On weak solutions to the Stefan problem
with Gibbs-Thomson correction

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**Abstract.** The paper investigates the well posedness of the quasi-stationary Stefan problem with the Gibbs-Thomson correction. The main result proves the existence of unique weak solutions provided the initial surface belongs to the $W_p^{2-3/p}$-Sobolev-Slobodeckij class for $p > n+3$, only. The proof is based on Schauder-type estimates in $L_p$-type spaces for a linearization of the original system in a rigid domain.

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*Key words:* Stefan problem with Gibbs-Thomson correction, parabolic equations of odd order, regular solutions, Schauder-type estimates

# 1 Introduction

The well posedness of the quasi-stationary one-phase Stefan problem with the Gibbs-Thomson correction is the subject of this paper. We investigate the following free boundary problem in sought domain $\Omega_t$

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \Omega_t, \\ p = a\kappa & \text{on} & \partial\Omega_t, \\ \frac{\partial p}{\partial \vec{n}} = -V_{\vec{n}} & \text{on} & \partial\Omega_t, \\ \Omega_t|_{t=0} = \Omega_0, & & \end{array} \tag{1.1}$$

where the unknown function $p$ in the classical thermodynamical interpretation represents the temperature; $\partial\Omega_t$ is the unknown phase transition surface being the boundary of $\Omega_t$; $\kappa$ is the mean curvature of boundary $\partial\Omega_t$, $V_{\vec{n}}$ describes the normal velocity of the free boundary; $a$ is a given positive constant; $\vec{n}$ is the outer normal vector to $\partial\Omega_t$ and $\Omega_0$ is the initial domain – required to be bounded.

System (1.1) belongs to the classical phase transition theory [Chal],[Sa]. It describes the motion of interface separating two states – here it is sought $\partial\Omega_t$. In the classical theory they are solid and liquid states of the same material. Similar models may be found in mechanics, biology or medicine [Chal],[G],[FR1]. Sometimes they are called the Hele-Shaw or Mullins-Sekerka systems.

The issue of existence to system (1.1) was a serious challenge for mathematicians. First, we had only partial results [Ch],[DR],[L] and the general answer has been given in [ChHY],[ES1]. The results are based on the parabolic character of the system and deliver classical solutions in the Hölder class. They use the fact that system (1.1) is the nonlocal parabolic system of the third order.

In our article we would like to consider the issue of optimal regularity of the initial surface. The question is related with finding weakest assumptions which imply uniqueness of solutions. In [ES1] the authors required $\partial\Omega_0 \in C^{2+a}$, next their result has been generalized in [EPS] to the $L_p$-framework for the evolutionary version of (1.1) with $\partial\Omega_0 \in W_p^{4-3/p}$ for sufficiently large $p$. In [Mu1] the initial boundary belongs to $W_p^{3-3/p}$. All these results were concerned with regular solutions, here we will investigate a weaker form of system (1.1), where boundary conditions $(1.1)_3$ will be studied in a distributional sense.

We shall mention about other approaches to problems represented by (1.1). In [LS],[R] authors analyze an reformulation of the system given by a phase function obtaining weak solutions. This approach, based on the variation technique, shows existence in very low class of regularity, however does not guarantee the uniqueness of obtained solutions. The geometrical interpretation is not also immediate, either.

Our result will be local in time, since we consider general initial domains. For some global in time stability results we refer to [CP],[ES2],[FR2].

Since the mathematical difficulties are located on the boundary we set

$$\dim \Omega_t = n+1 \qquad \text{and} \qquad \dim \partial\Omega_t = n. \tag{1.2}$$

Our main result states the well posedness of system (1.1). The initial boundary will be described in the Sobolev-Slobodeckij spaces $W_p^s$ and the $p$-function in the Besov spaces $B_{p1}^s$ (see section 2). The main theorem reads as follows.

**Theorem 1.1.** *Let $p > n+3$ and $\Omega_0$ is a bounded subdomain of $\mathbf{R}^{n+1}$ with $\partial\Omega_0 \in W_p^{2-3/p}$. There exists a unique solution to problem (1.1) on time interval $(0,T)$ for $T>0$ such that the following inclusions hold*

$$\begin{aligned} &\left(\textstyle\bigcup_{0<t<T} \partial\Omega_t \times \{t\}\right) \in W_p^{2,2/3} \quad \textit{as a submanifold in } \mathbf{R}^{n+2}, \\ &p \in L_p(0,T; B_{p1}^{1/p}(\Omega_t)) \quad \textit{and} \quad \tfrac{\partial p}{\partial \vec{n}} \in L_p(0,T; W_p^{-1}(\partial\Omega_t)). \end{aligned} \tag{1.3}$$

Theorem 1.1 constructs unique solutions to system (1.1) assuming possible lowest regularity of the initial surface. The technique of the proof is based on analysis of a linearization of the studied system. Considering a free boundary problem a way of linearization is not obvious. Here we transform the domain in a rigid given set preserving the measure. Thanks to that the form of the Laplacian in a new coordinate system is better to analyze the full nonlinear system, we gain "one derivative". However the result for the linear equations is not elementary. The method requires results for the Laplace operator in very weak regularity in Besov spaces $B_{p1}^{1/p}$, which guaranteeing existence of traces [Tr]. Basic tools are the Fourier analysis and Marcinkiewicz-type theorems. They are necessary to obtain optimal estimates of solutions in $L_p$-type spaces. This way we are able to concentrate our investigations to the surface, only.

Comparing the presented technique to the approach in [ChHY] and [ES1], we obtain improved information about the regularity of solutions. The theory of semigroups applied in [ES1] required initial surface in the Hölder class $C^{2+a}$ and here is needed $W_p^{2-3/p}$, only. It follows from the fact that this technique requires relatively high regularity to define suitable solutions to (1.1) and does not see the whole structure of the system. Theorem 1.1 improves the results from [Mu1] and the approach from [EPS], too.

In our case we obtain weak solutions defined as follows: *a pair $\{p, \Omega_t\}$ is the solution to weak system (1.1) iff $(p, \Omega_t)$ fulfills conditions (1.3) and the following integral identity* *holds*

$$\int_{\partial\Omega_t} V_{\vec{n}}\pi d\sigma + \int_{\partial\Omega_t} a\kappa \frac{\partial \pi}{\partial \vec{n}} d\sigma = \int_{\Omega_t} p\Delta\pi dx \tag{1.4}$$

*for each sufficiently smooth function* $\pi$, *e.g.* $\pi \in C^\infty(\bigcup_{0<t<T}\Omega_t)$.

Theorem 1.1 proves existence of unique solutions to problem (1.1) assuming relatively low regularity of initial surface $\partial\Omega_0$. The parabolic character of the system helps to control nonlinearities, however main difficulties are hidden in the geometry of (1.1). The main obstacle is to relax regularity of solutions is the fact that our approach requires the Hölder continuity of the normal vector. Regularity (1.3) implies that the normal vector filed $\vec{n} \in C^{a,a/3}$ as $p > n+3$. However lower regularity would cause the serious difficulties with the interpretation of solutions, since the trace would not be well defined. One point which is worthwhile to emphasize is that the curvature is only an $L_p$-function on the obtained surface and the initial curvature is a distribution, only.

