},\infty}\big)\hookrightarrow\widetilde{L}_1\big(0,t;\dot{B}^0_{\frac{n}{2},\infty}\big).$$

Since $\delta u|_{t=0}=0$, Lemma 10 will entail that $\delta u$ and $\nabla\delta\Pi$ have the required regularity.

We have already seen that $(f-(\partial_t+u_2\cdot\nabla)u_2)\delta\rho$ is in $L_1([0,T];L_{\frac{n}{2}})$. Next, because $\nabla u_1$ and $\nabla u_2$ are in $L_2(0,T;\dot{B}^0_{n,1})$ (argue by interpolation) we gather that $\nabla\delta u$ is in $L_2(0,T;L_n)$. Since $\rho_1u_1\in L_\infty(0,T;L_n)$ this implies that $\rho_1u_1\cdot\nabla\delta u$ is in $L_2(0,T;L_{\frac{n}{2}})$. Similar arguments yield $\rho_1\delta u\cdot\nabla u_2\in L_2(0,T;L_{\frac{n}{2}})$. Finally, because $(1-\rho_1)\in L_\infty(0,T;L_n)$ and $\partial_tu_1\in L_1(0,T;L_n)$, the last term in the right-hand side of the equation for $\delta u$ is also in $L_1(0,T;L_{\frac{n}{2}})$.

Now, bounding $\delta u$ relies on Lemma 10. Denoting

$$\delta U(t):=\|\partial_t\delta u\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{n}{2},\infty})}+\mu\|\delta u\|_{\widetilde{L}_1(0,t;\dot{B}^2_{\frac{n}{2},\infty})}+\|\delta u\|_{L_\infty(0,t;\dot{B}^0_{\frac{n}{2},\infty})}+\|\nabla\delta\Pi\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{n}{2},\infty})},$$

we get for some constant $C$ depending only on $n$,

$$\delta U(t)\leqslant C\big(\delta U_1(t)+\delta U_2(t)+\delta U_3(t)+\delta U_4(t)\big) \tag{4.29}$$

with

$$\delta U_1(t):=\big\|\delta\rho\big(f-(\partial_t+u_2\cdot\nabla)u_2\big)\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{n}{2},\infty})},\qquad\delta U_2(t):=\|\rho_1u_1\cdot\nabla\delta u\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{n}{2},\infty})},$$

$$\delta U_3(t):=\big\|\delta u\cdot(\rho_1\nabla u_2)\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{n}{2},\infty})},\qquad\delta U_4(t):=\big\|(1-\rho_1)\partial_t\delta u\big\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{n}{2},\infty})}.$$

For bounding $\delta U_1(t)$, we proceed as explained above. We get for all $t\in[0,T]$,

[6] In fact, this term belongs to $L_1(0,T;\dot{B}^0_{n,1})$ but this does not help in what follows.

$$\left\|\delta U_1(t)\right\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{n}{2},\infty})} \leqslant C\int_0^t \left\|f-(\partial_t+u_2\cdot\nabla)u_2\right\|_{L_n}\|\delta\rho\|_{L_n}\,d\tau.$$

Therefore, there exists some integrable function $V$ over $[0,T]$ such that

$$\delta U_1(t)\leqslant\int_0^t V(\tau)\left\|\delta\rho(\tau)\right\|_{L_n}d\tau. \tag{4.30}$$

In order to bound the other terms, one may resort to Lemma 5. Indeed, for $\delta U_2(t)$, it suffices to apply this lemma with $F=\nabla\delta u$, $G=\rho_1 u_1$ and $r_1=r_2=2$. This is indeed possible since having $u_1\in L_1(0,T;\dot{B}^2_{n,1})\cap L_\infty(0,T;\dot{B}^0_{n,1})$ implies (by interpolation) that $u_1\in L_2(0,T;\dot{B}^1_{n,1})$, whence $u_1\in L_2(0,T;L^\infty\cap\dot{W}^1_n)$ by embedding. Now, $\rho_1$ is in $L_\infty(0,T;\dot{W}^1_n\cap L_\infty)$ and $\dot{W}^1_n\cap L_\infty$ is an algebra, so $\rho_1u_1$ does belong to $L_2(0,T;\dot{W}^1_n\cap L_\infty)$. One can thus write that

$$\delta U_2(t)\leqslant A(t)\|\nabla\delta u\|_{\widetilde{L}_2(0,t;\dot{B}^0_{\frac{n}{2},\infty})}$$

for some bounded function $A:[0,T]\to\mathbb{R}_+$ such that $\lim_{t\to0}A(t)=0$. From straightforward interpolation arguments, we see that $\|\nabla\delta u\|_{\widetilde{L}_2(0,t;\dot{B}^0_{\frac{n}{2},\infty})}\leqslant\mu^{-\frac{1}{2}}\delta U(t)$. Hence, up to a change of the function $A$,

$$\delta U_2(t)\leqslant A(t)\delta U(t). \tag{4.31}$$

Similar arguments lead to

$$\delta U_3(t)\leqslant C\|\rho_1\nabla u_2\|_{L_1(0,t;L^\infty\cap\dot{W}^1_n)}\|\delta u\|_{L_\infty(0,t;\dot{B}^0_{\frac{n}{2},\infty})},$$

whence, as $\rho_1\nabla u_2\in L_1(0,T;L_\infty\cap W^1_n)$,

$$\delta U_3(t)\leqslant A(t)\delta U(t). \tag{4.32}$$

Finally, we have

$$\delta U_4(t)\leqslant C\|1-\rho_1\|_{L_\infty(0,t;L_\infty\cap\dot{W}^1_n)}\|\partial_t\delta u\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\frac{n}{2},\infty})}. \tag{4.33}$$

So plugging inequalities (4.30) to (4.33) in (4.29) and using the smallness condition (4.26) and the fact that $\lim_{t\to0}A(t)=0$, we get

$$\delta U(t)\leqslant\int_0^t V(\tau)\left\|\delta\rho(\tau)\right\|_{L_n}d\tau \tag{4.34}$$

for all $t$ in a small enough interval $[0,T_0]$.

At this stage, we are in trouble for bounding $\delta\rho$ through inequality (4.28) requires an $L_1(0,t;L_\infty)$ control of $\delta u$ which *is not* given by $\delta U(t)$. In order to overcome this, one may use the following logarithmic interpolation estimate, proved in Appendix A (see Proposition 9):

$$\|\delta u\|_{L_1(0,t;L_\infty)} \leqslant C\|\delta u\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\infty,\infty})} \log\left(e + \frac{\|\delta u\|_{L_1(0,t;L_n)} + \|\nabla\delta u\|_{L_1(0,t;L_\infty)}}{\|\delta u\|_{\widetilde{L}_1(0,t;\dot{B}^0_{\infty,\infty})}}\right). \tag{4.35}$$

Let us notice that, due to $u_i \in L_\infty(0,T;\dot{B}^0_{n,1}) \cap L_1(0,T;\dot{B}^2_{n,1})$, and to the embedding

$$\dot{B}^0_{n,1} \hookrightarrow L_n \quad \text{and} \quad \dot{B}^2_{n,1} \hookrightarrow \dot{W}^1_\infty,$$

the numerator is a bounded function over $[0,T]$. Remark also that $\widetilde{L}_1(0,t;\dot{B}^2_{\frac{n}{2},\infty}) \hookrightarrow \widetilde{L}_1(0,t;\dot{B}^0_{\infty,\infty})$ (see Proposition 4). So finally, inserting inequality (4.35) in (4.28), using that for all $\alpha > 0$, the map $r \mapsto r\log(e+\alpha/r)$ is nondecreasing, then coming back to (4.34), we conclude that for some positive constant $C$ and integrable function $V$ we have

$$\delta U(t) \leqslant \int_0^t V(\tau)\delta U(\tau)\log\left(e + \frac{C}{\delta U(\tau)}\right)d\tau.$$

Osgood's lemma thus guarantees that $\delta U \equiv 0$ on $[0,T_0]$.

