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3D STEADY COMPRESSIBLE NAVIER–STOKES EQUATIONS

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Abstract. We study the steady compressible Navier–Stokes equations in a bounded smooth three-dimensional domain, together with the slip boundary conditions. We show that for a certain class of the pressure laws, there exists a weak solution with bounded density (in L^∞ up to boundary).

1. Introduction; main result.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth connected boundary. We consider the system of PDE's describing the steady flow of a compressible Newtonian gas in Ω

$$\begin{aligned} \operatorname{div}(\varrho \boldsymbol{v}) &= 0 \\ \varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \operatorname{div}\big(2\mu \boldsymbol{D}(\boldsymbol{v}) + \nu(\operatorname{div} \boldsymbol{v})\boldsymbol{I} - \pi(\varrho)\boldsymbol{I}\big) &= \varrho \boldsymbol{F}, \end{aligned} \tag{1}$$

together with the boundary conditions on $\partial\Omega$

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{n} &= 0 \\ \boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{v}, \pi(\varrho)) \cdot \boldsymbol{\tau}_k + f\boldsymbol{v} \cdot \boldsymbol{\tau}_k &= 0, \quad k = 1, 2. \end{aligned} \tag{2}$$

Here, $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ represents the sought velocity field, $\pi(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}$, a given function of the sought density ϱ, is the pressure. The density is non-negative and the total mass of the fluid is given $\int_\Omega \varrho dx = M > 0$. The external volume force is denoted by $\boldsymbol{F}$. The stress tensor is given by

$$\boldsymbol{T}(\boldsymbol{v}, \pi(\varrho)) = 2\mu \boldsymbol{D}(\boldsymbol{v}) + \nu(\operatorname{div} \boldsymbol{v})\boldsymbol{I} - \pi(\varrho)\boldsymbol{I}. \tag{3}$$

Here, $\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$ stands for the symmetric part of the velocity gradient and μ and ν are the viscosity coefficients. From the thermodynamical considerations it follows that $\mu > 0$, $2\mu + 3\nu > 0$.

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The boundary $\partial\Omega$ is assumed to be fixed and impermeable; however, the tangent component of the velocity is in general nonzero; τ_k, $k=1,2$ are two perpendicular tangent vectors to $\partial\Omega$. The slip coefficient f is constant and non-negative; if $f=0$, special geometric restrictions on Ω are needed which is connected with the validity of Korn's inequality in the right form (the domain cannot be axially symmetric), see [8] for more information.

Next, we summarize our conditions on the pressure function. We assume that

$$\begin{array}{ll} \pi(t)=\int_0^t sR(s)ds \qquad \text{with } R(s)>0 \quad \text{on} \quad \mathbb{R}^+; \\ b_1+a_1 s^{\gamma^+-2}\geq R(s)\geq a_2 s^{\gamma_--2}+b_2, \qquad R(\cdot)\in C(\mathbb{R}_0^+); \\ a_1,a_2>0,\, b_1\geq b_2\geq 0, \quad \gamma^+\geq\gamma_->3, \quad \frac{2}{3}\gamma^++1<\gamma_-. \end{array} \tag{4}$$

Remark 1. Note that in the two-dimensional case we may take $\gamma^+\geq\gamma_->1$ with no further restriction. The proof follows the same lines as in [4], with several changes as presented below. We leave the proof to the kind reader.

Our main result is the following

Theorem 1. *Let $\Omega\in C^2$ be a bounded subdomain in $\mathbb{R}^3$, the constants f, μ, ν and M and the domain Ω satisfy conditions given above. Let $\pi(\cdot)$ satisfy (4) and $\boldsymbol{F}\in L^\infty(\Omega)$. Then there exists at least one weak solution to problem (1)–(3) such that $\varrho\in L^\infty(\Omega)$, $\boldsymbol{v}\in W^{1,p}(\Omega)$ for all $p<\infty$.*

Remark 2. We call a pair $(\varrho,\boldsymbol{v})$ such that $\varrho\in L^{\gamma^+}(\Omega)$, $\boldsymbol{v}\in W^{1,2}(\Omega)$ a weak solution to (1)–(3) iff

$$\begin{aligned} &\int_\Omega \varrho\boldsymbol{v}\cdot\nabla\eta=0 \qquad \forall\eta\in C^\infty(\overline{\Omega}) \\ &\int_\Omega \big(\varrho(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{\varphi}+2\mu\boldsymbol{D}(\boldsymbol{v}):\boldsymbol{D}(\boldsymbol{\varphi})+\nu(\operatorname{div}\boldsymbol{v})(\operatorname{div}\boldsymbol{\varphi})-\pi(\varrho)\operatorname{div}\boldsymbol{\varphi}\big)dx \\ &+\int_{\partial\Omega} f\big((\boldsymbol{v}\cdot\boldsymbol{\tau}_1)(\boldsymbol{\varphi}\cdot\boldsymbol{\tau}_1)+(\boldsymbol{v}\cdot\boldsymbol{\tau}_2)(\boldsymbol{\varphi}\cdot\boldsymbol{\tau}_2)\big)d\sigma=\int_\Omega\varrho\boldsymbol{F}\cdot\boldsymbol{\varphi}dx \\ &\forall\boldsymbol{\varphi}\in C^\infty(\overline{\Omega});\boldsymbol{\varphi}\cdot\boldsymbol{n}=0 \quad \text{at } \partial\Omega. \end{aligned}$$

The first global existence result for the steady compressible Navier–Stokes equations goes back to the book by Lions, see [2]. He considered mostly the Dirichlet boundary conditions and proved the existence of a (renormalized) weak solution for $\pi(\varrho)=\varrho^\gamma$ with $\gamma>1$ in two space dimensions and $\gamma>\frac{5}{3}$ in three space dimensions, in both cases for $\Omega\in C^2$. Next, based on the Feireisl's approach for the evolutionary problem, Novo and Novotný, see [6], proved the existence in three space dimensions for $\gamma>\frac{3}{2}$, however only for potential external forces with a small non potential perturbation. Moreover, they allowed also less regular domains (Lipschitz continuous). Already in the book [2] it can be found that if $\gamma>1$ (N=2) or $\gamma>3$ (N=3) the constructed solution belongs to $L^\infty_{loc}(\Omega)$ and $\boldsymbol{v}\in W^{1,p}_{loc}(\Omega)$ for all $p<\infty$; the proof requires quite heavy technique; see also [5] for similar results.

The slip boundary condition was rigorously studied in the paper [4]. However, only the simple form $\pi(\varrho)=\varrho^\gamma$ was considered. Here we present similar results for three space dimensions, for more general pressure functions. Our method is based on a special approximation scheme introduced in [4], see next section. The presented technique enables to control uniformly the L^∞-norm of approximative densities and it reduces a number of necessary tricks which appear in the classical theory [2].

