A caricature of a singular curvature flow in the plane

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Abstract. We study a singular parabolic equation of the total variation type in one dimension. The problem is a simplification of the singular curvature flow. We show existence and uniqueness of weak solutions. We also prove existence of weak solutions to the semi-discretization of the problem as well as convergence of the approximating sequences. The semi-discretization shows that facets must form. For a class of initial data we are able to study in details the facet formation and interactions and their asymptotic behavior. We notice that our qualitative results may be interpreted with the help of a special composition of multivalued operators.

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1 Introduction

Many free boundary problems involving the Gibbs-Thomson relation may be considered as a driven weighted mean curvature flow coupled through the forcing term to a diffusion equation (see [CR], [Ra], [L] [AW]). More explicitly, in the classical setting the modified Stefan problem with kinetic undercooling consists of (i) a diffusion equation for the temperature u,

$$u_t = \Delta u$$

in two regions separated by a free surface $\Gamma(t)$ and (ii) the curvature flow

$$\beta V = \kappa + u \tag{1.1}$$

on the free surface $\Gamma(t)$. Here V is the normal velocity of $\Gamma(t)$, β is a kinetic coefficient and κ is the mean curvature of Γ. Since the melting temperature u depends upon the curvature κ we obtain an additional explicit coupling of these two equations.

We have a considerable body of literature concerning these Stefan type problems for the Euclidean curvature, κ, of the interface, including the question of precise regularity of solutions treated by Escher, Prüss, Simonett and Mucha, see [EPS], [ES], [Mu]. On the other hand, less is known if the curvature appearing in the Gibbs-Thomson relation is singular, see e.g. [Ry]. In order to stress dependence of this curvature upon the anisotropy function $\bar\gamma$ we will denote it by κ_γ. The formal definition as follows,

$$\kappa_\gamma = \mathrm{div}_S \nabla_\xi \bar\gamma(\xi)|_{\xi=\mathbf{n}},$$

where $\bar\gamma : \mathbb{R}^2 \to \mathbb{R}$ is the anisotropy or energy density function, which is convex and positively one-homogeneous. The term 'singular' refers to the situation, when we do not assume any further regularity of $\bar\gamma$ beyond convexity, which implies Lipschitz continuity.

This line of research has been initiated by Taylor, [T], and independently by Angenent and Gurtin, [AG]. However, just solvability of equations of the singular curvature flow is interesting. Existence of the flow was obtained by Bellettini, Novaga, Paolini [BNP1], [BNP2] and by Chambolle [Ch]. Driven singular curvature flow was studied by M.-H.Giga, Y.Giga and Rybka, see [GG], [GR1], [GR2].

In fact, the existence and properties of solutions to the singular weighted mean curvature flow

$$V = \kappa_\gamma \qquad \text{on} \quad \Gamma(t), \tag{1.2}$$

are interesting in itself even in the plane and without forcing, i.e. with $u = 0$, especially when the anisotropy function is singular. Here, $\Gamma(t)$ is the unknown curve. Our ultimate goal would be to study existence and behavior of solutions to (1.2).

In its full generality problem (1.2) for an arbitrary initial curve is rather difficult. One source of difficulties is the geometry of the system, it is already present in the two-dimensional setting. Here, we want to concentrate only on the purely analytical difficulties appearing in (1.2). This is why we will restrict our attention to a simplified equation, which retains the singular character of the original problem, but its relation to the original is rather loose, this is why we call the system below 'a caricature of a curvature flow'.

Here is our postulated equation,

$$\begin{aligned} &\Lambda_t = \frac{\partial}{\partial s}\frac{d}{d\phi} J(s+\Lambda_s) \quad \text{in } S\times(0,T),\\ &\Lambda(s,0) = \Lambda_0(s) \quad \text{on } S,\\ &\Lambda(2\pi,t) = \Lambda(0,t), \quad t\ge 0, \end{aligned} \tag{1.3}$$

here S is the unit circle parameterized by interval $[0, 2\pi)$ and Λ is the sought function. We call it a 'caricature of a singular curvature flow', because it retains the main analytic difficulty of (1.2), while after geometric simplifications it is far from the original. Compared with (1.2) our new system has one analytical advantage. Namely, the domain of definition of $\Lambda(\cdot, t)$ is independent of time.

We present a justification of this equation in the Appendix. Here, we explain our notation. The variable s is the arclength parameter on the reference curve S, the subscript s denotes the differentiation with respect to s. We frequently refer to $\varphi = \Lambda_s + s$ as the angle between the x_1 axis and the outer normal to the curve. Such an interpretation helps drawing pictures, but the relation to the actual angle is rather loose.

We make a specific choice of J corresponding to the surface energy density functions. We want to study a situation which is already very singular yet tractable. In many instances of a great physical interest an anisotropy appears, which is merely convex, not even strictly convex (understood in a proper sense). As a result, we choose J, which is convex and piecewise linear. This is an independent source of difficulties. In order to avoid further technical troubles we will choose J corresponding to the situation where that curve minimizing the surface energy (which is the Wulff shape of the anisotropy function) is a square. We must stress again that the correspondence is at the level of ideas, because (1.3) is *not* a curvature flow, but its caricature. However, the obtained behavior of solutions to (1.3) is almost the same as for the equation (1.2) with the anisotropy function corresponding to a square, [Ch].

Thus, we pick J which suffers jumps of equal height $\frac{\pi}{2}$ at the equi-spaced angles

$$\mathcal{A} = \left\{ \alpha_k = -\frac{3\pi}{4} + k\Delta\alpha : \quad k = 0, 1, 2, 3, \qquad \text{with } \Delta\alpha = \frac{\pi}{2} \right\}. \tag{1.4}$$

Specifically, we put

$$J(\varphi) = \frac{\pi}{4} \left(|\varphi - \frac{3\pi}{4}| + |\varphi - \frac{\pi}{4}| + |\varphi + \frac{\pi}{4}| + |\varphi + \frac{3\pi}{4}| \right). \tag{1.5}$$

Since Λ is defined over the unit circle its graph over S is a closed curve. The meaning of the spacing between α_k's can be explained by looking at the equation

$$\frac{\partial}{\partial s} \frac{d}{d\varphi} J(s + \Lambda_s) = 1,$$

considered in [MRy] – see subsection 3.2, too. Roughly speaking, the spacing between α_k and α_{k+1} corresponds to the length of facets having the normal vector $\mathbf{n}$ with the normal angle α_k. The size of the jump of $\frac{d}{d\varphi} J(s + \Lambda_s)$ corresponds to the angle between the normals to the curve, which is a solution to the above equation, at a corner.

The chosen anisotropy function (1.5) is nowhere regular, hence we can expect nonstandard effects requiring new analytical tools. This has been observed by researchers working on the total variation flow, whose simplification is

$$u_t - \delta_0(u_x) u_{xx} = 0 \tag{1.6}$$

augmented with initial and boundary data. Here, δ_a is the Dirac measure concentrated at a.

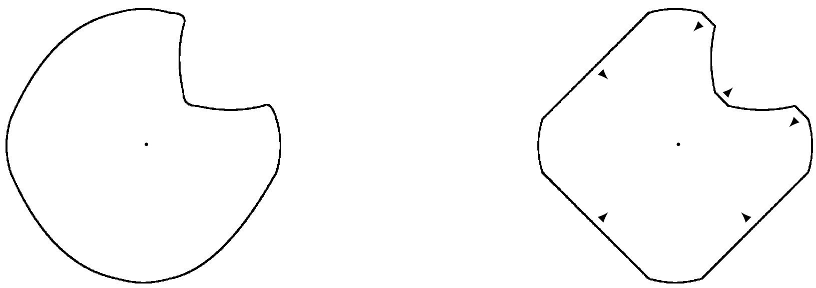
We noticed so far two main types of motivation to study the total variation flow,

$$u_t - \text{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0. \tag{1.7}$$

The first one is the image denoising and reconstruction introduced by Rudin and Osher, [RO], [ROF]. The second one is evolution of the facets of crystals. The bulk of the papers (see [ABC1], [ABC2], [BCN], [GK], [GGK], [ACM], [Mo]) uses the theory of nonlinear semigroups to establish existence. The last paper is particularly interesting, because it deals with the anisotropic total variation flow. Moreover, the notion of entropy solutions was introduced to deal with uniqueness of the total variation flow (see [ABC1], [BCN]). The tools of convex analysis were useful to make sense out of (1.6). The authors, mentioned above, paid special attention to piecewise constant initial data and they were interested in the asymptotic behavior, in particular the asymptotic shape was identified. M.-H.Giga, Y.Giga and R. Kobayashi, [GK], [GGK], also calculated the speed of flat facets. No matter what is the approach, it is apparent that the most important information is located in sets $\{u_x = 0\}$, where the singular dissipation starts to play a role and where the classical multivalued theory of function loses the meaning.

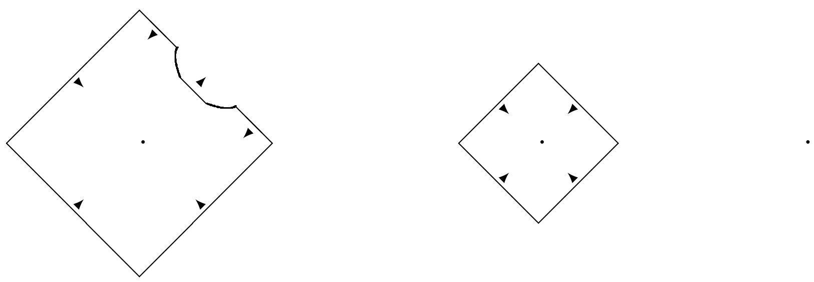
Our approach differs in many aspects. We prove existence by a regularizing procedure and passing to the limit with the regularizing parameter, this approach was used, e.g. by Feng and Prohl, see [FP]. The main difficulty is associated with studying the limit of the non-linear terms. We present a more detailed analysis of regularity of solutions permitting us to call them 'almost classical'. For generic data, our solutions are twice differentiable with respect to s, except a finite number of points (for fixed time). This will be explained in detail below. We mention here that we use the tools of the convex analysis, in particular we rely on the fact that for a convex function the subdifferential is well-defined everywhere. However, the classical theory of multivalued functions is not sufficient. We have to introduce a new definition of the composition of two multivalued functions to describe the meaning and qualitative properties of solutions to system (1.3) as well a class of the J-R functions, where regularity is described from the point of view of the properties of the function J. In our opinion the results we prove contribute to better understanding parabolic systems with measure coefficients.

Our technique requires a new look at the regularity of functions. We will generalize the meaning of the convexity defining a class of J-regular functions preserving some important properties of the convexity. Our main qualitative result says that any sufficiently regular initial curve evolving according to system (1.3), will eventually reach a minimal solution, which is called the asymptotic profile in the area of the total variation flow. The geometric interpretation is that the solution reaches its asymptotic shape, i.e. the square in our case. This may happen in infinite or finite time depending upon initial data. If this event occurs in finite time, then subsequently, the solution shrinks to a point. This behavior can be illustrated by the pictures below. The precise meaning is contained in Theorem 5.1.



Time $t = 0$ Time $t = t_1$

The evolution is determined by motion of facets defined by singularities of the J-function (the arrows show the direction of the evolution). In finite time we obtain a convex domain, which becomes a square converging to a point in finite time.



Time $t = t_2$ Time $t = t_3$ Time $t = t_{end}$

We have to underline that the illustrated evolution hides the novel idea of definition of singular term $\delta_0(u_x)u_{xx}$ being a multiplication of two Dirac deltas (as in (1.6)), however the nonlocal character will allow us to define this object. Additionally, by the uniqueness of solutions to our system we show that our novel definition is the only admissible. Formally, the dissipation caused by the Dirac delta coefficient is so strong that the changes of regularity (i.e. appearance of the facets) happen instantly.

We will state our results in the Section below, the proofs will be presented in the further Sections.

Here, we present the outline of the rest of the paper. We show the existence of weak solutions in Section 2, uniqueness is the content of that Section, too. The qualitative analysis is based on the semi-discretization which is performed in Section 3. Our goal is to make some of the properties more apparent. Namely, we want to show that facets (i.e. intervals where $\varphi = \Lambda_s + s$ has a constant value equal to one of the α_i's) form instantaneously. In Section 5, we show further geometric properties of solutions, namely the curve becomes convex (i.e. the angle φ becomes monotone) in finite time. In addition, we show that solutions become fully faceted in finite time, i.e. the solution is composed only of facets. These two events are not correlated in time. Finally, we show that our solutions converge to a special solution which we call minimal.

2 The main results

Here, we present our results. We begin by noticing that if J is given by (1.5), then the meaning of (1.3) is not clear at all because its right-hand-side formally becomes

$$\Lambda_t = \frac{\pi}{2}\sum_{k=0}^{3} \delta_{k\frac{\pi}{2}-\frac{3\pi}{4}}(s+\Lambda_s)(1+\Lambda_{ss}).$$

Hence the above equation can be viewed as a generalization of equation (1.6).

We will use the tools of the convex analysis to interpret it. Due to convexity of J its subdifferential is always well-defined. Since in general $\partial_\phi J(\phi)$ is not a singleton it is necessary to find its proper selection, in particular (1.3) takes the form,

$$\begin{array}{ll} \Lambda_t \in \dfrac{\partial}{\partial s}\partial_\varphi J(\Lambda_s + s), & \text{in } S\times(0,T), \\ \Lambda(s,0) = \Lambda_0(s), & \text{on } S, \\ \Lambda(2\pi,t) = \Lambda(0,t), & \text{for } t \geq 0, \end{array} \tag{2.1}$$

where S is the unit circle.

In other words, we have to find (weakly) differentiable selections of $\partial_\phi J(\Lambda_s + s)$. Thus, we are lead to the following notion of a weak solution to (1.3).

Definition 2.1. *We say that $\Lambda \in C([0,T]; L_2(S))$, such that $\Lambda_s \in L_\infty(0,T; TV(S))$ is a weak solution to (2.1), if there exists a function $\Omega \in L_1(0,T; W^1_1(S))$ such that $\Omega(s,t) \in \partial J(\Lambda_s + s)$ a.e., and for any function h in $C^\infty(S \times [0,T])$ it holds*

$$\int_S \Lambda_t h = -\int_S (\Omega - s) h_s + \int_S h$$

in the distributional sense on $[0,T)$.

With this definition we can show the following existence result.

Theorem 2.1. *Let us suppose that J is defined by(1.5), $\Lambda_0 \in L_1(S)$ and $\Lambda_{0,s} \in TV(S)$, then there exists $\Lambda \in C^\alpha(0,T; L_2(S))$ with $\alpha > 0$, additionally*

$$\Lambda_s \in L_\infty(0,T; TV(S)) \quad \text{and} \quad \Lambda_t \in L_2(0,T; L_2(S))$$

such that it is a unique weak solution to (2.1).

The proof will be achieved through an approximation procedure, it is performed in Section 3. Moreover, we show uniqueness of the solution constructed here, this is the content of Theorem 3.1 in Section 3.

However, our main goal is to describe precisely qualitative properties of solutions to (2.1). As a motivation, we present a special type of solutions, which we will call *minimal solutions*, which are given explicitly, one of them is given here, (see also §3.2),

$$\bar{\Lambda}(s,t) = \int_0^s \bar{\varphi}(u)\,du + t,$$

where

$$\bar{\varphi}(s) = \frac{\pi}{4}\chi_{[0,\frac{\pi}{2})}(s) + \frac{3\pi}{4}\chi_{[\frac{\pi}{2},\pi)}(s) + \frac{5\pi}{4}\chi_{[\pi,\frac{3\pi}{2})}(s) + \frac{7\pi}{4}\chi_{[\frac{3\pi}{2},2\pi)}(s). \tag{2.2}$$

It is a matter of an easy exercise to see that $\bar{\Lambda}$ defined above with $\bar{\Omega}(x,t) = x$ is indeed a weak solution to (2.1). In fact, this is an asymptotic profile, which can be reached in finite time.

We will keep in mind this example while developing the proper class of regular solution. The idea is that we want to extend properties of convex solutions to a more general class, hence we introduce a class of J-regular function, where restrictions on regularity depend on function J from (1.5).

Firstly, we define the space of functions which are helpful to describe the regularity of the derivative of our solutions. We recall that any function $\phi \in TV$ is a difference of two monotone functions. Thus, we shall call a multifunction $\phi : [0,2\pi) \to 2^{\mathbb{R}}$ a *maximal TV function* if it is a difference of two maximal monotone multifunctions and one of them is continuous.

Definition 2.2. *We say that a maximal TV multivalued function $\phi : [0,2\pi) \to \mathbb{R}$ is J-regular, i.e. $\phi \in$ J-R$[0,2\pi)$, provided that the set*

$$\Xi(\phi) = \{s \in [0,2\pi) : \phi(s) \ni \alpha_k \;\; \textit{for} \;\; k = 0,1,2,3\}$$

consists of a finite number of connected components, i.e. we allow only isolated intervals or isolated points. Additionally, on any connected subset $[0,2\pi) \setminus \Xi$ function ϕ takes its values in interval $(\alpha_k, \alpha_k + \frac{\pi}{2})$ for some $k = 0,\ldots,3$, modulo 2π – see (1.4).

For each $\phi \in$ J-R$[0,2\pi)$ we define a function $K :$ J-R$[0,2\pi) \to \mathbb{N}$ by the formula

$$K(\phi) = \textit{the number of connected components of the set } \Xi(\phi).$$

Additionally we put

$$||\phi||_{\text{J-R}[0,2\pi)} = ||\phi||_{TV[0,2\pi)} + K(\phi).$$

Let us note that the J-R class does not form a Banach space. It is not a linear space. Additionally let us underline the finiteness of the number $K(\cdot)$, this restriction excludes all functions with infinite oscillations of critical values of ∂J.

In order to formulate the meaning of solutions, first we define the *composition of* J-R *functions with ∂J*. Because of the complex structure the definition is long.