Note that the above proof works if we start from any time $t_0 \in [0,T]$ such that $u_2(t_0) = u_1(t_0)$. Therefore $\{t \in [0,T]\colon\ u_2(t) = u_1(t)\}$ is a nonempty open set of $[0,T]$. Let us also notice that if $(t_n)_{n\in\mathbb{N}}$ is a sequence of $[0,T]$ such that $u_2(t_n) = u_1(t_n)$ for all $n \in \mathbb{N}$ then, due to the fact that $u_1$ and $u_2$ are *continuous* with values in $\dot{B}^1_{n,1}$, we also have $u_2(t) = u_1(t)$. Hence the above set is also closed, and one can conclude that it equals $[0,T]$. □

### *4.3. The proof of existence in the critical case*

In order to prove the existence part of Theorem 1, we shall proceed as follows:

- first, we solve system (*INS*) for mollified data (taking advantage of Theorem 3) and state uniform bounds;
- second, we resort to compactness arguments in order to pass to the limit;
- third, we check that the limit is indeed a solution;
- last, we prove that it has the desired properties of regularity.

1. *Uniform bounds for the solution with smoothed out data*. Fix some $p > n$. From Proposition 3 and Remark 5, one can construct a sequence $(\rho_0^l, u_0^l, f^l)$ in[7]

$$\left(W^1_n \cap W^1_p\right) \times \left(\dot{B}^0_{n,1} \cap \dot{B}^0_{p,1}\right)^n \cap \left(L_1\left(\mathbb{R}_+;\dot{B}^0_{n,1} \cap \dot{B}^0_{p,1}\right)\right)^n$$

tending strongly to $(\rho_0, u_0, f)$ in

[7] For the density, consider the symmetric extension and use the fact that the set of smooth compactly supported functions in $\mathbb{R}^n$ is dense in $W^1_n(\mathbb{R}^n)$.

$$W^1_n \times \big(\dot B^0_{n,1}\big)^n \cap \big(L_1\big(\mathbb{R}_+;\dot B^0_{n,1}\big)\big)^n.$$

According to Theorem 3, system (*INS*) has a unique global solution $(\rho^l, u^l, \nabla\Pi^l)$ in $E_n \cap E_{p,\mathrm{loc}}$ satisfying in addition for all $t \geqslant 0$,

$$\begin{gathered}\big\|\rho^l(t)-1\big\|_{L_\infty} = \big\|\rho^l_0-1\big\|_{L_\infty}, \qquad \big\|\nabla\rho^l(t)\big\|_{L_n} \leqslant 2\big\|\nabla\rho^l_0\big\|_{L_n},\\ \big\|\nabla\Pi^l, \partial_t u^l, \mu\nabla^2 u^l\big\|_{L_1(0,t;\dot B^0_{n,1})} + \big\|u^l\big\|_{L_\infty(0,t;\dot B^0_{n,1})} \leqslant M\big(\big\|u^l_0\big\|_{\dot B^0_{n,1}} + \big\|f^l\big\|_{L_1(0,t;\dot B^0_{n,1})}\big).\end{gathered} \tag{4.36}$$

Note that all the terms of the right-hand sides may be bounded independently of $l$. Therefore $(\rho^l, u^l, \nabla\Pi^l)$ is bounded in the space $E_n$.

2. *Compactness.* We claim that sequence $(\rho^l, u^l, \nabla\Pi^l)_{l\in\mathbb{N}}$ converges weakly (up to an omitted extraction) to some distribution $(\rho, u, \nabla\Pi)$.

Compactness for the density stems from the fact that $(\partial_t\rho^l)_{l\in\mathbb{N}}$ is bounded in the space $L_2(\mathbb{R}_+; L_n)$. Indeed, we have $\partial_t\rho^l = -u^l\cdot\nabla\rho^l$ and we know that $(\nabla\rho^l)_{l\in\mathbb{N}}$ is bounded in $L_\infty(\mathbb{R}_+; L_n)$ and that (by interpolation and embedding) $(u^l)_{l\in\mathbb{N}}$ is bounded in $L_2(\mathbb{R}_+; L_\infty)$. Therefore $(\rho^l-1)_{l\in\mathbb{N}}$ is bounded in $\mathcal{C}^{\frac12}([0,T]; L_n)$ for all $T>0$. Now, because $(\rho^l-1)_{l\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}_+; W^1_n)$ and the embedding of $W^1_n$ in $L_n$ is locally compact, Ascoli's theorem combined with Cantor diagonal process ensures that, up to extraction, sequence $(\rho^l)_{l\in\mathbb{N}}$ tends to some function $\rho$ in $\mathcal{C}(\mathbb{R}_+; L_{n,\mathrm{loc}})$. From (4.36) and interpolation, one can thus conclude that

$$(\rho-1)\in L_\infty\big(\mathbb{R}_+; L_\infty\cap W^1_n\big) \quad \text{and satisfies (1.4)} \tag{4.37}$$

and that, up to an omitted extraction,

$$\lim_{l\to+\infty}\rho^l = \rho \quad \text{in } L_{\infty,\mathrm{loc}}\big(\mathbb{R}_+; W^s_{n,\mathrm{loc}}\big) \text{ for all } s\in[0,1). \tag{4.38}$$

Let us now turn to the study of the velocity. The important fact is that (4.36) implies that $(u^l)_{l\in\mathbb{N}}$ is bounded in $L_{1,\mathrm{loc}}(\mathbb{R}_+; W^2_n)$ and that $(\partial_t u^l)_{l\in\mathbb{N}}$ is bounded in $L_{1,\mathrm{loc}}(\mathbb{R}_+; L_n)$. This implies that $(u^l)_{l\in\mathbb{N}}$ is bounded in the space $W^1_{1,\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}^n_+)$ which is compactly embedded in $L_{p,\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}^n_+)$ for any $p<(n+1)/n$. Hence we gather that, up to extraction,

$$\lim_{l\to+\infty} u^l = u \quad \text{in } L_{p,\mathrm{loc}}\big(\mathbb{R}_+\times\mathbb{R}^n_+\big) \text{ for any } p<(n+1)/n. \tag{4.39}$$

Of course, the divergence free condition is preserved.

Finally, because $(\nabla\Pi^l)_{l\in\mathbb{N}}$ is bounded in $L_1(\mathbb{R}_+; L_n)$, there exists some distribution $Q$ such that, up to extraction, $(\nabla\Pi^l)_{l\in\mathbb{N}}$ tends weakly to $Q$. Of course, $Q$ is the gradient of some distribution $\Pi$.

3. *Passing to the limit in* (*INS*). Let us first show that $(\rho, u, \nabla\Pi)$ satisfies the conservative formulation of (*INS*).[8] Passing to the limit in the linear terms is straightforward. In order to pass to the limit in the nonlinear terms, we shall first state some properties of *strong* convergence.

[8] That is $u\cdot\nabla\rho$ is replaced by $\mathrm{div}(\rho u)$, and $\rho(\partial_t u + u\cdot\nabla u)$, by $\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u)$.

From step 1 and interpolation, we know that $(u^l)_{l\in\mathbb{N}}$ is bounded in $L_\infty(\mathbb{R}_+;L_n)\cap L_2(\mathbb{R}_+;L_\infty)$, hence, using the weak $*$ compactness properties of those spaces, we gather that

$$u^l \rightharpoonup u \text{ weak } * \quad \text{in } L_\infty\big(\mathbb{R}_+;L_n\big)\cap L_2\big(\mathbb{R}_+;L_\infty\big). \tag{4.40}$$

Interpolating with (4.39), we thus get

$$u^l \to u \quad \text{strongly in } L_{p,\mathrm{loc}}\big(\mathbb{R}_+\times\mathbb{R}^n_+\big) \text{ for some } p>2. \tag{4.41}$$

Since the density converges strongly in $L_\infty(0,T;L_{n,\mathrm{loc}})$, this is enough to pass to the weak limit in the terms $\mathrm{div}(\rho^l u^l)$, $\partial_t(\rho^l u^l)$ and $\mathrm{div}(\rho^l u^l\otimes u^l)$.