2. **Construction of the approximation.** Let $\varepsilon > 0$. Let $m > 0$ (sufficiently large). Let $K : \mathbb{R}_0^+ \to [0,1]$ be a smooth non-increasing function such that

$$K(t) = \begin{cases} 1 & \text{for} \quad t \le m-1 \\ \in (0,1) & \text{for} \quad m-1 < t < m \\ 0 & \text{for} \quad m \le t. \end{cases} \tag{5}$$

Moreover, $K'(t) < 0$ in $(m-1, m)$. Denote

$$P(t) = \int_0^t K(s)\pi'(s)ds;$$

(in particular $P(t) \ge K(t)\pi(t)$). Then we consider

$$\left.\begin{array}{r} \varepsilon h\boldsymbol{v} + \varepsilon\varrho\boldsymbol{v} + \frac{1}{2}\mathrm{div}\,(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}) + \frac{1}{2}K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} \\ -\mathrm{div}\,\big(2\mu\boldsymbol{D}(\boldsymbol{v}) + \nu(\mathrm{div}\,\boldsymbol{v})\boldsymbol{I} - P(\varrho)\boldsymbol{I}\big) = K(\varrho)\varrho\boldsymbol{F} \\ \varepsilon\varrho + \mathrm{div}\,(K(\varrho)\varrho\boldsymbol{v}) - \varepsilon\Delta\varrho = \varepsilon hK(\varrho) \end{array}\right\} \text{ in } \Omega$$
$$\left.\begin{array}{c} \frac{\partial\varrho}{\partial\boldsymbol{n}} = 0 \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \\ \boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{v}, P(\varrho))\cdot\boldsymbol{\tau}_k + f\boldsymbol{v}\cdot\boldsymbol{\tau}_k = 0,\ k = 1,2 \end{array}\right\} \text{ at } \partial\Omega. \tag{6}$$

Proposition 1. *Let $\varepsilon > 0$, $h = \frac{M}{|\Omega|}$, $K(\cdot)$ satisfy (5). Let assumptions of Theorem 1 be satisfied. Then there exists a solution to (6) such that $\boldsymbol{v}_\varepsilon \in W^{2,p}(\Omega)$, $\varrho_\varepsilon \in W^{2,p}(\Omega)$ for all $p < \infty$. Moreover, $0 \le \varrho_\varepsilon \le m$, $\int_\Omega \varrho_\varepsilon dx \le M$.*

Proof: The proof of Proposition 1 will be divided into several steps.
Step 1: We first show the a priori estimates for the density. In order to verify the L^1 bound, we simply integrate $(6)_2$ over Ω. Then

$$\varepsilon\int_\Omega \varrho dx + \int_{\partial\Omega} K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{n}d\sigma - \varepsilon\int_{\partial\Omega}\frac{\partial\varrho}{\partial\boldsymbol{n}}d\sigma = \varepsilon h\int_\Omega K(\varrho)dx.$$

Due to the regularity of ϱ and $\boldsymbol{v}$, together with the boundary conditions, the surface integrals are equal to zero which gives immediately the L^1 bound for the approximative density. Next, we integrate $(6)_2$ over the sets $\{x \in \Omega; \varrho(x) < 0\}$ and $\{x \in \Omega; \varrho(x) > m\}$, provided these sets are regular. (If this is not the case, we replace 0 and m by sequences $\varepsilon_n \to 0^-$ and $m_n \to m^+$, respectively.) Thus

$$\varepsilon\int_{\Omega^-} \varrho dx + \int_{\partial\Omega^-} K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{n}d\sigma - \varepsilon\int_{\partial\Omega^-}\frac{\partial\varrho}{\partial\boldsymbol{n}}d\sigma = \varepsilon h\int_{\Omega^-} K(\varrho)dx,$$

where $\Omega^- = \{x \in \Omega; \varrho(x) < 0\}$. Again, the boundary terms are zero, which, due to the properties of $K(\cdot)$ implies $|\{x \in \Omega; \varrho(x) < 0\}| = 0$. Similarly we prove $\varrho(x) \le m$; here the fact that $\frac{\partial\varrho}{\partial n} \le 0$ at the boundary of $\{x \in \Omega; \varrho(x) > m\}$ is used.
Step 2: Next we consider the approximative continuity equation. We denote for $p \in [1, \infty]$

$$M_p = \{\boldsymbol{w} \in W^{1,p}(\Omega); \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ at } \partial\Omega\}$$

and define $S : M_\infty \to W_p^2(\Omega)$, $1 \le p < \infty$, $S(\boldsymbol{v}) = \varrho$, where ϱ solves

$$\begin{aligned} -\varepsilon\Delta\varrho + \varepsilon\varrho + \mathrm{div}\,\big(K(\varrho)\varrho\boldsymbol{v}\big) &= \varepsilon hK(\varrho) &&\text{in } \Omega \\ \frac{\partial\varrho}{\partial\boldsymbol{n}} &= 0 &&\text{at } \partial\Omega. \end{aligned}$$

We have

Lemma 1. *Let assumptions of Proposition 1 be satisfied. Then S is well defined as the operator from M_∞ to $W^2_p(\Omega)$ for any $p<\infty$. Moreover, if $\varrho=S(\boldsymbol{v})$ then $0\le\varrho\le m$ in Ω, $\int_\Omega \varrho dx\le M$ and if $\|\boldsymbol{v}\|_{1,\infty}\le L$, $L>0$ then*

$$\|\varrho\|_{2,p}\le C(\varepsilon,p,\Omega)(1+L)h, \qquad 1\le p<\infty. \tag{7}$$

Proof: The a priori bounds $0\le\varrho\le m$ and the L^1 bound follow from the estimates presented in Step 1. We have to prove solvability of the approximative continuity equation. To this aim, for $t\in[0,1]$, we consider the following family of problems

$$\varrho_t=T_t(\xi), \qquad \xi\in W^{1,p}(\Omega),$$

where

$$\begin{aligned} -\varepsilon\Delta\varrho_t=-t\varepsilon\xi-t\mathrm{div}\,\big(K(\xi)\xi\boldsymbol{v}\big)+t\varepsilon hK(\xi) \qquad &\text{in } \Omega\\ \frac{\partial\varrho_t}{\partial\boldsymbol{n}}=0 \qquad &\text{at } \partial\Omega. \end{aligned} \tag{8}$$

First note that we look for solutions to (8) such that $\int_\Omega \varrho_t dx=h\int_\Omega K(\varrho_t)dx$. Thus the solution to (8) is uniquely determined. Especially, for $t=0$, the solution is $\varrho=A$ with A being the unique solution to

$$A=hK(A).$$

Note that $0<A<m$ for $h>0$. Further, we have to verify that there is $C=C(p,L,\varepsilon,\Omega,h)$ such that $\|\varrho_t\|_{1,p}\le C$ for any fixed point $\varrho_t=T_t(\varrho_t)$ which implies that $0\notin(I-T_t)(\partial B_C)$, where

$$B_C=\big\{\varrho\in W^{1,p}(\Omega); \|\varrho\|_{1,p}\le C, \int_\Omega \varrho dx=h\int_\Omega K(\varrho)dx\big\}.$$

Note that if $\varrho_t=T_t(\varrho_t)$ and $\varrho_t\in W^{1,p}(\Omega)$ with the bound $\|\varrho_t\|_{1,p}\le C$ then also $\nabla\varrho_t\in W^{1,p}(\Omega)$ and $\|\varrho_t\|_{2,p}\le C(p,L,\varepsilon,\Omega,h)$.