An interesting question is: is it possible to relax regularity of initial surface to obtain unique solutions, but without requirements of Hölder continuity of the normal vector field of $\partial\Omega_0$? A positive answer could enable us to prove a similar result for generalizations of model (1.1) as in [GiRy].

The paper is organized as follows. First we transform system (1.1) to problem in a rigid domain and state the result for the linearized system. In section 3 we prove some important facts for the Laplace operator in Besov spaces in very low regularity. Subsequently, we show the result for the linear system in the halfspace. In section 5 we prove the theorem for the linear system in the bounded domain. Finally, we prove Theorem 1.1.

# 2 Preliminaries

First, we transform $\Omega_t$ to a rigid domain $D$ requiring smoothness (even $C^\infty$) of its boundary. Moreover we assume that $dist(\partial\Omega_t, \partial D)$ is sufficiently small in the $W_p^{2-3/p}$-norm which is possible since we look for local in time solutions and the transform

$$\Phi_t : \Omega_t \to D \tag{2.1}$$

preserves the measure, i.e. the Jacobian of $\Phi_t$ is equal to 1. Existence of such transform is not immediate. Since our domains are bounded, so in particular $|\Omega_t|$ is finite. Hence one can find always one-parameter deformation from $D$ to $\partial\Omega_t$ preserving the total volume. Then using methods based on the Lagrangian coordinates (from the fluid mechanics) and properties of divergence-free vector fields - see [MZ] or [So] - we are able to find the sought transform with suitable regularity described by smoothness of $\partial\Omega_t$. And in our case $\partial\Omega_t \in W_p^{2,2/3}$ which implies $\Phi_t \in W_p^{2,2/3}$, too.

What is the basic consequence of Jacobian being equal 1? We have the following form of the Laplace operator in the new coordinate system

$$\Delta_x = \Phi_t^{-1}\Delta = \mathrm{div}_y A\nabla_y, \tag{2.2}$$

in a general case we would have $\Delta_x = A_1\nabla_y \cdot (A_2\nabla_y)$ which is more complex than (2.2). To see the above relation it is enough to consider the following identity

$$\int_D \mathrm{div}_x \nabla_x \bar{p}\, \bar{q} dy = \int_{\Omega_t} \Delta p\, q |J\Phi_t| dx = -\int_{\Omega_t} \nabla p\, \nabla q |J\Phi_t| dx = -\int_D \nabla_x \bar{p}\, \nabla_x \bar{q} dy \qquad (2.3)$$

for $p, q \in C_0^\infty(\Omega_t)$. Remembering that $|J\Phi_t| = 1$ we get (2.2).

Having this transform we apply it to system (1.1), thus we obtain

$$\begin{array}{lll} \Delta q = \mathrm{div}(I - A_t)\nabla q & \text{in} & D \times (0,T), \\ q = a\tilde{\kappa} & \text{on} & \partial D \times (0,T), \\ \frac{\partial q}{\partial \vec{n}} = -v + (\frac{\partial}{\partial \vec{n}} - \frac{\partial}{\partial \vec{n}_t})q & \text{on} & \partial D \times (0,T), \end{array} \qquad (2.4)$$

where $q = \Phi_t^* p$, $\tilde{\kappa} = \Phi_t^* \kappa$, $v = \Phi_t^* V_{\vec{n}}$ and $A_t$ is the transform matrix described by map $\Phi_t$ (the star $*$ means the standard pullback operator).

The free boundary $\partial\Omega_t$ is described by function $\psi$ defined on $\partial D$ as follows

$$\partial\Omega_t \ni x(y,t) = y + \psi(y,t)\vec{n}, \quad \text{where} \quad y \in \partial D \qquad (2.5)$$

in particular $\partial\Omega_0 = \{x(y) = y + \psi_0(y)\vec{n} \text{ for } y \in D\}$.

We remember that the choice of domain $D$ guarantees

$$||\psi_0||_{W_p^{2-3/p}} + ||\vec{n} - \vec{n}_0||_{C^a(\partial D)} \leq \epsilon \qquad (2.6)$$

for sufficiently small $\epsilon > 0$ and $0 < a < \frac{p-(n+3)}{p-3}$.

In the above parameterization system (1.1) (or (2.4)) reads

$$\begin{array}{lll} \Delta q = \mathrm{div}(Id - A_t(\psi))\nabla q & \text{in} & D \times (0,T), \\ q = a\tilde{\kappa} & \text{on} & \partial D \times (0,T), \\ \frac{\partial q}{\partial \vec{n}} = -\partial_t \psi \vec{n} \cdot \vec{n}_t + (\frac{\partial}{\partial \vec{n}} - \frac{\partial}{\partial \vec{n}_t})q & \text{on} & \partial D \times (0,T), \\ \psi|_{t=0} = \psi_0 & \text{on} & \partial D. \end{array} \qquad (2.7)$$

The mean curvature $\tilde{\kappa}$ (the r.h.s. of $(2.7)_2$) can be described in a local coordinate system by inducted metric $\{g_{ij}\}$ as follows

$$\tilde{\kappa} = g^{-1/2} \frac{\partial}{\partial y^i}(g^{1/2} g^{ij} \frac{\partial \psi}{\partial y^j}) + K(\psi, \nabla\psi),$$

where $K(\cdot,\cdot)$ is a smooth function.

By the properties of map $\Phi_t$, we deduce that

$$||Id - A_t(\psi)||_{L_\infty(\partial D \times (0,T))} \leq c||\nabla\psi||_{L_\infty(\partial D \times (0,T))}. \qquad (2.8)$$

Our analysis requires special coverings of domain $D$. We take two collections of open sets: $\{\omega^k\}$ and $\{\Omega^k\}$ such that $\overline{\omega^k} \subset \Omega^k \subset D$, $\bigcup_k \omega^k = \bigcup_k \Omega^k = D$ with $k \in \mathcal{M} \cup \mathcal{N}$. Index $k$ belongs to one of two sets: $k \in \mathcal{M}$ if $\overline{\Omega^k} \cap \partial D = \emptyset$ and $k \in \mathcal{N}$ if $\overline{\omega^k} \cap \partial D \neq \emptyset$. Moreover

$$\sup_k \mathrm{diam}\, \Omega^k \leq 2\lambda \qquad (2.9)$$

for a small number $\lambda$ which will be specified latter. The magnitude of $\lambda$ will be chosen sufficiently small comparing to the curvature of $\partial D$. The Lebesgue covering number for collections $\{\omega^k\}$ and $\{\Omega^k\}$ is independent of smallness of $\lambda$. These coverings define us a partition of unity for domain $D$. Let $\zeta^k : D \to [0,1]$ be a smooth function such that

$$\zeta^k(x) = \begin{cases} 1 & \text{for} \quad x \in \omega^k \\ \in [0,1] & \text{for} \quad x \in \Omega^k \setminus \omega^k \\ 0 & \text{for} \quad x \in D \setminus \Omega^k \end{cases} \tag{2.10}$$

and $|\nabla^\alpha \zeta^k| \le c/\lambda^{|\alpha|}$, $1 \le \sum_k (\zeta^k)^2 \le N_0$. With the help of functions $\zeta^k$ we define

$$\pi^k = \frac{\zeta^k}{\sum_l (\zeta^l)^2}, \quad \text{then} \sum_{k \in \mathcal{M} \cup \mathcal{N}} \pi^k \zeta^k = 1. \tag{2.11}$$

By the definition of $\zeta^k$, functions $\pi^k$ vanish for $x \in D \setminus \Omega^k$ as well as $|\nabla^\alpha \pi^k| \le c/\lambda^{|\alpha|}$. Functions $\pi^k \zeta^k$ define the partition of the unity .