Finally, the properties of regularity stated hitherto for $(\rho,u)$ are enough to show that $(\rho,u,\nabla\Pi)$ satisfies (*INS*). For instance, having $(\rho-1)\in\mathcal{C}(\mathbb{R}_+;L_{p,\mathrm{loc}})$ for all $p<\infty$, $\partial_t\rho\in L^2(\mathbb{R}_+;L_n)$ and $(\partial_t u^l)_{l\in\mathbb{N}}$ bounded in $L_1(\mathbb{R}_+;L_n)$ ensures that $\partial_t(\rho u)=\rho\partial_t u+u\partial_t\rho$ in the sense of distributions. Similar computations may be done for the other nonlinear terms.

4. *Regularity.* The bounds of step 1 combined with interpolation guarantee that $(u^l)_{l\in\mathbb{N}}$ is bounded in $L_r(\mathbb{R}_+;\dot B^{\frac{2}{r}}_{n,1})$ for all $r\geqslant 1$. Using the Fatou properties for those spaces *in the case* $r>1$ ensures that $u\in L_r(\mathbb{R}_+;\dot B^{\frac{2}{r}}_{n,1})$. We now want to show that

$$\nabla^2 u \quad \text{is in } L_1\big(\mathbb{R}_+;\dot B^0_{n,1}\big) \text{ and satisfies the bounds of step 1.} \tag{4.42}$$

Starting from the fact that $\nabla^2 u^l$ tends weakly to $\nabla^2 u$ and that functions $\nabla^2 u^l$ and $\nabla^2 u$ are in (say) $L_{5/4}(\mathbb{R}_+;\dot B^{-\frac{2}{5}}_{n,1}(\mathbb{R}^n_+))$, we extend $\nabla^2 u^l$ and $\nabla^2 u$ by 0 on the whole space. Since $1/n-1<-2/5<1/n$, Proposition 3 guarantees that the corresponding extensions $\widetilde{v}^l$ and $\widetilde{v}$ are in $L_{5/4}(\mathbb{R}_+;\dot B^{-\frac{2}{5}}_{n,1}(\mathbb{R}^n))$ and, obviously, we still have $\widetilde{v}^l\rightharpoonup\widetilde{v}$ in the weak sense.

Now, using the spectral truncation operator $E_k$ defined just above (4.23) and Bernstein inequality, one can assert that for any fixed $k\in\mathbb{N}$, sequence $(E_k\widetilde{v}^l)_{l\in\mathbb{N}}$ is bounded in $L_{5/4}(\mathbb{R}_+;\dot B^0_{n,1}(\mathbb{R}^n))$, thus also in $L_{1,\mathrm{loc}}(\mathbb{R}_+;\dot B^0_{n,1}(\mathbb{R}^n))$, and tends to $E_k\widetilde{v}$ in the weak sense. As $E_k\widetilde{v}$ is actually in $L_{1,\mathrm{loc}}(\mathbb{R}_+;\dot B^0_{n,1}(\mathbb{R}^n))$, one may write for all $T>0$ and $k\in\mathbb{N}$,

$$\int_0^T \|E_k\widetilde{v}\|_{\dot B^0_{n,1}(\mathbb{R}^n)}\,dt \leqslant \liminf_{l\to\infty}\int_0^T \big\|E_k\widetilde{v}^l\big\|_{\dot B^0_{n,1}(\mathbb{R}^n)}\,dt,$$

whence, according to (4.36),

$$\int_0^T \big\|E_k\widetilde{v}(t)\big\|_{\dot B^0_{n,1}(\mathbb{R}^n)}\,dt \leqslant C\big(\|u_0\|_{\dot B^0_{n,1}(\mathbb{R}^n_+)}+\|f\|_{L_1(0,T;\dot B^0_{n,1}(\mathbb{R}^n_+))}\big).$$

Arguing exactly as in the proof of Theorem 3, we then get $\widetilde{v}\in L_1(\mathbb{R}_+;\dot B^0_{n,1})$, whence $\nabla^2 u\in L_1(\mathbb{R}_+;\dot B^0_{n,1})$ (with the desired bound).

Getting regularity for $\nabla\Pi$ and $\partial_t u$ is more involved. As a first step, let us fix some bounded interval $I$ of $\mathbb{R}_+$ and let us prove that

$$\partial_t u \quad \text{belongs to } L_1(I; L_n). \tag{4.43}$$

In what follows we denote by $L_{p,\sigma}$ the completion of the set of smooth compactly supported divergence free vector fields over $\mathbb{R}^n_+$, for the $L_p(\mathbb{R}^n_+)$ norm.

Let us notice that (4.42) implies that $\nabla u \in L_1(I; L_\infty)$. Because we also know that $u \in L_\infty(I; L_n)$ and $\rho \in L_\infty(I \times \mathbb{R}^n_+)$, we deduce that $\rho u \cdot \nabla u \in L_1(I; L_n)$. Taking advantage of the momentum equation and of the fact that the Helmholtz projector $P_H$ onto divergence free vector fields maps $L_n$ onto $L_{n,\sigma}$ — see [21], we conclude that

$$h := P_H[\rho\partial_t u] \in L_1(I; L_n). \tag{4.44}$$

We claim that this implies that $\partial_t u$ itself is in $L_1(I; L_n)$. For proving that, let us introduce the operator

$$\Phi : w \mapsto P_H[\rho w]. \tag{4.45}$$

**Lemma 11.** *Let $p \in (1, \infty)$. There exists a constant $c$ such that if*

$$\|1 - \rho\|_{L_\infty(I\times\mathbb{R}^n_+)} \leqslant c \tag{4.46}$$

*then $\Phi$ is an invertible self-map on $L_1(I; L_{p,\sigma})$ and on $\mathcal{C}_b(I; L_{p,\sigma})$.*

**Proof.** Obviously it suffices to consider the stationary case (viz. proving that $\Phi$ is an invertible self-map of $L_{p,\sigma}$ if $\|1 - \rho\|_{L_\infty(\mathbb{R}^n_+)} \leqslant c$ for a small enough $c$).

First, it is clear that $\Phi$ is a linear bounded self-map on $L_{p,\sigma}$ and that (4.46) implies that it is one-to-one provided $c$ is sufficiently small. Next, a simple implementation of the Banach fixed point theorem guarantees the existence of solutions to the equation

$$P_H[\rho w] = g \quad \text{for arbitrary } g \in L_{p,\sigma},$$

with the estimate $\|w\|_{L_p} \leqslant 2\|g\|_{L_p}$. Indeed for solving this equation, one may consider the following iterative scheme:

$$w^0 := 0, \qquad w^{n+1} := g + P_H\big[(1 - \rho)w^n\big].$$

Lemma 11 is proved. □

We now plan to use the above lemma for showing that $\partial_t u \in L_1(I; L_n)$. We have already seen that $P_H(\rho\partial_t u) = h \in L_1(I; L_n)$. Hence it suffices to state that the distribution $\partial_t u$ coincides with the unique solution $\overline{\partial_t u}$ in $L_1(I; L_n)$ to the equation $\Phi(w) = h$. For that, we are going to show that $(\partial_t u^l)_{l\in\mathbb{N}}$ tends weakly to $\overline{\partial_t u}$, or, in other words, that for all $\phi \in \mathcal{C}_b(I; L_{n',\sigma})$ (with $n' = n/(n - 1)$) we have

$$\lim_{l\to\infty}\int_{I\times\mathbb{R}^n_+}\big(\partial_t u^l-\overline{\partial_t u}\big)\cdot\phi\,dx\,dt=0. \tag{4.47}$$

Fix some $\phi\in\mathcal{C}_b(I;L_{n',\sigma})$ and denote $\psi:=\Phi^{-1}(\phi)$. According to Lemma 11, we have $\psi\in\mathcal{C}_b(I;L_{n',\sigma})$. Now, using the definition of $\Phi$ and the fact that $\partial_t u^l-\overline{\partial_t u}$ and $\psi$ are divergence free, and that $P_H$ is symmetric, one may write

$$\begin{aligned}\int_{I\times\mathbb{R}^n_+}\big(\partial_t u^l-\overline{\partial_t u}\big)\cdot\phi\,dx\,dt&=\int_{I\times\mathbb{R}^n_+}\big(\partial_t u^l-\overline{\partial_t u}\big)\cdot(\rho\psi)\,dx\,dt\\&=\int_{I\times\mathbb{R}^n_+}\big\{\big(\partial_t u^l-\overline{\partial_t u}\big)\cdot\psi+(\rho-1)\big(\partial_t u^l-\overline{\partial_t u}\big)\cdot P_H\psi\big\}\,dx\,dt\\&=\int_{I\times\mathbb{R}^n_+}\big\{\big(\partial_t u^l-\overline{\partial_t u}\big)+P_H\big[(\rho-1)\big(\partial_t u^l-\overline{\partial_t u}\big)\big]\big\}\cdot\psi\,dx\,dt.\end{aligned}$$