Using as test function in $\varrho_t=T_t(\varrho_t)$ the function ϱ_t^β with $0<\beta\le 1$ we get

$$\|\varrho_t\|_{\beta+1}\le C(\beta,\Omega)h$$

for β chosen appropriately, for details see [8], Lemma 4.30. Next we rewrite our problem as

$$\begin{aligned} -\varepsilon\Delta\varrho_t=\mathrm{div}\,\boldsymbol{b} \qquad &\text{in } \Omega\\ \frac{\partial\varrho_t}{\partial\boldsymbol{n}}=\boldsymbol{b}\cdot\boldsymbol{n} \qquad &\text{at } \partial\Omega \end{aligned}$$

with $\boldsymbol{b}=\varepsilon t\mathcal{B}\big(K(\varrho_t)h-\varrho_t\big)-t\varrho_t\boldsymbol{v}K(\varrho_t)$, where $\mathcal{B}:\{f\in L^p(\Omega); \int_\Omega f dx=0\}\to W^{1,p}_0(\Omega)$ is the solution operator to $\mathrm{div}\,\boldsymbol{u}=f$ with homogeneous Dirichlet boundary condition, see below. Due to the regularity of the solutions to the Neumann problem (see [8], Lemma 4.27) and the bootstrap argument

$$\|\nabla\rho_t\|_p\le C(\varepsilon,p,\Omega)(1+L)h, \quad 1\le p<\infty$$

which finally gives

$$\|\varrho_t\|_{1,p}\le C(\varepsilon,p,\Omega)(1+L)h;$$

hence (see above)

$$\|\varrho_t\|_{2,p}\le C(\varepsilon,p,\Omega)(1+L)h.$$

Next, it is not difficult to see that for any $t\in[0,1]$ and any $B_R\subset W^1_p(\Omega)$ with $B_R(\Omega)=\{w\in W^1_p(\Omega); \|w\|_{1,p}\le R, \int_\Omega w dx=h\int_\Omega K(w)dx\}$ the set $T_t(B_R)$ is precompact in $W^{1,p}(\Omega)$. Finally, for any $s,t\in[0,1]$

$$\|T_t(\xi)-T_s(\xi)\|_{1,p}\le C(L,\varepsilon,\Omega,p)|t-s|(\|\xi\|_{1,p}+h), \quad 1\le p<\infty.$$

Thus the Leray–Schauder fixed point theorem finishes the proof of Lemma 1. □

Step 3: Next we concentrate ourselves on the Lamé system. We define $\mathcal{T}$: $M_\infty \to M_\infty$, where $\mathcal{T}(\boldsymbol{v}) = \boldsymbol{w}$ with $\boldsymbol{w}$ being the solution to

$$\begin{aligned}
&-\operatorname{div}\big(2\mu \boldsymbol{D}(\boldsymbol{w}) + \nu(\operatorname{div}\boldsymbol{w})\boldsymbol{I}\big) = -\varepsilon h\boldsymbol{v} - \varepsilon\varrho\boldsymbol{v} - \frac{1}{2}\operatorname{div}\big(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}\big)\\
&-\frac{1}{2}K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla P(\varrho) + K(\varrho)\varrho\boldsymbol{F} \qquad \text{in } \Omega\\
&\boldsymbol{w}\cdot\boldsymbol{n} = 0 \qquad \text{at } \partial\Omega\\
&\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{w},P(\varrho))\cdot\boldsymbol{\tau}_k + f\boldsymbol{w}\cdot\boldsymbol{\tau}_k = 0,\ k=1,2 \text{ at } \partial\Omega,
\end{aligned} \tag{9}$$

where $\varrho = S(\boldsymbol{v})$. Note that a fixed point of $\mathcal{T}$ is our sought solution to the approximative system.

First of all, for any right hand-side of (9) $\boldsymbol{G}$ (here, $\boldsymbol{G}$ denotes all the terms on the right hand-side of $(9)_1$) such that $\boldsymbol{G} \in W^{-1,p}(\Omega)$, $1 < p < \infty$, there is a solution to (9) in M_p. Moreover, if $\Omega \in C^2$ and $\boldsymbol{G} \in L^p(\Omega)$, the solution $\boldsymbol{w} \in W^{2,p}(\Omega) \cap M_p$ and

$$\|\boldsymbol{w}\|_{2,p} \le C\|\boldsymbol{G}\|_p.$$

This is a direct consequence of the fact that the Lamé system with our boundary conditions is an elliptic system in the sense of Agmon, Douglis and Nirenberg.
Step 4: We intend to apply the Leray–Schauder fixed point theorem. To this aim, we consider a family of operators $\mathcal{T}_t$: $M_\infty \to M_\infty$, $t \in [0,1]$ such that

$$\begin{aligned}
&-\operatorname{div}\big(2\mu \boldsymbol{D}(\boldsymbol{w}_t) + \nu(\operatorname{div}\boldsymbol{w}_t)\boldsymbol{I}\big) = -t\varepsilon h\boldsymbol{v} - t\varepsilon\varrho\boldsymbol{v} - \frac{1}{2}t\operatorname{div}\big(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}\big)\\
&-\frac{1}{2}tK(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - t\nabla P(\varrho) + tK(\varrho)\varrho\boldsymbol{F} \qquad \text{in } \Omega\\
&\boldsymbol{w}_t\cdot\boldsymbol{n} = 0 \qquad \text{at } \partial\Omega\\
&\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{w}_t,P(\varrho))\cdot\boldsymbol{\tau}_k + f\boldsymbol{w}_t\cdot\boldsymbol{\tau}_k = 0,\ k=1,2 \text{ at } \partial\Omega
\end{aligned} \tag{10}$$

and $\varrho = S(\boldsymbol{v})$. We first prove

Lemma 2. *Under the assumptions of Proposition 1, let $t \in [0,1]$ and $\boldsymbol{v}_t \in M_\infty$ be a fixed point $\boldsymbol{v}_t = \mathcal{T}_t(\boldsymbol{v}_t)$. Then*

$$\|\boldsymbol{v}_t\|_{1,2}^2 + t\|P(\varrho)\|_2^2 \le C(M,\Omega,\boldsymbol{F}), \tag{11}$$

where $\varrho = S(\boldsymbol{v}_t)$. Especially, the constant C is independent of ε, m and t.