Next, we introduce local coordinate systems related with each $\omega^k$ for $k \in \mathcal{N}$. We find maps $Z_k : \mathbf{R}^{n+1}_+ \to \mathbf{R}^{n+1}$ such that $Z_k^{-1}(\Omega^k) \subset \mathbf{R}^{n+1}_+$, $Z_k^{-1}(\Omega^k \cap \partial D) \subset \mathbf{R}^n$. Regularity of diffeomorphism $Z_k$ is controlled by smoothness of $\psi$. Considering truncations $Z_k|_{\mathbf{R}^n}$ we obtain an atlas of maps for $\partial D$.

To analyze the nonlinear system (2.7) we investigate the following linear system

$$\begin{array}{lll} \Delta p = \operatorname{div} A \nabla q & \text{in} & D \times (0,T), \\ p = a\Lambda\phi + G & \text{on} & \partial D \times (0,T), \\ \frac{\partial p}{\partial \vec{n}} = -\partial_t \phi + H & \text{on} & \partial D \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & \partial D, \end{array} \tag{2.12}$$

where

$$\Lambda\phi = \sum_{k \in \mathcal{N}} Z_k^*[\Delta' Z_k^{-1*}(\pi^k \zeta^k \phi)] \quad \text{and} \quad \Delta' = \frac{\partial^2}{\partial z_1^2} + \dots + \frac{\partial^2}{\partial z_n^2}; \tag{2.13}$$

$q$, $G$ and $H$ are given functions, $A$ is a fixed metric $(n+1) \times (n+1)$. The above setting of the elliptic operator defined on $\partial D$ will be suitable to our approach, since it is given directly on maps of $\partial D$.

Let us introduce the basic notation of function spaces. As a background we take books [BIN],[Tr]. By $L_p(\Omega)$ we denote the standard space of functions integrable with the $p$-th power and for $m \in \mathbf{N}$ space $W^m_p(\Omega)$ denotes the standard Sobolev space. However our approach requires the fractional spaces. For this purpose we introduce the Slobodeckij isotropic space on a domain $Q \subset \mathbf{R}^d$ given by the norm for $m \in \mathbf{R}_+$:

$$||u||_{W^m_p(Q)} = ||u||^p_{W_p^{[m]_-}(Q)} + < u >^p_{W^m_p(Q)}, \tag{2.14}$$

where $[t]_-$ is the biggest integer number less than $t$ and $< \cdot >$ denotes the main seminorm of the $W^m_p$-norm, i.e.

$$< u >^p_{W^m_p(Q)} = \sum_{|\alpha| = [m]_-} \int_Q dx \int_Q dx' \frac{|\partial^\alpha u(x) - \partial^\alpha u(x')|^p}{|x - x'|^{d + p(m - [m]_-)}}. \tag{2.15}$$

The evolutionary character of our problem requires function spaces defined on set $Q \times (0,T)$ with the distinguished time direction. That is the reason we define the anisotropic spaces $W^{m,n}_p$ for $m,n \in \mathbf{R}_+$ given by the norm

$$||u||^p_{W^{m,n}_p(Q\times(0,T))} = ||u||^p_{W^{[m]_-,[n]_-}_p} + \int_0^T < u(\cdot,t) >^p_{W^m_p} dt + \int_Q < u(x,\cdot) >^p_{W^n_p(0,T)} dx. \quad (2.16)$$

The $W^m_p$-spaces are not sufficient in our approach, we need $B^s_{p1}$-spaces. To define Besov spaces $B^s_{p1}(Q)$, first, we introduce them for $Q = \mathbf{R}^n$ with help of the Paley-Littlewood decomposition [Tr], i.e. we find a sequence of functions $\phi_k : \mathbf{R}^n \to \mathbf{R}$ such that $\phi_k \geq 0$ and $\sum_{k=0}^\infty \phi_k = 1$, additionally we require supp $\phi_k \subset \{2^{k-n} \leq |\xi| \leq 2^{k+n}\}$. Then we define the norm of the $B^s_{p1}$-space as follows

$$||u||_{B^s_{p1}(\mathbf{R}^n)} = \sum_{k=0}^\infty 2^{ks}||u_k||_{L_p(\mathbf{R}^n)}, \quad \text{where } u_k = \mathcal{F}^{-1}_x[\phi_k \mathcal{F}_x[u]] \quad (2.17)$$

and $\mathcal{F}_x[\cdot]$ denotes the Fourier transform on $\mathbf{R}^n$. To obtain the space defined on $Q$ we apply the localization or extension techniques. The main advantage of the above spaces is that the trace theorem holds in the critical case, i.e. $u|_{\partial Q} \in L_p(\partial Q)$ as $u \in B^{1/p}_{p1}(Q)$ – for details see [Tr,Chap. 2.9].

Dual spaces to Besov and Slobodeckij spaces are denoted by negative derivatives.

The basic result of our paper is the following theorem for linear system (2.12).

**Theorem 2.1.** *Let $p > n+3$, $\phi_0 \in W^{2-3/p}_p(\partial D)$ and*

$$q \in L_p(0,T; B^{1/p}_{p1}(D)), \quad \tfrac{\partial q}{\partial \vec{n}}|_{\partial D} \in L_p(0,T;W^{-1}_p(\partial D)), \quad A \in L_\infty(0,T;W^1_p(D)),$$
$$A|_{\partial D} \in L_\infty(0,T;W^1_p(\partial D)), \quad G \in L_p(\partial D \times (0,T)), \quad H \in L_p(0,T;W^{-1}_p(\partial D)).$$

*Then there exists a unique solution to system (2.12) such that*

$$\phi \in W^{2,2/3}_p(\partial D \times (0,T)) \text{ and } p \in L_p(0,T;B^{1/p}_{p1}(D))$$
$$\text{with } \partial_t\phi,\ \frac{\partial p}{\partial \vec{n}}|_{\partial D} \in L_p(0,T;W^{-1}_p(\partial D));$$

*moreover the following estimate is valid*

$$||\phi||_{W^{2,2/3}_p(\partial D\times(0,T))} + ||p||_{L_p(0,T;B^{1/p}_{p1}(D))} + ||\tfrac{\partial p}{\partial \vec{n}}|_{\partial D}||_{L_p(0,T;W^{-1}_p(\partial D))} + ||\partial_t\phi||_{L_p(0,T;W^{-1}_p(\partial D))} \leq$$
$$c(||A||_{L_\infty(0,T;W^1_p(D))} + ||A|_{\partial D}||_{L_\infty(0,T;W^1_p(\partial D))})(||q||_{L_p(0,T;B^{1/p}_{p1}(D))} + ||\tfrac{\partial q}{\partial \vec{n}}|_{\partial D}||_{L_p(0,T;W^{-1}_p(\partial D))})$$
$$+c(||G||_{L_p(\partial D\times(0,T))} + ||H||_{L_p(0,T;W^{-1}_p(\partial D))} + ||\phi_0||_{W^{2-3/p}_p(\partial D)}). \quad (2.18)$$