Because $u^l$ satisfies the momentum equation of (*INS*), we have

$$\begin{aligned}&\partial_t u^l-\overline{\partial_t u}+P_H\big[(\rho-1)\big(\partial_t u^l-\overline{\partial_t u}\big)\big]=h^l-h+P_H\big[\big(\rho-\rho^l\big)\partial_t u^l\big]\\&\quad\text{with } h^l=P_H\big[\mu\Delta u^l-\rho^l u^l\cdot\nabla u^l\big].\end{aligned}$$

Hence

$$\int_{I\times\mathbb{R}^n_+}\big(\partial_t u^l-\overline{\partial_t u}\big)\cdot\phi\,dx\,dt=\int_{I\times\mathbb{R}^n_+}\big(h^l-h\big)\cdot\psi\,dx\,dt+\int_{I\times\mathbb{R}^n_+}P_H\big[\big(\rho-\rho^l\big)\partial_t u^l\big]\cdot\psi\,dx\,dt.$$

The boundedness and convergence properties stated so far ensure that $(h^l)_{l\in\mathbb{N}}$ is bounded in $L_1(I;L_n)$ and tends to $h$ in the weak sense. As $h\in L_1(I;L_n)$, this ensures that the first term of the right-hand side of the above equality tends to 0 as $l$ goes to infinity. Next, combining (4.38) and the fact that $(\partial_t u^l)_{l\in\mathbb{N}}$ is bounded in $L_1(I;L_n)$, we readily get that $P_H[(\rho-\rho^l)\partial_t u^l]$ tends to zero in the sense of distributions. Because $P_H[(\rho-\rho^l)\partial_t u^l]$ is also bounded in $L_1(I;L_n)$ one can thus conclude that the second term of the above equality also tends to 0. This completes the proof of (4.47). Hence (4.43) is true for all bounded interval $I$ of $\mathbb{R}^+$.

Knowing that $\partial_t u$ is in $L_{1,\mathrm{loc}}(\mathbb{R}_+;L_n)$, that $\partial_t u^l$ tends weakly to $\partial_t u$ and that (4.36) is fulfilled, it is not difficult to show that $\partial_t u\in L_1(\mathbb{R}_+;\dot B^0_{n,1})$. It suffices to use the spectral truncation operator $E_k$ and to follow the proof of (4.42). Finally, from the momentum equation combined with Lemma 4 and Theorem 2, we deduce that $\nabla\Pi\in L_1(\mathbb{R}_+;\dot B^0_{n,1})$ and that $u\in\mathcal{C}_b(\mathbb{R}_+;\dot B^0_{n,1})$. Theorem 1 is proved.

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## Appendix A

**Proof of Proposition 3.** Let us denote for $\varepsilon > 0$ and $t \in \mathbb{R}$,

$$\eta_\varepsilon(t) := \begin{cases} 0 & \text{for } t < \varepsilon, \\ \frac{2}{\varepsilon}t - 2 & \text{for } \varepsilon \leqslant t \leqslant 3\varepsilon, \\ 1 & \text{for } t > 3\varepsilon. \end{cases}$$

We agree that $\eta_0$ stands for the characteristic function of $\mathbb{R}_+$ and for all function $u$ over $\mathbb{R}^n$, we denote $\Phi_\varepsilon(u) : x \mapsto \eta_\varepsilon(x_n)u(x)$.

Let us admit for a while the following lemma:

**Lemma 12.** *For all $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $-1 + 1/p < \sigma < 1/p$ the operator $\Phi_\varepsilon$ maps $\dot{B}^\sigma_{p,q}(\mathbb{R}^n)$ in $\dot{B}^\sigma_{p,q}(\mathbb{R}^n)$ uniformly with respect to $\varepsilon \geqslant 0$.*

*Moreover, if $q$ is finite then for all $u \in \dot{B}^s_{p,q}(\mathbb{R}^n)$, we have*

$$\Phi_\varepsilon(u) \to_{\varepsilon \to 0} \Phi_0(u) \quad \text{in } \dot{B}^s_{p,q}\bigl(\mathbb{R}^n\bigr).$$

Let $u \in \dot{B}^s_{p,q}(\mathbb{R}^n_+)$ (with $1 \leqslant p, q < \infty$ and $1/p - 1 < s < 1/p$). Then $u$ is the restriction to $\mathbb{R}^n_+$ of some function $\widetilde{u} \in \dot{B}^s_{p,q}(\mathbb{R}^n)$. Now, the above lemma ensures that $\Phi_\varepsilon(\widetilde{u})$ is supported in $\mathbb{R}^n_+$ and tends to $1_{\mathbb{R}^n_+}\widetilde{u}$ in $\dot{B}^s_{p,q}(\mathbb{R}^n)$. As $1_{\mathbb{R}^n_+}\widetilde{u}$ coincides with the extension of $u$ by 0, Proposition 3 is proved. □

For the sake of completeness, let us now prove Lemma 12.

**Proof of Lemma 12.** As a first step, let us state that $\Phi_\varepsilon$ maps $\dot{B}^\sigma_{p,1}(\mathbb{R}^n)$ in $\dot{B}^\sigma_{p,\infty}(\mathbb{R}^n)$ with uniform bounds with respect to $\varepsilon \geqslant 0$, whenever $0 < \sigma < 1/p$ and $1 \leqslant p < \infty$.

Because $\sigma \in (0, 1)$, one can use the definition of Besov norms in terms of finite differences (see Remark 4) to write that

$$\begin{aligned}\bigl\|\Phi_\varepsilon(u)\bigr\|_{\dot{B}^\sigma_{p,\infty}(\mathbb{R}^n)} \leqslant{}& \sum_{i=1}^{n-1} \sup_{h_i > 0} h_i^{-\sigma} \bigl\|\eta_\varepsilon(x_n)\bigl\{u(x_1, \dots, x_i + h_i, \dots, x_n) - u(x_1, \dots, x_n)\bigr\}\bigr\|_{L_p(\mathbb{R}^n)} \\ &+ \sup_{h_n > 0} h_n^{-\sigma} \bigl\|\eta_\varepsilon(x_n + h_n)u(x', x_n + h_n) - \eta_\varepsilon(x_n)u(x', x_n)\bigr\|_{L_p(\mathbb{R}^n)}.\end{aligned}$$

Because $|\eta_\varepsilon(x_n)| \leqslant 1$ for all $x_n \in \mathbb{R}$, we have

$$\begin{aligned}\bigl\|\Phi_\varepsilon(u)\bigr\|_{\dot{B}^\sigma_{p,\infty}(\mathbb{R}^n)} \leqslant{}& \|u\|_{\dot{B}^\sigma_{p,\infty}(\mathbb{R}^n)} + \sup_{h_n > 0} h_n^{-\sigma} \bigl\|\eta_\varepsilon(x_n + h_n)\bigl\{u(x', x_n + h_n) - u(x', x_n)\bigr\}\bigr\|_{L_p(\mathbb{R}^n)} \\ &+ \sup_{h_n > 0} h_n^{-\sigma} \bigl\|\bigl\{\eta_\varepsilon(x_n + h_n) - \eta_\varepsilon(x_n)\bigr\}u(x', x_n)\bigr\|_{L_p(\mathbb{R}^n)}\end{aligned}$$

whence, according to Hölder inequality,

$$\|\Phi_\varepsilon(u)\|_{\dot B^\sigma_{p,\infty}(\mathbb{R}^n)} \leqslant 2\|u\|_{\dot B^\sigma_{p,\infty}(\mathbb{R}^n)} + \sup_{h_n>0} h_n^{-\sigma}\|\eta_\varepsilon(\cdot+h_n)-\eta_\varepsilon\|_{L_{q^*}(\mathbb{R})}\|u\|_{L_{p^*}(\mathbb{R};L_p(\mathbb{R}^{n-1}))} \tag{A.1}$$

with $q^*=1/\sigma$ and $1/p^*=1/p-1/q^*$ (a choice which is in accordance with the assumption that $0<\sigma<1/p$).