Definition 2.3. We define the *composition $\partial J \bar{\circ} A$*,

$$\partial J \bar{\circ} A : [a,b) \to [e,f],$$

where $A : [a,b) \to [c,d]$ is an J-R function and $\partial J : [c,d] \to [e,f]$ as follows:

To begin with, we decompose the domain $[a, b)$ into three disjoint parts $[a, b) = \mathcal{D}_r \cup \mathcal{D}_f \cup \mathcal{D}_s$, where

$$\begin{gathered}\mathcal{D}_s = \{s \in [a, b] : A(s) = [c_s, d_s] \text{ and } c_s < d_s\}; \\ \mathcal{D}_f = \{\bigcup_k (a_k, b_k) : A|_{(a_k, b_k)} = c_k, \text{ where } c_k \text{ is a constant}\}; \quad \mathcal{D}_r = [a, b) \setminus (\mathcal{D}_s \cup \mathcal{D}_f).\end{gathered} \tag{2.3}$$

Then, the composition is defined in three steps:

1. For each $s \in \mathcal{D}_r$ the set $A(s)$ is a singleton, thus the composition is given in the classical way

$$\partial J \bar{\circ} A(s) = \partial J(A(s)) \qquad \text{for } \ s \in \mathcal{D}_r. \tag{2.4}$$

2. In the case $s \in \mathcal{D}_f$ the definition is "unnatural". For a given set $(a_k, b_k) \subset \mathcal{D}_f$ we have $A|_{(a_k, b_k)} = c_k$. If $\partial J(c_k)$ is single-valued, then for $s \in (a_k, b_k)$ we have,

$$\partial J \bar{\circ} A(s) = \{\frac{dJ}{d\phi}(c_k)\}.$$

However, if $\partial J(c_k)$ is multivalued, i.e. $\partial J(c_k) = [\alpha_k, \beta_k]$, then the definition is not immediate. We have to consider four cases related to the behavior of multifunction A in a neighborhood of interval (a_k, b_k). The regularity properties of the J-R class imply the necessity to consider the following four cases (for small $\epsilon > 0$):

(i) A is increasing, i.e. $A(s) < c_k$ for $s \in (a_k - \epsilon, a_k)$ and $A(s) > c_k$ for $s \in (b_k, b_k + \epsilon)$;

(ii) A is decreasing, i.e. $A(s) > c_k$ for $s \in (a_k - \epsilon, a_k)$ and $A(s) < c_k$ for $s \in (b_k, b_k + \epsilon)$;

(iii) A is convex, i.e. $A(s) > c_k$ for $s \in (a_k - \epsilon, a_k)$ and $A(s) > c_k$ for $s \in (b_k, b_k + \epsilon)$;

(iv) A is concave, i.e. $A(s) < c_k$ for $s \in (a_k - \epsilon, a_k)$ and $A(s) < c_k$ for $s \in (b_k, b_k + \epsilon)$.

Let us emphasize that these are the only possibilities for $\phi \in$ J-R, because we explicitly excluded all functions with oscillatory behavior. Indeed, the set $\Xi(\phi)$ is permitted to have only a finite number of components – see Definition 2.2.

In the case (i) we put

$$\partial J \bar{\circ} A(t) = x_k(t - b_k) + y_k(t - a_k) \qquad \text{for } \ t \in (a_k, b_k), \tag{2.5}$$

where $x_k = \frac{\alpha_k}{a_k - b_k}$ and $y_k = \frac{\beta_k}{b_k - a_k}$.

For case (ii) we put

$$\partial J \bar{\circ} A(t) = x_k(t - b_k) + y_k(t - a_k) \qquad \text{for } \ t \in (a_k, b_k), \tag{2.6}$$

where $x_k = \frac{\beta_k}{a_k - b_k}$ and $y_k = \frac{\alpha_k}{b_k - a_k}$.

When we deal with case (iii) we set

$$\partial J \bar{\circ} A(t) = \beta_k \qquad \text{for } \ t \in (a_k, b_k). \tag{2.7}$$

Finally, if (iv) holds, then we put

$$\partial J \bar{\circ} A(t) = \alpha_k \qquad \text{for } \ t \in (a_k, b_k). \tag{2.8}$$

3. In the last case, if $s \in \mathcal{D}_s$ our definition is just a consequence of first two steps. Since set $\mathcal{D}_s$ consists of a countable number of points we consider each of them separately. We have $A(d_k) = [e_k, f_k]$ with $e_k \neq f_k$, then

$$\partial J \bar{\circ} A(d_k) = [\limsup_{t \to d_k^-} \partial J \bar{\circ} A(t), \liminf_{t \to d_k^+} \partial J \bar{\circ} A(t)]. \tag{2.9}$$

Definition 2.3 is complete.

Thanks to the J-R regularity of A, the above limits are well defined. As a result, we are able to omit point from $\mathcal{D}_s$ in (2.3). We note that the above construction guarantees that

$$\partial J \bar{\circ} A : [a,b) \to [e,f] \quad \text{is a J-R function.}$$

After having completed the definition we make additional comments on step 2. Formulae (2.5)-(2.8) are immediate consequences of the pointwise approximation of the considered function by smooth functions. The presented composition agrees with the results from [MRy], where a stationary version of the problem has been considered. In particular, our definition follows from a requirement: if A is maximal monotone then we expect

$$A^{-1} \bar{\circ} A = Id.$$

Moreover, the composition of two maximal increasing functions is maximal increasing. Another point, which should be emphasized, is the nonlocal character of the above definition. Step 3 depends on step 2, so steps 1 and 2 should be performed at the very beginning.

Now we are prepared to introduce the main definition.

Definition 2.4. *We say that a function $\Lambda : S \to \mathbb{R}$ is an almost classical solution to system (1.3) iff Λ is a weak solution with $\Omega = \partial J \bar{\circ} [\Lambda_s + s]$, $\Lambda_s + s \in L_\infty(0,T; \text{J-R}[0,2\pi))$ and*

$$\begin{array}{lll} \Lambda_t = \frac{d}{ds} \partial J \bar{\circ} [\Lambda_s + s] & \text{in} & [S \times ((0,T) \setminus N)] \setminus \bigcup_{0<t<T} \partial \Xi(\Lambda_s(\cdot,t) + s) \times \{t\}, \\ \Lambda|_{t=0} = \Lambda_0 & \text{on} & S, \end{array} \tag{2.10}$$

where N is finite and ∂E denotes the boundary of set E.

The main point of Definition 2.3 is to determine the composition appearing on the RHS of the equation on sets, where the solution and ∂J are singular. Note that equation $(2.10)_1$ is fulfilled in the classical sense except for finite number of point for each $t \in (0,T) \setminus N$. This is so due to the definition of set $\Xi(\Lambda_s(\cdot,t) + s)$ implying that its boundary consists of finite number of points. It is easy to see that the minimal solutions (2.2) fulfills Definition 2.4.

The main result of our considerations is the following.

Theorem 2.2. *Let Λ_0 be such that $\Lambda_{0,s} + s \in \text{J-R}[0,2\pi)$, then there exists a unique almost classical solution to system (2.1) conforming to Definition 2.4.*

In fact this is a statement about the regularity of weak solutions. Theorem 2.2 is a result of the semi-discretization of system (1.3). At this level, we will be able to show that facets must appear, as suggested by the pictures in the Introduction. The semi-discretization will determine the RHS of (2.10) on sets where the solution falls into the singular part of ∂J. We will obtain that on these sets the term ∂J is constant on each connected part (or time dependent for the evolutionary system). Next, by the elementary means we will show that the semi-discretization tends uniformly to the solutions obtained by Theorem 2.1. However, performing a rigorous proof that we indeed constructed an almost normal solution requires more work on the structure of weak solutions, which is the content of Section 5. Thus, it will be postponed until the end of this part.

At the end, in Section 5, we deeply go into the qualitative analysis of the evolution showing the convexification effect and convergence to the minimal solutions. Since we know that facets must appear and the solutions are unique we are in a position to construct quite explicit solutions. We are able to follow their qualitative changes. This is made precise in Theorem 5.1. In particular we show instantaneous creation of facets. For the sake of this study we show a comparison principle in subsection 5.1. Moreover, we show that the evolution of facets is governed by a system of ODE's which are coupled if the facets interact, this is explained in Section 5. A conclusion from our analysis is existence of a sequence of instances at which our solution gets simplified before it gets the final form of the asymptotic profile, i.e. the minimal solution.

3 Existence of solutions

In this Section we show an existence and uniqueness of weak solutions of (1.3). We use the tools of the convex analysis to interpret it. In particular, we shall make the gradient flow structure of (1.3) transparent. However, the existence is shown by the method of regularization. Some of the statements are easier to interpret if they are written in the language of the 'angle' $\varphi = \Lambda_s + s$. Here, φ plays the role of the angle between the normal to the curve and the x_1–axis. Thus, for convex closed curves φ must be increasing, but we shall not require that, instead we admit φ being a functions of bounded total variations, i.e., $\varphi(\cdot, t) \in TV(S)$, in particular $\varphi \in L^\infty(S)$ and it may be discontinuous though.

3.1 The proof of the general existence result

We present a proof of our existence result, Theorem 2.1. It will be achieved through an approximation procedure. For any $\epsilon > 0$ we set

$$J^\epsilon(x) := J \star \rho_\epsilon(x) + \frac{\epsilon^2}{2}x^2, \tag{3.1}$$

where ρ_ϵ is a standard mollifier kernel, with support in $(-\epsilon, \epsilon)$. Let us note properties of the approximation J^ϵ:

(a) $J^\epsilon \in C^\infty(\mathbb{R})$;

(b) $\frac{d}{dx}J^\epsilon$ is strictly monotone;

(c) $\frac{d^2}{dx^2}J^\epsilon \geq \epsilon$;

(d) $\frac{d}{dx}J^\epsilon(x) - \epsilon x = \frac{d}{dx}J(x)$ for x such that $|x - \alpha_k| > \epsilon$ for $k = 0, 1, 2, 3$.

We start with existence of the regularized system.

Lemma 3.1. *Let us suppose that J^ϵ is defined by (3.1) and Λ_0^ϵ is smooth and 2π-periodic. Then, for any $T > 0$ there exists a unique, smooth solution to the regularized problem,*

$$\begin{aligned} &\Lambda_t^\epsilon = \frac{\partial}{\partial s}\frac{d}{d\varphi}J^\epsilon(\Lambda_s^\epsilon + s), \qquad \textit{in } S \times (0, T), \\ &\Lambda^\epsilon(s, 0) = \Lambda_0^\epsilon(s), \qquad \textit{on } S, \\ &\Lambda^\epsilon(s + 2\pi, t) = \Lambda^\epsilon(s, t), \qquad \textit{for} \quad t > 0. \end{aligned} \tag{3.2}$$

Proof. By properties (a), (b) (c) and (d) of J^ϵ, see (3.1), the existence and uniqueness of smooth solutions to (3.2), is guaranteed by the standard theory of parabolic systems, see [LSU]. □

We now study properties of established solutions.

Lemma 3.2. *Let us suppose that* Λ^ϵ *is a smooth solution to (3.2).*
(a) If for $a, b \in \mathbb{R}$ *and the initial datum satisfies* $a \le (\Lambda^\epsilon_{0,s}(s) + s) \le b$*, then, for all* $t < T$ *we have*

$$a \le (\Lambda^\epsilon_s(s,t) + s) \le b.$$

(b) If moreover, $(\Lambda^\epsilon_{0,s}(s) + s)_s \in L_1(0, 2\pi)$*, then, for all* $t < T$ *we have*

$$(\Lambda^\epsilon_s(s,t) + s)_s \in L_\infty(0, T; L_1(0, 2\pi)).$$

Proof. We use the maximum principle. First of all, we differentiate (3.2) with respect to s,

$$\Lambda^\epsilon_{st} = \frac{d}{ds}\left(\frac{\partial^2 J^\epsilon}{\partial\varphi^2}(s + \Lambda^\epsilon_s)(s + \Lambda^\epsilon_s)_s\right).$$

We notice $(s + \Lambda^\epsilon_s)_t = \Lambda^\epsilon_{st}$. We set $w = (s + \Lambda^\epsilon_s)$, hence we obtain the equation for w,

$$w_t = \frac{d}{ds}(a(s,t)w_s), \tag{3.3}$$

where by (3.1) we have $a(s,t) = \frac{\partial^2 J^\epsilon}{\partial\varphi^2}(s + \Lambda^\epsilon_s) \ge \epsilon > 0$. Hence, by the maximum principle we obtain (a).

To prove (b) we note that from (3.3) we obtain

$$w_{st} = \frac{d^2}{ds^2}(a(s,t)w_s). \tag{3.4}$$

By Lemma 3.1 our solutions are smooth. In order to finish the proof of (b) it is enough to integrate (3.4) over sets $\{w_s > 0\}$ and $\{w_s < 0\}$ to reach,

$$\frac{d}{dt}\int_{\{w_s>0\}} w_s dx \le 0 \quad \text{and} \quad \frac{d}{dt}\int_{\{w_s<0\}} w_s dx \ge 0. \tag{3.5}$$

□

Having established this Lemma, we will obtain L_∞ estimates for the spatial derivative of solution Λ.

Corollary 3.1. *There is a constant* M *independent of* ϵ *and* T *such that*

$$\|\varphi^\epsilon\|_{L_\infty(S\times(0,T))} \le M, \qquad \|\varphi^\epsilon(\cdot,t)\|_{L_\infty(0,T;TV[0,2\pi))} \le M.$$

Proof. The first part follows from Lemma 3.2 (a) directly, because $\varphi^\epsilon = \Lambda^\epsilon_s + s$. The second part is the result of Lemma 3.2 (b), combined with the properties of approximation of TV functions in L_1. □

We want to show that the estimates for Λ^ϵ will persist after passing to the limit with ϵ.

Lemma 3.3. *Let us suppose that* Λ^ϵ *converges weakly in* $L^2(S \times (0,T))$ *to* Λ*. If* $(\Lambda^\epsilon_s + s)_s \ge 0$ *in* $\mathcal{D}'(S)$*, then* $(\Lambda_s + s)_s \ge 0$ *as well in* $\mathcal{D}'(S)$*.*

Proof. Indeed, if $h \in \mathcal{D}(S)$ is positive, then $0 \le \int_S (\Lambda^\epsilon_s + s)h_s$. The inequality holds after taking the limit. □

Lemma 3.4. *There is a constant independent of* ϵ *such that*

$$\int_0^T\int_0^{2\pi}(\Lambda^\epsilon)^2\,dxdt \le M, \qquad \int_0^T\int_0^{2\pi}[(\Lambda^\epsilon_x)^2 + (\Lambda^\epsilon_t)^2]\,dxdt \le M.$$

Proof. The bound on $\int_0^T\int_0^{2\pi}(\Lambda^\epsilon)^2$ is trivial, due to L^∞ estimates established in previous lemmas. Similarly, the bounds in Corollary 3.1 imply that $\int_0^T\int_0^{2\pi}(\Lambda^\epsilon_x)^2 \le M$. We shall calculate the last integral with the help of integration by parts,

$$\begin{aligned}\int_0^T\int_0^{2\pi}(\Lambda^\epsilon_t)^2\,dsdt \qquad &= -\int_0^T\int_0^{2\pi}\Lambda^\epsilon_{st}\frac{d}{d\varphi}J^\epsilon(\Lambda^\epsilon_s+s)\,dsdt + \int_0^T\Lambda^\epsilon_t\frac{d}{d\varphi}J^\epsilon(\Lambda^\epsilon_s+s)|_{s=0}^{s=2\pi}\,dt\\ &= \int_0^{2\pi}J^\epsilon(\varphi_0(s))\,ds - \int_S J^\epsilon(\varphi(s,T))\,ds\\ &\quad+\int_0^T\Lambda^\epsilon_t(0,t)\left(\frac{d}{d\varphi}J^\epsilon(\Lambda^\epsilon_s(0,t)+2\pi)-\frac{d}{d\varphi}J^\epsilon(\Lambda^\epsilon_s(0,t))\right)dt,\end{aligned}$$

Here, we also exploited periodicity of Λ. We notice that the difference $\frac{d}{d\varphi}J^\epsilon(\Lambda^\epsilon_s(0,t)+2\pi) - \frac{d}{d\varphi}J^\epsilon(\Lambda^\epsilon_s(0,t))$ equals exactly 2π. Hence,

$$\int_0^T\int_0^{2\pi}(\Lambda^\epsilon_t)^2\,dsdt \le \int_0^{2\pi}J^\epsilon(\varphi_0(s))\,ds + 2\pi(\Lambda^\epsilon(0,T)-\Lambda^\epsilon(0,0)) \le M$$

due to the Corollary 3.1. □

Remark. We want to stress that the above estimate on Λ_t is one of the most important differences between (1.2) and (2.1).

Now, we have enough information to select a weakly convergent subsequence, with properties announced in the theorem.

Proposition 3.1. *There exists a subsequence $\{\epsilon_k\}$ converging to zero, such that*
(a) $\Lambda^{\epsilon_k} \rightharpoonup \Lambda$ in $W^1_2(S\times(0,T))$; $\qquad$ *$\varphi^{\epsilon_k}_s \rightharpoonup \varphi_s$ as measures in $S\times(0,T)$.*
(b) $\Lambda\in C([0,T),L_2(S))$.

Proof. The first part of (a) is implied by Lemma 3.4. The second part of (a) follows from $\varphi^\epsilon = \Lambda^\epsilon_s + s$, and Lemmas 3.2, 3.4. Part (b) follows from Lemma 3.2 and 3.4 and the embedding theorem (we have already proved $\Lambda^\epsilon\in L_2(0,T;W^1_2(0,2\pi))\cap W^1_2(0,T;L_1(0,2\pi))$). □

The next step is to show that that the limit is indeed a solution. In particular, we have to pass to the limit in the non-linear term. First of all, we shall change the notation in order to make more transparent what we are doing. We want to find $w(s,t)$ such that $w_s(s,t)=\varphi(s,t)$. By a simple integration of this formula and the definition of φ, we can see

$$w(s,t)=\frac{1}{2}s^2+\Lambda(s,t),$$

where we set $w(0,t)=\Lambda(0,t)$. Hence, $w_s=\varphi$ and we can re-write the evolution problem as a gradient system

$$\begin{aligned}&w_t\in\frac{d}{ds}\partial J(w_s), \qquad \text{in } S\times(0,T),\\ &w(s,0)=\frac{1}{2}s^2+\Lambda_0(s), \qquad \text{for } s\in S,\\ &w(s,t)-\frac{1}{2}s^2 \text{ is periodic for } \qquad t\in(0,T).\end{aligned}\tag{3.6}$$

If $\varphi(\cdot,0)$ is increasing, then due to Lemma 3.2 (b) and Lemma 3.3 $\varphi(\cdot,t)$ is increasing as well, hence $w(\cdot,t)$ is convex. Obvious changes are required to write the system for the regularization $w^\epsilon(s,t)=\frac{1}{2}s^2+\Lambda^\epsilon(s,t)$.

