Stability of 2D incompressible flows in $\mathbf{R}^3$

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Abstract. We investigate the global in time stability of regular solutions with large velocity vectors to the evolutionary Navier-Stokes equation in $\mathbf{R}^3$. The class of stable flows contains all two dimensional weak solutions. The only assumption which is required is smallness of the L_2-norm of initial perturbation or its derivative with respect to the 'z'-coordinate in the same norm. The magnitude of the rest of the norm of initial datum is not restricted.

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1 Introduction

The paper examines global in time regular solutions to the evolutionary Navier-Stokes equations in the whole three dimensional space. Since the problem of regularity of weak solutions is open and stays one of main challenges of the present mathematics, the study in this area is directed mainly onto finding special classes of solutions with large velocity vectors or developing the theory of so-called conditional regularity begun by Serrin [21] – see also [11],[20],[22],[27],[28].

The known theory allows to find nontrivial classes of global in time solutions to the Navier-Stokes equations with full regularity [9],[17],[23],[24],[29] guaranteeing the uniqueness. Here we will follow this direction. One approach is to consider the issue of stability of known generic solutions. The problem is well investigated for the equations in bounded domains. Thanks to the Poincare inequality it is possible to improve information following from the dissipation of the system [3],[15],[18]. The method in these cases is just modifications of techniques for problems for "pure" small data. However in more complex cases as in [10],[19],[26] the idea of proofs is not so elementary.

In the whole space the problem is more advance, there is no Poincare inequality and there is a need to find extra tools. We point two approaches to this case. In the first one [2],[14], authors assume sufficiently largeness of the vorticity, then using not standard theory as the Navier-Stokes equations they are able to obtain a large class of nontrivial regular solutions. The second type [7],[8] is a consequence of development of the theory of semigroups. Thanks to it we are able to prove existence of global in time solution for a class of initial data even with linear growth.

Our paper will show that two dimensional solutions are stable in $\mathbf{R}^3$ under a small perturbation of the L_2-norm - or even under weaker assumptions. The main idea is based on a "reduction" of the original problem to the two dimensional case. We will follow the old idea of Olga Ladyzhenskaya [12],[13]. In main steps the imbedding $H^1(\mathbf{R}^2) \subset BMO(\mathbf{R}^2)$ and the Marcinkiewicz-type interpolation for general spaces will play an essential role. This enables a two dimensional point of view on the estimation for the case in the whole $\mathbf{R}^3$. The class of generic solutions can be extended, but two dimensional solutions with finite Dirichlet integrals (in $\mathbf{R}^2$) seem to be the best identification of this set. Note that, because of a geometrical structure, the total energy of obtained solutions is infinite. An exception is the zero solution. We then obtain a wide class of global solutions with finite energy. This case is the motivation of our main result and it is carefully examined in section 2. Our results show a new large class of globally in time regular solutions with large velocity vectors. This way a new argument for the regularity of weak solutions in the general case is pointed, again.

From the mechanical point of view our result can give an interpretation that flows with two dimensional symmetry are stable independently from the magnitude of the constant in the Poincare inequality for the considered domain.

The subject of the paper is the evolutionary Navier-Stokes equations in the whole three dimensional space

$$\begin{array}{lll} v_{,t} + v \cdot \nabla v - \nu \Delta v + \nabla p = F & \text{in} & \mathbf{R}^3 \times (0,T), \\ \text{div}\, v = 0 & \text{in} & \mathbf{R}^3 \times (0,T), \\ v|_{t=0} = v_0 & \text{on} & \mathbf{R}^3, \end{array} \tag{1.1}$$

where $v = (v^x, v^y, v^z)$ is the sought velocity of the fluid, p its pressure, ν is the constant positive viscous coefficient, F – represents the external data, v_0 is an initial datum of the sought velocity which by $(1.1)_2$ is required to satisfies the compatibility condition $\text{div}\, v_0 = 0$ and comma ',' denotes the differentiation.

The solutions to system (1.1) are viewed in the form

$$v = w + u, \tag{1.2}$$

where w is a known smooth solution and u is a perturbation of it. Our analysis will concentrate on the system describing function u. From system (1.1) we obtain

$$\begin{array}{lll} u_{,t} + v \cdot \nabla u - \nu \Delta u + \nabla p = -u \cdot \nabla w & \text{in} & \mathbf{R}^3 \times (0,T), \\ \text{div}\, u = 0 & \text{in} & \mathbf{R}^3 \times (0,T), \\ u|_{t=0} = u_0 & \text{on} & \mathbf{R}^3, \end{array} \tag{1.3}$$

where initial datum $u_0 = v_0 - w|_{t=0}$.

Let us define the class of generic solutions w.

Definition. *We say that $w \in \Xi$ is a generic solution to system (1.1) iff:*
w – is a smooth solution to the Navier-Stokes equations (1.1) with external force F such that

$$\nabla w \in L_2((0,\infty)_t; L_2(\mathbf{R}^2_{xy}); L_\infty(\mathbf{R}_z)). \tag{1.4}$$

We distinguish one direction in $\mathbf{R}^3$ prescribed by the z-coordinate (we denote $\bar{x} = (x, y, z)$). A good identification of the class Ξ is the set of two dimensional solutions i.e.