Proof: We use as test function in (10) the solution $\boldsymbol{v}_t$. Thus

$$\begin{aligned}
&\varepsilon\int_\Omega (th|\boldsymbol{v}_t|^2 + t\varrho|\boldsymbol{v}_t|^2)dx + \int_\Omega (2\mu|\boldsymbol{D}(\boldsymbol{v}_t)|^2 + \nu(\operatorname{div}\boldsymbol{v}_t)^2)dx\\
&+\int_{\partial\Omega} f\big((\boldsymbol{v}_t\cdot\boldsymbol{\tau}_1)^2 + (\boldsymbol{v}_t\cdot\boldsymbol{\tau}_2)^2\big)d\sigma + t\int_\Omega \boldsymbol{v}_t\cdot\nabla P(\varrho)dx\\
=&t\int_\Omega K(\varrho)\varrho\boldsymbol{F}\cdot\boldsymbol{v}dx.
\end{aligned} \tag{12}$$

Using the fact that (see (4))

$$\boldsymbol{v}_t\cdot\nabla P(\varrho) = (\boldsymbol{v}_t\cdot\nabla\varrho)K(\varrho)R(\varrho)\varrho = \varrho K(\varrho)\boldsymbol{v}_t\cdot\nabla\Big(\int_0^\varrho R(s)ds\Big),$$

we get, after employing the fact that $\varrho = S(\boldsymbol{v}_t)$

$$\int_\Omega \boldsymbol{v}_t\cdot\nabla P(\varrho)dx = \varepsilon\int_\Omega \Big(\varrho\Big(\int_0^\varrho R(s)ds\Big) + |\nabla\varrho|^2R(\varrho) - hK(\varrho)\Big(\int_0^\varrho R(s)ds\Big)\Big)dx.$$

The term with the negative sign can be easily controlled by the positive terms on the l.h.s. of (12). It remains to estimate the term on the r.h.s. of (12). To do this, we consider $\boldsymbol{\psi}$, a suitable solution to

$$\begin{aligned} \operatorname{div} \boldsymbol{\psi} = P(\varrho) - \frac{1}{|\Omega|}\int_\Omega P(\varrho)dx \quad &\text{in } \Omega \\ \boldsymbol{\psi} = \mathbf{0} \quad &\text{at } \partial\Omega, \end{aligned} \tag{13}$$

such that

$$\|\nabla\boldsymbol{\psi}\|_q \leq C(q,\Omega)\|P(\varrho)\|_q,$$

see [1] or [8]. Multiplying $(10)_1$ by $\boldsymbol{\psi}$ and integrating over Ω we get

$$\begin{aligned} &t\int_\Omega |P(\varrho)|^2 dx \\ =&\frac{t}{|\Omega|}\Big(\int_\Omega P(\varrho)dx\Big)^2 + \varepsilon h t\int_\Omega \boldsymbol{v}_t\cdot\boldsymbol{\psi}dx + \varepsilon t\int_\Omega \varrho\boldsymbol{v}_t\cdot\boldsymbol{\psi}dx \\ &-\frac{1}{2}t\int_\Omega K(\varrho)\varrho(\boldsymbol{v}_t\otimes\boldsymbol{v}_t):\nabla\boldsymbol{\psi}dx + \frac{1}{2}t\int_\Omega K(\varrho)\varrho(\boldsymbol{v}_t\cdot\nabla\boldsymbol{v}_t)\cdot\boldsymbol{\psi}dx \\ &+\mu\int_\Omega \nabla\boldsymbol{v}_t:\nabla\boldsymbol{\psi}dx + (\mu+\nu)\int_\Omega(\operatorname{div}\boldsymbol{v}_t)(\operatorname{div}\boldsymbol{\psi})dx - t\int_\Omega K(\varrho)\varrho\boldsymbol{F}\cdot\boldsymbol{\psi}dx. \end{aligned}$$

As the most restrictive term is the convective one ($C.T.$), we get for $\gamma_- \geq 3$

$$\begin{aligned} |C.T.| \leq& t\|\boldsymbol{v}_t\|_{1,2}^2\|\nabla\boldsymbol{\psi}\|_2\|K(\varrho)\varrho\|_6 \\ \leq& Ct\|\boldsymbol{v}_t\|_{1,2}^2\|P(\varrho)\|_2\Big(\int_\Omega |P(\varrho)|^{\frac{6}{\gamma_-}}dx\Big)^{\frac{1}{6}} \\ \leq& C(M)t\|\boldsymbol{v}_t\|_{1,2}^2\|P(\varrho)\|_2^{1+\frac{6\gamma^+-\gamma_-}{6\gamma^+-3}\frac{1}{\gamma_-}}. \end{aligned}$$

Hence

$$\sqrt{t}\|P(\varrho)\|_2 \leq C_1\sqrt{t}\|\boldsymbol{v}_t\|_{1,2}^{\frac{(6\gamma^+-3)\gamma_-}{3\gamma^+(\gamma_--1)-\gamma_-}} + C_2\sqrt{t}$$

and due to the Korn inequality (thus the additional assumptions on Ω are needed for $f=0$) we get

$$\|\boldsymbol{v}_t\|_{1,2} \leq C_1\|\boldsymbol{v}_t\|_{1,2}^{\frac{6\gamma^+-5\gamma_-}{3\gamma^+(\gamma_--1)-\gamma_-}} + C_2;$$

it provides the desired estimate of $\boldsymbol{v}_t$ and $P(\varrho)$ if $\frac{6\gamma^+-5\gamma_-}{3\gamma^+(\gamma_--1)-\gamma_-} < 1$, which is evidently satisfied. ☐

With (11) at hand, using the estimates from Lemma 1, it is not difficult to see that there exists $C = C(\varepsilon,\Omega,M,\boldsymbol{F})$, independent of t, such that for any fixed point of $\mathcal{T}_t$

$$\|\boldsymbol{v}_t\|_{1,\infty} \leq C, \qquad t\in[0,1]$$

and thus

$$0 \notin (I-\mathcal{T}_t)(\partial B_C)$$

with $B_C = \{\boldsymbol{u}\in M_\infty; \|\boldsymbol{u}\|_{1,\infty}\leq C\}$.

Step 5: To finish the proof of Proposition 1 we recall that for $t=0$ the unique fixed point of $\boldsymbol{v}_0 = \mathcal{T}_0(\boldsymbol{v}_0)$ is $\boldsymbol{v}_0 = \mathbf{0}$. Moreover, for any $t\in[0,1]$, $\mathcal{T}_t(B)$ is a precompact set in M_∞, where B is a ball of center 0 in M_∞. Finally

$$\|\mathcal{T}_t(\boldsymbol{v}) - \mathcal{T}_s(\boldsymbol{v})\|_{1,\infty} \leq C(\Omega,B,h,\boldsymbol{F})|t-s|, \qquad \boldsymbol{v}\in\overline{B}$$

and the Leray–Schauder fixed point finishes the proof of Proposition 1. □

Next we will prove certain ε-independent estimates for ϱ_ε and $\boldsymbol{v}_\varepsilon$, our solutions to the approximative system. To get them, we basically follow [4], however, due to the general pressure law we will present some more details here. To simplify the notation, in what follows, we skip writing the subindeces ε.