Let us bound the last term in (A.1). On the one hand, for $h_n \geqslant 2\varepsilon$ one may write owing to the fact that $\eta_\varepsilon(\cdot+h_n)-\eta_\varepsilon$ is supported in $[\varepsilon-h_n, 3\varepsilon]$ and valued in $[0,1]$,

$$\|\eta_\varepsilon(\cdot+h_n)-\eta_\varepsilon\|_{L_{q^*}(\mathbb{R})} = \left(\int_{\varepsilon-h_n}^{3\varepsilon}|\eta_\varepsilon(t+h_n)-\eta_\varepsilon(t)|^{q^*}\,dt\right)^{1/q^*} \leqslant Ch_n^{1/q^*}.$$

On the other hand, if $h_n<2\varepsilon$ then one may split the integral over $[\varepsilon-h_n,3\varepsilon]$ into integrals over $[\varepsilon-h_n,\varepsilon]$, $[\varepsilon,3\varepsilon-h_n]$ and $[3\varepsilon-h_n,3\varepsilon]$. As the first and last intervals have length $h_n$, the corresponding integrals may be bounded by $h_n^{1/q^*}$. As for the second integral, one may write

$$\left(\int_\varepsilon^{3\varepsilon-h_n}|\eta_\varepsilon(t+h_n)-\eta_\varepsilon(t)|^{q^*}\,dt\right)^{1/q^*} = \left(\int_\varepsilon^{3\varepsilon-h_n}\left(\frac{2h_n}{\varepsilon}\right)^{q^*}dt\right)^{1/q^*} \leqslant Ch_n^{1/q^*}\left(\frac{h_n}{\varepsilon}\right)^{1-\frac{1}{q^*}} \leqslant Ch_n^{1/q^*}.$$

Finally, let us notice that according to e.g. [35, Chapter 2.8] or [5, Chapter 18], we have

$$\dot B^\sigma_{p,1}(\mathbb{R}^n) \hookrightarrow L_{p^*}\big(\mathbb{R}_{x_n}; L_p(\mathbb{R}^{n-1}_{x'})\big).$$

Hence, putting together the above two inequalities and (A.1), one ends up with

$$\|\Phi_\varepsilon(u)\|_{\dot B^\sigma_{p,\infty}(\mathbb{R}^n)} \leqslant C\|u\|_{\dot B^\sigma_{p,1}(\mathbb{R}^n)} \quad \text{uniformly in } \varepsilon \geqslant 0. \tag{A.2}$$

One can now deduce that for all $0<\sigma<1/p$ and $1\leqslant p,q\leqslant\infty$, the map $\Phi_\varepsilon$ is (uniformly) bounded from $\dot B^\sigma_{p,q}(\mathbb{R}^n)$ to $\dot B^\sigma_{p,q}(\mathbb{R}^n)$. Indeed, one may find $\delta>0$ so small as to satisfy $0<\delta<\min\{\sigma,\frac{1}{p}-\sigma\}$. Hence we have, uniformly with respect to $\varepsilon\geqslant 0$,

$$\Phi_\varepsilon : \dot B^{\sigma+\delta}_{p,1}(\mathbb{R}^n) \to \dot B^{\sigma+\delta}_{p,\infty}(\mathbb{R}^n) \quad \text{and} \quad \Phi_\varepsilon : \dot B^{\sigma-\delta}_{p,1}(\mathbb{R}^n) \to \dot B^{\sigma-\delta}_{p,\infty}(\mathbb{R}^n).$$

So we get the result by interpolation.

Let us now state the properties of convergence of $\Phi_\varepsilon(u)$ in the case $0<\sigma<1/p$ and $1\leqslant q<\infty$. We already know that for all $\sigma'\in(\sigma,1/p)$, operator $\Phi_\varepsilon : \dot B^{\sigma'}_{p,q}(\mathbb{R}^n) \to \dot B^{\sigma'}_{p,q}(\mathbb{R}^n)$ is uniformly bounded. In addition, as $p$ is finite, Lebesgue theorem ensures that for all function $u$ in $L^p(\mathbb{R}^n)$ then $\Phi_\varepsilon(u)$ tends to $\Phi_0(u)$ in $L^p(\mathbb{R}^n)$. These two properties combined with an interpolation argument will enable us to prove that for all $u\in\dot B^\sigma_{p,q}(\mathbb{R}^n)$ we have $\Phi_\varepsilon(u)\to\Phi_0(u)$ in $\dot B^\sigma_{p,q}(\mathbb{R}^n)$. Indeed, fix $\sigma'\in(\sigma,1/p)$ and set $\theta=\sigma/\sigma'$. For $m\in\mathbb{N}$, denote

$$u_m^1 := \sum_{j<-m} \Delta_j u, \qquad u_m^2 := \sum_{|j|\leqslant m} \Delta_j u \quad \text{and} \quad u_m^3 := \sum_{j>m} \Delta_j u. \tag{A.3}$$

Note that $u_m^2$ belongs to $L^p \cap \dot B^{\sigma'}_{p,r}$. Therefore, owing to the uniform bounds for $\Phi_\varepsilon$ stated previously and to an obvious interpolation inequality, one may write that $I_\varepsilon := \|\Phi_\varepsilon(u) - \Phi_0(u)\|_{\dot B^\sigma_{p,q}}$ satisfies

$$\begin{aligned} I_\varepsilon &\leqslant \|\Phi_\varepsilon(u_m^1) - \Phi_0(u_m^1)\|_{\dot B^\sigma_{p,q}} + \|\Phi_\varepsilon(u_m^2) - \Phi_0(u_m^2)\|_{\dot B^\sigma_{p,q}} + \|\Phi_\varepsilon(u_m^3) - \Phi_0(u_m^3)\|_{\dot B^\sigma_{p,q}} \\ &\leqslant C\big(\|u_m^1\|_{\dot B^\sigma_{p,q}} + \|\Phi_\varepsilon(u_m^2) - \Phi_0(u_m^2)\|_{L^p}^{1-\theta} \|\Phi_\varepsilon(u_m^2) - \Phi_0(u_m^2)\|^\theta_{\dot B^{\sigma'}_{p,q}} + \|u_m^3\|_{\dot B^\sigma_{p,q}}\big) \\ &\leqslant C\big(\|u_m^1\|_{\dot B^\sigma_{p,q}} + \|\Phi_\varepsilon(u_m^2) - \Phi_0(u_m^2)\|_{L^p}^{1-\theta} \|u_m^2\|^\theta_{\dot B^{\sigma'}_{p,q}} + \|u_m^3\|_{\dot B^\sigma_{p,q}}\big). \end{aligned}$$

Because $q<\infty$ the terms $\|u_m^1\|_{\dot B^\sigma_{p,q}}$ and $\|u_m^3\|_{\dot B^\sigma_{p,q}}$ tend to 0 when $m$ goes to $+\infty$. Next, for fixed $m$ we have $u_m^2 \in L^p$ so that

$$\|\Phi_\varepsilon(u_m^2) - \Phi_0(u_m^2)\|_{L_p} \underset{\varepsilon\to 0}{\to} 0.$$

Putting those two results together, it is now easy to conclude that $\Phi_\varepsilon(u)$ tends to $\Phi_0(u)$ in $\dot B^\sigma_{p,q}$.

**Remark 7.** Note that the above convergence result holds true for $q=\infty$ if we assume in addition that $u$ belongs to the completion of the Schwartz class of the $B^\sigma_{p,\infty}(\mathbb{R}^n)$ norm. Indeed, in this case the terms $\|u_m^1\|_{\dot B^\sigma_{p,\infty}}$ and $\|u_m^3\|_{\dot B^\sigma_{p,\infty}}$ defined in (A.3) tend to 0 when $m$ goes to $-\infty$ and $+\infty$ respectively.