$$w(t, x, y, z) = \tilde{w}(t, x, y), \tag{1.5}$$

where $\tilde{w}$ is a solution to the two dimensional Navier-Stokes equations

$$\begin{array}{lll} \tilde{w}_{,t} + \tilde{w} \cdot \tilde{\nabla}\tilde{w} - \nu\tilde{\Delta}\tilde{w} + \tilde{\nabla}\tilde{p} = \tilde{F} & \text{in} & \mathbf{R}^2 \times (0,T), \\ \tilde{\operatorname{div}}\, \tilde{w} = 0 & \text{in} & \mathbf{R}^2 \times (0,T), \\ \tilde{w}|_{t=0} = \tilde{w}_0 & \text{on} & \mathbf{R}^2 \end{array} \tag{1.6}$$

with an analogical description as for system (1.1). If $\tilde{F} \in L_2(0,\infty; \dot{H}^{-1}(\mathbf{R}^2))$, then the energy estimate for solutions to (1.6) yields the inclusion

$$\tilde{\nabla}\tilde{w} \in L_2(0,\infty; L_2(\mathbf{R}^2_{xy})). \tag{1.7}$$

In the force-free case the description of properties of solution $\tilde{w}$ can be better precise. The results from [25], [1] imply that

$$\tilde{\nabla}\tilde{w} \in L_1(0,\infty; L_\infty(\mathbf{R}^2)), \tag{1.8}$$

provided suitable assumptions on the initial datum $\tilde{w}_0$. The class defined by (1.8) is the kernel of the set of generic solutions. As we will see, we will be able to "extend" feature (1.8) on the whole class of functions from set Ξ.

The main result of the paper is the following.

Theorem 1. *Let $w \in \Xi$. If $u_0 \in H^1(\mathbf{R}^3) \cap W^{2-2/4}_4(\mathbf{R}^3)$. Additionally one of two below conditions is satisfies:*

(i) $||u_0||_{L_2(\mathbf{R}^3)}$ *is sufficiently small;*

or

(ii) $||u_{0,z}||_{L_2(\mathbf{R}^3)}$ *is sufficiently small, provided*

$$\begin{array}{c} ||w_{,z}||_{L_5(\mathbf{R}^3\times(0,\infty))} \quad \textit{and} \quad ||\nabla w_{,z}||_{L_{5/2}(\mathbf{R}^3\times(0,\infty))} \\ \textit{sufficiently small comparing to the norm } ||u_0||_{H^1(\mathbf{R}^3)\cap W^{2-1/2}_4(\mathbf{R}^3)}; \end{array} \tag{1.9}$$

then there exists regular unique global in time solution to equations (1.1) in form (1.2), where u is the solution to system (1.3) such that $u \in W^{2,1}_{4(loc)}(\mathbf{R}^3 \times (0,\infty))$ and

$$< u >_{W^{2,1}_4(\mathbf{R}^3\times(0,\infty))} := ||u_t||_{L_4(\mathbf{R}^3\times(0,\infty))} + ||\nabla^2 u||_{L_4(\mathbf{R}^3\times(0,\infty))} \le DATA, \tag{1.10}$$

where $DATA$ depends on norms of initial datum v_0 and vector field w.

The above result points a large class of regular global in time solutions to the Navier-Stokes equations in $\mathbf{R}^3$. From (1.10) – by the classical results [23] – solutions delivered by Theorem 1 become smooth provided smoothness of initial data. In particular by (1.7) we obtain that any two dimensional weak solution – being sufficiently smooth – is stable in the whole three dimensional space. Obviously smallness of a possible perturbation depends on the magnitude of the whole norm of the perturbed flow, however it is restricted to cases (i) or (ii), and only one of them have to be fulfilled. Comparing to results from [18] where stability of two dimensional flows were considered, too, our assumption (1.4) admits a larger class of generic flows. In [18] the authors required $\nabla w \in L_4(0,\infty; L_2(\Omega))$, additionally the whole $H^1(\mathbf{R}^3)$-norm of the initial datum has been assumed to be sufficiently small.

The proof of Theorem 1 is based on the classical energy method, however the novel idea is to reduce the view of the nonlinear term with respect to the geometrical structure of a given flow w. The energy method allows us to obtain an information about solutions omitting the influence of the nonlinear convective term $v \cdot \nabla v$. Similar techniques for simpler versions of the presented problem have been applied in [15], [16].

An alternative approach can be given by the theory of semigroups. However this technique requires the smallness of the whole norm and in the most optimal case in the three dimensions – by Kato's results [9] we ought to assume smallness of the L_3-norm of the initial datum. Here for any given initial norm in space $H^1(\mathbf{R}^3) \cap W^{2-1/2}_4(\mathbf{R}^3)$ we describe the required smallness of the L_2-norm. In particular the L_3-norm (even any $L_{2+\epsilon}$) can be arbitrary large.

Theorem 1 can be stated in spaces $W^{2,1}_p$ with $p \geq 2$ defined by the norm

$$\begin{gathered} ||u||_{W^{2,1}_p(\mathbf{R}^3\times(0,T))} = ||u||_{L_p(\mathbf{R}^n\times(0,T))} + < u >_{W^{2,1}_p\mathbf{R}^3\times(0,T))} \\ = (\int_0^T \int_{\mathbf{R}^3} |u|^p dxdt)^{1/p} + (\int_0^T \int_{\mathbf{R}^3} |u_t|^p dxdt)^{1/p} + (\int_0^T \int_{\mathbf{R}^3} |\nabla^2_x u|^2 dxdt)^{1/p}. \end{gathered} \tag{1.11}$$

The trace of a function from the $W^{2,1}_p$-space for fixed time as for $t = 0$ belong to the Besov $W^{2-2/p}_p$-class introduced by the norm (for $p > 2$)

$$\begin{gathered} ||u||_{W^{2-2/p}_p(\mathbf{R}^3)} = ||u||_{L_p(\mathbf{R}^3)} + < u >_{W^{2-2/p}_p(\mathbf{R}^3)} \\ = (\int_{\mathbf{R}^3} |u|^p dx)^{1/p} + (\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|\nabla_x u(x) - \nabla_x u(y)|^p}{|x-y|^{3+(2-2/p)p}} dxdy)^{1/p}. \end{gathered} \tag{1.12}$$

Such regularity would also guantantee smoothness of solutions – however in our considerations the case $p = 4$ is distinguish and simplifies our calculations. Since we are interested in smooth solutions we will not relax regularity of initial data.