Lemma 3. *Under the assumptions of Proposition 1 we have for $\gamma_- \geq 3$*

$$\|\boldsymbol{v}\|_{1,2}^2 + \|P(\varrho)\|_2^2 + \varepsilon\|\nabla\varrho\|_2^2 \leq C,$$
$$\|\boldsymbol{v}\|_{1,q} + \|P(\varrho)\|_q \leq C_1 + C_2 m^{\gamma^+(1-\frac{2}{q})} \quad \textit{for} \quad 2 < q < \infty,$$

where C is independent of m, ε, however, it depends on q, M, $\|\boldsymbol{F}\|_\infty$ and Ω.

Proof: The proof of

$$\|\boldsymbol{v}\|_{1,2}^2 + \|P(\varrho)\|_2^2 \leq C$$

is a consequence of Lemma 2 (with $t = 1$) and was carried out above.

Next, we multiply the approximative continuity equation $(6)_2$ by ϱ and integrate over Ω; it gives

$$\varepsilon\int_\Omega(\varrho^2 + |\nabla\varrho|^2)dx + \int_\Omega \operatorname{div}\boldsymbol{v}\Big(\int_0^\varrho K(t)t dt\Big)dx = \varepsilon h\int_\Omega K(\varrho)\varrho dx.$$

As $\gamma_- \geq 3$ we get due to estimates proved above

$$\sqrt{\varepsilon}\|\nabla\varrho\|_2 \leq C,$$

independently of m and ε.

Finally, we prove the estimates which depend on the approximation parameter m. First, due to the standard interpolation we have

$$\|P(\varrho)\|_q \leq \|P(\varrho)\|_2^{\frac{2}{q}}\|P(\varrho)\|_\infty^{1-\frac{2}{q}} \leq C_1 m^{\gamma^+(1-\frac{2}{q})} + C_2. \tag{14}$$

Next, we rewrite the approximative momentum equation $(6)_1$ to the form

$$\left.\begin{aligned} -\operatorname{div}\big(2\mu\boldsymbol{D}(\boldsymbol{v}) + \nu(\operatorname{div}\boldsymbol{v})\boldsymbol{I}\big) = -\frac{1}{2}\operatorname{div}\left(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}\right) - \frac{1}{2}K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} \\ -\varepsilon h\boldsymbol{v} - \varepsilon\varrho\boldsymbol{v} - \nabla P(\varrho) + K(\varrho)\varrho\boldsymbol{F} \end{aligned}\right\} \text{ in } \Omega$$
$$\left.\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{n} = 0 \\ \boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{v}, P(\varrho))\cdot\boldsymbol{\tau}_k + f\boldsymbol{v}\cdot\boldsymbol{\tau}_k = 0, \quad k = 1,2 \end{aligned}\right\} \text{ at } \partial\Omega \tag{15}$$

and we use the elliptic theory. First for $q \leq 3$ and then for $q > 3$, using

$$\|\varrho\|_r \leq C\|P(\varrho)\|_{\frac{r}{\gamma_-}}^{\frac{1}{\gamma_-}} \leq C_1 m^{\frac{\gamma^+}{\gamma_-}(1-\frac{2\gamma_-}{r})} + C_2 \quad \text{with} \quad r > 2\gamma_-,$$

we get after standard, however tedious calculations that the most restrictive term is $\nabla P(\varrho)$. Thus

$$\|\nabla\boldsymbol{v}\|_q \leq C_1\|P(\varrho)\|_q + C_2 \leq C_1 m^{\gamma^+(1-\frac{2}{q})} + C_2.$$

The lemma is proved. □

3. **Limit process — a priori estimates.** Next we pass with $\varepsilon \to 0^+$. Using the a priori bounds from Lemma 3, together with the L^∞– bound for the density we obtain a subsequence such that:

$$\begin{array}{rll} \boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v} & \text{in } W^{1,p}(\Omega) \quad \text{for} & 1 \le p < \infty \\ \boldsymbol{v}_\varepsilon \to \boldsymbol{v} & \text{in } L^q(\Omega) \quad \text{for} & 1 \le q \le \infty \\ \varrho_\varepsilon \rightharpoonup^* \varrho & \text{in } L^\infty(\Omega) & \\ P(\varrho_\varepsilon) \rightharpoonup^* \overline{P(\varrho)} & \text{in } L^\infty(\Omega) & \\ K(\varrho_\varepsilon)\varrho_\varepsilon \rightharpoonup^* \overline{K(\varrho)\varrho} & \text{in } L^\infty(\Omega). & \end{array} \tag{16}$$

Passing to the limit, using also the properties of continuity equation in the momentum equation we end up with

$$\begin{gathered} \overline{K(\varrho)\varrho}\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \operatorname{div}\big(2\mu \boldsymbol{D}(\boldsymbol{v}) + \nu(\operatorname{div}\boldsymbol{v})\boldsymbol{I} - \overline{P(\varrho)}\boldsymbol{I}\big) = \overline{K(\varrho)\varrho}\boldsymbol{F} \\ \operatorname{div}\big(\overline{K(\varrho)\varrho}\boldsymbol{v}\big) = 0, \end{gathered} \tag{17}$$

together with the boundary conditions, satisfied in the weak sense.

To get that the limit is, indeed, a weak solution to our problem, we need to verify:

(i) $\overline{K(\varrho)\varrho} = \varrho$ (i.e. $\varrho \le m-1$ a.e.)
(ii) $\overline{P(\varrho)} = P(\varrho)$.

First, we will concentrate on problem (i). We introduce the Helmholtz decomposition of the velocity

$$\boldsymbol{v} = \nabla\varphi + \operatorname{curl}\boldsymbol{A}, \tag{18}$$

where the divergence-free part of the velocity satisfies the elliptic problem

$$\begin{gathered} \operatorname{curl}\operatorname{curl}\boldsymbol{A} = \operatorname{curl}\boldsymbol{v} = \boldsymbol{\omega} \quad \text{in} \quad \Omega \\ \operatorname{div}\operatorname{curl}\boldsymbol{A} = 0 \quad \text{in} \quad \Omega \\ \operatorname{curl}\boldsymbol{A}\cdot\boldsymbol{n} = 0 \quad \text{at } \partial\Omega. \end{gathered} \tag{19}$$

Thus (see e.g. [9])

$$\|\nabla \operatorname{curl}\boldsymbol{A}\|_q \le C\|\boldsymbol{\omega}\|_q \quad \text{and} \quad \|\nabla^2 \operatorname{curl}\boldsymbol{A}\|_q \le C\|\boldsymbol{\omega}\|_{1,q},$$

for $1 < q < \infty$. The same can be written for the approximative velocity $\boldsymbol{v}_\varepsilon$, just adding the index ε to $\boldsymbol{A}$, φ and $\boldsymbol{\omega}$.