Proposition 3.2. *For any fixed $t \geq 0$ and a sequence $\{\epsilon_k\}$ converging to zero there exists its subsequence $\{\epsilon_k\}$ (not relabeled), such that for each $x \in [0, 2\pi)$ the limit*

$$\lim_{\epsilon \to 0} \frac{d}{d\varphi}(J^\epsilon)(\varphi^\epsilon)(x,t) = \Omega(x,t)$$

exists. Moreover, $\Omega(x,t) \in \partial J(\varphi(x,t))$ for almost every $x \in [0, 2\pi)$.

Remark. It is important for us to make the selection of the subsequence independently of t.

Proof. Indeed, once we fix $t > 0$, we may recall that $\varphi^\epsilon(\cdot, t) \in TV$ as well as $\frac{d}{d\varphi} J^\epsilon(\varphi^\epsilon)(x,t) \in TV$. Hence, by Helly's convergence theorem there exists a subsequence ϵ_k such that these sequences converge. Using the new notation, we write,

$$\lim_{\epsilon \to 0} \varphi^\epsilon(x,t) = w_x(x,t), \qquad \lim_{\epsilon \to 0} \frac{d}{d\varphi} J^\epsilon(w_x^\epsilon(x,t)) = \Omega(x,t).$$

Now, we shall show that for each point x the number $\Omega(x,t)$ belongs to $\partial J(w_x(x,t))$. Since the functions J^ϵ are convex, we have the inequality

$$\int_0^{2\pi} J^\epsilon(w_x^\epsilon(x,t) + h_x(x)) - J^\epsilon(w_x^\epsilon(x,t))\, dx \geq \int_0^{2\pi} \frac{d}{d\varphi} J^\epsilon(w_x^\epsilon(x,t)) h_x(x)\, dx,$$

for each $h \in C_0^\infty(0, 2\pi)$. We know that w^ϵ and $\frac{d}{d\varphi} J^\epsilon(w_x^\epsilon(x,t))$ have pointwise limits, which are bounded, hence after passing to limit our claim will follow,

$$\int_0^{2\pi} J(w_x(x,t) + h_x(x)) - J(w_x(x,t))\, dx \geq \int_0^{2\pi} \Omega(x,t) h_x(x)\, dx. \qquad \square$$

We finish the *proof of Theorem 2.1.* By previous Lemmas there exists a sequence Λ^ϵ which converges weakly in $W_2^1(S \times (0,T))$. In particular, if $h \in C_0^\infty(0, 2\pi)$, $t > 0$ and $\tau > 0$ is arbitrary, then we see

$$\int_{t-\tau}^{t+\tau} \int_S \Lambda_t^\epsilon h\, ds dt' = \int_{t-\tau}^{t+\tau} \int_S \frac{\partial}{\partial s} \frac{\partial}{\partial \varphi} J^\epsilon(\Lambda_s^\epsilon + s) h\, ds dt' = -\int_{t-\tau}^{t+\tau} \int_S \frac{\partial}{\partial \varphi} J^\epsilon(\Lambda_s^\epsilon + s) h_s\, ds dt'.$$

Since $\frac{\partial}{\partial\varphi} J^\epsilon(\Lambda_s^\epsilon + s)$ is bounded, it converges weak-$*$ in $L_\infty((0,2\pi) \times (0,T))$ to Ω. We have to show that $\Omega(s,t) \in \partial J(\Lambda_s + s)$. First we notice that we may pass to the limit in the above integral identity,

$$\int_{t-\tau}^{t+\tau} \int_S \Lambda_t(s,t') h(s)\, ds dt' = -\int_{t-\tau}^{t+\tau} \int_S \Omega(s,t') h_s(s)\, ds dt'.$$

By the Lebesgue differentiation theorem we deduce,

$$\int_S \Lambda_t(s,t) h(s)\, ds = -\int_S \Omega(s,t) h_s(s)\, ds \tag{3.7}$$

for a.e. $t \in [0,T]$ for $h \in W_2^1(S)$ (we used the fact that 0 is not distinguished on S). In principle, the set $G = \{t \in [0,T] : \text{ (3.7) holds}\}$ depends upon h, i.e. $G = G(h)$. We shall see, that in fact we can choose G independently of h. Let us recall that $W_2^1(S)$ is separable and let us suppose that D is a dense, countable subset of $W_2^1(S)$. Of course, $\mathcal{G} = \bigcap_{h \in D}^\infty G(h)$ is a set of full measure. Let us then take $t \in \mathcal{G}$ and $h \in C^\infty(S)$. Let us suppose that $\{h_n\}$ is a sequence in $C^\infty(S)$ converging to h in the $W^1_2(S)$-norm. Then,

$$\int_S \Lambda_t(s,t)h_n(s)\,ds = -\int_S \Omega(s,t)(h_n)_s(s)\,ds$$

for all $t \in \mathcal{G}$. We may pass to the limit with n on both sides, thus we reach,

$$\int_S \Lambda_t(s,t)h(s)\,ds = -\int_S \Omega(s,t)h_s(s)\,ds.$$

In other words, (3.7) holds for all $h \in C^\infty(S)$ and all $t \in \mathcal{G}$.

If we now fix $t \in \mathcal{G}$, we next apply Proposition 3.2 to deduce that $\Omega(s,t) \in \partial J(\Lambda_s(s,t)+s)$. Hence the limit, Λ, is indeed a weak solution. □

Now, we are going to prove uniqueness.

Theorem 3.1. *If Λ^i, $i = 1,2$ are two solutions with $\Lambda^1(s,0) = \Lambda^2(s,0)$, then $\Lambda^1(s,t) = \Lambda^2(s,t)$, for $t \leq T$.*

Proof. If Λ^i, $i = 1,2$, are weak solutions, then by the definition of weak solutions we have

$$\int_S \Lambda^i_t h\,ds = -\int_S (\Omega^i - s)h_s\,ds + \int_S h\,ds,$$

where $w^i \in -\partial J$ and h is in H^1. We subtract these two identities for Λ^2 and Λ^1, then we take $(\Lambda^1 - \Lambda^2)$ as a the test function. Finally, the integration over $(0,\bar{t})$, $\bar{t} < T$ yields

$$\int_0^{\bar{t}}\int_S \frac{1}{2}\frac{d}{dt}(\Lambda^1 - \Lambda^2)^2\,dsdt = -\int_0^{\bar{t}}\int_S (\Omega^1 - \Omega^2)(\Lambda^1 - \Lambda^2)_s\,dsdt.$$

Monotonicity of ∂J implies that $\frac{1}{2}\|\Lambda^1 - \Lambda^2\|^2_{L^2(S)}(\bar{t}) \leq 0$. Hence, $\|\Lambda^1 - \Lambda^2\|^2_{L^2(S)}(\bar{t}) = 0$ for any $\bar{t} < T$. □

3.2 Minimal solutions

It is well-known that important information about the studied system is provided by special solutions, like traveling waves, self-similar solutions and other symmetry solutions. We can not talk about self-similar solutions because our systems lacks direct geometrical interpretation, however we may look for special ones, which we named minimal solutions.

In the theory of curvature flows it is natural to anticipate existence of curves such that their curvature is constant, but may change in time. Here, we ask if there exists such a solution $\bar{\varphi}$ to (1.3) that

$$\frac{d}{ds}\partial J(\bar{\varphi}) \ni k, \quad \text{hence } \partial J(\bar{\varphi}) \ni ks + s^*, \tag{3.8}$$

where s^* is appropriately chosen, e.g. $s^* = \frac{\pi}{4}$ and $|k| = 1$. The last restriction is of geometric nature, namely we want that for any $a \in \mathbb{R}$ the image of S by ∂J be contained in an interval no longer than 2π.

In fact, we may come up with explicit formulas. One for $k = 1$ is provided by formula (2.2). It is then obvious that

$$\bar{\varphi}(s) := (\partial J)^{-1}(s + s^*), \tag{3.9}$$

as in [MRy] and in Section 2. Moreover, $\int_0^{2\pi}\bar\varphi(s)\,ds = 2\pi^2 = \int_0^{2\pi} s\,ds$. By the reversal of the orientation, we immediately obtain the solution for $k=-1$,

$$\bar\varphi_{-1}(s) = \frac{7\pi}{4}\chi_{[0,\frac{\pi}{2})}(s) + \frac{5\pi}{4}\chi_{[\frac{\pi}{2},\pi)}(s) + \frac{3\pi}{4}\chi_{[\pi,\frac{3\pi}{2})}(s) + \frac{\pi}{4}\chi_{[\frac{3\pi}{2},2\pi)}(s).$$

We choose $\bar\varphi(s) := \bar\varphi_1(s)$, which is given by (2.2), because we prefer to have $\bar\varphi$ an increasing function.

As a result, $\bar\Lambda$ defined by $\bar\Lambda(s,t) = \int_0^s \bar\varphi(u)\,du + F(t)$ is indeed 2π periodic in s and it is a solution to (1.3). Here, we must take $F(t) = A + t$. One can check in a straightforward manner that indeed $\bar\Lambda$ solves (3.9). This is indeed so, because we have found $\bar\varphi(s) = \Lambda_s(s) + s$ and Ω is a section of $\partial I(\bar\varphi)$, namely, $\Omega(s,t) = s$, which satisfies (3.9). If we take $A = 0$, then $\bar\Lambda$ satisfies the initial condition: $\bar\Lambda(s,0) = \int_0^s \bar\varphi(u)\,du$.

4 The semi-discretization

In this part we examine the semi-discretization of (2.1). Our goals are not only to establish existence for the presented scheme, but also to show qualitative properties of the obtained solutions. In particular our considerations will explain the appearance of facets. Finally, we prove the convergence of solutions of the semi-discretization to the solutions obtained in Section 3.

We define the semi-discretization in time of system (2.1) as follows

$$\frac{\lambda_h^k(s) - \lambda_h^{k-1}(s)}{h} \in \frac{d}{ds}\partial J[\lambda_{h,s}^k(s) + s] \tag{4.1}$$

and $\lambda_h^k(0) = \lambda_h^k(2\pi)$ and $(\lambda_h^0)_s = \phi_0$ for $k = 1,\dots,[T/h]$; or equivalently equation (4.1) can be stated

$$\lambda_h^k(s) - h\frac{d}{ds}\partial J[\lambda_{h,s}^k(s) + s] \ni \lambda_h^{k-1}(s). \tag{4.2}$$

We establish existence of solution to this problem.

Lemma 4.1. *Let us suppose that an absolutely continuous function v is such that $v_s = \varphi \in TV[0,2\pi)$, then there exists $u \in AC([0,2\pi))$ such that $u_s \in TV$, which is a solutions to (4.1), i.e.*

$$u - v \in h\frac{d}{ds}\partial J(u_s) \tag{4.3}$$

with $u(0) + \frac{1}{2}(2\pi)^2 = u(2\pi)$ and the following bound is valid

$$||u_s||_{TV} \le ||v_s||_{TV}. \tag{4.4}$$

Remark. Our understanding of (4.3) is the same as that of (2.1), i.e., there exists $\omega \in W_1^1([0,2\pi))$, such that $\omega(x) \in \partial J(u_s)$ and $u - v = h\frac{d}{ds}\omega$.

We also note that u and v appearing in this Lemma need not be periodic, on the other hand $\Lambda(\cdot,t)$ and $\lambda_h^k(\cdot)$ are periodic.

Proof. Let us notice that if u is a solution to (4.3), then 0 belongs to the subdifferential of the functional

$$\mathcal{J}(u) = \int_0^{2\pi}[hJ(u_s) + \frac{1}{2}(u-v)^2], \quad \text{for } u \in AC([0,2\pi)),\ u_s \in TV.$$

i.e. u is a minimizer of $\mathcal{J}$. To be precise, we define $\mathcal{J}$ on $L_2(S)$ by the above formula for $u \in AC([0,2))\pi$ with $u_s \in TV$ and we put $\mathcal{J}(u) = +\infty$ for u belonging to the complement of this set.

In order to solve (4.3), we consider a family of regularized problems,

$$\mathcal{J}_\epsilon(u) = \int_0^{2\pi} [hJ_\epsilon(u_s) + \frac{1}{2}(u-v)^2],$$

where J_ϵ is the same regularization of J that we used in (3.1).

The functional $\mathcal{J}_\epsilon$ is well-defined, convex and coercive on the standard Sobolev space $W_1^2(0,2\pi)$, thus it possesses a unique minimizer u^ϵ. Now, we apply again the methods used in Section 3.1 to show existence of a weak solution of the evolution problem (1.3). The regularization of system (4.3) leads to the following equation

$$u^\epsilon_{ss} - \frac{d^2}{ds^2}(\frac{\partial J_\epsilon}{\partial \varphi^2}(u^\epsilon_s)u^\epsilon_{ss}) = v^\epsilon_{ss}.$$

By repeating the argument for (3.4),we get $||u^\epsilon_{ss}||_{L_1} \leq ||v^\epsilon_{ss}||_{L_1}$. Passing to the limit with $\epsilon \to 0$ yields (4.4).

In addition we have the following bounds $\int_0^{2\pi}(u^\epsilon)^2\,dx \leq M$, $\int_0^{2\pi}(u^\epsilon_x)^2\,dx \leq M$. In order to prove them we follow the lines of reasoning of Corollary 3.1 and Lemma 3.4. These bounds suffice to show existence of a subsequence $\{\epsilon_k\}$ converging to zero, such that
(a) $u^{\epsilon_k} \rightharpoonup u$ in $W^1_2(0,2\pi)$; $\qquad u^{\epsilon_k}_{ss} \rightharpoonup u_{ss}$ as measures.

Subsequently, by Helly's theorem we conclude existence of the pointwise limits (for another subsequence $\{\epsilon_k\}$, not relabeled)

$$\lim_{t\to\infty} \varphi^\epsilon(x) = u_x(x), \qquad \lim_{t\to\infty} \frac{d}{d\varphi} J^\epsilon(u^\epsilon_x(x)) = \Omega(x).$$

Moreover, $\Omega(x) \in \partial J(u_x(x))$ for each $x \in [0,2\pi)$.

Now, we show uniqueness of solutions, constructed in Lemma 4.1.

Lemma 4.2. *Let $v \in AC([0,2\pi))$, $v_s \in TV([0,2\pi))$, then there exists at most one weak solution $u \in AC([0,2\pi))$, $u_s \in TV(S)$ to problem (4.3)*

Proof. Let us suppose that there are two solutions to (4.3), u^i, $i = 1,2$. By the definition, there are two functions $\omega_i \in \partial J(u^i_s)$, $i = 1,2$, such that

$$u^i - v = h\frac{d}{ds}\omega_i, \qquad i = 1,2.$$

After subtracting these two equations and multiplying them by $u_1 - u_2$ and integrating over $[0,2\pi)$ we see

$$\|u^1 - u^2\|^2 - \int_0^{2\pi} h(\frac{d}{ds}\omega_1 - \frac{d}{ds}\omega_2)(u^1 - u^2)\,ds = 0.$$

The integration by parts leads us to

$$0 = \|u^1 - u^2\|^2 + \int_0^{2\pi} h(\omega_1 - \omega_2)(u^1_s - u^2_s)\,ds \geq \|u^1 - u^2\|^2 \geq 0.$$

As a result $u^1 = u^2$. □

In order to finish our preparations, we introduce the sets of preferred orientation which dominate the behavior of solutions. Let us suppose, that w is absolutely continuous and $w_s \in TV$, then at any point s, the left derivative w_s^-, as well as the right derivative w_s^+ are well-defined, hence we may set

$$\partial w(s) = \{\tau w_s^- + (1-\tau) w_s^+ :\ \tau \in [0,1]\}. \tag{4.5}$$

Under our assumptions on w the set $\partial w(s)$ is the Clarke differential of w and equality holds in (4.5) due to [Cg, Section 2, Ex. 1]. If w is convex, then ∂w is the well-known subdifferential of w.

Now, for each $l = 0, 1, 2, 3$, we set

$$\Xi_l(w_s) = \{s \in [0, 2\pi] : w \text{ is differentiable at } s \text{ and } w_s(s) = \alpha_k \quad \text{or} \quad \alpha_k \in \partial w(s)\} \tag{4.6}$$

Furthermore, we set $\Xi(w_s) = \bigcup_{l=0}^{3} \Xi_l(w_s)$.

The result, delivering the main properties of solutions, is the following.

Theorem 4.1. *Let $\phi_0 = \lambda^0_{h,s} + s \in$ J-R$[0,\pi)$. Then a solution $\{\lambda^h_k\}$ to problem (4.1) exists, it is unique and it satisfi es the following bound*

$$||\lambda^h_{k,s} + s||_{\text{J-R}[0,2\pi)} \le ||\lambda_{0,s} + s||_{\text{J-R}[0,2\pi)}. \tag{4.7}$$

Moreover, we have

$$\Xi(\lambda^h_{k-1,s} + s) \subset \Xi(\lambda^h_{k,s} + s) \quad \textit{and} \quad K(\lambda^k_{h,s} + s) \le K(\lambda^{k-1}_{h,s} + s) \tag{4.8}$$

and

$$\sup_k \sup_{l=0,1,2,3} |\Xi^k_l \setminus \Xi^{k-1}_l| \le C(V(h) + h^{1/2}). \tag{4.9}$$

where $V(s) \to 0$ as $s \to 0$ and V is determined by the initial datum ϕ_0. Moreover, on connected components of the set $\Xi^{k-1} \setminus (\bigcup_{l=0}^{3} \Xi^k_l \setminus \Xi^{k-1}_l)$

$$\frac{d}{ds}\partial J[\lambda^k_{h,s} + s] \quad \textit{is constant.} \tag{4.10}$$

Proof. By Lemmas 4.1 and 4.2 we conclude existence of the sequence of solutions to the semi-discretization, the solutions are such that λ^h_s belong to $TV(S)$. It is enough to restate the equation (4.2) as follows:

$$u - h\frac{d}{ds}\partial J[u_s] = v \tag{4.11}$$

with $u = \lambda^k_h + \frac{1}{2}s^2$ and $v = \lambda^{k-1}_h + \frac{1}{2}s^2$, and boundary condition $u(0) + 2\pi^2 = u(2\pi)$.