Throughout the paper we try to use the standard notation [13], [23]. Generic constants are denoted by the same letter C.

The paper is organized as follows. First we show a particular case of Theorem 1 for $w \equiv 0$. In section 3 we construct the main estimate for case (i). Next, we show an analogical bound for case (ii). And in section 5 we present a proof of global in time existence in both cases.

2 Motivation

The aim of this section is to show the main idea and tools of the techniques which will be applied to prove Theorem 1. We analyze a special case of system (1.1), we consider system (1.3) for the trivial solution $w \equiv 0$ with $F \equiv 0$ for case (ii) from Theorem 1

$$\begin{array}{lll} v_t + v\cdot\nabla v - \nu\Delta v + \nabla p = 0 & \text{in} & \mathbf{R}^3\times(0,T), \\ \mathrm{div}\; v = 0 & \text{in} & \mathbf{R}^3\times(0,T), \\ v|_{t=0} = v_0 & \text{on} & \mathbf{R}^3. \end{array} \tag{2.1}$$

The initial datum is required to be sufficiently smooth – in particular $v_0 \in H^1(\mathbf{R}^3)$. Additionally the compatibility condition $\mathrm{div}\, v_0 = 0$ is assumed. We want to show the following version of Theorem 1.

Theorem 2. *Let $v_0 \in H^1(\mathbf{R}^3)$. If*

$$||v_{0,z}||_{L_2(\mathbf{R}^3)} \quad \textit{is sufficiently small,} \tag{2.2}$$

then there exists global in time regular (unique) solution to system (2.1).

Proof. We skip the proof of existence. Its idea is the same as in the one presented in section 5, where the general system will be considered – see also [16]. We concentrate only on a proof of the control of smallness of $||v_{,z}||_{L_\infty(0,\infty;L_2(\mathbf{R}^3))}$. The theory guarantees us existence of weak solutions defined globally in time. Hence, provided sufficient smoothness of them, we find a suitable a priori estimate controlling smallness of mentioned quantity.

Write the energy identity for solutions to system (2.1)

$$\frac{d}{dt}||v||^2_{L_2(\mathbf{R}^3)} + 2\nu||\nabla v||^2_{L_2(\mathbf{R}^3)} = 0. \tag{2.3}$$

From (2.3) we conclude that

$$||v||_{L_\infty((0,\infty)_t;L_2(\mathbf{R}^3))} + ||\nabla v||_{L_2(\mathbf{R}^3\times(0,\infty))} \le C||v_0||_{L_2(\mathbf{R}^3)}. \tag{2.4}$$

In our considerations we distinguish a one space direction, say, the z-direction. Let us differentiate system (2.1) with respect to this coordinate, getting

$$\begin{array}{lll} v_{,zt} + v\cdot\nabla v_{,z} - \nu\Delta v_{,z} + \nabla p_{,z} = -v_{,z}\cdot\nabla v & \text{in} & \mathbf{R}^3\times(0,\infty), \\ \mathrm{div}\, v_{,z} = 0 & \text{in} & \mathbf{R}^3\times(0,\infty), \\ v_{,z}|_{t=0} = v_{0,z} & \text{on} & \mathbf{R}^3. \end{array} \tag{2.5}$$

The energy method yields the following differential inequality

$$\frac{d}{dt}||v_{,z}||^2_{L_2(\mathbf{R}^3)} + 2\nu||\nabla v_{,z}||^2_{L_2(\mathbf{R}^3)} \le 2\int_{\mathbf{R}^3} |v_{,z}\cdot\nabla v v_{,z}|\, d\bar{x}, \tag{2.6}$$

where $d\bar{x} = dxdydz$. Hence integrating inequality (2.6) over $(0,\infty)$ we obtain

$$\begin{aligned} &||v_{,z}||_{L_\infty((0,\infty)_t;L_2(\mathbf{R}^3))} + ||\nabla v_{,z}||_{L_2((0,\infty)_t;L_2(\mathbf{R}^3))} \\ &\le C\left(\left(\int_0^\infty dt\int_{\mathbf{R}^3} d\bar{x}|v_{,z}|^2|\nabla v|\right)^{1/2} + ||v_{0,z}||_{L_2(\mathbf{R}^3)}\right). \end{aligned} \tag{2.7}$$

To simplify our notation let us introduce the following quantities

$$\begin{aligned} I &= ||v||_{L_\infty((0,\infty)_t;L_2(\mathbf{R}^3))} + ||\nabla v||_{L_2(\mathbf{R}^3)}, \\ J &= ||v_{,z}||_{L_\infty((0,\infty)_t;L_2(\mathbf{R}^3))} + ||\nabla v_{,z}||_{L_2(\mathbf{R}^3\times(0,\infty)_t)}. \end{aligned} \tag{2.8}$$

Taking into account information from (2.4) and (2.7), assuming finiteness of I and J – we concentrate our attention only on finding the estimates, so above quantities are assumed to be finite – we conclude that

$$\nabla v \in L_2((0,\infty)_t; L_2(\mathbf{R}^2_{xy}); H^1(\mathbf{R}_z)). \tag{2.9}$$

From the imbedding theorem ($H^1(\mathbf{R}) \subset L_\infty(\mathbf{R})$) we have the following inequality

$$||\nabla v||_{L_2((0,\infty)_t;L_2(\mathbf{R}^2_{xy});L_\infty(\mathbf{R}_z))} \le CI^{1/2}J^{1/2}. \tag{2.10}$$