In order to complete the proof, let us now focus on the case of negative index of regularity (that is $\frac{1}{p} - 1 < \sigma < 0$). First, we want to prove that $\Phi_\varepsilon : \dot B^\sigma_{p,q}(\mathbb{R}^n) \to \dot B^\sigma_{p,q}(\mathbb{R}^n)$ uniformly. The basic idea is that the *negative* space $\dot B^\sigma_{p,q}(\mathbb{R}^n)$ can be represented as the dual (or pre-dual) of the *positive* space $\dot B^{-\sigma}_{p',q'}(\mathbb{R}^n)$ where $p'$ and $q'$ are the conjugate exponents of $p$ and $q$ (see e.g. [7] for the homogeneous framework). More precisely, one may write

$$\|\Phi_\varepsilon(u)\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)} = \sup\left\{ \int_{\mathbb{R}^n} \eta_\varepsilon u f\,dx\colon\ f \in \mathcal{S}_0(\mathbb{R}^n) \text{ and } \|f\|_{\dot B^{-\sigma}_{p',q'}(\mathbb{R}^n)} \leqslant 1 \right\}. \tag{A.4}$$

However, as $0 < -\sigma < 1 - \frac{1}{p} = \frac{1}{p'}$, the previous steps ensure that $\eta_\varepsilon f \in \dot B^{-\sigma}_{p,q}(\mathbb{R}^n)$ and that

$$\|\eta_\varepsilon f\|_{\dot B^{-\sigma}_{p',q'}(\mathbb{R}^n)} \leqslant C\|f\|_{\dot B^{-\sigma}_{p',q'}(\mathbb{R}^n)}.$$

Therefore, one can conclude that

$$\|\Phi_\varepsilon(u)\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)} \leqslant C\|u\|_{\dot B^\sigma_{p,q}(\mathbb{R}^n)}. \tag{A.5}$$

A similar argument works for proving that $\Phi_\varepsilon(u)$ tends to $\Phi_0(u)$ in $\dot B^\sigma_{p,q}(\mathbb{R}^n)$. The case $q=1$ stems from Remark 7.

Finally, the remaining case $\sigma=0$ follows by interpolation. This completes the proof of the lemma. □

**Proof of Lemma 2.** Let us first consider the case $q=1$ and $s\in(0,1)$. We have to show that if $h\in \dot B^{s-\frac1p}_{p,1}(\mathbb{R}^{n-1})$ then $H:=\mathcal{F}^{-1}_{x'}[e^{-|\xi|x_n}\mathcal{F}_{x'}h]$ satisfies

$$\|H\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}\leqslant C\|h\|_{\dot B^{s-1/p}_{p,1}(\mathbb{R}^{n-1})}.$$

Taking advantage of an interpolation argument (see [35, Section 2.5]), we can write

$$\dot B^s_{p,1}\big(\mathbb{R}^n_+\big)=\dot B^s_{p,1}\big(\mathbb{R}^{n-1};L_p(\mathbb{R}_+)\big)\cap \dot B^s_{p,1}\big(\mathbb{R}_+;L_p\big(\mathbb{R}^{n-1}\big)\big). \tag{A.6}$$

So let us first bound $H$ in $\dot B^s_{p,1}(\mathbb{R}^{n-1};L_p(\mathbb{R}_+))$. For $\xi\in\mathbb{R}^{n-1}$, denote $\varphi'_k(\xi)=\phi(2^{-k}|\xi|)$ and $\widetilde{\varphi}'_k(\xi)=\widetilde{\phi}(2^{-k}|\xi|)$ where $\widetilde{\phi}$ is a smooth function on $\mathbb{R}_+$, with support in $\{1/3\leqslant r\leqslant 3\}$ and value 1 on a neighborhood of $\operatorname{supp}\phi$, and $\phi$ defined in (2.1).

Denote by $(\Delta'_k)_{k\in\mathbb{Z}}$ the corresponding Littlewood–Paley decomposition on $\mathbb{R}^{n-1}$. By definition of the Besov norm, one can write that

$$\|H\|_{\dot B^s_{p,1}(\mathbb{R}^{n-1};L_p(\mathbb{R}_+))}=\sum_{k\in\mathbb{Z}}2^{sk}\big\|\mathcal{F}^{-1}_{x'}\big[\varphi'_k e^{-|\xi|x_n}\mathcal{F}_{x'}h\big]\big\|_{L_p(\mathbb{R}^n_+)}.$$

Now, it is not difficult to check that

$$\xi\mapsto\widetilde{\varphi}'_k(\xi)e^{(2^{k-2}-|\xi|)x_n} \tag{A.7}$$

is a multiplier with bounds *independent* of $k\in\mathbb{Z}$ and $x_n\in\mathbb{R}_+$.

Therefore, the Marcinkiewicz theorem (see [16]) ensures that

$$\begin{aligned}\|H\|_{\dot B^s_{p,1}(\mathbb{R}^{n-1};L_p(\mathbb{R}_+))}&\leqslant C\sum_{k\in\mathbb{Z}}2^{ks}\big\|e^{-2^{k-2}x_n}\big\|_{L_p(\mathbb{R}_+)}\big\|\Delta'_k h\big\|_{L_p(\mathbb{R}^{n-1})}\\&\leqslant C\sum_{k\in\mathbb{Z}}2^{ks}2^{-\frac1p k}\big\|\Delta'_k h\big\|_{L_p(\mathbb{R}^{n-1})}\\&\leqslant C\|h\|_{\dot B^{s-1/p}_{p,1}(\mathbb{R}^{n-1})}.\end{aligned}$$

Next, we want to show that $H\in\dot B^s_{p,1}(\mathbb{R}_+;L_p(\mathbb{R}^{n-1}))$. Because $s\in(0,1)$, the norm $\|\cdot\|_{\dot B^s_{p,1}(\mathbb{R}_+;L_p(\mathbb{R}^{n-1}))}$ may be expressed in terms of finite differences of order one (see [35]):

$$\|H\|_{\dot B^s_{p,1}(\mathbb{R}_+;L_p(\mathbb{R}^{n-1}))}=\int_0^\infty\frac{dw}{w^{1+s}}\big\|H(\cdot,\cdot)-H(\cdot,\cdot+w)\big\|_{L_p(\mathbb{R}^n_+)}=:I. \tag{A.8}$$

To estimate the right-hand side of (A.8), we use the obvious inequality

$$\|z\|_{L_p(\mathbb{R}^n_+)} \leqslant \sum_{k\in\mathbb{Z}} \left\| \mathcal{F}_{x'}^{-1}\left[\varphi'_k \mathcal{F}_{x'} z\right]\right\|_{L_p(\mathbb{R}^n_+)}.$$

Hence, using again the Marcinkiewicz theorem for the multiplier defined in (A.7), we discover that

$$\begin{aligned}\left\|H(\cdot,\cdot)-H(\cdot,\cdot+w)\right\|_{L_p(\mathbb{R}^n_+)} &\leqslant C\sum_{k\in\mathbb{Z}}\left\|\mathcal{F}_{x'}^{-1}\left[\varphi'_k\left(e^{-|\xi|x_n}-e^{-|\xi|(x_n+w)}\right)\mathcal{F}_{x'}h\right]\right\|_{L_p(\mathbb{R}^n_+)}\\ &\leqslant C\sum_{k\in\mathbb{Z}}2^{-\frac{k}{p}}\left\|\mathcal{F}_{x'}^{-1}\left[\varphi'_k\left(1-e^{-|\xi|w}\right)\mathcal{F}_{x'}h\right]\right\|_{L_p(\mathbb{R}^{n-1})}.\end{aligned}$$

Now, returning to (A.8) we get

$$I \leqslant C\sum_{k\in\mathbb{Z}}\int_0^\infty \frac{dw}{w^{1+s}}\,2^{-\frac{k}{p}}\left(1-e^{-2^k w}\right)\left\|\mathcal{F}_{x'}^{-1}\left[\psi_k\varphi'_k\mathcal{F}_{x'}h\right]\right\|_{L_p(\mathbb{R}^{n-1})}$$

$$\text{with } \psi_k(\xi)=\widetilde{\phi}'_k(\xi)\frac{1-e^{-|\xi|w}}{1-e^{-2^k w}}.$$

Again, it turns out that $\psi_k$ is a multiplier with bounds independent of $k$ and $w$. So combining the Marcinkiewicz theorem and the change of variables $u=2^k w$, we get

$$I \leqslant C\sum_{k\in\mathbb{Z}}\int_0^\infty \frac{du}{u^{1+s}}\,2^{k(s-\frac{1}{p})}\left(1-e^{-u}\right)\left\|\Delta'_k h\right\|_{L_p(\mathbb{R}^{n-1})},$$

whence

$$I \leqslant C\left(\int_0^\infty \frac{1-e^{-u}}{u^{1+s}}\,du\right)\|h\|_{\dot{B}^{s-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})}.$$

This completes the proof in the case $s\in(0,1)$ and $q=1$.