The set, where function $J[u_s]$ is singular, i.e. $\Xi(u_s)$, plays the key role. Our first task is to prove the inclusion from (4.8). Note that in a neighborhood of any point $s \notin \Xi(u_s)$ function $\partial J[u_s(\cdot)]$ is constant, hence we get $u(s) = v(s)$. Thus, we point the first feature of solutions to (4.11)

$$u(s) = v(s) \qquad \text{for} \quad s \in (0, 2\pi) \setminus \Xi(u_s). \tag{4.12}$$

From (4.12) we deduce that if $s \notin \Xi(v_s)$, then $s \notin \Xi(u_s)$. Subsequently, we get $\Xi(v_s) \subset \Xi(u_s)$ which proves the inclusion from (4.8). Thus, the isolated elements stay isolated or merge with other elements. From this we obtain that $K(v_s) \ge K(u_s)$ what ends the proof of line (4.8).

By properties (4.12), (4.8) and the estimate from Lemma (4.1), we immediately deduce estimate (4.7). In particular, what we gain is a uniform bound in $L_\infty(S)$ on $\{\lambda^k_{h,s}\}$.

The set $\Xi(u_s)$ is defined as the sum of $\bigcup_{l=0}^{3} \Xi_l(u_s)$, thus without loss of generality we can concentrate our attention on one of them, e.g. on the set $\Xi_2(u_s)$ – see (4.6). From the J-R-regularity of u_s set $\Xi_2(u_s)$ is a sum of closed intervals, so we take one of them, say,

$$[a_-, a_+] \subset \Xi_2(u_s) \quad \text{ and } u_s|_{(a_-,a_+)} = \frac{\pi}{4}. \tag{4.13}$$

Recalling the required regularity of the functions in the J-R-class, we find $\epsilon > 0$ such that one of the four following possibilities holds:

$$\begin{array}{rll} (i) & u_s|_{(a_- - \epsilon, a_-)} > \frac{\pi}{4}, & u_s|_{(a_+, a_+ + \epsilon)} < \frac{\pi}{4}, \\ (ii) & u_s|_{(a_- - \epsilon, a_-)} < \frac{\pi}{4}, & u_s|_{(a_+, a_+ + \epsilon)} > \frac{\pi}{4}, \\ (iii) & u_s|_{(a_- - \epsilon, a_-)} > \frac{\pi}{4}, & u_s|_{(a_+, a_+ + \epsilon)} > \frac{\pi}{4}, \\ (iv) & u_s|_{(a_- - \epsilon, a_-)} < \frac{\pi}{4}, & u_s|_{(a_+, a_+ + \epsilon)} < \frac{\pi}{4}. \end{array} \tag{4.14}$$

Subsequently, we integrate (4.11) over $(a_- - \epsilon, a_+ + \epsilon)$ to get

$$\int_{a_- - \epsilon}^{a_+ + \epsilon} u ds - h(\partial J[u_s]|_{a_- - \epsilon}^{a_+ + \epsilon}) = \int_{a_- - \epsilon}^{a_+ + \epsilon} v ds. \tag{4.15}$$

After passing with $\epsilon \to 0^+$, we obtain – according to the above four cases (4.11) – the following identities

$$\begin{array}{ccc} (i) & \int_{a_-}^{a^+} u ds - h\frac{\pi}{2} = \int_{a_-}^{a^+} v ds & \text{(convexity)}, \\ (ii) & \int_{a_-}^{a^+} u ds + h\frac{\pi}{2} = \int_{a_-}^{a^+} v ds & \text{(concavity)}, \\ (iii) \text{ and } (iv) & \int_{a_-}^{a^+} u ds = \int_{a_-}^{a^+} v ds & \text{(monotonicity)}. \end{array} \tag{4.16}$$

In our present analysis, we essentially use the fact that the energy density function J is defined by a square. Due to the definition of J, see (1.5), formula (4.16) exhausts all the possibilities of the behavior of u_s. For more complex polygons, we would have to discuss more possible types of facets – here, there are just four of them.

We keep considering the interval $[a_-, a_+] \subset \Xi_2(u_s)$, see (4.13). Let us introduce a set

$$\Pi = ([a_-, a_+] \cap \Xi(v_s)) \setminus (\Xi_0(u_s) \cup \Xi_1(u_s) \cup \Xi_3(u_s)), \tag{4.17}$$

then by the properties of sets Ξ, we deduce that

$$(u - v)|_{\Pi} = C_h \text{ is constant}. \tag{4.18}$$

The sign of constant C_h is determined by the geometrical properties of cases in (4.16). We have

$$C_h > 0 \text{ for } (i), \qquad C_h < 0 \text{ for } (ii) \text{ and } C_h = 0 \text{ for } (iii) \text{ and } (iv). \tag{4.19}$$

Also identity (4.18) and equation (4.11) yield

$$\frac{d}{ds}\partial J[u_s]\bigg|_{\Pi} = \frac{C_h}{h} \quad \text{and} \quad \frac{d}{ds}\partial J[u_s]\bigg|_{(0,2\pi)\setminus\Xi(u_s)} = 0. \tag{4.20}$$

Thus, we proved (4.10).

Next, we are going to study (4.9). From the analysis of (4.11), we conclude that

$$||u - v||_{L_1(S)} \leq h\frac{\pi}{2}K(\phi_0). \tag{4.21}$$

Additionally, from (4.8) we have also that $u, v \in W^1_\infty(S)$, thus simple considerations lead us to the following bound

$$||u - v||_{L_\infty(S)} \le h^{1/2} C(\phi_0). \tag{4.22}$$

In order to measure the set $\Xi_l(u_s) \setminus \Xi_l(v_s)$ we split it into two parts

$$\Xi_l(u_s) \setminus \Xi_l(v_s) = [(\Xi_l(u_s) \setminus \Xi_l(v_s)) \cap \Xi(v_s)] \ \cup \ (\Xi_l(u_s) \setminus \Xi(v_s)) = \Pi_1 \cup \Pi_2. \tag{4.23}$$

Let us consider Π_1. On this set we watch the evolution of the intersection of facets. Thanks to the full information about the direction of this facet, we deduce immediately that

$$|\Pi_1| \le C(\phi_0) h^{1/2} \tag{4.24}$$

The number of possible intersections is controlled by $K(\phi_0)$.

To estimate Π_2, let us note that this set is a subset of $\Xi(\lambda_{0,s} + s)$, thus in the general case we can say only

$$|\Pi_2| \le V(h), \tag{4.25}$$

where $V(s) \to 0$ as $s \to 0$ and V is determined by the initial datum. Assuming strict convexity of initial domain we would obtain $V(h) \sim h^{1/3}$ – see the example at the end of subsection 5.2. $\square$

Theorem 4.1 is proved.

Next, we show that sequences $\{\lambda^k_h\}$ converge to solutions of the original problem. We will compare solutions given by Theorem 2.1 and Lemma 4.1, in particular, all assumptions of Theorem 2.1 are not required. We follow the standard procedure which is valid for parabolic operators (see [MRa]).

Our next task is to show the following lemma.

Lemma 4.3. *Let Λ and $\{\lambda^k_k\}$ be solutions to problems (2.1) and (4.1) respectively, then*

$$||\Lambda(s,t) - \sum_{k=0}^{[T/h]} \lambda^k_h(s) \chi_{[k,k+1)}(t)||_{L_1(0,T;L_2(S))} \to 0 \qquad as \quad h \to 0^+. \tag{4.26}$$

If the initial datum fulfi lls the assumptions of Theorem 2.1, i.e. $\Lambda_{0,s} \in TV(S)$, then

$$||\Lambda(s,t) - \sum_{k=0}^{[T/h]} \lambda^k_h(s) \chi_{[k,k+1)}(t)||_{L_p(0,T;W^{2-\epsilon}_1(S))} \to 0 \qquad as \quad h \to 0^+ \tag{4.27}$$

for any $1 < p < \infty$ and $\epsilon > 0$.

Proof. From the properties of solutions to problem (2.1), we know that $\Lambda_t \in L_2(0,T;L_2(S))$. It follows that

$$\left|\left|\frac{\Lambda(s,t) - \Lambda(s,t-h)}{h} - \Lambda_t(s,t)\right|\right|_{L_1(h,T;L_2(S))} \to 0 \ \text{ as } \ h \to 0^+. \tag{4.28}$$

For fixed $h > 0$ we denote

$$R_h(s,t) = \frac{\Lambda(s,t) - \Lambda(s,t-h)}{h} - \Lambda_t(s,t), \tag{4.29}$$

then the equation $(2.1)_1$ can be restated as follows

$$\int_S \frac{\Lambda(s,t)-\Lambda(s,t-h)}{h}\pi ds = -\int_S \Omega(s,t)\pi_s + R_h(s,t)\pi ds \tag{4.30}$$

for each π in $C^\infty(S\times(0,T))$ and each selection $\Omega(s,t)$ of multivalued function $\partial J[\Lambda_s(s,t)+s]$.

We want to compare the above system with the semi-discretization given in Section 4.

$$\int_S \frac{\lambda_h^k(s)-\lambda^{k-1}(s)}{h}\pi ds = -\int_S \omega(s,t)\pi ds \tag{4.31}$$

where $t\in[kh,(k+1)h)$ and $\omega(s,t)$ is any section of $\partial J[\lambda_{h,s}^k(s)+s]$.

Let us define

$$A^k(s,t) = \Lambda(s,t)-\lambda_h^k(s,t), \tag{4.32}$$

provided $t\in[kh,(k+1)h)$, then from (4.30) and (4.31) we deduce

$$\int_S \frac{A^k(s,t)-A^{k-1}(s,t-h)}{h}\pi ds = -\int_S \{\Omega(s,t)-\omega(s,t)\}\pi_s + R_h(s,t)\pi ds. \tag{4.33}$$

Taking in (4.33) as a test function $A^k(t,s)$, we get

$$\begin{gathered}\int_{(0,2\pi)} |A^k(s,t)|^2 ds = \int_{(0,2\pi)} A^k(s,t)A^{k-1}(s,t-h)ds \\ -h\int_{(0,2\pi)} (\Omega(s,t)-\omega(s,t))\left(\Lambda_s(s,t)-\lambda_{h,s}^k(s,t)\right)ds \\ +h\int_{(0,2\pi)} R_h(s,t)A^k(s,t)ds,\end{gathered} \tag{4.34}$$

but the monotonicity of ∂J implies

$$\int_{(0,2\pi)} (\Omega(s,t)-\omega(s,t))\left(\Lambda_s(s,t)-\lambda_s^k(s,t)\right)ds \geq 0. \tag{4.35}$$

So, defining $\alpha^k(t) = ||A^k(\cdot,t)||_{L_2(S)}$, by the Schwarz inequality, we get from (4.29) the following inequality

$$\alpha^k(t) \leq \alpha^{k-1}(t-h) + hr_h^k(t) \qquad \text{for} \quad t\in[kh,(k+1)h), \tag{4.36}$$

where $r_h^k(t) = ||R_h(\cdot,t)||_{L_2(0,2\pi)}$. Thus (4.36) yields

$$\alpha^k(t) \leq \alpha^0(t-kh) + \sum_{l=1}^k hr_h^l(t-(k-l)h) \quad \text{with} \quad t\in[kh,(k+1)h). \tag{4.37}$$

Integrating (4.37) over $t\in[kh,(k+1)h)$, we get

$$\int_{kh}^{(k+1)h} \alpha^k(t)dt \leq \int_0^h \alpha^0(\tau)d\tau + h\int_0^T ||R_h(\cdot,t)||_{L_2(0,2\pi)}dt \tag{4.38}$$

for $T > (k+1)h$. Introducing function $\tilde\alpha(t) = \sum_{l=0}^L \alpha^l(t)\chi_{[lh,(l+1)h)}(t)$ with $L = [T/h]$, from (4.38) we get

$$\int_0^T \tilde\alpha(t)dt \leq h^{1/2}T||\Lambda_t||_{L_2(0,T;L_2(0,2\pi))} + T||R_h||_{L_1(0,T;L_2(0,2\pi))}, \tag{4.39}$$

because the first term of the right-hand-side (RHS) of (4.39) is a consequence of the following estimate

$$\begin{aligned}&\frac{1}{h}\int_0^h \alpha^0(t)dt \le \frac{1}{h}\int_0^h ||\int_0^t \Lambda_t(\cdot,\tau)d\tau||_{L_2(0,2\pi)}dt\\ \le &\frac{1}{h}(\int_0^h t^2dt)^{1/2}(\int_0^h ||\Lambda_t||^2_{L_2(0,2\pi)}dt)^{1/2} \le Ch^{1/2}||\Lambda_t||_{L_2(0,T;L_2(0,2\pi))}.\end{aligned} \tag{4.40}$$

Hence, from (4.39) and (4.28) we conclude $||\tilde{\alpha}_h||_{L_1(0,T;L_2(0,2\pi))} \to 0$ as $h \to 0^+$ and we get (4.26). From the interpolation estimates and the results of Theorem 2.1, we deduce that for any $p < \infty$ and $\epsilon > 0$ the convergence (4.27) is valid. Lemma 4.3 is proved. □

5 Analysis of solutions

The semi-discretization process proves that the set $\Xi(u^k_{h,s})$ grows with k. One can show that the set $\bigcup_{t\ge 0}\Xi(u_{h,s}(\cdot,t))$ may be estimated from below to show that it survives the limiting process as $h \to 0$. This may be achieved by the analysis of the semi-discretization procedure, but this seems tedious. We propose an alternative approach by the construction of an explicit solution to (1.3) for data in $\varphi_0 \in$ J-R. By uniqueness result, see Theorem 2.1, this is the solution.

We shall assume in this Section that $\varphi \equiv w_s$ belongs to J-R and this is the case for the initial data φ_0 of system (2.1). As a result of the definition of the J-R class we see

$$\Xi(\varphi_0) = \bigcup_{l=0}^{3} \Xi_l(\varphi_0) = \bigcup_{k=1}^{N_0}[\xi^-_k, \xi^+_k], \tag{5.1}$$

where

$$\xi^-_k \le \xi^+_k \quad \text{and} \quad \xi^+_k \le \xi^-_{k+1}, \quad k = 1,\dots,N_0, \tag{5.2}$$

(with the understanding $\xi^-_{N_0+1} = \xi^-_1 + 2\pi$). Moreover, each interval $[\xi^-_k, \xi^+_k]$ is a connected component of one of the sets $\Xi_l(\varphi_0)$, $l = 0,1,2,3$. We shall also adopt the convention that $0 \le \xi^-_1$ and possibly $\xi^+_{N_0} > 2\pi$, but $\xi^+_{N_0} - 2\pi \le \xi^-_1$.

If $[\xi^-_k, \xi^+_k]$ is one of the connected components of $\Xi_l(\varphi)$, then we will call by a *facet* the set $F = F_k(\xi^-_k, \xi^+_k) = \{(x,y) \in \mathbb{R}^2 :\ y = w(x),\ x \in [\xi^-_k, \xi^+_k]\}$. The interval $[\xi^-_k, \xi^+_k]$ will be called the pre-image of facet F_k. Let us stress that we admit $\xi^-_k = \xi^+_k$, i.e. a facet degenerated to a point as well as $\xi^+_{k-1} = \xi^-_k$, i.e. we expect interaction of facets. We shall see that the generic initial data lead to the facet creation (from the degenerate ones) and their interaction. We show that facets are formed instantaneously from the data. At this point we mention that creation of interacting facets leads to additional difficulties and this process is handled separately.

We will come up with an explicit formula. Once we check that indeed this formula yields a solution to equation (2.1), we will be assured that this is the unique solution we seek. Subsequently, we shall see that solutions get convexified, i.e. after some finite time the angle becomes increasing, hence w becomes convex. Finally, we study interaction of facets. We will prove that $w(\cdot,t)$ becomes a minimal solution at the limit time.

It will be also convenient to say that a facet $F_k(\xi^-_k, \xi^+_k)$, has *zero curvature*, if $[\xi^-_k, \xi^+_k]$ is a connected component of $\Xi(\varphi)$ and there exists an open interval (A,B), containing $[\xi^-_k, \xi^+_k]$ such that w_s is not monotone on any interval (a,b), satisfying

$$[\xi^-_k, \xi^+_k] \subset (a,b) \subset (A,B).$$

Furthermore, we say that a facet $F_k = F_k(\xi_k^-, \xi_k^+)$ is *regular* if $\xi_k^- < \xi_k^+$. Otherwise, we say that F_k is *degenerate*. If $w_s \in$ J-R is such that the graph of w contains degenerate facets, then we say that facets are created in solutions to (2.1).

Finally, we say that facets $F_l, \ldots, F_{l+r}$ for $r > 0$, *interact* (or are *interacting*) if $F_k \cap F_{k+1}$, $k = l, \ldots, l+r-1$, is a singleton. We call a single facet F_k *non-interacting*, if it is not true that it interacts with any other facet.

Thus, we have the total of eight combinations, we will treat each case separately.

5.1 A comparison principle

We are going to establish that solutions to equation (2.1) enjoy the expected comparison principle. This result is interesting for its own sake but also it is a useful tool analysis. We will apply it to show creation of interacting facets.

We first recall the basic result (see, e.g. [S]).

Proposition 5.1. *Let us suppose that u_1, u_2 are smooth solutions to a strongly parabolic equation*

$$u_t = (a(x, u_x))_x \quad \text{in } S \times (0, T)$$

and $u_2(x, 0) \geq u_1(x, 0)$, then $u_2(x, t) \geq u_1(x, t)$ for all $t \in (0, T)$. □

With this result we may deduce the following comparison principle.