Employing the interpolation inequality from the theory from [4], we get

$$\begin{aligned} &||v_{,z}||_{L_4(\mathbf{R}^2_{xy}\times(0,\infty)_t;L_2(\mathbf{R}_z))} \\ &\le C||v_{,z}||^{1/2}_{L_\infty(0,\infty)_t;L_2(\mathbf{R}^2_{xy});L_2(\mathbf{R}_z))}||v_{,z}||^{1/2}_{L_2((0,\infty)_t;BMO(\mathbf{R}^2_{xy});L_2(\mathbf{R}_z))} \le CJ. \end{aligned} \tag{2.11}$$

To get the above inequality it is enough to note that $H^1(\mathbf{R}^2) \subset BMO(\mathbf{R}^2)$, then the interpolation relation implies

$$\begin{aligned} &L_4(\mathbf{R}^2_{xy}\times(0,\infty)_t;L_2(\mathbf{R}_z)) = \\ &\big((L_\infty((0,\infty)_t;L_2(\mathbf{R}^2_{xy});L_2(\mathbf{R}_z)), L_2((0,\infty)_t;BMO(\mathbf{R}^2_{xy});L_2(\mathbf{R}_z))\big)_{1/2}, \end{aligned} \tag{2.12}$$

since $\frac{1}{4} = \frac{1-1/2}{\infty} + \frac{1/2}{2}$ and $\frac{1}{4} = \frac{1-1/2}{2} + \frac{1/2}{BMO}$ – the constant in (2.11) depends on interpolation parameters. Note that in (2.11) the classical Ladyzhenskaya inequality from [12] is hidden. This inequality guarantees the solvability of the regularity problem in two dimensions.

Now we are prepared to examine the first term in the r.h.s. of (2.7) which is the only difficulty in inequality (2.7). We have

$$\begin{aligned} &\left[\int_0^\infty dt\int_{\mathbf{R}^3}|v_{,z}|^2|\nabla v|d\bar{x}\right]^{1/2} \\ &\le C\left[\int_0^\infty dt\int_{\mathbf{R}^2}||\nabla v(t,x,y,\cdot)||_{L_\infty(\mathbf{R}_z)}||v_{,z}(t,x,y,\cdot)||^2_{L_2(\mathbf{R}_z)}dxdy\right]^{1/2} \\ &\le C\left[||\nabla v||_{L_2(\mathbf{R}^2_{xy}\times(0,\infty)_t;L_\infty(\mathbf{R}_z))}||v_{,z}||^2_{L_4(\mathbf{R}^2_{xy}\times(0,\infty)_t;L_2(\mathbf{R}_z))}\right]^{1/2} \\ &\le C[I^{1/2}J^{1/2}J^2]^{1/2} = CI^{1/4}J^{5/4}. \end{aligned} \tag{2.13}$$

Hence using (2.8) we state inequality (2.7) as follows:

$$J \le A_0 I^{1/4}J^{5/4} + J_0, \tag{2.14}$$

where $J_0 = C||v_{0,z}||_{L_2(\mathbf{R}^3)}$. If J_0 is so small that $A_0 I^{1/4}(2J_0)^{1/4} < \frac{1}{2}$, then from (2.14) we conclude

$$J \leq 2J_0. \tag{2.15}$$

Thus, the smallness of initial J_0 implies the global in time smallness of norms controlled by J – see (2.8). Here we stop the considerations for Theorem 2, since the rest of the proof is almost the same as in the proof of the main theorem. Hence we claim that Theorem 2 has been proved.

Remark. From the imbedding theorem in $\mathbf{R}^3$ we have

$$||w||_{L_6(\mathbf{R}^3)} \leq C||w_{,x}||^{1/3}_{L_2(\mathbf{R}^3)}||w_{,y}||^{1/3}_{L_2(\mathbf{R}^3)}||w_{,z}||^{1/3}_{L_2(\mathbf{R}^3)}. \tag{2.16}$$

Smallness of J_0 may imply that the L_6-norm of initial datum v_0 is small, too. Next, the interpolation estimate may follow the L_3-norm is small, too. However it is not the case. The initial datum taken in Theorem 2 (or in Theorem 1) may be chosen in that way the L_3-norm is arbitrary large (even $L_{2+\epsilon}$) and the only restriction is posed on the L_2-norm of the derivative with respect to z. It has to be sufficiently small comparing to the magnitude of the “rest” of the norm of the initial datum.

3 Control of the L_2-norm

In this part we show the basic a priori estimate of the L_2-norm of solutions to system (1.3). Precisely, we prove the estimate to part (i) of Theorem 1.

Lemma 3. *Let $w \in \Xi$, then sufficiently smooth solutions to (1.3) fulfill the following estimate*

$$||u||_{L_\infty(0,\infty;L_2(\mathbf{R}^3))} + ||\nabla u||_{L_2(\mathbf{R}^3\times(0,\infty)))} \leq C||u_0||_{L_2(\mathbf{R}^3)}. \tag{3.1}$$

Proof. For any given w fulfilling (1.4) and any given $\epsilon > 0$ we are able to find a smooth function $Q : (0,\infty) \times \mathbf{R}^3 \to \mathbf{R}$ such that

$$||Q - |\nabla w|\,||_{L_2(\mathbf{R}^3_{xy}\times(0,\infty)_t;L_\infty(\mathbf{R}_z))} \leq \epsilon \tag{3.2}$$

and

$$Q \in L_1(0,\infty; L_\infty(\mathbf{R}^3)). \tag{3.3}$$

We treat system (1.1) as a perturbation of a special flow w. Multiplying $(1.3)_1$ by u, integrating over $\mathbf{R}^3$, we get

$$\frac{d}{dt}||u||^2_{L_2(\mathbf{R}^3)} + 2\nu||\nabla u||^2_{L_2(\mathbf{R}^3)} = -2\int_{\mathbf{R}^3} u\cdot\nabla w u d\bar{x}. \tag{3.4}$$