However, we must be much more careful with the estimates of the vorticity than in the two-dimensional case; note that the slip boundary conditions are "responsible" for the improved estimates up to the boundary. We have

$$\begin{gathered} -\mu\Delta\boldsymbol{\omega}_\varepsilon = \operatorname{curl}\big(K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{F} - K(\varrho_\varepsilon)\varrho_\varepsilon(\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon) - \varepsilon h\boldsymbol{v}_\varepsilon - \varepsilon\varrho_\varepsilon\boldsymbol{v}_\varepsilon \\ -\frac{1}{2}\varepsilon hK(\varrho_\varepsilon)\boldsymbol{v}_\varepsilon + \frac{1}{2}\varepsilon\varrho_\varepsilon\boldsymbol{v}_\varepsilon\big) - \operatorname{curl}\big(\frac{1}{2}\varepsilon\Delta\varrho_\varepsilon\boldsymbol{v}_\varepsilon\big) := H_1 + H_2 \\ \left.\begin{array}{c} \boldsymbol{\omega}_\varepsilon\cdot\boldsymbol{\tau}_1 = -(2\chi_2 - f/\mu)\boldsymbol{v}_\varepsilon\cdot\boldsymbol{\tau}_2 \\ \boldsymbol{\omega}_\varepsilon\cdot\boldsymbol{\tau}_2 = (2\chi_1 - f/\mu)\boldsymbol{v}_\varepsilon\cdot\boldsymbol{\tau}_1 \\ \operatorname{div}\boldsymbol{\omega}_\varepsilon = 0 \end{array}\right\} \text{ at } \partial\Omega \end{gathered} \tag{20}$$

with χ_k – main curvatures of the boundary. The boundary relations $(20)_{2,3}$ follow from properties of slip boundary conditions $(2)_2$ — see [3] or [11]. (It is enough to differentiate $(2)_1$ with respect to tangent directions and apply the suitable condition $(2)_2$.) The last boundary condition is just the consequence of $\boldsymbol{\omega}_\varepsilon = \operatorname{curl}\boldsymbol{v}_\varepsilon$.

Lemma 4. *We have* $\boldsymbol{\omega}_\varepsilon = \boldsymbol{\omega}_\varepsilon^0 + \boldsymbol{\omega}_\varepsilon^1 + \boldsymbol{\omega}_\varepsilon^2$ *with*

$$\|\boldsymbol{\omega}_\varepsilon^0\|_{1,q} + \|\boldsymbol{\omega}_\varepsilon^1\|_{1,q} \le C_1 m^{1+\gamma^+(\frac{4}{3}-\frac{2}{q})} + C_2 \quad \text{for } q \ge \frac{3}{2}$$
$$\|\boldsymbol{\omega}_\varepsilon^2\|_q \le C\varepsilon^{\frac{1}{2}} \quad \text{for } q \le 2.$$

Proof. We denote by $\boldsymbol{\omega}_\varepsilon^0$ the solution to

$$\begin{aligned} -\mu\Delta\boldsymbol{\omega}_\varepsilon^0 = \mathbf{0} \quad \text{in } \Omega \\ \left.\begin{aligned} \boldsymbol{\omega}_\varepsilon^0 \cdot \boldsymbol{\tau}_1 = -(2\chi_2 - f/\mu)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{\tau}_2 \\ \boldsymbol{\omega}_\varepsilon^0 \cdot \boldsymbol{\tau}_2 = (2\chi_1 - f/\mu)\boldsymbol{v}_\varepsilon \cdot \boldsymbol{\tau}_1 \\ \operatorname{div}\boldsymbol{\omega}_\varepsilon^0 = 0 \end{aligned}\right\} \text{ at } \partial\Omega. \end{aligned} \tag{21}$$

Let $\boldsymbol{\alpha}^0$ denote the divergence-free extension to the tangent boundary values of $\boldsymbol{\omega}_\varepsilon^0$ (constructed e.g. as a solution to the Stokes problem with zero right-hand side and additionally $\boldsymbol{\alpha}^0 \cdot \boldsymbol{n} = 0$ at $\partial\Omega$). Thus we have

$$\begin{aligned} -\mu\Delta(\boldsymbol{\omega}_\varepsilon^0 - \boldsymbol{\alpha}^0) = \mu\Delta\boldsymbol{\alpha}^0 \quad \text{in } \Omega \\ \left.\begin{aligned} (\boldsymbol{\omega}_\varepsilon^0 - \boldsymbol{\alpha}^0) \cdot \boldsymbol{\tau}_1 = 0 \\ (\boldsymbol{\omega}_\varepsilon^0 - \boldsymbol{\alpha}^0) \cdot \boldsymbol{\tau}_2 = 0 \\ \operatorname{div}(\boldsymbol{\omega}_\varepsilon^0 - \boldsymbol{\alpha}^0) = 0 \end{aligned}\right\} \text{ at } \partial\Omega. \end{aligned} \tag{22}$$

Solvability of the above system can be found in [9] or [10] (here the assumption of connectness of the boundary is required) with the following estimates for solutions to (22)

$$\|\boldsymbol{\omega}_\varepsilon^0\|_{1,q} \le C\|\boldsymbol{\alpha}^0\|_{1,q} \le C\|\boldsymbol{v}_\varepsilon\|_{1,q} \le \begin{cases} C & q \le 2 \\ C_1 m^{\gamma^+(1-\frac{2}{q})} + C_2 & q > 2. \end{cases}$$

Next, we consider problem for $\boldsymbol{\omega}_\varepsilon^1$. It solves problem of the type (20) with zero boundary data and right-hand side H_1. Thus we get as above (the most restrictive term is the convective one)

$$\|\boldsymbol{\omega}_\varepsilon^1\|_{1,q} \le C(1 + \|\varrho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon\|_q) \le C_1 m^{1+\gamma^+(\frac{4}{3}-\frac{2}{q})} + C_2.$$

Finally, for the last term, we cannot expect estimate in $W^{1,q}(\Omega)$. However, for $q \le 2$ we have

$$\|\boldsymbol{\omega}_\varepsilon^2\|_q \le C\|\varepsilon\Delta\varrho_\varepsilon \boldsymbol{v}_\varepsilon\|_{-1,q} \le C\varepsilon\big(\|\nabla\varrho_\varepsilon \boldsymbol{v}_\varepsilon\|_q + \|\nabla\varrho_\varepsilon \nabla\boldsymbol{v}_\varepsilon\|_{\frac{6}{5}}\big) \le C(m)\varepsilon^{\frac{1}{2}}.$$

□

Next, we introduce the effective viscous flux. We define it (see (18)) as the potential part of the momentum equation

$$\begin{aligned} G_\varepsilon &= P(\varrho_\varepsilon) - (2\mu + \nu)\Delta\varphi_\varepsilon \\ G &= \overline{P(\varrho)} - (2\mu + \nu)\Delta\varphi. \end{aligned} \tag{23}$$

Lemma 5. *We have for* G *and* G_ε *defined above*

(i) $\|G\|_\infty \le C_1 + C_2(\eta) m^{1+\frac{2\gamma^+}{3}+\eta}$, $\eta > 0$, *arbitrarily small*
(ii) $G_\varepsilon \to G$ *in* $L^q(\Omega)$, $q \le 2$.