To extend the result for any $s\in\mathbb{R}_+\setminus\mathbb{N}$, it suffices to differentiate $[s]$ times the expression of $H$, and to repeat the above proof with exponent $s-[s]$.

Finally, because the Besov spaces are an interpolation family, namely

$$\dot{B}^s_{p,q}=\left(\dot{B}^{s_1}_{p,1},\dot{B}^{s_2}_{p,1}\right)_{\theta,q}\quad\text{with } s=\theta s_1+(1-\theta)s_2,$$

one gets the desired result for all $s>0$ and $q\in[1,\infty]$. Lemma 2 is proved. □

**Proof of Lemma 3.** Take $p\in(1,\infty)$ and $s\in(-1+1/p,1/p)$. We consider the case $q=p$, the general case will follow from interpolation. Using the properties of duality of Besov spaces (see [4]), one can write (with $p'$ conjugate exponent of $p$)

$$\|F_n|_{x_n=0}\|_{\dot B^{s-1/p}_{p,p}(\mathbb{R}^{n-1})} = \sup\left\{ \int\limits_{\mathbb{R}^{n-1}} F_n\phi\,dx' \colon \phi\in\mathcal{S}_0\big(\mathbb{R}^{n-1}\big) \text{ and } \|\phi\|_{\dot B^{-s+1/p}_{p',p'}(\mathbb{R}^{n-1})}\leqslant 1\right\}.$$

Because $\operatorname{div} F=0$, we have

$$\int\limits_{\mathbb{R}^{n-1}} F_n\phi\,dx' = \int\limits_{\mathbb{R}^n_+} F\cdot\nabla(E\phi)\,dx,$$

where $E\phi$ is the extension of $\phi$ in $\dot B^{-s+1}_{p',p'}(\mathbb{R}^n_+)$ given by Lemma 2 — the assumptions guarantee that $-s+1>0$. Next, $\nabla(E\phi)\in\dot B^{-s}_{p',p'}(\mathbb{R}^n_+)$ with $-s\in(-1+1/p',1/p')$. Then, thanks to Proposition 3, both functions $\nabla(E\phi)$ and $F$ can be extended by zero for $x_n<0$. We thus get

$$\int\limits_{\mathbb{R}^{n-1}} F_n\phi\,dx' \leqslant C\|\nabla E\phi\|_{\dot B^{-s}_{p',p'}(\mathbb{R}^n_+)}\|F\|_{\dot B^s_{p,p}(\mathbb{R}^n_+)} \leqslant C\|\phi\|_{\dot B^{-s+1/p}_{p',p'}(\mathbb{R}^{n-1})}\|F\|_{\dot B^s_{p,p}(\mathbb{R}^n_+)}.$$

This completes the proof of Lemma 3. □

**Proof of Lemma 4.** Let us first prove the result in the case when $F$ and $G$ are defined on the whole space $\mathbb{R}^n$. Because $G$ belongs to the Besov space $\dot B^1_{q,\infty}(\mathbb{R}^n)$ and $F\in\dot B^0_{n,1}(\mathbb{R}^n)$, the equalities

$$F=\sum_{j\in\mathbb{Z}}\dot\Delta_j F \quad\text{and}\quad G=\sum_{j\in\mathbb{Z}}\dot\Delta_j G$$

make sense in the set of tempered distributions.

Therefore, one can decompose the product $FG$ according to the following *homogeneous Bony decomposition* (see the original paper [6] by J.-M. Bony, and [4] for the homogeneous framework):

$$FG=\dot T_F G+\dot R(F,G)+\dot T_G F.$$

Above, the paraproduct operator $\dot T$ is defined by

$$\dot T_F G := \sum_{k\in\mathbb{Z}} S_k F\Delta_k G \quad\text{with } S_k := \sum_{j\leqslant k-3}\Delta_j,$$

and the remainder operator $\dot R$ is defined by

$$\dot R(F,G) := \sum_{k\in\mathbb{Z}}\Delta_k F\widetilde\Delta_k G \quad\text{with } \widetilde\Delta_k := \sum_{|i|\leqslant 2}\Delta_{k+i}.$$

We claim that

$$\|\dot{T}_F G\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)} \leqslant C\|F\|_{\dot{B}^0_{n,1}(\mathbb{R}^n)}\|G\|_{\dot{B}^1_{q,\infty}(\mathbb{R}^n)}, \tag{A.9}$$

$$\|\dot{T}_G F\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)} \leqslant C\|G\|_{L_\infty(\mathbb{R}^n)}\|F\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)}, \tag{A.10}$$

$$\big\|\dot{R}(F,G)\big\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)} \leqslant C\|F\|_{\dot{B}^0_{n,1}(\mathbb{R}^n)}\|G\|_{\dot{B}^1_{q,\infty}(\mathbb{R}^n)}, \tag{A.11}$$

where the constant $C$ in the first two inequalities depends only on the dimension, and depends continuously on $q>1$ in the last inequality.

As an example, let us prove inequality (A.9). By definition of the norm in $\dot{B}^0_{q,1}(\mathbb{R}^n)$ and of the paraproduct, we have

$$\|\dot{T}_F G\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)} = \sum_{l\in\mathbb{Z}}\bigg\|\sum_{j\in\mathbb{Z}}\sum_{k\leqslant j-3}\Delta_l(\Delta_k F\Delta_j G)\bigg\|_{L_q(\mathbb{R}^n)}.$$

Note that, by virtue of the support properties of the function $\phi$ defined in (2.1), the second sum may be restricted to those $j$ such that $|j-l|\leqslant 3$. Hence we have, for some constant $C$ depending only on $\phi$ and $n$,

$$\|\dot{T}_F G\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)} \leqslant C\sum_{l\in\mathbb{Z}}\sum_{|j-l|\leqslant 3}\sum_{k\leqslant j-3}\|\Delta_k F\|_{L_\infty(\mathbb{R}^n)}\|\Delta_j G\|_{L_q(\mathbb{R}^n)}.$$

From Bernstein inequalities, we get

$$\|\Delta_k F\|_{L_\infty(\mathbb{R}^n)} \leqslant C2^k\|\Delta_k F\|_{L_n(\mathbb{R}^n)} \quad \text{and} \quad \|\Delta_j G\|_{L_q(\mathbb{R}^n)} \leqslant C2^{-j}\|\nabla\Delta_j G\|_{L_q(\mathbb{R}^n)}$$

so that the above inequality becomes

$$\|\dot{T}_F G\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)} \leqslant C\sum_{l\in\mathbb{Z}}\sum_{|j-l|\leqslant 3}\sum_{k\leqslant j-3}2^{k-j}\|\Delta_k F\|_{L_n(\mathbb{R}^n)}\|\nabla\Delta_j G\|_{L_q(\mathbb{R}^n)}.$$

Applying convolution inequalities for series thus yields (A.9). The proof of inequalities (A.10) and (A.11) goes along the same lines (for more details, refer to e.g. [4, Chapter 2]).

Finally, putting inequalities (A.9), (A.10) and (A.11) together, one can conclude that

$$\|FG\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)} \leqslant C\big(\|F\|_{\dot{B}^0_{n,1}(\mathbb{R}^n)}\|\nabla G\|_{\dot{B}^0_{q,\infty}(\mathbb{R}^n)} + \|F\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)}\|G\|_{L_\infty(\mathbb{R}^n)}\big), \tag{A.12}$$

with $C$ depending only on $n$ and on $q_0$ (if $q\geqslant q_0>1$).