Let us consider the r.h.s. of (3.4). Since the regularity or rather vanishing conditions on function w are too weak, we apply a trick with function Q in the following way

$$\left|\int_{\mathbf{R}^3} u\cdot\nabla w u d\bar{x}\right| \leq \int_{\mathbf{R}^3}|Q||u|^2 d\bar{x} + \int_{\mathbf{R}^3}||Q - |\nabla w(t,x,y,\cdot)|\,||_{L_\infty(\mathbf{R}_z)}|u|^2 d\bar{x}.$$

Hence the identity (3.4) yields the following inequality

$$\frac{d}{dt}\left[||u||^2_{L_2(\mathbf{R}^3)}\exp\left\{-\int_0^t ||Q||_{L_\infty(\mathbf{R}^3)}ds\right\}\right] + 2\nu||\nabla u||^2_{L_2(\mathbf{R}^3)}\exp\left\{-\int_0^t ||Q||_{L_\infty(\mathbf{R}^3)}ds\right\}$$
$$\leq C\int_{\mathbf{R}^3} ||Q - |\nabla w|\,||_{L_\infty(\mathbf{R}_z)}|u|^2 \exp\left\{-\int_0^t ||Q||_{L_\infty(\mathbf{R}^3)}ds\right\} d\bar{x}. \tag{3.5}$$

Let us introduce an auxiliary function redefining our sought function

$$\mathcal{U} = u\exp\left\{-\frac{1}{2}\int_0^t ||Q||_{L_\infty(\mathbf{R}^3)}ds\right\}. \tag{3.6}$$

And again, the same as in section 2, we introduce

$$K = ||\mathcal{U}||_{L_\infty(0,\infty;L_2(\mathbf{R}^3))} + ||\nabla\mathcal{U}||_{L_2(\mathbf{R}^3\times(0,\infty))}. \tag{3.7}$$

Then inequality (3.5) can be stated as follows

$$K^2 \leq C\int_0^\infty\int_{\mathbf{R}^3} ||Q - |\nabla w|\,||_{L_\infty(\mathbf{R}_z)}|\mathcal{U}|^2 d\bar{x} + ||u_0||^2_{L_2(\mathbf{R}^3)}. \tag{3.8}$$

Assuming finiteness of K the same as for (2.11)-(2.12) we conclude

$$\mathcal{U} \in L_4(\mathbf{R}^2_{xy}\times(0,\infty)_t; L_2(\mathbf{R}_z)).$$

Take the first term from the r.h.s. of (3.8)

$$\int_0^\infty dt\int_{\mathbf{R}^3} ||Q - |\nabla w|\,||_{L_\infty(\mathbf{R}_z)}|\mathcal{U}|^2 dxdydz \leq \int_0^\infty dt\int_{\mathbf{R}^2} ||Q - |\nabla w|\,||_{L_\infty(\mathbf{R}_z)}||\mathcal{U}||^2_{L_2(\mathbf{R}_z)}dxdy$$
$$\leq C\left(\int_0^\infty\int_{\mathbf{R}^2} ||Q - |\nabla w|\,|^2_{L_\infty(\mathbf{R}_z)}dtdxdy\right)^{1/2}\left(\int_0^\infty\int_{\mathbf{R}^2} ||\mathcal{U}||^4_{L_2(\mathbf{R}_z)}dtdxdy\right)^{1/2}. \tag{3.9}$$

So by (3.2) and (3.9) inequality (3.8) takes the following form

$$K^2 \leq C\epsilon K^2 + K_0^2 \tag{3.10}$$

with $K_0 = C||u_0||_{L_2(\mathbf{R}^3)}$. Since C in (3.10) is an absolute constant, we can choose ϵ – see (3.2) – such that inequality (3.10) yields

$$K \leq 2K_0. \tag{3.11}$$

From the definition of K – see (3.7) – we deduce (3.1), since by (3.3) integral $\int_0^\infty ||Q||_{L_\infty}ds$ is finite and given. Lemma 3 is proved.

The obtained estimate stays independently from the magnitude of initial datum K_0. Hence if K_0 is small, then K is small, too. Lemma 3 applied to case (i) from Theorem 1 guarantees that uniformally in time the smallness of the L_2-norm is controlled.

Another advantage of Lemma 3 is that it does not require smallness of the L_2-norm of initial datum u_0, hence it works in case (ii) of Theorem 1, too. Thus, the next section starts with information given by (3.1).

4 Differentiation with respect to "z"

In this section we show the main estimate of the proof of the second part of Theorem 1. We prove.

Lemma 4. *Let assumptions of Theorem 1 – case (ii) with conditions (1.9) be fulfilled, then sufficiently smooth solutions to system (1.3) satisfy the following bound*

$$||u_{,t}||_{L_\infty(0,\infty;L_2(\mathbf{R}^3))} + ||\nabla u||_{L_2(\mathbf{R}^3\times(0,\infty))} \le C(||u_{0,z}||_{L_2(\mathbf{R}^3)} + \sigma ||u||_{L_2(\mathbf{R}^3)}). \tag{4.1}$$

where σ describes smallness of norms mentioned in condition (1.9).