Proof. As $\nabla G = \overline{K(\varrho)\varrho}\boldsymbol{F} - \overline{K(\varrho)\varrho}\boldsymbol{v} \cdot \nabla\boldsymbol{v} + \mu\Delta\operatorname{curl}\boldsymbol{A}$, we have

$$\|\nabla G\|_q \le \|\overline{K\varrho)\varrho}\boldsymbol{F}\|_q + \|\overline{K(\varrho)\varrho}\boldsymbol{v} \cdot \nabla\boldsymbol{v}\|_q + \|\mu\Delta\operatorname{curl}\boldsymbol{A}\|_q \le C_1 + C_2 m^{1+\gamma^+(\frac{4}{3}-\frac{2}{q})}.$$

(Note that, using proof of Lemma 4

$$\|\Delta\operatorname{curl}\boldsymbol{A}\|_q \le C\|\boldsymbol{\omega}\|_{1,q} \le C\big(\|\boldsymbol{v}\|_{1,q} + \|\overline{K\varrho)\varrho}\boldsymbol{F}\|_q + \|\overline{K(\varrho)\varrho}\boldsymbol{v} \cdot \nabla\boldsymbol{v}\|_q\big).$$

The corresponding power of m is a consequence of Lemma 4.) Using the estimate above with $q > 3$, sufficiently close to 3, together with

$$\Big|\int_\Omega G dx\Big| = \Big|\int_\Omega \big(\overline{P(\varrho)} - (2\mu+\nu)\mathrm{div}\,\boldsymbol{v}\big)dx\Big| \leq C,$$

we get by the imbedding theorem ($\frac{4}{3} - \frac{2}{q} = \frac{2}{3} + (\frac{2}{3} - \frac{2}{q})$)

$$\|G\|_\infty \leq C\Big(\|\nabla G\|_q + \Big|\int_\Omega G dx\Big|\Big) \leq C_1 m^{1+\frac{2}{3}\gamma^+ + \eta} + C_2$$

with η positive, arbitrarily small.

Next, for the difference $G_\varepsilon - G$ we have

$$\begin{aligned}\nabla(G_\varepsilon - G) =& \big(K(\varrho_\varepsilon)\varrho_\varepsilon - \overline{K(\varrho)\varrho}\big)\boldsymbol{F} + \overline{K(\varrho)\varrho\boldsymbol{v}}\cdot\nabla\boldsymbol{v} - K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon\\ &- \varepsilon h\boldsymbol{v}_\varepsilon - \varepsilon\varrho_\varepsilon\boldsymbol{v}_\varepsilon + \frac{1}{2}\varepsilon\varrho_\varepsilon\boldsymbol{v}_\varepsilon - \frac{1}{2}\varepsilon\Delta\varrho_\varepsilon\boldsymbol{v}_\varepsilon - \frac{1}{2}\varepsilon hK(\varrho_\varepsilon)\boldsymbol{v}_\varepsilon\\ &+ \mu\Delta(\mathrm{curl}\,\boldsymbol{A}_\varepsilon - \mathrm{curl}\,\boldsymbol{A}).\end{aligned}$$

Similarly as in the estimate above we prove that $\nabla(G - G_\varepsilon) = B^1_\varepsilon + B^2_\varepsilon + B^3_\varepsilon$, where $B^1_\varepsilon \to 0$ in $L^q(\Omega)$, $B^2_\varepsilon \rightharpoonup 0$ in $L^q(\Omega)$ and $B^3_\varepsilon \to 0$ in $W^{-1,q}(\Omega)$ for any $q \leq 2$. Recalling that the mean value of $G - G_\varepsilon$ goes to zero, the second statement of the lemma is also proved. □

We can start to solve problem (i). We have

Theorem 2. *Let m from the definition of function $K(\cdot)$ – see (5) – be sufficiently large. Then there is a number $\kappa > 0$ such that $\|G\|_\infty^{\frac{1}{\gamma_-}} \leq \kappa < m - 2$ and, up to a subsequence, it holds*

$$\lim_{\varepsilon\to 0^+} |\{x \in \Omega; \varrho_\varepsilon(x) > \kappa\}| = 0. \tag{24}$$

Especially, for this subsequence, $\overline{K(\varrho)\varrho} = \varrho$.

Proof. Let us take e.g. $\kappa = m - 3$ and define a non-increasing function

$$M(t) = \begin{cases} 1 & \text{for} \quad t \leq \kappa \\ \in (0,1) & \text{for} \quad \kappa < t < \kappa+1 \\ 0 & \text{for} \quad \kappa+1 \leq t, \end{cases}$$

such that $M'(t) < 0$ for $t \in (\kappa, \kappa+1)$.

Take $l \in \mathbb{N}$ (later on we require l sufficiently large) and multiply the approximative continuity equation $(6)_2$ by $M^l(\varrho_\varepsilon)$. As

$$\varepsilon\int_\Omega M^l(\varrho_\varepsilon)\Delta\varrho_\varepsilon dx = -\varepsilon l\int_\Omega M^{l-1}(\varrho_\varepsilon)M'(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2 dx \geq 0,$$

we get

$$\int_\Omega M^l(\varrho_\varepsilon)\mathrm{div}\,(K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon)dx \geq -\varepsilon\int_\Omega \varrho_\varepsilon M^l(\varrho_\varepsilon)dx + \varepsilon h\int_\Omega K(\varrho_\varepsilon)M^l(\varrho_\varepsilon)dx := R_\varepsilon,$$

where $R_\varepsilon \to 0$ as $\varepsilon \to 0^+$, uniformly in l. Thus

$$-l\int_\Omega M^{l-1}(\varrho_\varepsilon)M'(\varrho_\varepsilon)K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\varrho_\varepsilon dx \geq R_\varepsilon.$$

As $M(t) = 0$ for $t \geq \kappa+1$, we get

$$l\int_\Omega\Big(\int_0^{\varrho_\varepsilon} tM^{l-1}(t)M'(t)dt\Big)\mathrm{div}\,\boldsymbol{v}_\varepsilon dx \geq R_\varepsilon. \tag{25}$$

From the definition of G_ε – see (23), recalling that $M'(t) = 0$ for $t < \kappa$ (we know that $\varrho_\varepsilon \leq m$), inequality (25) yields

$$\begin{aligned}&-\kappa\int_\Omega\Big(\int_0^{\varrho_\varepsilon} lM^{l-1}(t)M'(t)dt\Big)P(\varrho_\varepsilon)dx\\ \leq & m\Big|\int_\Omega\Big(\int_0^{\varrho_\varepsilon} lM^{l-1}(t)M'(t)dt\Big)G_\varepsilon dx\Big| + (2\mu+\nu)|R_\varepsilon|.\end{aligned}$$