Theorem 2.1 delivers solutions in the following sense: *a pair $(p,\phi)$ is a weak solution to problem (2.12) iff $(p,\phi)$ fulfills regularity prescribed in Theorem 2.1 and the following integral identity holds*

$$\int_{\partial D}\left(\partial_t\phi\pi + a\Lambda\phi\frac{\partial\pi}{\partial\vec{n}}\right)d\sigma - \int_D p\Delta\pi dx = \int_{\partial D}(H\pi - G\frac{\partial\pi}{\partial\vec{n}})d\sigma + \int_D \operatorname{div}(A\nabla q)\pi dx \quad (2.19)$$

*for each $\pi \in C^\infty(D\times(0,T))$.*

# 3 The Laplace operator

The aim of this section is to establish some results for the Laplace operator for very weak solutions. We study the following system in a smooth domain $\Omega \subset \mathbf{R}^{n+1}$:

$$\begin{array}{lll} \Delta p = \mathrm{div} A\nabla q & \text{in} & \Omega, \\ \frac{\partial p}{\partial \vec{n}} = \vec{n} \cdot A\nabla q & \text{on} & \partial\Omega, \end{array} \qquad \int_\Omega p dx = 0, \tag{3.1}$$

where $A$ is a given matrix $(n+1) \times (n+1)$. The compatibility condition is fulfilled by the form of the r.h.s. of (3.1).

Our purpose in this section is to prove the following theorem.

**Theorem 3.1.** *Let $p > n+3$, $A \in W^1_p(\Omega)$, $A|_{\partial\Omega} \in W^1_p(\partial\Omega)$, $q \in B^{1/p}_{p1}(\Omega)$, $\nabla q|_{\partial\Omega} \in W^{-1}_p(\partial\Omega)$, then there exists a weak solution to problem (3.1) such that $p \in B^{1/p}_{p1}(\Omega)$ and $\frac{\partial p}{\partial \vec{n}}\big|_{\partial\Omega} \in W^{-1}_p(\partial\Omega)$, moreover*

$$\begin{aligned} &||p||_{B^{1/p}_{p1}(\Omega)} + ||\tfrac{\partial p}{\partial \vec{n}}|_{\partial\Omega}||_{W^{-1}_p(\partial\Omega)} \\ &\leq c(||A||_{W^1_p(\Omega)} + ||A|_{\partial\Omega}||_{W^1_q(\partial\Omega)}) \cdot \Big(||q||_{B^{1/p}_{p1}(\Omega)} + ||\tfrac{\partial q}{\partial \vec{n}}|_{\partial\Omega}||_{W^{-1}_p(\partial\Omega)}\Big). \end{aligned} \tag{3.2}$$

**Proof.** First, we show existence of very weak solutions in $L_2(\Omega)$. By the standard method we deduce the following existence:

*We say that $p \in L_2(\Omega)$ is a very weak solution to problem (3.1) iff the following identity*

$$\int_\Omega p\Delta\phi dx = \int_\Omega q\, div\,(A^T\nabla\phi)dx - \int_{\partial\Omega} (\vec{n}\cdot A\nabla q)\phi\, d\sigma \tag{3.3}$$

*for every $\phi \in W^2_2(\Omega) \cap \{\frac{\partial\phi}{\partial\vec{n}}|_{\partial\Omega} = 0$ and $\int_\Omega \phi dx = 0\}$.*

Inserting a test function $\phi = \phi_0$ such that $\Delta\phi_0 = p$ in $\Omega$ and $\frac{\partial\phi_0}{\partial\vec{n}}|_{\partial\Omega} = 0$ on $\partial\Omega$, remembering that $p_* > 2$, we calculate the following basic estimate

$$||p||_{L_2(\Omega)} \leq c(||A||_{W^1_p(\Omega)} + ||A|_{\partial\Omega}||_{W^1_p(\partial\Omega)}) \cdot \left(||q||_{B^{1/p}_{p1}(\Omega)} + ||\frac{\partial q}{\partial\vec{n}}|_{\partial\Omega}||_{W^{-1}_p(\partial\Omega)}\right). \tag{3.4}$$

Then a tool such as the Galerkin method implies the existence of a distributional solution to system (3.1) in the sense of the definition given by (3.3).

Now, to prove Theorem 3.1, it is enough to improve the regularity by the localization of the problem. Using the partition of unity from section 2 we get two cases. Let us choose $k \in \mathcal{N}$, at first. Then we consider the localized problem in the halfspace

$$\begin{array}{lll} \Delta Z^*_k(\zeta^k p) = (\Delta - \Delta_x)Z^*_k(\zeta^k p) + \mathrm{div}\,\tilde{A}\nabla Z^*_k(\zeta^k q) + S_1 & \text{in} & \mathbf{R}^{n+1}_+, \\ Z^*_k(\zeta^k p)_{,z_{n+1}}|_{z_{n+1}=0} = (\tilde{A}\nabla Z^*_k(\zeta^k q))^{(n+1)} & \text{on} & \mathbf{R}^n, \end{array} \tag{3.5}$$

where $S_1$ is a more regular part, being a consequence of the localization, belonging at least to $B^{1/p-1}_{p1}$. To get the suitable form of the r.h.s. of (3.5) we assumed that each transformation $Z_k$ preserves the normal vector. Since in our case the boundary is smooth this choice does not restrict our examination. Elementary calculations lead to the following form of term $S_1$

$$S_1 = \mathrm{div}(B_0(\lambda)\nabla Z_k^*(\zeta^k p)) + \mathrm{div}(B_1 p) + \mathrm{div}(B_2 q) + pB_3 + qB_4. \tag{3.6}$$

The transformation into the halfspace guarantees that $(B_0)^{(n+1)} \equiv 0$ and $B_1^{(n+1)} = B_2^{(n+1)} = 0$, additionally properties of map $Z_k$ implies $|B_0(\lambda)| \leq c\lambda$ on supp $\bar{\zeta}^k$ - see (2.9). Coefficients $B_1$ and $B_3$ are smooth, and $B_2$ and $B_4$ depend on matrix $A$.

To obtain the whole information about the regularity of solutions to problem (3.5) we need the below lemma.