Proof. Differentiating system (1.3) with respect to the z-coordinate we get from the first (momentum) equation the following one

$$\begin{aligned} &u_{,zt} + v\cdot\nabla u_{,z} - \nu\Delta u_{,z} + \nabla p_{,z} = \\ &\qquad -u_{,z}\cdot\nabla u + w_{,z}\cdot\nabla u - u_{,z}\cdot\nabla w - u\cdot\nabla w_{,z} \quad \text{in } \mathbf{R}^3\times(0,\infty). \end{aligned} \tag{4.2}$$

Multiplying (4.2) by $u_{,z}$, integrating over $\mathbf{R}^3$, we get

$$\begin{aligned} \frac{d}{dt}||u_{,z}||^2_{L_2(\mathbf{R}^3)} + 2\nu||\nabla u_{,z}||^2_{L_2(\mathbf{R}^3)} \le C\bigg(\int_{\mathbf{R}^3}|u_{,z}\cdot\nabla w u_{,z}|d\bar{x} + \int_{\mathbf{R}^3}|u_{,z}\cdot\nabla u u_{,z}|d\bar{x} \\ + \int_{\mathbf{R}^3}|w_{,z}\cdot\nabla u u_{,z}|d\bar{x} + \int_{\mathbf{R}^3}|u\cdot\nabla w_{,z}u_{,z}|d\bar{x}\bigg) = I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.3}$$

In the case as generic solution w is generated by a two dimensional flow integrals I_3 and I_4 vanish and conditions (1.9) are trivially fulfilled (σ in (4.1) is equal zero).

The same as in Lemma 3 we introduce

$$\mathcal{U}_{,z} = u_{,z}\exp\{-\frac{1}{2}\int_0^t ||Q||_{L_\infty(\mathbf{R}^3)}ds\}.$$

Thus from (4.3) and properties of function Q – (3.2) and (3.3) – we get

$$\begin{aligned} \frac{d}{dt}||\mathcal{U}_{,z}||^2_{L_2(\mathbf{R}^3)} + 2\nu||\nabla\mathcal{U}_{,z}||^2_{L_2(\mathbf{R}^3)} \le C\bigg(\int_{\mathbf{R}^3}|Q - |\nabla w||\,|\mathcal{U}_{,z}|^2 d\bar{x} \\ + \int_{\mathbf{R}^3}|\mathcal{U}_{,z}\cdot\nabla u\,\mathcal{U}_{,z}|d\bar{x} + \int_{\mathbf{R}^3}|w_{,z}\cdot\nabla\mathcal{U}\,\mathcal{U}_{,z}|d\bar{x} + \int_{\mathbf{R}^3}|\mathcal{U}\cdot\nabla w_{,z}\mathcal{U}_{,z}|d\bar{x}\bigg) \end{aligned} \tag{4.4}$$

which leads the following inequality

$$\begin{aligned} L \le C\Bigg[\bigg(\int_0^\infty dt\int_{\mathbf{R}^3}|Q - |\nabla w||\,|\mathcal{U}_{,z}|^2 d\bar{x}\bigg)^{1/2} \\ + \bigg(\exp\{\frac{1}{2}\int_0^\infty ||Q||_{L_\infty(\mathbf{R}^3)}ds\}\int_0^\infty dt\int_{\mathbf{R}^3}|\mathcal{U}_{,z}\cdot\nabla\mathcal{U}\,\mathcal{U}_{,z}|d\bar{x}\bigg)^{1/2} \\ + \bigg(\int_0^\infty dt\int_{\mathbf{R}^3}|w_{,z}\cdot\nabla\mathcal{U}\,\mathcal{U}_{,z}|d\bar{x}\bigg)^{1/2} + \bigg(\int_0^\infty dt\int_{\mathbf{R}^3}\mathcal{U}\cdot\nabla w_{,z}\mathcal{U}_{,z}|d\bar{x}\bigg)^{1/2}\Bigg] + L_0 \\ = A_1 + A_2 + A_3 + A_4 + L_0, \end{aligned} \tag{4.5}$$

where

$$L = ||\mathcal{U}_{,z}||_{L_\infty((0,\infty)_t;L_2(\mathbf{R}^3))} + ||\nabla\mathcal{U}_{,z}||_{L_2(\mathbf{R}^3\times(0,\infty))}. \tag{4.6}$$

and $L_0 = C||u_{0,z}||_{L_2(\mathbf{R}^3)}$. Again, applying (3.2) and the method for estimation (3.9) the first integral from the r.h.s. of (4.5) is bounded as follows

$$A_1 \leq C||Q - |\nabla w|\,||_{L_2(\mathbf{R}^2_{xy}\times(0,\infty)_t;L_\infty(\mathbf{R}_z))}||\mathcal{U}_z||^2_{L_4(\mathbf{R}^3_{xy}\times(0,\infty)_t;L_2(\mathbf{R}_z))} \leq \epsilon L. \tag{4.7}$$

To estimate A_2 we repeat exactly steps from section 2 – estimation (2.9)-(2.13) – getting

$$\begin{aligned} A_2 &\leq \left[\exp\{\frac{1}{2}\int_0^\infty ||Q||_{L_\infty(\mathbf{R}^3)}ds\}\int_0^\infty\int_{\mathbf{R}^3}|\mathcal{U}_{,z}\cdot\nabla\mathcal{U}\,\mathcal{U}_{,z}|d\bar{x}\right]^{1/2} \\ &\leq C\exp\{\frac{1}{4}\int_0^\infty ||Q||_{L_\infty(\mathbf{R}^3)}dx\}K^{1/4}L^{5/4}, \end{aligned} \tag{4.8}$$

where K is defined by (3.7) and by our assumptions and Lemma 3 is already given.