Next, assume that $F$ and $G$ are defined only on the half-space $\mathbb{R}^n_+$. Then, according to Proposition 3 the *symmetric* extensions $\widetilde{F}$ and $\widetilde{G}$ of $F$ and $G$ are in $\dot{B}^0_{n,1}(\mathbb{R}^n)\cap\dot{B}^0_{q,1}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)\cap\dot{B}^1_{q,\infty}(\mathbb{R}^n)$ respectively and satisfy

$$\|\widetilde{F}\|_{\dot{B}^0_{n,1}(\mathbb{R}^n)} \leqslant 2\|F\|_{\dot{B}^0_{n,1}(\mathbb{R}^n_+)}, \qquad \|\widetilde{F}\|_{\dot{B}^0_{q,1}(\mathbb{R}^n)} \leqslant 2\|F\|_{\dot{B}^0_{q,1}(\mathbb{R}^n_+)},$$

$$\|\widetilde{G}\|_{L_\infty(\mathbb{R}^n)} = \|G\|_{L_\infty(\mathbb{R}^n_+)}, \qquad \|\nabla\widetilde{G}\|_{\dot{B}^0_{q,\infty}(\mathbb{R}^n)} \leqslant 2\|\nabla G\|_{\dot{B}^0_{q,\infty}(\mathbb{R}^n_+)}.$$

As $\|FG\|_{\dot B^0_{q,1}(\mathbb{R}^n_+)} \leqslant \|\widetilde F\widetilde G\|_{\dot B^0_{q,1}(\mathbb{R}^n)}$, plugging the above inequalities in (A.12) completes the proof. □

**Proof of Lemma 5.** As for the previous lemma, it suffices to prove the result for functions defined on $I\times\mathbb{R}^n$ so that one may resort to Bony's decomposition. Note that the continuity results for the paraproduct and the remainder in $\widetilde L_r(I;\dot B^s_{p,q})$ spaces are the same as for $\dot B^s_{p,q}$. The time Lebesgue exponents just behave according to Hölder inequality (see e.g. [4]). So we have

$$\|T_GF\|_{\widetilde L_r(I;\dot B^0_{q,\infty})} \leqslant C\|F\|_{\widetilde L_{r_1}(I;\dot B^0_{q,\infty})}\|G\|_{L_{r_2}(I;L_\infty)},$$

and, because $q<\infty$,

$$\|T_FG\|_{\widetilde L_r(I;\dot B^0_{q,\infty})} \leqslant C\|F\|_{\widetilde L_{r_1}(I;\dot B^0_{q,\infty})}\|\nabla G\|_{\widetilde L_{r_2}(I;\dot B^0_{n,\infty})}.$$

Finally, since $q>1$, we have

$$\big\|R(F,G)\big\|_{\widetilde L_r(I;\dot B^0_{q,\infty})} \leqslant C\|F\|_{\widetilde L_{r_1}(I;\dot B^0_{q,\infty})}\|\nabla G\|_{\widetilde L_{r_2}(I;\dot B^0_{n,\infty})}.$$

This gives the result. □

**Remark 8.** In the above two statements, the assumption $q>1$ comes into play only for bounding the remainder term. Actually, even if the time is not involved the lemma is false for $q=1$ because if it were true, we could for instance multiply a $W^1_n\cap L_\infty$ function by a Dirac mass (which belongs to $B^0_{1,\infty}$). That the case $q=1$ is false is the main reason why one cannot prove uniqueness in dimension two if no additional regularity assumptions on the density.

**Proposition 9.** *There exists a constant $C$ such that for any $z\in L_1(0,T;L_n\cap L_\infty(\mathbb{R}^n_+))$ with $\nabla z\in L_1(0,T;L_\infty(\mathbb{R}^n_+))$ the following inequality holds true*:

$$\|z\|_{L_1(0,T;L_\infty(\mathbb{R}^n_+))} \leqslant C\|z\|_{\widetilde L_1(0,T;\dot B^0_{\infty,\infty}(\mathbb{R}^n_+))}\log\biggl(e+\frac{\|z\|_{L_1(0,T;L_n(\mathbb{R}^n_+))}+\|\nabla z\|_{L_1(0,T;L_\infty(\mathbb{R}^n_+))}}{\|z\|_{\widetilde L_1(0,T;\dot B^0_{\infty,\infty}(\mathbb{R}^n_+))}}\biggr).$$

**Proof.** Let us first point out that $L_1((0,T);L_\infty(\mathbb{R}^n_+))\subset\widetilde L_1(0,T;\dot B^0_{\infty,\infty}(\mathbb{R}^n_+))$ so that the above inequality makes sense. Let $\widetilde z$ be the symmetric extension to $z$. Obviously, $\widetilde z\in L_1((0,T);L_n\cap L_\infty(\mathbb{R}^n))$, $\nabla\widetilde z\in L_1((0,T);L_\infty(\mathbb{R}^n))$ and we have

$$\|\widetilde z\|_{L_1(0,T;L_p(\mathbb{R}^n))} = 2^{\frac1p}\|z\|_{L_1(0,T;L_p(\mathbb{R}^n_+))}\quad\text{for } p=n,\infty,$$
$$\|\nabla\widetilde z\|_{L_1(0,T;L_\infty(\mathbb{R}^n))} = \|\nabla z\|_{L_1(0,T;L_\infty(\mathbb{R}^n_+))},$$
$$\|\widetilde z\|_{\widetilde L_1(0,T;\dot B^0_{\infty,\infty}(\mathbb{R}^n_+))} \leqslant 2\|z\|_{\widetilde L_1(0,T;\dot B^0_{\infty,\infty}(\mathbb{R}^n_+))},$$

so it suffices to state the desired inequality for $\widetilde z$ in the whole space.

In order to do so, one may split $\widetilde{z}$ into low, medium and high frequencies according to Littlewood–Paley decomposition. More precisely, for any nonnegative integer $m$, we have[9]

$$\|\widetilde{z}\|_{\widetilde{L}_1(0,T;L_\infty)} \leqslant \sum_{k\leqslant -m} \|\Delta_k\widetilde{z}\|_{L_1(0,T;L_\infty)} + \sum_{|k|<m} \|\Delta_k\widetilde{z}\|_{L_1(0,T;L_\infty)} + \sum_{k\geqslant m} \|\Delta_k\widetilde{z}\|_{L_1(0,T;L_\infty)}.$$

Therefore, taking advantage of Bernstein inequality and of the definition of $\|\cdot\|_{\widetilde{L}_1(0,T;\dot{B}^0_{\infty,\infty})}$,

$$\|\widetilde{z}\|_{\widetilde{L}_1(0,T;L_\infty)} \leqslant C\bigg(\sum_{k\leqslant -m} 2^k\|\Delta_k\widetilde{z}\|_{L_1(0,T;L_n)} + (2m-1)\|\widetilde{z}\|_{\widetilde{L}_1(0,T;\dot{B}^0_{\infty,\infty})} + \sum_{k\geqslant m} 2^{-k}\|\Delta_k\nabla\widetilde{z}\|_{L_1(0,T;L_\infty)}\bigg).$$

Because

$$\|\Delta_k\widetilde{z}\|_{L_1(0,T;L_n)} \leqslant C\|\widetilde{z}\|_{L_1(0,T;L_n)} \quad \text{and} \quad \|\Delta_k\widetilde{\nabla z}\|_{L_1(0,T;L_\infty)} \leqslant C\|\widetilde{\nabla z}\|_{L_1(0,T;L_\infty)},$$

one can thus conclude that

$$\|\widetilde{z}\|_{\widetilde{L}_1(0,T;L_\infty)} \leqslant C\big(2^{-m}\|\widetilde{z}\|_{L_1(0,T;L_n)} + (2m-1)\|\widetilde{z}\|_{\widetilde{L}_1(0,T;\dot{B}^0_{\infty,\infty})} + 2^{-m}\|\widetilde{\nabla z}\|_{L_1(0,T;L_\infty)}\big).$$

Choosing for $m$ the closest positive integer to $\log_2\big(\frac{\|\widetilde{z}\|_{L_1(0,T;L_n)}+\|\nabla\widetilde{z}\|_{L_1(0,T;L_\infty)}}{\|\widetilde{z}\|_{\widetilde{L}_1(0,T;\dot{B}^0_{\infty,\infty})}}\big)$ yields the desired inequality. □

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