**Lemma 3.1.** *Let* $\bar{A} \in W_p^1(\mathbf{R}_+^{n+1})$, $\bar{A}|_{z_{n+1}=0} \in W_p^{-1}(\mathbf{R}^n)$, $\bar{q} \in B_{p1}^{1/p}(\mathbf{R}_+^{n+1})$ *and* $\bar{q}_{,z_{n+1}}|_{z_{n+1}=0} \in W_p^1(\mathbf{R}^n)$, *then the solutions to problem*

$$\begin{array}{lll} \Delta\bar{p} = \mathrm{div}\bar{A}\nabla\bar{q} & in & \mathbf{R}_+^{n+1}, \\ \bar{p}_{,z_{n+1}}|_{z_{n+1}=0} = (\bar{A}\nabla\bar{q})^{(n+1)} & on & \mathbf{R}^n \end{array} \tag{3.7}$$

*satisfy the following inclusions:* $\bar{p} \in B_{p1}^{1/p}(\mathbf{R}_+^{n+1})$ *and* $\bar{p}_{,z_{n+1}}|_{z_{n+1}=0} \in W_p^{-1}(\mathbf{R}^n)$ *with the following estimate*

$$\begin{gathered} ||\bar{p}||_{B_{p1}^{1/p}(\mathbf{R}_+^{n+1})} + ||\bar{p}_{,z_{n+1}}|_{z_{n+1}=0}||_{W_p^{-1}(\mathbf{R}^n)} \\ \leq c(||\bar{A}||_{W_p^1(\mathbf{R}_+^{n+1})} + ||\bar{A}|_{z_{n+1}=0}||_{W_p^1(\mathbf{R}^n)})(||\bar{q}||_{B_{p1}^{1/p}(\mathbf{R}_+^{n+1})} + ||\bar{q}_{,z_{n+1}}|_{z_{n+1}=0}||_{W_p^{-1}(\mathbf{R}^n)}). \end{gathered} \tag{3.8}$$

**Proof.** Our first task is to remove inhomogenuity from the r.h.s. of boundary condition $(3.7)_2$. Thanks to the structure of boundary data can be viewed in the following form

$$(\bar{A}\nabla q)^{(n+1)} = \sum_{k=1}^{n+1}(\bar{A}_k^{n+1}q)_{,z_k} - q\sum_{k=1}^{n+1}\bar{A}_{k,z_k}^{n+1}.$$

The last term in the r.h.s. is of better regularity, it belongs to $L_{p/2}$, hence we will omit considerations for it.

We want to construct a function from $B_{p1}^{1/p}(\mathbf{R}_+^{n+1})$ fulfilling condition $(3.7)_2$.

First, introduce a function $Q_{n+1} = A_{n+1}^{n+1}q$. It belongs to $B_{p1}^{1/p}(\mathbf{R}_+^{n+1})$ and by the definition it satisfies the following boundary condition

$$Q_{n+1,z_{n+1}}|_{z_{n+1}=0} = (A_{n+1}^{n+1}q)_{,z_{n+1}}|_{z_{n+1}=0}.$$

Next, let us consider problems for $k = 1, ..., n$

$$\begin{array}{lll} -\Delta Q_k + Q_k = 0 & \mathrm{in} & \mathbf{R}_+^{n+1}, \\ Q_k = -\mathcal{F}_z^{-1}[\frac{1}{\sqrt{1+|\xi|^2}}\mathcal{F}_z[(\bar{A}_k^{n+1}\cdot\bar{q})_{,z_k}]] & \mathrm{on} & \mathbf{R}^n. \end{array} \tag{3.9}$$

The definition of the dual space $W_p^{-1}(\mathbf{R}^n)$ implies that the r.h.s. of $(3.9)_2$ belongs to $L_p$.

But $(3.9)_2$ can be viewed as follows $Q_k = -\mathcal{F}_z^{-1}[\frac{i\xi_k}{\sqrt{1+|\xi|^2}}\mathcal{F}_z[\bar{A}_k^{n+1}\cdot\bar{q}]]$, too. Since $\frac{i\xi_k}{\sqrt{1+|\xi|^2}}$ is a pseudodifferential operator of order zero, we conclude that a function (defined on the whole halfspace)

$$-\mathcal{F}_z^{-1}\left[\frac{i\xi_k}{\sqrt{1+|\xi|^2}}\mathcal{F}_z[\bar{A}_k^{n+1}\cdot\bar{q}]\right] \in B_{p1}^{1/p}(\mathbf{R}_+^{n+1}),$$

what gives us a good extension of boundary datum $(3.9)_2$. The solvability of this problem follows from the extention theorem and the method of symmetry. Then we get that

$$Q_k \in B^{1/p}_{p1}(\mathbf{R}^{n+1}_+) \qquad \text{and} \qquad ||Q_k||_{B^{1/p}_{p1}(\mathbf{R}^{n+1}_+)} \leq c||(\bar{A}^{n+1}_k \bar{q})_{,z_k}||_{W^{-1}_p(\mathbf{R}^n)}. \tag{3.10}$$

Moreover, the explicit formula of the solution of system (3.9)

$$\mathcal{F}_z Q_k(\xi, z_{n+1}) = \frac{1}{\sqrt{1+|\xi|^2}} \mathcal{F}_z[(\bar{A}^{n+1}_k \bar{q})_{,z_k}] e^{-\sqrt{1+|\xi|^2} z_{n+1}}$$

implies that $Q_{k,z_{n+1}}|_{z_{n+1}=0} = (\bar{A}^{n+1}_k \bar{q})_{,z_k}|_{z_{n+1}=0}$ on $\mathbf{R}^n$. So defining $u = \sum_{k=0}^{n+1} Q_k$, where $Q_0$ corresponds to term $q \sum_{k=1}^{n+1} \bar{A}^{n+1}_{k,z_{n+1}}$, we get

$$u \in B^{1/p}_{p1}(\mathbf{R}^{n+1}_+) \qquad \text{and} \qquad ||u||_{B^{1/p}_{p1}(\mathbf{R}^{n+1}_+)} \leq c||(\bar{A}\nabla \bar{q})^{(n+1)}||_{W^{-1}_p(\mathbf{R}^n)}. \tag{3.11}$$

Moreover, the construction implies that

$$u_{,z_{n+1}}|_{z_{n+1}=0} = (\bar{A}\nabla\bar{q})^{(n+1)}|_{z_{n+1}=0} \qquad \text{on} \quad \mathbf{R}^n, \tag{3.12}$$

which specifies the regularity of the l.h.s. of (3.12), too.

Now, we modify the original problem (3.7). Let $w = \bar{p} - u$, then this function satisfies

$$\begin{array}{lll} \Delta w = \operatorname{div}(\bar{A}\nabla\bar{q} - \nabla u) & \text{in} & \mathbf{R}^{n+1}_+, \\ w_{,z_{n+1}}|_{z_{n+1}=0} = 0 & \text{on} & \mathbf{R}^n. \end{array} \tag{3.13}$$

To solve the above system we apply the method of symmetry.

Using the following symmetry

$$\bar{\cdot}(z, z_{n+1}) = \begin{cases} \cdot(z, z_{n+1}) & \text{for} \quad z_{n+1} \geq 0 \\ \cdot(z, -z_{n+1}) & \text{for} \quad z_{n+1} < 0 \end{cases} \tag{3.14}$$

we get the following problem in the whole space

$$\Delta \bar{w} = \overline{\operatorname{div}\,(\bar{A}\nabla\bar{q} - \nabla u)} \qquad \text{in} \ \mathbf{R}^{n+1}. \tag{3.15}$$

If we prove existence to the above equation, the symmetry will imply that $w_{,z_{n+1}} = 0$ on set $\{z_{n+1} = 0\}$ as a distribution. Moreover one point which is unclear is whether the r.h.s. of (3.15) belongs to $B^{1/p-2}_{p1}(\mathbf{R}^{n+1})$. Note that $\overline{\operatorname{div}\,(\bar{A}\nabla\bar{q} - \nabla u)} = \operatorname{div} \tilde{H}$, where $\tilde{H}^{(i)} = \overline{\bar{A}\nabla\bar{q} - \nabla u}^{(i)}$ for $i = 1, .., n$ and

$$\tilde{H}^{(n+1)} = \begin{cases} (\bar{A}\nabla\bar{q} - \nabla u)^{(n+1)}(z, z_{n+1}) & \text{for} \quad z_{n+1} \geq 0 \\ -(\bar{A}\nabla\bar{q} - \nabla u)^{(n+1)}(z, -z_{n+1}) & \text{for} \quad z_{n+1} < 0 \end{cases} \tag{3.16}$$