To estimate A_3 and A_4 we apply extra assumptions given by (1.9) having

$$\begin{aligned} A_3 &\leq C[||w_{,z}||_{L_5(\mathbf{R}^3\times(0,\infty))}||\nabla\mathcal{U}||_{L_2(\mathbf{R}^3\times(0,\infty))}||\mathcal{U}_{,z}||_{L_{10/3}(\mathbf{R}^3\times(0,\infty))}]^{1/2} \leq C\sigma K^{1/2}L^{1/2}, \\ A_4 &\leq C[||\nabla w_{,z}||_{L_{5/2}(\mathbf{R}^3\times(0,\infty))}||\mathcal{U}||_{L_{10/3}(\mathbf{R}^3\times(0,\infty))}||\mathcal{U}_{,z}||_{L_{10/3}(\mathbf{R}^3\times(0,\infty))}]^{1/2} \leq C\sigma K^{1/2}L^{1/2}, \end{aligned} \tag{4.9}$$

where we applied the parabolic imbedding into $L_{10/3}(\mathbf{R}^3\times(0,\infty))$.

Summing up estimates (4.5)-(4.9), remembering that Q is given and fulfills (3.3), thus the integral in the r.h.s. of (4.8) is given, too, we obtain the following inequality

$$L \leq \epsilon L + C\exp\{\frac{1}{4}\int_0^\infty ||Q||_{L_\infty(\mathbf{R}^3)}dx\}K^{1/4}L^{5/4} + C\sigma K^{1/2}L^{1/2} + L_0. \tag{4.10}$$

Smallness of ϵ – see (3.2) – and σ – see (1.9) – reduces (4.10) to the following form

$$L \leq A_1K^{1/4}L^{5/4} + \sigma K + 2L_0. \tag{4.11}$$

Provided σ and L_0 such that $A_1K^{1/4}[4(L_0+\sigma K)]^{1/4} < \frac{1}{2}$, controlling K by Lemma 3 and (3.11), we conclude that

$$L \leq 4(L_0 + \sigma K_0). \tag{4.12}$$

Hence by (4.12) we get bound (4.1) guaranteeing us smallness of the l.h.s. in this estimate. Lemma 4 is proved.

Now we are prepared to show estimate (1.10) from Theorem 1.

5 The existence

In this section we show a proof of existence of regular global in time solutions to system (1.3) guaranteeing by Theorem 1. Local in time results for these systems follow from the standard approach and detailed proofs can be found e.g. in [15],[16],[24]. Hence to obtain global in time solutions a priori estimates in a suitable high class of regularity is required, only. Here it will be the $W_4^{2,1}$-space – see (1.10) and (1.11). First we consider case (ii) which seems to be more advanced than (i).

A key element of our technique will be an application of information about global smallness of quantity L controlling by Lemma 4. A direct method seems to be not so effective, but by the imbedding theorem we get a more suitable quantity. By (2.16) we conclude

$$||u(\cdot,t)||_{L_6(\mathbf{R}^3)} \leq C||u_{,z}(\cdot,t)||^{1/3}_{L_2(\mathbf{R}^3)}||u_{,x}(\cdot,t)||^{1/3}_{L_2(\mathbf{R}^3)}||u_{,y}(\cdot,t)||^{1/3}_{L_2(\mathbf{R}^3)} \tag{5.1}$$

which leads us to the following inequality

$$||u||_{L_3(0,\infty;L_6(\mathbf{R}^3))} \leq C||u_{,z}||^{1/3}_{L_\infty(0,\infty;L_2(\mathbf{R}^3))}||\nabla u||^{2/3}_{L_2(\mathbf{R}^3\times(0,\infty))}. \tag{5.2}$$

Next, let us note that the interpolation between L_p spaces implies

$$L_4(0,\infty;L_4(\mathbf{R}^3)) = (L_3(0,\infty;L_6(\mathbf{R}^3)), L_\infty(0,\infty;L_2(\mathbf{R})^3))_{1/4}\,. \tag{5.3}$$

Hence remembering that the energy norm (3.1) is controlled by Lemma 3 by given data, from (5.2) and (3.1) we obtain

$$||u||_{L_4(\mathbf{R}^3\times(0,\infty))} \leq C||u_{,z}||^{1/3(1-1/4)}_{L_\infty(0,\infty;L_2(\mathbf{R}^3))}, \tag{5.4}$$

where C in (5.4) contains the energy norm given by Lemma 3. That is the reason we choose the $W_4^{2,1}$-space to show existence of regular solutions to (1.3). Obviously we can repeat the proof for any $W_p^{2,1}$ with general p – see [15].

Now we estimate solutions in higher norms. We restate problem (1.3) in the following form

$$\begin{array}{lll} u_{,t} - \nu\Delta u + \nabla p = -u\cdot\nabla u - w\cdot\nabla u - u\cdot\nabla w & \text{in} & \mathbf{R}^3\times(0,T), \\ \text{div}\, u = 0 & \text{in} & \mathbf{R}^3\times(0,T), \\ u|_{t=0} = u_0 & \text{on} & \mathbf{R}^3. \end{array} \tag{5.5}$$

Time T – above – describes the lifespan of the maximal solution given by the local result. Our goal is to show that we will be able to prolong this time to $T=\infty$ at the end of our analysis.

By the classical results [6],[16],[24] for the Stokes system in the whole space (the l.h.s. of (5.5)) the following L_p-Schauder type estimate for solutions to (5.5) is known

$$\begin{aligned} &||u_{,t}||_{L_p(\mathbf{R}^3\times(0,T))} + ||\nabla^2 u||_{L_p(\mathbf{R}^3\times(0,T))} \\ &\leq C\left(||\text{r.h.s.of}(5.5)_1||_{L_p(\mathbf{R}^3\times(0,T))} + ||u_0||_{W_p^{2-2/p}(\mathbf{R}^3)}\right), \end{aligned} \tag{5.6}$$

where C does not depend on T, so we can put $T=\infty$ in estimate (5.6). In our case we consider bound (5.6) for $p=4$.

To apply estimate (5.6) there is a need to find bound on the r.h.s. of (5.5) in the L_4-norm.