Proposition 5.2. *Let us suppose that Λ_1, Λ_2 are weak solutions to (2.1) and $\Lambda_1(x, 0) \leq \Lambda_2(x, 0)$, then $\Lambda_2(x, t) \geq \Lambda_1(x, t)$ for all $t \in (0, T)$.*

Proof. Since $\Lambda_1(x, 0) \leq \Lambda_2(x, 0)$, we deduce that $\Lambda_1^\epsilon(x, 0) \leq \Lambda_2^\epsilon(x, 0)$, where Λ_i^ϵ, $i = 1, 2$ are solutions to the regularized system (3.2). Application of the preceding result yields

$$\Lambda_1^\epsilon(x, t) \leq \Lambda_2^\epsilon(x, t).$$

Since the point-wise limit exists we conclude that our proposition holds. □

We stress that no information about Ω_i, $i = 1, 2$ is needed in the proof of the above result.

5.2 Facet formation

We shall see below that the evolution of a facet F_k separated from other facets is governed by an ODE for its end-points, see (5.14) below. In the case of interacting facets their evolution is described by a system of ODE's (5.19).

As we mentioned we admit facets F_k degenerated to a single point at the initial instance $t_0 = 0$. In this case the single ODE (5.11) and system ODE (5.19) become singular. While we can resolve satisfactorily the singularity of the single ODE, the analysis of the system is more difficult. In fact, we circumvent this problem by using the comparison principle to show creation of interacting facets.

We shall use the notions and notation introduced above. In addition, in order to facilitate our construction we shall write

$$x \mapsto \alpha_k(x - s_k) + \tau_k =: l_k(x, s_k, \tau_k),$$

where $\alpha_k \in \mathcal{A}$, $s_k \in \Xi(\varphi)$, $\tau_k \in \mathbb{R}$.

Theorem 5.1. *Let us assume that $\varphi_0 = w_{0,s} \in$ J-R and w is the unique solution to (2.1). We also assume that the set $\Xi(w_{0,s}) = \bigcup_{k=1}^{N_0}[\xi^-_{k0}, \xi^+_{k0}]$ fulfills conditions (5.1) and (5.2). Then, there exists a finite sequence of time instances $0 \le t_0 < t_1 < \ldots < t_M < \infty$ and a finite sequence of continuous functions*

$$\begin{aligned} \xi^\pm_k &: [t_i, t_{i+1}] \to \mathbb{R}, \qquad i = 0, \ldots t_{M-1}, \quad k = 1, \ldots N_i, \\ \xi^\pm_k &: [t_M, \infty) \to \mathbb{R}, \qquad k = 1, \ldots N_M = 4, \end{aligned}$$

where $N_0 \ge N_1 \ge \ldots \ge N_M = 4$.

The functions $\xi^-_k(\cdot)$, $\xi^+_k(\cdot)$ satisfying (5.2) have the following properties:
(a) $\xi^\pm_k(0) = \xi^\pm_{k0}$;
(b) $0 \le \xi^-_1(t) \le \xi^+_1(t) \le \xi^-_2(t) \le \ldots \le \xi^-_N(t) \le \xi^+_N(t) \le \xi^-_1 + 2\pi$, $t \in [t_i, t_{i+1})$;
(c) for $t \in [t_i, t_{i+1})$ we have $\Xi(\varphi(\cdot, t)) = \bigcup_{k=1}^{N_i}[\xi^-_k(t), \xi^+_k(t)]$, and each interval $[\xi^-_k(t), \xi^+_k(t)]$ is a connected component of one of the sets $\Xi_l(\varphi(\cdot, t))$, $l = 0, 1, 2, 3$.

There exist functions $\tau_k : [t_i, t_{i+1}) \to \mathbb{R}$, $i = 0, \ldots t_M$, $k = 1, \ldots N_i$, and $t_{M+1} = \infty$. They are such that the unique solution to (2.1) with initial data $\varphi(x, 0) = \varphi_0(x)$ is given by the following formula for $t \in [t_i, t_{i+1})$, $i = 0, \ldots, M$

$$w(x,t) = \begin{cases} w_0(x) & \text{if } x \in [0, 2\pi) \setminus \bigcup_{k=1}^{N_i}[\xi^-_k(t), \xi^+_k(t)] \\ l_k(x, \xi^+_k(t_i), \tau_k(t)) + w(\xi^+_k(t_i), t_i) & \text{if } x \in [\xi^-_k(t), \xi^+_k(t)], \quad k = 1, \ldots, N_i \end{cases} \tag{5.3}$$

Moreover, $w_x(\cdot, t)$ is well-defined a.e., ∂w defined by (4.5) belongs to J-R and $\|\partial w(\cdot, t)\|_{\text{J-R}} \le \|\partial w_0\|_{\text{J-R}}$.

In addition, at each time instant t_i, $i = 0, \ldots, M$, one of the following happens:

(i) One or more zero-curvature facets disappear, i.e. if one facet disappears at t_i, then

$$\xi^+_{k_0-1}(t) \le \xi^-_{k_0}(t) < \xi^+_{k_0}(t) \le \xi^-_{k_0+1}(t), \quad \text{for } t_i < t < t_{i+1}$$

and

$$\lim_{t \to t^-_{i+1}} \xi^-_{k_0-1}(t) = \xi^-_{l_0}(t_{i+1}), \qquad \lim_{t \to t^-_{i+1}} \xi^-_{k_0+1}(t) = \xi^+_{l_0}(t_{i+1}),$$

where $[\xi^-_{l_0}(t_{i+1}), \xi^+_{l_0}(t_{i+1})]$ is a subset of a connected component of $\Xi_l(\varphi(t_{i+1}))$, as a result $N_{i+1} < N_i$.

(ii) One pair or more pairs of facets begin to interact, i.e. $\xi^+_{k-1}(t) < \xi^-_k(t)$ for $t_i < t < t_{i+1}$ and

$$\lim_{t \to t^-_{i+1}} \xi^+_{k-1}(t) = \xi^+_{k-1}(t_{i+1}) = \xi^-_k(t_{i+1}) = \lim_{t \to t^-_{i+1}} \xi^-_k(t).$$

The proof is achieved in a number of steps. Its major parts are separated as Lemmas. We start with constructing the $\xi^\pm_k$'s. We first consider non-interaction during creation of facets, i.e.

$$\text{if } \xi^-_k = \xi^+_k, \quad \text{then} \quad \xi^+_{k-1} < \xi^-_k \text{ and } \xi^+_k < \xi^-_{k+1}. \tag{5.4}$$

However, the lemma below is valid without this restriction.

Lemma 5.1. *Let us suppose that* $w_s = \varphi \in$ J-R *and* $[\xi_k^-, \xi_k^+]$ *is a connected component of* $\Xi_l(\varphi)$ *and* s_k *is its member. We assume that* $F_k(\xi_k^-, \xi_k^+)$ *is not a zero curvature facet.*
(a) If $\xi_k^+ < \xi_{k+1}^-$*, then for sufficiently small* τ_k *of a proper sign, there exist* $\xi_k^\pm(\tau_k)$ *such that*

$$w(\xi_k^\pm(\tau_k)) = l_k(\xi_k^\pm(\tau_k), \xi_k^\pm, \tau_k) + w(\xi_k^\pm) \qquad \text{and } \xi_k^\pm(0) = \xi_k^\pm. \tag{5.5}$$

Moreover, the functions $\tau_k \mapsto \xi_k^\pm(\tau_k)$ *are Lipschitz continuous, provided that* $w_s(\xi_k^\pm) \neq \alpha_k$*. Otherwise,* $\xi_k^\pm(\tau_k)$ *are locally Lipschitz continuous. In addition,*

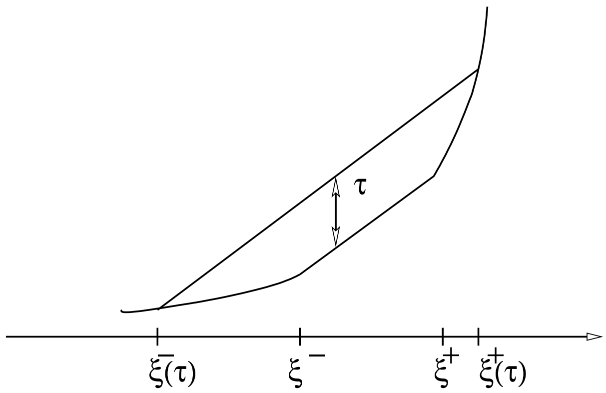
$$\frac{d\xi_k^+}{d\tau_k}(\tau_k) = \frac{1}{w_s(\xi_k^+) - \alpha_k}, \quad \frac{d\xi_k^-}{d\tau_k}(\tau_k) = \frac{1}{w_s(\xi_k^-) - \alpha_k} \qquad \text{for } a.e.\ |\tau_k| \in [0, \epsilon). \tag{5.6}$$

(b) If $\xi_l^+ < \xi_{l+1}^- \leq \xi_{l+1}^+ = \xi_{l+2}^- \leq \xi_{l+2}^+ = \xi_{l+3}^- \cdots \leq \xi_{l+r}^+ < \xi_{l+r+1}^-$*, (in particular we admit* $\xi_1^- = \xi_{N_0}^+ - 2\pi$*), then*

$$w(\xi_{k-1}^+) + l_{k-1}(\xi_{k-1}^+(\tau_{k-1}, \tau_k)), \xi_{k-1}^+, \tau_{k-1}) = w(\xi_k^+) + l_k(\xi_k^+(\tau_k, \tau_{k+1})), \xi_k^+, \tau_k) \tag{5.7}$$

for $k = l+1, \ldots, l+r$*. Moreover, the functions* $(\tau_k, \tau_{k+1}) \mapsto \xi_k^\pm(\tau_k, \tau_{k+1})$*,* $k = l+1, \ldots, l+r-1$ *are Lipschitz continuous.*

Proof. Before proceeding to the formal proof we will explain the situation by drawing a picture (where the subscript k is suppressed). The graph of $w(\cdot)$ and the line containing $F(\xi^-, \xi^+)$ moved vertically by τ intersect at $x = \xi^-(\tau)$ and at $x = \xi^+(\tau)$.



(a) Since F_k is not of zero curvature then by the fact that $\varphi \in$ J-R it follows that w in a neighborhood of $[\xi_k^-, \xi_k^+]$ is either convex or concave. Let us consider the case of w being convex on $(a, b) \supset [\xi_k^-, \xi_k^+]$, the other case is similar. By convexity, any chord is above the graph of w. Thus, the line $l_k(\cdot, \xi_k^+, \tau_k) + w(\xi_k^+)$ for sufficiently small $\tau_k > 0$ intersects the graph of w at exactly two points, i.e. for $\tau_k > 0$ equation (5.5) has exactly two solutions. One of them, which is greater than ξ_k^+ is called $\xi_k^+(\tau_k)$, the other one, smaller than ξ_k^- is dubbed $\xi_k^-(\tau_k)$. The function

$$x \mapsto w(x) - l_k(x, \xi_k^+, \tau_k) - w(\xi_k^+) =: F_k^+(x) \tag{5.8}$$

is increasing for $x \in [\xi_k^+, b)$ and this interval is maximal with this property, while the function

$$x \mapsto w(x) - l_k(x, \xi_k^-, \tau_k) - w(\xi_k^-) =: F_k^-(x) \tag{5.9}$$

and decreasing for $x \in (a, \xi_k^-]$ and again this interval is maximal with this property. One can see this by taking the derivative of (5.8) and (5.9), because we have

$$\frac{d}{dx}(w(x) - l_k(x, \xi_k^+, \tau_k)) = w'(x) - \alpha_k \geq w'(\xi_k^+) - \alpha_k > 0 \qquad \text{for } a.e.\ x \in [\xi_k^+, b)$$

and

$$\frac{d}{dx}(w(x) - l_k(x, \xi_k^-, \tau_k)) = w'(x) - \alpha_k \leq w'(\xi_k^-) - \alpha_k < 0 \qquad \text{for } a.e.\ x \in (a, \xi_k^-].$$

Thus, the function $[\xi_k^+, b) \in x \mapsto F_k^+(x)$ (resp. $(a, \xi_k^-] \mapsto F_k^-(x)$ has a continuous inverse. As a result, for any τ_k belonging to $[0, \delta) \subset F_k^+([\xi_k^+, b)) \cap F_k^-((a, \xi_k^-])$, $\delta > 0$, we may set $\xi_k^+(\tau_k) = (F_k^+)^{-1}(\tau_k)$ and $\xi_k^-(\tau_k) = (F_k^-)^{-1}(\tau_k)$. Moreover,

$$\frac{d\xi_k^\pm}{d\tau_k}(\tau_k) = \frac{1}{w_s(\xi_k^\pm(\tau_k)) - \alpha_k}, \quad a.e.$$

This formula combined with monotonicity of w_s yields,

$$\frac{1}{\alpha_k - w_s^-(b)} \leq \frac{d\xi_k^+}{d\tau_k}(\tau_k) \leq \frac{1}{w_s^+(\xi_k^+(\tau_k)) - \alpha_k}, \quad \frac{1}{w_s^+(a) - \alpha_k} \leq \left|\frac{d\xi_k^-}{d\tau_k}(\tau_k)\right| \leq \frac{1}{w_s^-(\xi_k^-(\tau_k)) - \alpha_k} \tag{5.10}$$

for a.e τ_k. If $w_s(\xi_k^+) \neq \alpha_k$, then it follows that $\xi_k^+(\cdot)$ is Lipschitz continuous on $[0, \epsilon]$, for some $\epsilon > 0$. A similar statement is valid for $\xi_k^-(\cdot)$.

(b) Functions $\xi_l^+(\tau_l, \tau_{l+1}), \ldots, \xi_{l+r-1}^+(\tau_{l+r-1}, \tau_{l+r})$ are defined as unique solutions to the decoupled system of linear equations (5.7) for any given $\tau_l, \ldots, \tau_{l+r}$. This is indeed possible because $\alpha_k \neq \alpha_{k+1}$. The solution ξ_k^+ depends linearly upon τ_k, τ_{k+1}. Subsequently, we set $\xi_{k+1}^- := \xi_k^+(\tau_k, \tau_{k+1})$, $k = l, \ldots, l + r - 1$. □

Remark. In the case (a) the derivatives $\frac{d}{d\tau_k}\xi_k^\pm$ are never zero. They may converge to infinity at $t = t_i$, as well as at $t = t_i + t^*$, if at that time instance $w_s(\xi_k^\pm) = \alpha_k$.

The lemma above expressed the evolution of the pre-images of facets in terms of τ_k, i.e. the amount of vertical shift of the line $l_k(\cdot, \xi^-, w(\xi^-))$. However, in order to render (5.3) meaningful, we have to figure out the time dependence of τ_k. At the same time we have to construct Ω. We begin with an explicit case.

Lemma 5.2. *Let us suppose that $F_k(\xi_k^-, \xi_k^+)$ is neither of zero-curvature nor interacting and it may be degenerate. Then there exist Ω_k^-, $\Omega_k^+ \in \partial J(\alpha_k)$ and a unique solution $\tau_k : [t_*, t_* + T_{max}) \to \mathbb{R}$ to the equation*

$$\frac{d\tau_k}{dt} = \frac{\Omega_k^+ - \Omega_k^-}{\xi_k^+(\tau_k) - \xi_k^-(\tau_k)}, \qquad \tau_k(t_*) = 0. \tag{5.11}$$

They are such that the function

$$\Omega(x, t) = \frac{\Omega_k^+ - \Omega_k^-}{\xi_k^+(t) - \xi_k^-(t)}(x - \xi_k^-(t)) + \Omega_k^- \tag{5.12}$$

and w defined by (5.3) satisfy

$$\frac{\partial w}{\partial t}(s, t) = \frac{\partial \Omega}{\partial s}(s, t) \qquad \textit{for } t \in [t_*, t_* + T_{max}), \quad s \in (\xi_k^-(\tau_k(t)), \xi_k^+(\tau_k(t)).$$

Proof. The non-interaction assumption implies that

$$\alpha_{k-1} \equiv \alpha_k - \Delta\alpha < w_{0,s}^-(\xi^-) \le \alpha_k \le w_{0,s}^+(\xi^+) < \alpha_{k+1} \equiv \alpha_k + \Delta\alpha$$

or

$$\alpha_{k-1} > w_{0,s}^-(\xi^-) \ge \alpha_k \ge w_{0,s}^+(\xi^+) > \alpha_{k+1}.$$

Keeping this in mind we set

$$\Omega_k^+ = \lim_{x\to(\xi_k^+(t))^+} \frac{\partial J}{\partial \varphi}(w_{0,x}(x)) \qquad \Omega_k^- = \lim_{x\to(\xi_k^-(t))^-} \frac{\partial J}{\partial \varphi}(w_{0,x}(x)). \tag{5.13}$$

Of course $\Omega_k^-, \Omega_k^+ \in \partial J(\alpha_k)$. We notice that both quantities are well-defined for regular as well as degenerate facets.

Now, we turn our attention to equation (5.11), we notice that this equation states that the time derivative of τ_k equals the slope of the straight line passing trough the points $(\xi_k^-, \Omega(\xi_k^-))$ and $(\xi_k^+, \Omega(\xi_k^+))$. This line provides a section of ∂J, necessary to construct solutions to (2.1).

The numerator of (5.11) is constant and if $\xi_k^+(\cdot)$, $\xi_k^-(\cdot)$ are Lipschitz continuous and $\xi_k^+(\tau_k) > \xi_k^-(\tau_k)$ for all the values of τ_k, then (5.11) has a unique solution. If however, $\xi_k^+(0) = \xi_k^-(0)$, then (5.11) is singular and this equation requires special attention. A similar situation arises when $w_s(\xi_k^\pm) = \alpha_k$. Fortunately, due to a simple structure of (5.11) we may resolve these issues.

The ODE (5.11) governing the behavior of a non-interacting facet F_k is obtained by taking the time derivative of (5.5),

$$\frac{d}{dt}(w(\xi_k^+(t)) - \alpha_k\xi_k^+(t)) = \frac{d}{dt}\tau_k(t), \qquad \frac{d}{dt}(w(\xi_k^-(t)) - \alpha_k\xi_k^-(t)) = \frac{d}{dt}\tau_k(t). \tag{5.14}$$

In reality, we do not assume that w is differentiable everywhere, but its one-sided derivatives do exist at each point. Due to monotonicity of $\xi_k^\pm$ the one-sided derivatives suffice in the formula above.