We can restrict our attention only to symmetric test functions, since each function can be split on two parts: symmetric and anti-symmetric, and the last part vanishes by definition (3.14). So we have

$$\int_{\mathbf{R}^{n+1}} \operatorname{div}\tilde{H}\Phi dz = -\int_{\mathbf{R}^{n+1}} \tilde{H} \cdot \nabla\Phi dz - 2\int_{\mathbf{R}^n} (\tilde{H})^{n+1}(z', 0)\Phi(z', 0) dz', \tag{3.17}$$

but by construction (3.11) we have $(\tilde{H})^{(n+1)}|_{z_{n+1}=0} = 0$. Recalling that we consider only symmetric test function – so $\Phi_{,z_{n+1}} = 0$ – we get that $\overline{\text{div } (\bar{A}\nabla\bar{q} - \nabla u)} \in B^{1/p-2}_{p1}(\mathbf{R}^{n+1})$.

The Marcinkiewicz multipliers theorem [Tr, Chap.2.2] together with the definition of Besov spaces implies that any solution to (3.15) belongs to $B^{1/p}_{p1}$ and the symmetry implies that $w_{,z_{n+1}}|_{z_{n+1}=0} = 0$ in the distributional sense. Summing up all the estimates we infer that the solution to problem (3.7) obeys bound (3.8). Lemma 3.1 is proved.

Applying Lemma 3.1 and the form of the r.h.s. of (3.5) we conclude that following estimates for solutions to system (3.5) are valid

$$\begin{aligned} ||\zeta^k p||_{B^{1/p}_{p1}(\Omega)} \leq c(\lambda||\zeta^k p||_{B^{1/p}_{p1}(\Omega)} + \epsilon||p||_{B^{1/p}_{p1}(supp\,\zeta^k)} + c(\epsilon)||p||_{L_2(supp\,\zeta^k)} \\ +||A||_{W^1_p(supp\,\zeta^k)}||q||_{B^{1/p}_{p1}(supp\,\zeta^k)} + ||A|_{\partial\Omega}||_{W^1_p(\partial\Omega\cap supp\,\zeta^k)}||\tfrac{\partial p}{\partial n}|_{\partial\Omega}||_{W^{-1}_p(\partial\Omega\cap supp\,\zeta^k)}). \end{aligned} \tag{3.18}$$

Taking suitable small $\lambda$ we remove the first term of the r.h.s. of (3.18).

To finish the proof of Theorem 3.1 we consider interior regularity. This case is simpler than the first one. The localization by $\zeta^k$ with $k \in \mathcal{M}$ (see section 2) leads to the following problem

$$\Delta(\zeta^k p) = \text{div} A\nabla(\zeta^k q) + S_2 \qquad \text{in } \mathbf{R}^{n+1}, \tag{3.19}$$

where term $S_2$ follows from the localization procedure and its regularity of it is higher than the regularity of the first term of the r.h.s. of (3.19), but with the same structure as $S_1$. Applying the same methods as for (3.18) we obtain suitable estimates for $\zeta^k p$. So finally summing over $k \in \mathcal{M} \cup \mathcal{N}$ we conclude

$$\begin{aligned} ||p||_{B^{1/p}_{p1}(\Omega)} \leq \epsilon||p||_{B^{1/p}_{p1}(\Omega)} + c(\epsilon)||A||_{W^{1+1/p}_p(\Omega)}||q||_{B^{1/p}_{p1}(\Omega)} \\ +||A|_{\partial\Omega}||_{W^1_p(\partial\Omega)}||\tfrac{\partial p}{\partial \vec{n}}|_{\partial\Omega}||_{W^{-1}_p(\partial\Omega)}. \end{aligned} \tag{3.20}$$

From (3.18) taking the sum over all possible indeces and (3.20) we obtain Theorem 3.1.

As a corollary to Theorem 3.1 we prove the following result for the Dirichlet problem:

**Theorem 3.2.** *Let $g \in B^{1/p}_{p1}(\Omega)$, then the solution to the following problem:*

$$\begin{array}{lll} \Delta p = 0 & in & \Omega, \\ p = g & on & \partial\Omega \end{array} \tag{3.21}$$

*exists uniquely and belongs to $B^{1/p}_{p1}(\Omega)$, moreover $\frac{\partial p}{\partial \vec{n}}|_{\partial\Omega} \in W^{-1}_p(\partial\Omega)$. Additionally the following estimate holds*

$$||p||_{B^{1/p}_{p1}(\Omega)} + ||\frac{\partial p}{\partial \vec{n}}|_{\partial\Omega}||_{W^{-1}_p(\partial\Omega)} \leq c||g||_{L_p(\partial\Omega)}. \tag{3.22}$$

**Proof.** First, we show existence of a distributional solution in $L_2(\Omega)$. Elementary theory guarantees us existence of the solution which satisfies the following integral identity:

$$\int_\Omega p\Delta\phi dx = \int_{\partial\Omega} g\frac{\partial\phi}{\partial\vec{n}}d\sigma \qquad \text{for each } \phi \in W^2_2(\Omega) \cap \{\phi|_{\partial\Omega} = 0\}. \tag{3.23}$$

Next, we increase the regularity of this solutions the same as in the proof of Theorem 3.1. For each $k \in \mathcal{N}$ we consider the following problem

$$\begin{aligned} &\Delta Z_k^*[\zeta^k p] = (\Delta - \Delta_x) Z_k^*[\zeta^k p] + Z_k^*[2\nabla\zeta^k \nabla p + \Delta\zeta p] && \text{in} \quad \mathbf{R}_+^{n+1}, \\ &Z_k^*[\zeta^k p] = Z_k^*[\zeta^k g] && \text{on} \quad \mathbf{R}^n. \end{aligned} \tag{3.24}$$

The properties of the transformation imply that the r.h.s. of (3.24) has the following form

$$RHS(3.24)_1 = \text{div } A_0(\lambda)\nabla Z_k^*[\zeta^k p] + \text{div } (A_1 p) + A_2 p, \tag{3.25}$$

where $(A_0(\lambda))^{(n+1)} \equiv 0$, $(A_1)^{(n+1)} = 0$ on $\{z_{n+1} = 0\}$ moreover $A_1, A_2, A_3$ are smooth and $|A_0(\lambda)| \leq c\lambda$. To simplify system (3.24) we consider two problems:

$$\begin{aligned} &\Delta p_1 = \text{div } A_0(\lambda)\nabla Z_k^*[\zeta^k p] + \text{div } (A_1 p), \quad \Delta p_2 = A_2 p && \text{in} \quad \mathbf{R}_+^{n+1}, \\ &p_{1,z_{n+1}}|_{z_{n+1}=0} = 0, \qquad\qquad p_2|_{z_{n+1}=0} = 0 && \text{on} \quad \mathbf{R}^n. \end{aligned} \tag{3.26}$$