Thus

$$\frac{\kappa}{m}\int_{\{\varrho_\varepsilon>\kappa\}}(1-M^l(\varrho_\varepsilon))P(\varrho_\varepsilon)dx \leq \int_{\{\varrho_\varepsilon>\kappa\}}(1-M^l(\varrho_\varepsilon))|G_\varepsilon|dx + \frac{(2\mu+\nu)}{m}|R_\varepsilon|. \quad (26)$$

Due to (4) inequality (26) yields

$$\begin{aligned}&\Big(\frac{\kappa}{m}\Big(\frac{a_2}{\gamma_-}\kappa^{\gamma_-}+\frac{b_2}{2}\kappa^2\Big)-\|G\|_\infty\Big)|\{\varrho_\varepsilon>\kappa\}|\\ \leq & \frac{\kappa}{m}\|P(\varrho_\varepsilon)\|_{L^2(\{\varrho_\varepsilon>\kappa\})}\|M^l(\varrho_\varepsilon)\|_{L^2(\{\varrho_\varepsilon>\kappa\})} + \|G-G_\varepsilon\|_1 + \frac{2\mu+\nu}{m}|R_\varepsilon|. \quad (27)\end{aligned}$$

Now, under the assumption that m is sufficiently large, we find, using Lemma 5 and the assumption $\frac{2}{3}\gamma^+ + 1 < \gamma_-$, number κ such that

$$\Big(\frac{\kappa}{m}\Big(\frac{a_2}{\gamma_-}\kappa^{\gamma_-}+\frac{b_2}{2}\kappa^2\Big)-\|G\|_\infty\Big) \geq 1$$

and therefore (27) implies

$$|\{\varrho_\varepsilon>\kappa\}| \leq C\Big(\|M^l(\varrho_\varepsilon)\|_{L^2(\{\varrho_\varepsilon>\kappa\})} + \|G-G_\varepsilon\|_1 + |R_\varepsilon|\Big). \quad (28)$$

We estimate the r.h.s. of (28). Fix $\delta > 0$. There is $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$

$$C\Big(\|G-G_\varepsilon\|_1 + |R_\varepsilon|\Big) < \frac{\delta}{2}.$$

Moreover, as sequence $\{M^l(\rho_\epsilon)\}_{l\in\mathbb{N}}$ for $\{x\in\Omega : \rho_\epsilon(x) < \kappa\}$ tends to zero pointwise monotonely, there is $l = l(\delta,\varepsilon)$ such that

$$C\|M^l(\varrho_\varepsilon)\|_{L^2(\{\varrho_\varepsilon>\kappa\})} < \frac{\delta}{2} \qquad \text{and therefore} \qquad \lim_{\varepsilon\to 0^+}|\{\varrho_\varepsilon>\kappa\}| \leq \delta.$$

As $\delta > 0$ can be chosen arbitrarily small, the theorem is proved. □

4. **Convergence of the pressure.** In this part we finish the proof of Theorem 1. We show that the convergence of approximative density can be improved to the strong one in any L^q-space for $q < \infty$. Thus we solve our problem (ii).

Lemma 6. *We have*

$$\int_\Omega \overline{P(\varrho)\varrho}dx \leq \int_\Omega G\varrho dx.$$

Proof. For a fixed $\delta > 0$ we multiply the approximative continuity equation $(6)_2$ by $\ln m - \ln(\varrho_\varepsilon + \delta)$ and integrate over Ω which, after straightforward calculations, gives

$$\int_\Omega K(\varrho_\varepsilon)\varrho_\varepsilon\frac{\boldsymbol{v}_\varepsilon\cdot\nabla\varrho_\varepsilon}{\varrho_\varepsilon+\delta}dx \geq \varepsilon\int_\Omega\big(\varrho_\varepsilon K(\varrho_\varepsilon)-\varrho_\varepsilon)(\ln m - \ln(\varrho_\varepsilon+\delta))dx. \quad (29)$$

We pass with $\delta \to 0^+$. Following the idea presented in [4], we get

$$\int_\Omega K(\varrho_\varepsilon)\boldsymbol{v}_\varepsilon\cdot\nabla\varrho_\varepsilon \geq \varepsilon C(m)$$

and using Theorem 2 we conclude that

$$-\int_\Omega \varrho_\varepsilon \operatorname{div} \boldsymbol{v}_\varepsilon \geq R_\varepsilon, \tag{30}$$

where $R_\varepsilon \to 0$ for $\varepsilon \to 0^+$. Hence from $(23)_1$ and (30) we obtain

$$\int_\Omega P(\varrho_\varepsilon)\varrho_\varepsilon dx \leq \int_\Omega G_\varepsilon \varrho_\varepsilon dx - (2\mu + \nu)R_\varepsilon. \tag{31}$$

Passing with $\varepsilon \to 0^+$ in inequality (31) we get the result. □

Next we prove

Lemma 7. *We have*

$$\int_\Omega \overline{P(\varrho)}\varrho dx = \int_\Omega G\varrho dx. \tag{32}$$

Proof. Using the fact that we can approximate ϱ in suitable spaces by a sequence of smooth functions ϱ_n in such a way that $0 \leq \varrho_n \leq m$ and $\boldsymbol{v} \cdot \nabla \varrho_n \to \boldsymbol{v} \cdot \nabla \varrho$ (see [7] for more details – here is hidden Friedrich's lemma) and the fact that

$$\int_\Omega (\varrho_n \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \varrho_n) dx = 0,$$

we get, passing with $n \to \infty$,

$$\int_\Omega (\varrho \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \varrho) dx = 0.$$

Next, using as a test function for the continuity equation: $-\ln \frac{\delta}{\varrho_n + \delta}$ and passing with $n \to \infty$ and with $\delta \to 0^+$ we get

$$\int_\Omega \boldsymbol{v} \cdot \nabla \varrho dx = 0. \tag{33}$$

Thus from $(23)_2$ and (33) we conclude the sought identity (32), which finishes the proof of the lemma. □

To conclude we recall that the assumptions on $P(\cdot)$ imply that the function is monotone on $\mathbb{R}_0^+$. Thus, (details e.g. Lemma 3.35 from [8]) we have

$$\overline{P(\varrho)\varrho} \geq \overline{P(\varrho)}\varrho \quad \text{ a.e. in } \Omega. \tag{34}$$

This, together with Lemmae 6 and 7 gives

$$\overline{P(\varrho)\varrho} = \overline{P(\varrho)}\varrho \quad \text{ a.e. in } \Omega. \tag{35}$$

Then, using standard methods as Lemma 3.39 from [8], information from (34) and (35) yields $\overline{P(\varrho)} = P(\varrho)$. Moreover, using the Lebesgue dominated convergence theorem and Theorem 2 we conclude the strong convergence of approximative densities $\rho_\epsilon \to \rho$ in $L^q(\Omega)$ for any $q < \infty$. It implies that limit found by (16) satisfies the definition of the weak solution – see Remark 2 – to the original system (1)–(3). Theorem 1 is proved.

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