By the definition of $\xi_k^\pm$ we rewrite (5.11) as follows

$$((F_k^+)^{-1}(\tau_k) - (F_k^-)^{-1}(\tau_k))\frac{d\tau_k}{dt} = \Delta\Omega_k.$$

Here, due to the definition of J and (5.13), we have

$$\Delta\Omega = \Delta\Omega_k = \Omega^+ - \Omega^- = \frac{\pi}{2}.$$

Since $\xi_k^+(\tau_k) > \xi_k^-(\tau_k)$ as long as $\tau_k \neq 0$, then we deduce that G, the primitive function of $(F_k^+)^{-1}(\tau_k) - (F_k^-)^{-1}(\tau_k)$ such that $G(0) = 0$, is strictly increasing. Thus (5.11) takes the form

$$\frac{d}{dt}\left(G(\tau_k)\right) = \Delta\Omega$$

or $G(\tau_k) = \Delta\Omega t$. As a result function τ_k is given uniquely by the formula

$$\tau_k(t) = G^{-1}(\Delta\Omega t)$$

and $\tau_k(0) = 0$.

If we now set Ω by formula (5.12), then by the convexity of the set $\partial J(\alpha_k)$, we conclude that $\Omega(x,t) \in \partial J(\alpha_k)$. Moreover, for w defined by (5.3), the following equality holds by the definition of Ω and τ_k,

$$\frac{\partial w}{\partial t}(x,t) = \frac{d\tau_k}{dt} = \frac{\Omega_k^+ - \Omega_k^-}{\xi_k^+(t) - \xi_k^-(t)} = \frac{\partial \Omega}{\partial x}(x,t)$$

for $t \in [t_*, t_* + T_{max})$, $x \in (\xi_k^-(t), \xi_k^+(t))$.

We note that Ω, which we so far constructed, belongs to $W_1^1([0, 2\pi))$ for each $t > t_*$, if however the facet does not degenerate, then $\Omega(\cdot, t_*) \in W_1^1([0, 2\pi))$ too.

□

We can infer the following observation from Lemma 5.1 and (5.11).

Corollary 5.1. *Let us suppose that w_s is increasing (resp. decreasing) in a neighborhood of the pre-image $[\xi_k^-, \xi_k^+]$ of a non-interacting facet. Then, there exists a positive δ, such that for $t \in [t_k, t_k + T)$:*
(a) if $\xi_k^+ < \xi_{k+1}^-$, then $\frac{d}{dt}\xi_k^+(\tau_k(t)) \geq \delta > 0$ a.e. (resp. $\frac{d}{dt}\xi_k^+(\tau_k(t)) \leq \delta < 0$ a.e.).
(b) if $\xi_{k-1}^+ < \xi_k^-$, then $\frac{d}{dt}\xi_k^-(\tau_k(t)) \leq \delta < 0$ a.e. (resp. $\frac{d}{dt}\xi_k^-(\tau_k(t)) \geq \delta > 0$ a.e.).

Proof. The chain formula yields $\frac{d}{dt}\xi_k^+ = \frac{d\xi_k^+}{d\tau_k}\frac{d\tau_k}{dt}$ $a.e.$ In the case (a), by the geometry of the problem, we deduce that $\frac{d\xi_k^+}{d\tau_k} > 0$ (see (5.6)) as well as $\frac{d\tau_k}{dt} > 0$ (see (5.11)). Moreover, formulas (5.6) and (5.11) imply that none of the factors may vanish, in fact they are separated from zero.

The remaining cases are handled in the same way. □

We shall state a result corresponding to Lemma 5.2 for a set of interacting facets. It will be somewhat more tedious.

Lemma 5.3. *Let us suppose that non-degenerate facets $F_l, \ldots, F_{l+r}$, $r > 0$ interact, while $\xi_{l-1}^+ < \xi_l^-$ and $\xi_{l+r}^+ < \xi_{l+r+1}^-$. Then, there exist continuous functions $\xi_k^\pm : [t_*, t_* + T) \to \mathbb{R}$, $k = l, \ldots, l + r$, such that they are locally Lipschitz continuous on $(t_*, t_* + T)$ satisfying (5.19) below and there are C^1 functions $\tau_k : [t_*, t_* + T) \to \mathbb{R}$, $k = l, \ldots, l + r$, and $\Omega(\cdot, t) \in W_1^1(\xi_l^-(t), \xi_{l+r}^+(t))$. They are all such that w defined by (5.3) satisfies*

$$\frac{\partial w}{\partial t}(s, t) = \frac{\partial \Omega}{\partial s}(s, t) \qquad \text{for } t \in [t_*, t_* + T_{max}), \quad s \in (\xi_l^-(t), \xi_{l+r}^+(t)). \tag{5.15}$$

Remark. The above Lemma includes the case when the set $S \setminus \Xi(w_{0,s})$ consists of a single component.

Proof. By our assumption the pairs of facets F_{l-1}, F_l and F_{l+r}, F_{l+r+1} do not interact. Thus, the evolution of the end points ξ_l^- and ξ_{l+r}^+ is determined as for a single non-interacting facet. This remain applicable, unless $\Xi(w_s(\cdot, t_i)) = [0, 2\pi)$. We proceed as in Lemma 5.2, but we have to determine $\xi_{l+i}^\pm$, τ_{l+i}, $i = 1, \ldots, r$ and Ω simultaneously. We keep in mind that $\xi_{l+i}^- = \xi_{l+i-1}^+$, $i = 1, \ldots, r$. In order to obtain their time evolution, we differentiate (5.7) with respect to time. This yields,

$$\alpha_k \dot{\xi}_k^+ + \dot{\tau}_k = \alpha_{k+1}\dot{\xi}_{k+1}^+ + \dot{\tau}_{k+1}. \tag{5.16}$$

The equation for τ_k should be similar to (5.11), if so we have to select $\Omega_{l+i}^\pm$, $i = 1, \ldots, r$. We define Ω_l^- and Ω_{l+r}^+ as in (5.13), i.e.

$$\Omega_{l+r}^+ = \lim_{x \to (\xi_{l+r}^+(t))^+} \frac{\partial J}{\partial \varphi}(w_{0,x}(x)), \qquad \Omega_l^- = \lim_{x \to (\xi_k^-(t))^-} \frac{\partial J}{\partial \varphi}(w_{0,x}(x)). \tag{5.17}$$

We have to define the remaining $\Omega_k^\pm$'s while keeping in mind $\Omega_k^+ = \Omega_{k+1}^-$. By the properties of derivative $\frac{\partial J}{\partial \varphi}(\varphi)$ and the subdifferential $\partial J(\alpha_k)$ the number Ω_l^- is one endpoint of the interval $\partial J(\alpha_k)$, thus we inductively define Ω_k^+ as follows,

$$\Omega_{k+1}^+ = \begin{cases} \Omega_k^+, & \text{if the facet } F_k \text{ has zero curvature} \\ \text{the other endpoint of the interval } \partial J(\alpha_k), & \text{otherwise.} \end{cases}$$

We have to check that Ω^+_{l+r} defined in this way agrees with $(5.17)_2$. We prove this by induction with respect to r, the number of interacting facets. If $r = 1$, then the claim follows from the preceding considerations. Let us suppose validity of the claim for some $r \geq 1$, we will show it for $r + 1$. Let us suppose that w_0 corresponds to a group of $r + 1$ interacting facets satisfying the assumptions of the Lemma. We consider such a mollification w^ϵ_0 of w_0 in a neighborhood of $\xi^+_{l+r} = \xi^-_{l+r+1}$ that $w^\epsilon_0 = w_0$ for x satisfying $|x - \xi^+_{l+r}| \geq \epsilon$ and w^ϵ_0 is smooth. Moreover, we require that $w_{0,s}$ and $w^\epsilon_{0,s}$ are simultaneously increasing or decreasing. Thus the facets corresponding to w^ϵ_0 are $F_l, \ldots, \tilde F_{l+r}$, $\tilde F_{l+r+1}$. We notice that facet $\tilde F_{l+r+i}$ is of zero curvature iff facet F_{l+r+i} is of zero curvature, $i = 0, 1$. Moreover, facets $\tilde F_{l+r}$, $\tilde F_{l+r+1}$ do not interact. By the inductive assumption $\tilde\Omega^+_{r+l} = \Omega^+_{r+l}$ is equal to Ω^-_{r+l+1}. At the same time $\tilde\Omega^+_{r+l} = \tilde\Omega^-_{r+l+1}$ is determined from $\tilde\Omega^+_{r+l+1} = \Omega^+_{r+l+1}$ and $w_{0,s}$ as in Lemma 5.2. The two ways of course coincide, due to formulae (5.13). Our claim follows.

We now write equations for τ_k, $k = l, \ldots, l + r$, they are as (5.11),

$$\frac{d\tau_k}{dt} = \frac{\Omega^+_k - \Omega^-_k}{\xi^+_k(\tau_k) - \xi^-_k(\tau_k)}, \qquad \tau_k(t_*) = 0 \qquad \text{for} \quad k = l, \ldots, l + r. \tag{5.18}$$

Since we do not admit degenerate facets, these equations are not singular. We combine them with (5.13) and after writing $\eta = (\xi^+_l, \ldots, \xi^+_{l+r-1})$, we arrive at

$$A\dot\eta = B(\eta), \tag{5.19}$$

where

$$A = \begin{bmatrix} \alpha_l & -\alpha_{l+1} & \ddots & 0 \\ 0 & \alpha_{l+1} & -\alpha_{l+2} & 0 \\ \ldots & \ddots & \vdots & -\alpha_{l+r-1} \\ \ddots & 0 & 0 & \alpha_{l+r-1} \end{bmatrix},$$

$$B(\eta)_k = -\frac{\Omega^+_k - \Omega^-_k}{\eta^+_k - \eta^+_{k-1}} + \frac{\Omega^+_{k+1} - \Omega^-_{k+1}}{\eta^+_{k+1} - \eta^+_k}, \qquad k = l + 1, \ldots, l + r - 2,$$

$$B(\eta)_{l+r-1} = \alpha_{l+r}\frac{d}{dt}\xi^+_{l+r} - \frac{\Omega^+_{l+r-1} - \Omega^-_{l+r-1}}{\eta^+_{l+r-1} - \eta^+_{l+r-2}} + \frac{\Omega^+_{l+r} - \Omega^-_{l+r}}{\xi^+_{l+r} - \eta^+_{l+r-1}}.$$

Under our assumptions, there is a separate equation for $\frac{d}{dt}\xi^+_{l+r}$ i.e. (5.11). Due to the assumption of absence of degenerate interacting facets, this system is uniquely solvable on $[t_*, t_* + T)$.

We have to define Ω, it will be a continuous piece-wise linear function,

$$\Omega(x, t) = \frac{\Omega^+_{l+i} - \Omega^-_{l+i}}{\xi^+_{l+i}(t) - \xi^-_{l+i}(t)}(x - \xi^-_{l+i}(t)) + \Omega^-_{l+i}. \tag{5.20}$$

Moreover, w and Ω satisfy (5.15). □

We claim in Theorem 5.1 that the number of facets decreases in time. The result below explains that certain phenomena are forbidden. Namely, no facet with non-zero curvature may degenerate.

Proposition 5.3. *In any group of interacting facets F_k, $k = l, \ldots, l + r$, $r > 0$ only a facet with zero curvature may degenerate.*

Proof. Let us suppose that $F_l, \ldots, F_{l+r}$, $r > 0$ is a maximal group of interacting facets with non-zero curvature. For the sake of definiteness, we will proceed while assuming that w_s is increasing on $(a,b) \supset [\xi_l^-, \xi_{l+r}^+]$.
Step 1. Let us observe that for a facet F_k to disappear, it is necessary, (but not sufficient) that one of neighboring facets moves upward faster than F_k, i.e. either $V_{k+1} = \frac{d\tau_{k+1}}{dt} > \frac{d\tau_k}{dt} = V_k$ or $V_{k-1} = \frac{d\tau_{k-1}}{dt} > \frac{d\tau_k}{dt} = V_k$. Indeed, the position of F_k is defined by the intersection of the lines containing F_k, F_{k+1} moved vertically by τ_k and respectively by τ_{k+1} and the intersection of lines containing F_k, F_{k-1} moved vertically by τ_k and respectively by τ_{k-1}. Thus, if the lines containing F_{k+1} and F_{k-1} are moved up so much that their intersection is above the line containing F_k moved vertically by τ_k, then facet F_k is going to disappear. This situation may occur only if $V_{k+1} > V_k$ or $V_{k-1} > V_k$.
Step 2. Let us suppose that facets F_k, F_{k-1} interact, hence by (5.19)

$$\alpha_k \dot{\xi}_k - \alpha_{k-1} \dot{\xi}_{k-1} = \dot{\tau}_{k-1} - \dot{\tau}_k. \tag{5.21}$$

By the monotonicity assumption on w_s we notice that $\dot{\tau}_{k-1}$ and $\dot{\tau}_k$ are positive. If the length of F_k, which is equal to $\xi_k - \xi_{k-1}$, stays bounded on $[t_*, t_* + T)$ while the length of F_{k-1} vanishes at $t = t_* + T$, then in a neighborhood of $t_* + T$ we have $\dot{\tau}_{k-1} - \dot{\tau}_k < 0$. Thus, by (5.21) we can see that

$$\alpha_{k-1}(\dot{\xi}_k - \dot{\xi}_{k-1}) + (\alpha_k - \alpha_{k-1})\dot{\xi}_k < 0$$

and by (5.18) the left-hand-side (LHS) converges to $-\infty$ when t tends to $t_* + T$. Since $\dot{\xi}_k - \dot{\xi}_{k-1}$ must be bounded from above, we deduce that $\dot{\xi}_k < 0$ for t close to $t_* + T$.
Step 3. Since always $\dot{\xi}_{l-1}^- < 0$ and $\dot{\xi}_{l+r+1}^+ > 0$ (unless $\Xi(\varphi) = [0, 2\pi)$), we conclude that not all of the facets vanish simultaneously at $t = t_* + T$. As a result we may assume the length $\ell(F_{l-1})$ of F_{l-1} is greater than $d > 0$ on $[t_i, T)$. Thus, we conclude by step 1, that for t close to $t_* + T$ we have $V_l > V_{l-1}$. By induction we obtain that

$$V_{k+1} > V_k, \qquad k = l, \ldots, j + r - 1. \tag{5.22}$$

We notice that we have the following possibilities for facet F_{l+r}: (a) there is an adjacent zero-curvature facet F_{l+r+1}; (b) ξ_{l+r+1}^+ is defined as $(F_{l+r+1}^+)^{-1}(\tau_{l+r+1})$ (see the proof of Lemma 5.1). In case (a) we can see that $\tau_{l+r+1} = 0$ while in (b) $\tau_{l+r+1} > 0$.

The condition (5.22) combined with (5.18) implies that

$$\xi_l^+ - \xi_l^- > \ldots > \xi_{r+l+1}^+ - \xi_{r+l+1}^-.$$

Hence, the endpoints of F_k, $k = l, \ldots, j + r - 1$ converge to a common limit p. But by step 2

$$\xi_{r+l+1}^+(t) > \xi_{r+l+1}^+(t_*) > \xi_{l+1}^+(t_*) > \xi_{l+1}^+(t).$$

This is a contradiction, our claim follows. □

This observation shows that the initial time $t_0 = 0$ is special. If the data are poor from the view-point of dynamics, but still acceptable, then they get immediately regularized. That is all non-zero curvature degenerate facet become regular.

We are now ready for the proof of the main result.

Proof of Theorem 5.1. PART A. We start with data free from degenerate interacting facets. We set $t_0 = 0$, we have to define time instance t_i, $i = 1, \ldots, M$ postulated by the theorem. We shall proceed iteratively.

It follows from Proposition 5.3, that degenerate, non-zero curvature facets are possible only at $t = 0$, i.e. at the initial time instance.

Let us suppose that $[\xi_k^-(t_i), \xi_k^+(t_i)]$ is a connected component of $\Xi_l(w_s(t_i))$. We have six possibilities for $F_k = F_k(\xi_k^-(t_i), \xi_k^+(t_i))$:
(a) F_k is regular, does not have zero curvature, is non-interacting;
(b) F_k is regular, does not have zero curvature, is interacting;
(c) F_k is regular, has zero curvature, is non-interacting;
(d) F_k is regular, has zero curvature, is interacting;
(e) F_k is degenerate, does not have zero curvature, is non-interacting;
(f) F_k is degenerate, has zero curvature, is non-interacting.

Cases (a) and (e) are solved in Lemma 5.2, where corresponding $\xi_k^\pm$ are constructed.

The construction of $\xi_k^\pm$ corresponding to (b), (d) is performed in Lemma 5.3. We stress that in all these cases τ_k, is given by (5.11).

The definition of $\xi_k^\pm$ is simple if (c) or (f) holds, we just set

$$\xi_k^-(t) = \xi_k^-, \quad \xi_k^+(t) = \xi_k^+, \quad \tau_k(t) = 0. \tag{5.23}$$

We have to define Ω. By the very definition of zero-curvature facets the intersection $\partial J(\xi_k^+ + \epsilon) \cap \partial J(\xi_k^- - \epsilon)$ is a singleton $\{\alpha\}$ for any positive $\epsilon < \min\{\xi_{k+1}^- - \xi_k^+, \xi_k^- - \xi_{k-1}^+\}$. Moreover, $\alpha \in \mathcal{A}$, hence we set

$$\Omega(x,t) = \alpha, \quad \text{for } x \in [\xi_k^-(t), \xi_k^+(t)]. \tag{5.24}$$

Thus, we have specified evolution of $\xi_k^\pm$ for every configuration. In all these cases the functions $\xi_k^\pm$, $k = 1, \ldots, N_i$ are defined on maximal intervals $[t_i, t_i + T_k^\pm]$. The numbers $T_k^\pm$ are defined as follows.

In (a) and (e) the positive number T_k^+ (resp. T_k^-) is such that $\xi_k^+(t) < \xi_{k+1}^-(t)$ (resp. $\xi_{k-1}^+(t) < \xi_k^-(t)$) for $t < t_i + T_k^+$ (resp. $t < t_i + T_k^-$), while equality occurs at $t = T_k^+$ (resp. $t = T_k^-$), i.e. the facet begins to interact with its neighbor. By Corollary 5.1 $T_k^\pm$ are finite.