By Lemma 3.1 and classical results we conclude that

$$||p_1||_{B_{p1}^{1/p}(\Omega)} \leq c\lambda||\zeta^k p||_{B_{p1}^{1/p}(\Omega)} + c||p||_{L_p(\Omega)} \quad \text{and} \quad ||p_2||_{W_p^2(\Omega)} \leq c||p||_{L_p(\Omega)}. \tag{3.27}$$

Considering solutions to (3.21) in the following form $Z_k^*[\zeta^k p] = p_0 + p_1 + p_2$, we reduce our consideration to the following problem

$$\begin{aligned} &\Delta p_0 = 0 && \text{in} \quad \mathbf{R}_+^{n+1}, \\ &p_0 = Z[\zeta^k g] - p_1 - p_2 && \text{on} \quad \mathbf{R}^n. \end{aligned} \tag{3.28}$$

By (3.27) w get

$$||Z_k^*[\zeta^k g] - p_1 - p_2||_{L_p(\mathbf{R}^n)} \leq c\lambda||Z_k^*[\zeta^k p]||_{B_{p1}^{1/p}(\mathbf{R}_+^{n+1})} + ||p||_{L_p(\Omega)} + ||g||_{L_p(\partial\Omega)}. \tag{3.29}$$

Applying methods from the proof of Lemma 3.1 one can show that

$$p_{0,z_{n+1}}|_{z_{n+1}=0} \in W_p^{-1}(\mathbf{R}^n) \quad \text{and} \quad p_0 \in B_{p1}^{1/p}(\mathbf{R}_+^{n+1}). \tag{3.30}$$

The same information we get for interior regularity (localized by $\zeta^k$ with $k \in \mathcal{M}$). Summing over $k \in \mathcal{M} \cup \mathcal{N}$, taking suitable small $\lambda$ we obtain (3.22). Theorem 3.2 is proved.

# 4 Half space

In this section we prove Theorem 2.1 in the halfspace. In this case problem (2.12) reads

$$\begin{aligned} &\Delta p = \text{div } A\nabla q + S && \text{in} \quad \mathbf{R}_+^{n+1} \times (0,\infty), \\ &p = a\Delta'\phi + g && \text{on} \quad \mathbf{R}^n \times (0,\infty), \\ &p_{,z_{n+1}} = -\partial_t\phi + h && \text{on} \quad \mathbf{R}^n \times (0,\infty), \\ &\phi|_{t=0} = \phi_0 && \text{on} \quad \mathbf{R}^n, \\ &p, \phi \to 0 && \text{as} \quad |x| \to \infty. \end{aligned} \tag{4.1}$$

The main result of this section is the following.

**Theorem 4.1.** *Let* $q \in L_p(0,\infty; B^{1/p}_{p1}(\mathbf{R}^{n+1}_+))$, $q_{,z_{n+1}}|_{z_{n+1}=0} \in L_p(0,\infty; W^{-1}_p(\mathbf{R}^n))$, $A \in L_\infty(0,\infty; W^1_p(\mathbf{R}^{n+1}_+))$, $A|_{z_{n+1}=0} \in L_\infty(0,\infty; W^1_p(\mathbf{R}^n))$, $S \in L_\infty(0,\infty; B^{1/p-1}_{p1}(\mathbf{R}^{n+1}_+))$, $g \in L_p(\mathbf{R}^n \times (0,\infty))$, $h \in W^{-1,-1/3}_p(\mathbf{R}^n \times (0,\infty))$ *and* $\phi_0 \in W^{2-3/p}_p(\mathbf{R}^n)$*; then there exists a unique solutions to (4.1) such that* $\phi \in W^{2,2/3}_p(\mathbf{R}^n \times (0,\infty))$, $p \in L_p(0,\infty; B^{1/p}_{p1}(\mathbf{R}^{n+1}_+))$ *and* $p_{,z_{n+1}}|_{z_{n+1}=0} \in L_p(0,\infty; W^{-1}_p(\mathbf{R}^n))$*, moreover the following estimate holds*

$$\begin{aligned}
&< \phi >_{W^{2,2/3}_p(\mathbf{R}^n\times(0,\infty))} + < p >_{L_p(0,\infty;B^{1/p}_{p1}(\mathbf{R}^{n+1}_+))} + ||p_{,z_{n+1}}|_{z_{n+1}=0}||_{L_p(0,\infty;W^{-1}_p(\mathbf{R}^n))}\\
&\quad\le c(||A||_{L_\infty(0,\infty;W^1_p(\mathbf{R}^{n+1}_+))} + ||A|_{z_{n+1}=0}||_{L_\infty(0,\infty;W^1_p(\mathbf{R}^n))})\cdot\\
&\quad\cdot(||q||_{L_p(0,\infty;B^{1/p}_{p1}(\mathbf{R}^{n+1}_+))} + ||q_{,z_{n+1}}|_{z_{n+1}=0}||_{L_p(0,\infty;W^{-1}_p(\mathbf{R}^n))})\\
&+c(||S||_{L_\infty(0,\infty;B^{1/p-1}_{p1}(\mathbf{R}^{n+1}_+))} + ||g||_{L_p(\mathbf{R}^n\times(0,\infty))} + ||h||_{W^{-1,-1/3}_p(\mathbf{R}^n\times(0,\infty))} + ||\phi_0||_{W^{2-3/p}_p(\mathbf{R}^n)}).
\end{aligned} \tag{4.2}$$

The considerations from the previous section and results from [Mu1] give us possibility to simplify system (4.1) by removing the inhomogeneity from the r.h.s. of $(4.1)_1$. Considering systems:

$$\begin{aligned}
&\Delta p_1 = \operatorname{div} A\nabla q; \qquad && \Delta p_2 = S \quad \text{in} \quad \mathbf{R}^{n+1}_+ \times \{t\},\\
&p_{1,z_{n+1}} = (A\nabla q)^{(n+1)}; && p_2 = 0 \quad \text{on} \quad \mathbf{R}^n \times \{t\},
\end{aligned} \tag{4.3}$$

we easily conclude that $p_1(\cdot,t), p_2(\cdot,t) \in B^{1/p}_{p1}(\mathbf{R}^{n+1}_+)$ and $p_{1,z_{n+1}}(\cdot,t)|_{z_{n+1}=0}$, $p_{2,z_{n+1}}(\cdot,t)|_{z_{n+1}=0} \in W^{-1}_p(\mathbf{R}^n)$. The regularity of $p_2$ is even higher ($p_2 \in B^{1+1/p}_{p1}$), however this information is not needed. Thus we look for solutions of (4.1) in the following form: $p_{old} = p_{new} + p_1 + p_2$. Then from (4.1) we get the reduced problem

$$\begin{array}{lll}
\Delta p = 0 & \text{in} & \mathbf{R}^{n+1}_+ \times (0,\infty),\\
p = a\Delta'\phi + g & \text{on} & \mathbf{R}^n \times (0,\infty),\\
p_{,z_{n+1}} = -\partial_t\phi + h & \text{on} & \mathbf{R}^n \times (0,\infty),\\
\phi|_{t=0} = \phi_0 & \text{on} & \mathbf{R}^n,\\
p,\phi \to 0 & \text{as} & |x| \to \infty,
\end{array} \tag{4.4}$$

where $g_{new} = g_{old} - p_1$ and $h_{new} = h_{old} - (p_1+p_2)_{,z_{n+1}}$ on $\mathbf{R}^n \times (0,\infty)$. Now, we are able to solve equation $(4.4)_1$. The structure of system (4.4) allows to apply the Fourier transform with respect to tangential directions. From $(4.4)_1$ and $(4.4)_5$ we obtain the form of the p-function

$$p(z, z_{n+1}, t) = \mathcal{F}^{-1}_z[e^{-|\xi|z_{n+1}}\mathcal{F}_z[P]] \tag{4.5}$$

for a certain function $P : \mathbf{R}^n \times (0,\infty) \to \mathbf{R}$. Taking into account (4.5) boundary relations $(4.4)_{2,3}$ take the following form

$$P = a\Delta'\phi + g, \qquad \mathcal{F}^{-1}_z[-|\xi|\mathcal{F}_z[P]] = -\partial_t\phi + h \qquad \text{on} \quad \mathbf{R}^n \times (0,\infty). \tag{4.6}$$