The imbedding theorem [5, Chap. 11] yields the following inclusions

$$W_4^{2,1}(\mathbf{R}^3\times(0,T)) \subset L_{12}(\mathbf{R}^3\times(0,T)), \quad \nabla W_4^{2,1}(\mathbf{R}^3\times(0,T)) \subset L_6(\mathbf{R}^3\times(0,T)),$$

moreover there exists a function $c(\cdot)$ such that $c(\sigma) \to \infty$ as $\sigma \to 0$ and

$$||u||_{L_{12}(\mathbf{R}^3\times(0,T))} + ||\nabla u||_{L_6(\mathbf{R}^3\times(0,T))} \leq \sigma < u >_{W_4^{2,1}(\mathbf{R}^3\times(0,T))} + c(\sigma)||u||_{L_4(\mathbf{R}^3\times(0,T))}, \quad (5.7)$$

where $< \cdot >_{W_4^{2,1}}$ denotes the main seminorm of space $W_4^{2,1}(\mathbf{R}^3 \times (0,T))$ – see (1.11).

Applying estimate (5.7) to terms of the r.h.s. of (5.6) we get

$$\begin{aligned} ||u\cdot\nabla u||_{L_4(\mathbf{R}^3\times(0,T))} &\leq C||u||_{L_{12}(\mathbf{R}^3\times(0,T))}||\nabla u||_{L_6(\mathbf{R}^3\times(0,T))} \\ &\leq \sigma^2 < u >^2_{W_4^{2,1}(\mathbf{R}^3\times(0,T))} + c(\sigma)||u||^2_{L_4(\mathbf{R}^3\times(0,T))} \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} ||w\cdot\nabla u||_{L_4(\mathbf{R}^3\times(0,T))} &\leq \sigma < u >_{W_4^{2,1}(\mathbf{R}^3\times(0,T))} + c(\sigma, ||w||_{L_\infty(\mathbf{R}^3\times(0,T))})||u||_{L_4(\mathbf{R}^3\times(0,T))}, \\ ||u\cdot\nabla w||_{L_4(\mathbf{R}^3\times(0,T))} &\leq C||\nabla w||_{L_\infty(\mathbf{R}^3\times(0,T))}||u||_{L_4(\mathbf{R}^3\times(0,T))}. \end{aligned} \quad (5.9)$$

Inserting (5.8) and (5.9) to estimate (5.6), remembering about (5.4), we obtain

$$\begin{aligned} < u >_{W_4^{2,1}(\mathbf{R}^3\times(0,T))} &\leq \sigma < u >_{W_4^{2,1}(\mathbf{R}^3\times(0,T))} + \sigma^2 < u >^2_{W_4^{2,1}(\mathbf{R}^3\times(0,T))} \\ &+ c(\sigma, ||w||_{W^1_\infty(\mathbf{R}^3\times(0,\infty))})||u||_{L_4(\mathbf{R}^3\times(0,\infty))} + C < u_0 >_{W_4^{2-1/2}(\mathbf{R}^3)}. \end{aligned} \quad (5.10)$$

Provided smallness of σ, remembering that the L_4-norm of u by bound (5.4) is sufficiently small by (5.10), from (5.10) we obtain

$$< u >_{W_4^{2,1}(\mathbf{R}^3\times(0,T))} \leq DATA. \quad (5.11)$$

Note that to obtain (5.11) smallness of $< u_0 >_{W_4^{2-1/2}(\mathbf{R}^3)}$ is not required, the only condition on

$$(1-\sigma)^2 > 4\sigma^2 \left[c(\sigma, ||w||_{W^1_\infty(\mathbf{R}^3\times(0,\infty))})||u||_{L_4(\mathbf{R}^3\times(0,\infty))} + C < u_0 >_{W_4^{2-1/2}(\mathbf{R}^3)}\right]. \quad (5.12)$$

But the choice of σ is arbitral, additionally it prescribes the smallness of the L_4-norm of u by (5.4), thus the r.h.s. of (5.12) can be arbitrary small.

$DATA$ in (5.11) are bounded by all given data, in general case it may not be small.

However, first of all the r.h.s. of (5.11) does not depend on T, hence we are able to extend our estimate on $T = \infty$, getting the desired global in time solutions with sufficiently high regularity guaranteeing the smoothness. Thus, we proved case (ii) for Theorem 1.

Let us briefly look on case (i). This part of Theorem 1 is similar to case (ii), so we point a reduction of this case to the first considered one.

From Lemma 3 and the parabolic imbedding we immediately obtain smallness of the $L_{10/3}$-norm, i.e.

$$||u||_{L_{10/3}(\mathbf{R}^3\times(0,\infty))} \leq C||u_0||_{L_2(\mathbf{R}^3)}. \quad (5.13)$$

Additionally the theory from [5, Chap. 18] guarantees us an analogical estimate (5.7), but with the $L_{10/3}$-norm, i.e. there exists a function $c(\sigma) \to \infty$ and $\sigma \to 0$ such that

$$||u||_{L_{12}(\mathbf{R}^3\times(0,\infty))} + ||\nabla u||_{L_6(\mathbf{R}^3\times(0,\infty))} \leq \sigma < u >_{W_4^{2,1}(\mathbf{R}^3\times(0,\infty))} + c(\sigma)||u||_{L_{10/3}(\mathbf{R}^3\times(0,\infty))}.$$

Thus, remembering (5.13), the whole estimation (5.6)-(5.11) is almost the same. Concluding in a similar way we are able to show

$$< u >_{W_4^{2,1}(\mathbf{R}^3\times(0,T))} \leq DATA. \tag{5.14}$$

The same for (5.11) we can obtain bound (5.14) on time interval $(0, \infty)$.

The proof of Theorem 1 is done.

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