If a group of interacting facets $F_l, \ldots F_{l+r}$ does not contain any zero-curvature facet, then by Proposition 5.3 it may not vanish and its maximal existence time is defined as in (a) for ξ_{l+r}. Thus, at T_{r+l}^+ the group begins to interact with another facet. On the other hand, if this group of interacting facets $F_l, \ldots F_{l+r}$ contains a zero-curvature facet, say F_p, then T_p^+ is defined as the extinction time of F_p, i.e. $\xi_p^-(t) < \xi^+(t)$ for $t \in [t_i, t_i + T_p^+)$, while $\xi_p^-(t_i + T_p^+) = \xi^+(t_i + T_p^+)$. Thus, the number of facets drops by one.

Cases (c) and (f) do not contribute to the definition of t_{i+1}, because (5.23) is valid for all $t \geq t_i$.

We have to define also $\Omega(x,t)$. An attempt to do so reveals another difficulty related to construction of $\xi_k^\pm$ starting from $t = 0$. Let us consider two interacting facets $F_k(\xi_k^-, \xi_k^+)$, $F_{k'}(\xi_{k'}^-, \xi_{k'}^+)$, where

$$[\xi_k^-, \xi_k^+] \subset \Xi_l(w_{0,s}), \quad [\xi_{k'}^-, \xi_{k'}^+] \subset \Xi_r(w_{0,s}). \tag{5.25}$$

It is obvious that for any $s \in [\xi_k^-, \xi_k^+]$ and $s' \in [\xi_{k'}^-, \xi_{k'}^+]$ the intersection

$$\partial J(w_{0,s}(s)) \cap \partial J(w_{0,s}(s'))$$

is non-empty if and only if $|l - r| = 1$. If the above intersection is non-empty, we can construct the desired $\Omega(x,t)$. On the other hand, if this intersection is void, then we have no chance to construct a W_1^1 section of $\partial J(w_s)$.

Let us suppose then that (5.25) holds and $|l - r| = p + 1$, $p > 0$. Let us suppose for simplicity that $l < r$. Thus, a single point ξ is a connected component of $\Xi_j(w_{0,s})$, $j = l, l+1, \ldots, r$, i.e.

$$\xi = \xi_j^- = \xi_j^+, \quad j = l, l+1, \ldots, r.$$

In other words, we have a number of degenerate, interacting facets at ξ. The system of ODE's (5.19) is singular. The problem of evolution of interacting degenerate facets shall be dealt with below in Part B of the proof. It occurs only at $t = 0$.

Finally, we check that $w(x,t)$ and $\Omega(x,t)$ fulfill the conditions postulated in the definition of the weak solution. They satisfy the equation

$$w_t(x,t) = \Omega_x(x,t) \tag{5.26}$$

and the initial and boundary conditions are satisfied. Integral identity in Definition 2.1 follows.

PART B. After finishing part A, i.e. the case of data satisfying (5.4), we consider the interaction of facets during creation, i.e. (5.4) is no longer valid. We have two cases to consider:
(g) F_k is degenerate, with nonzero curvature and interacting;
(h) F_k is degenerate, with zero curvature and interacting.

We begin with (g). Let us suppose that w_0 violates (5.4) at some ξ. Thus, we are dealing with the situation when one sided derivatives of w_0 differ at ξ, i.e.,

$$w_{0,s}^-(\xi) < \alpha_k < w_{0,s}^+(\xi)$$

for some $a_k \in \mathcal{A}$. It may as well happen that the reverse inequalities occur, however for the sake of definiteness we shall stick to the above choice.

We shall construct two functions w_ϵ, w^ϵ such that their derivatives belong to J-R, $w_\epsilon(x) < w_0(x) < w^\epsilon(x)$ and

$$|w_\epsilon(x) - w_0(x)|, \quad |w^\epsilon(x) - w_0(x)| < \epsilon. \tag{5.27}$$

We set

$$w^\epsilon(x) = \max\{w_0(x), l_k(x, \xi, w_0(\xi) + \delta)\},$$

where $\delta > 0$ is so chosen to guarantee (5.27). We also define

$$w_\epsilon(x) = \max\{w_0(x) - \epsilon, l_k(x, \xi, w_0(\xi) + \delta)\},$$

where $\delta \in (0, \epsilon)$ is arbitrary. Of course, (5.27) holds.

If the newly constructed w^ϵ and w_ϵ do not satisfy (5.4), we repeat the above process until they do. Subsequently, we apply the results of Part A to w^ϵ and w_ϵ. We deduce from that existence of interacting facets at ξ. By the comparison principle, the non-zero interacting facets exist for $t > t_0$.

Finally, we study (h). We notice that at such an instance F_k cannot interact with two neighboring facets, because this would mean that F_{k-1} and F_{k+1} lay on the same line, that is, F_k is their common end point. Thus, the three facets F_{k-1}, F_k and F_{k+1} form a single facet $\tilde{F}_k$ with the pre-image $[\xi_{k-1}^-, \xi_{k+1}^+]$. On the other hand it may happen that F_k is a degenerate, zero curvature facet interacting with just one neighbor, say F_{k+1}. Since F_k is degenerate, i.e., $\xi_k^+ = \xi_k^- =: \xi_k$, due to its interaction with F_{k+1} we have $\xi_{k+1}^- = \xi_k$. Moreover, $w_{0,s}^-(\xi_k) = \alpha_{k-1}$ and $w_{0,s}^+(\xi_k) = \alpha_k$ where α_{k-1}, $\alpha_k \in \mathcal{A}$ and we may assume that $\alpha_{k-1} < \alpha_k$, (the other case is handled similarly) and $w_s(\xi_{k-1}^+, \xi_k) \subset (\alpha_{k-2}, \alpha_{k-1})$. A similar situation occurs when F_k interacts with F_{k-1}.

In order to determine the evolution of the system we have to take into account if F_{k+1} has zero-curvature or not. In the former case $\dot{\tau}_{k+1} = 0$, hence we set $\tau_k \equiv 0$. In latter case we have $\dot{\tau}_{k+1} > 0$ (it may not occur $\dot{\tau}_{k+1} < 0$). Thus, F_k disappears instantly. As a result, we agree to disregard F_k and diminish N_0 by 1.

PART C. We have to deal with the points outside of $\Xi(w_s(\cdot,t)) \equiv \bigcup_{i=1}^{N_k}[\xi_k^-, \xi_k^+]$. By the definition of $\Xi(w_s(\cdot,t))$, its complement is open

$$[0, 2\pi] \setminus \Xi(w_s(\cdot,t)) = \bigcup_{l=1}^{N_i} (\xi_k^+, \xi_{k+1}^-).$$

where $(\xi_{N_i}^+, \xi_{N_i+1}^-)$ should be understood as $(\xi_{N_i}^+, 2\pi] \cup [0, \xi_1^-)$, (with the understanding that $0 \le \xi_k^\pm \le 2\pi$, $k = 1, \dots, N_i$). Using again the definition of Ξ, we come to the conclusion that, if x belongs to any of the intervals (ξ_k^+, ξ_{k+1}^-), then either $w_s(x,t)$ exists or $w_s^+(x,t) \ne w_s^-(x,t)$. In either case, the set $\partial w(x,t)$ (see (4.5)) does not intersect $\mathcal{A}$. Since $\partial w(x,t)$ is an interval, we deduce that there exists $\alpha_k \in \mathcal{A}$ such that

$$\partial w(x,t) \subset (\alpha_k, \alpha_{k+1}). \tag{5.28}$$

We have to make sure that the choice of α_k, in the formula above, depends only on the interval (ξ_k^+, ξ_{k+1}^-), but it is independent from a specific point $x \in (\xi_k^+, \xi_{k+1}^-)$. Indeed, by the definition of the J-R class $\partial w = M - f$ or $\partial w = f - M$, where f is a continuous increasing function and M a maximal monotone operator. Thus, the images $f(\xi_k^+, \xi_{k+1}^-)$ and $M(\xi_k^+, \xi_{k+1}^-)$ are connected intervals, so is the image $\partial w(\xi_k^+, \xi_{k+1}^-)$, which is disjoint from $\mathcal{A}$. Our claim follows.

As a result, our definition of $w(x,t)$ for $x \notin \Xi(w_s(\cdot,t))$ is as follows,

$$w(x,t) = w(x,t_k) \quad \text{and} \quad \Omega(x,t) = \frac{dJ}{d\varphi}(w_s(y,t_k)) \qquad \text{for } x \in (\xi_i^+, \xi_{i+1}^-).$$

where $y \in (\xi_i^+, \xi_{i+1}^-)$ is any differentiability point of $w(\cdot, t_k)$.

PART D. We have to define t_{k+1}. We do this inductively. Once t_k is given, we set

$$t_{k+1} = t_k + \min\{\min_i T_i^+, \min_i T_i^-\}.$$

Thus at t_{k+1} two facets begin to interact, due to the shrinkage of $[\xi_i^+, \xi_{i+1}^-]$ to a point or due to the disappearance of a facet. By Proposition 5.3, we know that only zero-curvature facets may disappear. We set

$$N_{i+1} = N_i - m,$$

where m is the number of removed degenerate, interacting, zero-curvature facets at $t = t_{i+1}$.

The last thing to show is the estimate $\|w_s(\cdot,t)\|_{\text{J-R}} \le \|w_s(\cdot,s))\|_{\text{J-R}}$, whenever $t > s$. By the construction above, the number of connected components of $\Xi(w_s(\cdot,t))$ drops at time instances t_k, $k = 1, \dots, M_N$, hence $K(w_s(\cdot,t)) \le K(w_s(\cdot,s))$, whenever $s \le t$. It remains to show that $\|w_s(\cdot,t)\|_{TV(S)} \le \|w_s(\cdot,s)\|_{TV(S)}$, where we denoted by $\|f\|_{TV(E)}$ the total variation of function f over set E.

We first consider the case $t > s$ such that $\Xi(w_s(\cdot,t)) \ne S$, we know that we always have $\Xi(w_s(\cdot,t)) \supset \Xi(w_s(\cdot,s))$ for $s < t$. By the general properties of the total variation, we notice that

$$\|w_s(\cdot,t)\|_{TV(S)} = \|w_s(\cdot,t)\|_{TV(\Xi(t))} + \|w_s(\cdot,t)\|_{TV(S\setminus\Xi(t))},$$

where we wrote $\Xi(\sigma)$ for $\Xi(w_s(\cdot,\sigma)$. Now, by the definition of $w(x,t)$, we notice that

$$\|w_s(\cdot,t)\|_{TV(S\setminus\Xi(t))} = \|w_s(\cdot,s)\|_{TV(S\setminus\Xi(t))} \le \|w_s(\cdot,s)\|_{TV(S\setminus\Xi(s))}.$$

We turn our attention to $\|w_s(\cdot,t)\|_{TV(\Xi(t))}$. On the intervals forming $\Xi(t)$ function $w_s(\cdot,t)$ is piece-wise constant. The jumps occur at the endpoint of these intervals. They are no bigger and no more numerous than the jumps of $w_s(\cdot,s)$. Thus our claim follows in the considered case of t. In fact, the case of t such that $\Xi(t) = S$ is not much different. Finally, we can see that w_s is a difference of two monotone functions and one of them is continuous, the other one a maximal monotone operator.

Our theorem is proved. □

We close this subsection with a formula, which might be called "morphing a circle into a square".

Example. Let us suppose that $\phi_0(s) = s$ or $w_0(s) = \frac{1}{2}s^2$. Due to the high symmetry of the problem, it is sufficient to consider just formation of one facet. Then, $w(x,t)$, the unique solution to (2.1), is given by the formula,

$$w(x,t) = \begin{cases} \frac{1}{2}s^2 & s \in [0,\xi_1^-(t)] \cup [\xi_1^+(t), \frac{\pi}{2}], \\ \frac{\pi}{4}s - \frac{\pi^2}{32} + \tau_1(t) & s \in [\xi_1^-(t), \xi_1^+(t)]. \end{cases}$$

Here, $\xi_1^{\pm} = \frac{\pi}{4} \pm \sqrt{2\tau_1}$ and $\tau_1 = \left(\frac{\sqrt{2}\pi}{12}t\right)^{2/3}$. Let us note that at $T_1 = \pi^2/2^6$ we have $\Omega^+ - \Omega^- = \xi^+ - \xi^-$, so for later times $\dot\tau = 1$. So $V(t)$ from (4.25) is given by $\xi_1^+ - \xi_1^- = \left(\frac{\pi}{3}t\right)^{1/3}$.

We can make this observation more general.

Proposition 5.4. *Let us suppose that the assumptions of Theorem 5.1 are satisfied. Then, there exist T_{fa}, such that if $t > T_{fa}$, then $w(\cdot,t)$ is fully faceted, i.e. $w_s(\cdot,t)$ is piece-wise linear. More precisely, for $\Xi(w_s(\cdot,t)) \subset [0,2\pi)$ for $t < T_{fa}$ and $\Xi(w_s(\cdot,t)) = [0,2\pi)$ for $t \ge T_{fa}$.*

Proof. Let us consider w_0. It is fully faceted or not. If not, then by the proof of Theorem 5.1, we deduce that after at some t_{i_0} we have $\Xi(t_{i_0}) = [0,2\pi)$ and our claim follows. □

5.3 Convexification

We show that after some depending upon the initial data, the solution becomes such that $w_s = \varphi$ is monotone decreasing or increasing. We shall call this process by convexification.

Proposition 5.5. *Let us suppose that the assumptions of Theorem 5.1 are satisfied. Then, there exist T_{cx}, such that if $t \ge T_{cx}$, then $w_s(\cdot,t)$ is monotone, while this is not true for $t < T_{cx}$.*

Proof. If $w_{0,s}$ is monotone, then we are done. Otherwise, let us suppose that t_j is the largest time such that at t_j a zero curvature facet disappears. Since the zero-curvature facets cannot persist because their endpoints necessarily move, it follows that $T_{cx} = t_j$ has the desired properties. □

Remark. All possibilities can be realized $T_{cx} > T_{fa}$ as well as $T_{cx} < T_{fa}$.

5.4 Asymptotic behavior of facets

Here, we consider the last stage of evolution, when $t \geq t_M$ and $N_M = 4$. In this case, it is sufficient to specify only ξ_k^+, $k = 1, 2, 3, 4$. Furthermore, the system for interacting facets, (5.19) takes the form,

$$\begin{aligned}
&\alpha_1\dot{\xi}_1 - \alpha_2\dot{\xi}_2 = \dot{\tau}_1 - \dot{\tau}_2 \\
&\alpha_2\dot{\xi}_2 - \alpha_3\dot{\xi}_3 = \dot{\tau}_2 - \dot{\tau}_3 \\
&\alpha_3\dot{\xi}_3 - \alpha_4\dot{\xi}_4 = \dot{\tau}_3 - \dot{\tau}_4 \\
&\alpha_4\dot{\xi}_4 - \alpha_1\dot{\xi}_1 = \dot{\tau}_4 - \dot{\tau}_1 \\
&\xi_k(t_M) = \xi_k, \quad k = 1, 2, 3, 4.
\end{aligned} \tag{5.29}$$

We notice that the stationary points of (5.29) are such that $\dot{\tau}_1 = \ldots = \dot{\tau}_4$. This occurs if and only if $\Omega_k^+ - \Omega_k^- = \xi_k - \xi_{k-1}, \quad k = 1, 2, 3, 4$, where by ξ_0 we understand ξ_4. Moreover, due to our assumptions on J we have $\Omega_k^+ - \Omega_k^- = \Delta\Omega$, $k = 1, 2, 3, 4$.

Additionally, system (5.29) possesses a Liapunov functional. Namely, let us write

$$F(\vec{\xi}) = \sum_{k=1}^{4} \ln(\xi_k - \xi_{k-1})^{\Delta\Omega},$$

with the understanding of ξ_0 as above. By direct calculation, we check that

$$\frac{d}{dt}F(\vec{\xi}) = \nabla_\xi F \cdot \frac{d}{dt}\vec{\xi} < 0.$$

This derivative vanishes if and only if ξ is the only equilibrium point. Thus, we have a complete picture of the asymptotic behavior of Λ.

Theorem 5.2. *Let us assume that* $\varphi_0 \in$ J-R *and* w *is the corresponding unique solution to (2.1). Then, there exists* T_1, $\max\{T_{cx}, T_{fa}\} \leq T_1 \leq \infty$ *with the following property:*
(a) If $T_1 < \infty$*, then* $\xi_l(t) = -\frac{3\pi}{4} + \frac{\pi}{2}l + \alpha$*, for some* $\alpha \geq 0$*,* $l = 0, \ldots, 3$*, and* $t \geq T_1$*, in other words,* w *is the minimal solution for* $t > T_1$*;*
(b) If $T_1 = \infty$*, then* $\lim_{t\to\infty} \xi_l(t) = -\frac{3\pi}{4} + \frac{\pi}{2}l + \alpha$*,* $l = 0, \ldots, 3$ *for some* $\alpha \geq 0$*.*

□

5.5 Proof of Theorem 2.2

In the course of proof of Theorem 5.1, we exhibited a quite explicit construction of the weak solution with such initial data that $\varphi_0 \in$ J-R. Now, we have to show that is has all the postulated properties of the almost classical solution. We have already noticed that $w_s = \Lambda_s + s$ belongs to the J-R class, furthermore $\|w_s(\cdot, t)\|_{\text{J-R}} \leq \|w_s(\cdot, 0)\|_{\text{J-R}}$. The key point, however, is to realize that

$$\Omega = \partial J \bar{\circ} \partial w, \tag{5.30}$$

where ∂w is the multivalued map whose section is w_s. We defined ∂w in (4.5). Checking that (5.30) indeed holds requires recalling the steps of construction of Ω, we will do this below. Finally, after we set $N = \{0, t_1, \ldots, t_M\}$ we see that

$$\Lambda_t = \frac{\partial}{\partial s}\partial J \bar{\circ} (\Lambda_s + s),$$

holds for all $t \in (0, +\infty) \setminus N$ in the L^1 sense, more precisely it holds pointwise except $x \in [0, 2\pi) \setminus \{\xi_i^\pm : i = 1, \ldots, N_k\}$. Indeed, the definitions (5.11), (5.18), (5.23) of $\tau_k(t)$ were such that $\frac{d}{dt}\tau_k(t) = \frac{\partial \Omega}{\partial s}$. Moreover, $\frac{\partial \Lambda}{\partial t} = \frac{d}{dt}\tau_k(t)$, see Lemma 5.2, Lemma 5.3 and eq. (5.26). We recall that by definition functions $\tau_k(\cdot)$ are continuous on $[t_i, t_{i+1}]$ and differentiable in (t_i, t_{i+1}). Moreover, the right derivative of $\tau_k(t)$ is well-defined for all t, except possibly $t = t_0$. Hence, $\frac{\partial \Lambda}{\partial t}$ is defined everywhere, except the points t_i, $i = 0, \ldots, M$, but the right time derivative $\frac{\partial \Lambda^+}{\partial t}$ is defined for all $t > 0$.

We will check below that Ω, constructed in the course of proof of Theorem 5.1, coincides with $\partial J \bar{\circ} \partial w$, — see (5.12), (5.20), (5.24), where $w_s(s,t) = \Lambda_s(s,t) + s$. In order to see that we examine the steps of the construction of Ω and compare it with the definition of the composition $\bar{\circ}$. Let us fix $t \in (t_k, t_{k+1})$, at the end we will consider $t = t_{k+1}$, then we compose $\partial w(\cdot, t) : [0, 2\pi] \to [a, b]$ with $\partial J : \mathbb{R} \to \mathbb{R}$. We have to identify the sets $\mathcal{D}_s$, $\mathcal{D}_f$ and $\mathcal{D}_r$ appearing in the Definition 2.3. For our choice of t we have

$$\mathcal{D}_s(t) = \{s \in [0, 2\pi] : w_x^+(s,t) \neq w_x^-(s,t)\}.$$

In particular, $\mathcal{D}_s(t)$ contains all points $\xi_i^\pm(t)$, $i = 1, \ldots, N_k$. We can see that

$$\mathcal{D}_f(t) = \bigcup_{i=1}^{N_k} (\xi_i^-(t), \xi_i^+(t)),$$

i.e., it is the sum of interiors of intervals contained in $\Xi(w_s(\cdot, t))$. Finally, by the definition

$$\mathcal{D}_r(t) = [0, 2\pi] \setminus (\mathcal{D}_s(t) \cup \mathcal{D}_f(t)).$$

We shall consider these cases separately.

1^o case $\mathcal{D}_r$. If $s \in \mathcal{D}_r(t)$, then w is differentiable at s and $w_s(s,t) \notin \mathcal{A}$. Thus, by (2.4) $\partial J \bar{\circ} \partial w(s,t) = \frac{dJ}{d\varphi}(w_s(s,t))$. We notice that $\mathcal{D}_r(t) \subset [0, 2\pi] \setminus \Xi(w_s(\cdot, t))$, hence by Part C of the proof of Theorem 5.1 we immediately see that $\partial J \bar{\circ} \partial w(s,t)$ equals $\Omega(s,t)$ on $\mathcal{D}_r$.

2^o case $\mathcal{D}_f$. By its definition $\mathcal{D}_f(t)$ is the sum of interiors of pre-images of facets, as noticed above. Moreover, on each interval $(\xi_i^-(t), \xi_i^+(t))$, the set $\partial w(x,t)$ is a singleton equal to $\{\alpha_k\} \subset \mathcal{A}$. Then, the cases of the Definition 2.3, see formulas (2.5)–(2.8) have their counterparts in the formulas (5.12), (5.20) and (5.24).

3^o case $\mathcal{D}_s$. We notice that, if $t > 0$, then the set $\Xi(w_s(\cdot, t))$ has no component, which is a singleton. Thus, if $s \in \mathcal{D}_s(t)$, then the set $\partial w(s,t)$ does not intersect $\mathcal{A}$. As a result, formula (2.9) for the composition yields a singleton, because on the RHS of (2.9) the limit of constant functions are taken. This in agreement with the discussion of Part C.

Finally we have to deal with the case $t = t_{k+1}$. On one hand $\Omega(\cdot, t_{k+1})$ is defined by the left time continuity of Ω, on the other hand we have to check that $\Omega = \partial J \bar{\circ} \partial w$.

By the very definition of t_{k+1} (see Part D of the proof of Theorem 5.1), at this time instant a zero-curvature curvature facet disappears or two facets begin to interact or merge, i.e.,

$$\lim_{t \to t_{k+1}^-} \xi_i^+(t) = a = \lim_{t \to t_{k+1}^-} \xi_{i+1}^-(t).$$

We have then two possibilities, either $a = \xi_j^+(t_{k+1}) = \xi_{j+1}^-(t_{k+1})$ or $a \in (\xi_j^-(t_{k+1}), \xi_j^+(t_{k+1}))$ where this interval is a connected component of $\Xi(w_s(\cdot, t_{k+1}))$. Once we realize this, it is clear that $\Omega(\cdot, t_{k+1}) = \partial J \bar{\circ} \partial w(\cdot, t_{k+1})$. $\square$

6 Appendix

6.1 Motivation of equation (1.3)

Here, we consider closed curves, we view them as graphs over a smooth, convex reference closed curve $\mathcal{M}$. We do not make here any attempt to consider non-smooth reference curves, which is reasonable because this would add up difficulties while not giving advantages.

Let us suppose that $x_0(s)$ is an arc-length parameterization of $\mathcal{M}$, which is assumed here to be the unit circle, and $\mathbf{e}_t(s)$, $\mathbf{e}_n(s)$ are unit tangent and normal vectors, respectively, such that $(\mathbf{e}_n(s), \mathbf{e}_t(s))$ is positively oriented. Then all points in a neighborhood of $\mathcal{M}$ can be uniquely written as $x = x_0(s) + \mathbf{e}_n\Lambda$, as a result we can parameterize our curve $\Gamma(t)$ as

$$x(s,t) = x_0(s) + \mathbf{e}_n(s)\Lambda(s,t).$$

Since $\mathcal{M}$ is convex we may write $\mathbf{e}_n$ uniquely as $\mathbf{e}_n(\varphi(s)) = (\cos\varphi(s), \sin\varphi(s))$, where φ is the measure of the angle between the x_1 axis and $\mathbf{e}_n$. Moreover,

$$\frac{d}{ds}\mathbf{e}_n(\varphi(s)) = \mathbf{e}_t(\varphi(s))\frac{d\varphi}{ds} = \kappa\mathbf{e}_t(\varphi(s)),$$

where κ is the Euclidean curvature of the curve $\mathcal{M}$.

We note

$$\frac{\partial x}{\partial s}(s,t) = \mathbf{e}_t(1+\kappa\Lambda) + \mathbf{e}_n\Lambda_s,$$

because $|\dot{x}_0(s)| = 1$. With this formula at hand, we can write the expression for the tangent and normal to $\Gamma(t)$, they are $\tau = \frac{1}{W}(\mathbf{e}_t(1+\kappa\Lambda) + \mathbf{e}_n\Lambda_s)$, $\mathbf{n} = \frac{1}{W}(-\Lambda_s\mathbf{e}_t + (1+\kappa\Lambda)\mathbf{e}_n)$, where $W^2 = (1-\kappa\Lambda)^2 + \Lambda_s^2$. Thus, the system $(\mathbf{n}, \tau)$ is positively oriented.

We look at equation (1.1) with $u = 0$,

$$\beta V = \kappa_\gamma, \tag{6.1}$$

where all the quantities have been already defined.

We notice that, the LHS of (6.1) takes the form

$$\beta V = \beta\frac{dx}{dt}\cdot\mathbf{n} = \frac{1}{W}(1+\kappa\Lambda)\Lambda_t.$$

In our paper [MRy], we have shown that κ_γ is equal to

$$\kappa_\gamma = \frac{d}{ds}\left(\frac{\partial}{\partial\varphi}I_\theta(\varphi)\right).$$

We defined $I_\theta(\varphi)$ as follows, $I_\vartheta(\varphi) = \bar{\gamma}(\mathbf{n}(\varphi)) + \int_\vartheta^\varphi d\psi \int_\vartheta^\psi \bar{\gamma}(\mathbf{n}(t))dt$. We noted that this function is convex iff the stored energy function $\bar{\gamma}$ is convex. However, in general I_ϑ does not enjoy higher regularity properties. It is not differentiable at angles corresponding to the normals to the Wulff shape.

Finally, equation (6.1) takes the form

$$\beta\mathbf{n}\cdot\mathbf{e}_n\Lambda_t = \frac{d}{ds}\left(\frac{\partial}{\partial\alpha}I_\theta(\alpha)\right), \tag{6.2}$$

where α is the measure of the angle between the x_1 axis and $\mathbf{n}$.

One may study evolution of convex curves defined by their angle parameterization. We notice $\alpha = \varphi + \psi$, where ψ is the measure of the angle between τ and $\mathbf{e}_t$. We notice that $\tau \cdot \mathbf{e}_n = \sin\psi = \frac{\Lambda_s}{W}$, $\mathbf{n} \cdot \mathbf{e}_n = \cos\psi = \frac{1+\kappa\Lambda}{W}$. Thus, we see that $\psi = Arg(\tau \cdot \mathbf{e}_n + i\mathbf{n} \cdot \mathbf{e}_n)$. If we recall that $\mathcal{M}$ is a unit circle, then $\kappa = 1$ and in fact we have $\psi = \arctan\left(\frac{\Lambda_s}{1+\Lambda}\right)$, where we choose a suitable branch of $\arctan$. !!!!!!!!!! Thus, (6.1) takes the form

$$\beta \mathbf{n} \cdot \mathbf{e}_n \Lambda_t = \frac{d}{ds}\left(\frac{\partial}{\partial \phi} I_\theta \left(\varphi + \arctan\left(\frac{\Lambda_s}{1+\Lambda}\right)\right)\right).$$

This equation is rather involved, we prefer to simplify it by dropping the terms which at this stage we deem not important, i.e. we take s for φ, (which is correct for the unit circle), we approximate $\arctan\left(\frac{\Lambda_s}{1+\Lambda}\right)$ by Λ_s which is reasonable, provided that Λ_s and Λ are small. We also simplify I_θ by J. Thus we come to (1.3).

6.2 Other choices of function J

We may also consider any properly chosen piecewise linear, convex J,

$$J_l(\varphi) = \sum_{i=1}^{N} b_i |\varphi - \alpha_i|. \tag{6.3}$$

We require that $N \geq 4$, $b_i > 0$ and $\alpha_0 < \alpha_1 < \ldots < \alpha_N < \alpha_0 + 2\pi$, we will write $S = [\alpha_0, \alpha_0 + 2\pi)$. In order to stick to geometrically relevant data, we also impose the condition that $\sum_{i=1}^N b_i = \pi$, which guarantees that $\partial I(S)$ is an interval of length 2π. In addition, we assume that the following function yields an angle parameterization of closed curve, which encompasses a convex region. Namely, we set

$$\Omega_j = \sum_{i=1}^{j} b_i - \sum_{i=j+1}^{N} b_i, \qquad j = 0, \ldots, N, \tag{6.4}$$

with the convention that the summation over an empty set of parameters yields zero. Then we define $\Phi : [\alpha_0, \alpha_0 + 2\pi) \to \mathbb{R}$ by the formula

$$\Phi(s) = \sum_{i=0}^{N} \Omega_i \chi_{[\alpha_i, \alpha_{i+1})}, \tag{6.5}$$

(with the convention $\alpha_{N+1} = \alpha_0 + 2\pi$) is an angle parameterization of closed curve. We notice that our assumptions imply that $\Omega_0 + 2\pi = \Omega_N$.

The analysis of behavior of solutions presented in Section 5 is valid also for J given by (1.5) and J_l, however the actual calculations for J_l are more lengthy. In addition we may show existence of weak solution for a general, piecewise smooth, convex J, but in this case we cannot offer detailed analysis of solutions, yet.

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References

[AW] F.Almgren and L.Wang, Mathematical existence of crystal growth with Gibbs–Thomson curvature effects, *J. Geom. Anal.*, **10**, (2000), 1-100.

[ABC1] F.Andreu, C.Ballester, V.Caselles, J.M.Mazón, The Dirichlet problem for the total variation flow, *J. Funct. Anal.*, **180**, (2001), 347–403.

[ABC2] F.Andreu, C.Ballester, V.Caselles, J.M.Mazón, Minimizing total variation flow, *Differential Integral Equations*, **14**, (2001), 321–360.

[ACM] F.Andreu, V.Caselles, J.M.Mazón, "Parabolic Quasilinear Equations Minimizing Linear Growth Functionals", Progress in Mathematics, 223. Birkhäuser Verlag, Basel, 2004.

[AG] S. B. Angenent and M. E. Gurtin, Multiphase thermomechanics with interfacial structure 2. Evolution of an isothermal interface, *Arch. Rational Mech. Anal.*, **108** (1989), 323-391.

[BCN] G.Bellettini, V.Caselles, M.Novaga, The total variation flow in $\mathbb{R}^N$, *J.Differential Equations*, **184** (2002), 475–525.

[BNP1] G.Bellettini, M.Novaga, M.Paolini, Characterization of facet breaking for nonsmooth mean curvature flow in the convex case, *Interfaces and Free Boundaries*, **3**, (2001), 415–446.

[BNP2] G.Bellettini, M.Novaga, M.Paolini, On a crystalline variational problem, part I: First variation and global L^∞ regularity, *Arch. Rational Mech. Anal.*, **157**, (2001), 165–191.

[BNP3] G.Bellettini, M.Novaga, M.Paolini, On a crystalline variational problem, part II: BV regularity and structure of minimizers on facets, *Arch. Rational Mech. Anal.*, **157**, (2001), 193–217.

[Ch] A.Chambolle, An algorithm for mean curvature motion *Interfaces Free Bound.*, **6**, (2004), 195–218.

[Cg] K.-Ch. Chang, Variational methods for non-differentiable functionals and their applications to Partial Differential Equations, *J. Math. Anal. Appl.*, **80**, (1981), 102–129.

[CR] X.Chen and F.Reitich, Local existence and uniqueness of solutions to the Stefan problem with surface tension and kinetic undercooling, *J. Math. Anal. Appl.*, **164**, (1992), 350–362.

[EPS] J. Escher, J. Prüss, G. Simonett, Analytic solutions for a Stefan problem with Gibbs-Thomson cthe high orrection, *J. Reine. Angew. Math.*, **563** (2003), 1–52.

[ES] J. Escher, G. Simonett, Classical solutions of multidimensional Hele-Shaw models, *SIAM J. Math. Anal.*, **28**, (1997), 1028–1047.

[FP] X.Feng, A.Prohl, Analysis of total variation flow and its finite element approximations. *M2AN Math. Model. Numer. Anal.* **37**, (2003), 533–556.

[FG] T. Fukui, Y.Giga, Motion of a graph by nonsmooth weighted curvature, *in* "World congress of nonlinear analysts '92", vol I, ed. V.Lakshmikantham, Walter de Gruyter, Berlin, 1996, 47-56.

[GG] M.-H.Giga, Y.Giga, A subdifferential interpretation of crystalline motion under nonuniform driving force. Dynamical systems and differential equations, vol. **I** (Springfield, MO, 1996). *Discrete Contin. Dynam. Systems,* Added Volume **I**, 276–287 (1998).

[GK] Y.Giga, R.Kobayashi, Equations with singular diffusivity, *J.Stat.Physics*, **95**, (1999), 1187-1220.

[GGK] M.-H.Giga, Y.Giga, R.Kobayashi, Very singular diffusion equation, *Advanced Studies in Pure Mathematics*, **31** (2001), 93-125.

[GR1] Y.Giga, P.Rybka, Facet bending in the driven crystalline curvature flow in the plane, *The Journal of Geometric Analysis*, **18**, (2008), 109-147.

[GR2] Y.Giga, P.Rybka, Facet bending driven by the planar crystalline curvature with a generic nonuniform forcing term, *preprint.*

[L] S.Luckhaus, Solutions for the two-phase Stefan problem with the Gibbs–Thomson law for the melting temperature, *Euro. J. Appl. Math.,* **1**, (1990), 101–111.

[Mo] J.S.Moll, The anisotropic total variation flow. *Math. Ann.*, **332**, (2005), 177–218.

[Mu] P.B.Mucha, On the Stefan problem with surface tension in the L_p framework, *Adv. Differential Equations* **10**, (2005), 861–900.

[MRa] P.B.Mucha, R.Rautmann, Convergence of Rothe's scheme for the Navier-Stokes equations with slip conditions in 2D domains. *Z. Angew. Math. Mech.*, **86**, (2006), 691–701.

[MRy] P.B.Mucha, P.Rybka, A new look at equilibria in Stefan type problems in the plane, *SIAM J. Math. Analysis*, **39**, No.4, 1120-1134 (2007).

[LSU] O.A.Ladyženskaja, V.A.Solonnikov, N.N.Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Am. Math. Soc., Providence, R. I., 1968.

[Ra] E.V.Radkevich, The Gibbs–Thompson correction and condition for the existence of classical solution of the modified Stefan problem, *Soviet Math. Dokl.*, **43**, (1991), 274–278.

[RO] L. Rudin, S. Osher, Total variation based image restoration with free local constraints, *in* "Proc. of the IEEE ICIP-94 Austin, TX," Vol. 1, pp. 31-35, 1994.

[ROF] L. Rudin, S. Osher, and E. Fatemi, Nonlinear total variation based noise removal, *Physica D*, **62**, (1992), 259-268.

[Ry] P.Rybka, The crystalline version of the modified Stefan problem in the plane and its properties, *SIAM J.Math. Anal.*, **30** (1999), 736–786.

[S] J.Smoller, "Shock waves and reaction-diffusion equations. Second edition". Grundlehren der Mathematischen Wissenschaften, 258. Springer-Verlag, New York, 1994.

[T] J.E. Taylor, Constructions and conjectures in crystalline nondifferential geometry, *in*: Differential Geometry (eds. B. Lawson and K. Tanenblat), Proceedings of the Conference on Differential Geometry, Rio de Janeiro, Pitman Monographs Surveys in Pure and Applied Math. **52** (1991) pp.321-336, Pitman, London.