Removing function $P$ from (4.6) we obtain the following parabolic problem

$$\begin{array}{lll}
\partial_t\phi + \mathcal{F}_z[|\xi|^3\mathcal{F}_z[\phi]] = h + \mathcal{F}^{-1}_z[|\xi|\mathcal{F}_z[g]] & \text{in} & \mathbf{R}^n \times (0,\infty),\\
\phi|_{t=0} = \phi_0 & \text{on} & \mathbf{R}^n.
\end{array} \tag{4.7}$$

The above system describes the character of problem (1.1), it shows that we deal with a parabolic system of the third order. To solve (4.7) we recall the following result.

**Lemma 4.1.** *Let* $m \in W_p^{-1,-1/3}(\mathbf{R}^n \times (0,\infty))$ *and* $\phi \in W_p^{2-3/p}(\mathbf{R}^n)$*, then there exists a unique solution to problem (4.7) such that* $\phi \in W^{2,2/3}_{p(loc)}(\mathbf{R}^n \times (0,\infty))$ *and the following estimate is valid*

$$< \phi >_{W_p^{2,2/3}} \leq c(||m||_{W_p^{-1,-1/3}(\mathbf{R}^n\times(0,\infty))} + < \phi_0 >_{W_p^{2-3/p}(\mathbf{R}^n)}). \tag{4.8}$$

We skip the proof of this lemma. It follows from the abstract theory presented in [GiSo,Th.2.3]. However it can be proved directly using methods from [Mu1,L.5.1].

To finish the proof of Theorem 4.1 we note that by Lemma 4.1 and (4.6) we see that $P \in L_p(\mathbf{R} \times (0,\infty))$ which follows that $p \in L_p(0,\infty; B^{1/p}_{p1}(\mathbf{R}^{n+1}_+))$, moreover by (4.6) $p_{,z_{n+1}}|_{z_{n+1}=0} \in L_p(0,\infty; W_p^{-1}(\mathbf{R}^n))$. Theorem 4.1 is proved.

# 5 The problem in domain $D$

This section proves Theorem 2.1. To simplify our considerations we remove inhomogeneity from the r.h.s. of $(2.12)_{1,2}$. We consider two systems:

$$\begin{array}{llll} \Delta p_0 = \mathrm{div} A\nabla q, & \Delta p_1 = 0 & \text{in} & D \times \{t\}, \\ \frac{\partial p_0}{\partial \vec{n}} = \vec{n} \cdot A\nabla q, & p_1 = G - p_0|_{\partial D} & \text{on} & \partial D \times \{t\}. \end{array} \tag{5.1}$$

By assumptions of Theorem 3.1 function $p_0 \in L_p(0,T; B^{1/p}_{p1}(D))$, but by the imbedding theorem $p_0|_{\partial D} \in L_p(\partial D \times (0,T))$. Applying Theorem 3.2 to the second system we get $p_1 \in L_p(0,T; B^{1/p}_{p1}(D))$ and $\frac{\partial p_0}{\partial \vec{n}} \in L_p(0,T; W_p^{-1}(\partial D)) \subset W_p^{-1,-1/3}(\partial D \times (0,T))$. Then we consider the $p$ - solution to problem (2.12) in the following form

$$p_{old} = p_{new} + p_0 + p_1. \tag{5.2}$$

So the system turns into the following one

$$\begin{array}{lll} \Delta p = 0 & \text{in} & D \times (0,T), \\ p = a\Lambda\phi & \text{on} & \partial D \times (0,T), \\ \frac{\partial p}{\partial \vec{n}} = -\partial_t \phi + H & \text{on} & \partial D \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & \partial D, \end{array} \tag{5.3}$$

where $H_{new} = H_{old} - \frac{\partial p_0}{\partial \vec{n}} - \frac{\partial p_1}{\partial \vec{n}}$ with suitable estimates.

The above system can be viewed as a local version of a nonlocal parabolic equation of third order on $\partial D \times (0,T)$ stated symbolically as follows

$$\begin{array}{lll} L\phi = H & \text{in} & \partial D \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & \partial D. \end{array} \tag{5.4}$$

**Theorem 5.1.** *Let* $\phi_0 \in W_p^{2-3/p}(\partial D)$ *and* $H \in W_p^{-1,-1/3}(\partial D \times (0,T))$*, then there exists a unique solution to problem (5.3) (or (5.4)) such that* $\phi \in W_p^{2,2/3}(\partial D \times (0,T))$ *and*

$$||\phi||_{W_p^{2,2/3}(\partial D\times(0,T))} \leq C(T)(||H||_{W_p^{-1,-1/3}(\partial D\times(0,T))} + ||\phi_0||_{W_p^{2-3/p}(\partial D)}). \tag{5.5}$$

**Corollary 5.1.** *Let all assumptions of Theorem 5.1 be fulfilled and additionally* $H \in L_p(0,T;W_p^{-1}(\partial D))$, *then* $\partial_t\phi \in L_p(0,T;W_p^{-1}(\partial D))$ *and*

$$||\partial_t\phi||_{L_p(0,T;W_p^{-1}(\partial D))} \leq C(T)(||H||_{L_p(0,T;W_p^{-1}(\partial D))} + ||\phi_0||_{W_p^{2-3/p}(\partial D)}).$$

To prove Theorem 5.1 we introduce the regularizer (see – [LSU]).

*Let us introduce* $\phi^k = R^k(\zeta^k H, \zeta^k\phi_0)$, *where* $\phi^k = Z_k^{-1*}[\bar\phi^k]$ *and* $\bar\phi^k$ *is the solutions to the following problem*

$$\begin{array}{lll} \Delta\bar p = 0 & in & \mathbf{R}_+^{n+1}\times(0,T), \\ \bar p = a\Delta'\bar\phi^k & on & \mathbf{R}^n\times(0,T), \\ \bar p_{,z_{n+1}} + \partial_t\bar\phi^k = Z_k^*[\zeta^k H] & on & \mathbf{R}^n\times(0,T), \\ \bar\phi^k|_{t=0} = Z_k^*[\zeta^k\phi_0] & on & \mathbf{R}^n. \end{array} \tag{5.6}$$

*Then an operator* $R: W_p^{-1,-1/3}(\partial D\times(0,T))\times W_p^{2-3/p}(\partial D)\to W_p^{2,2/3}(\partial D\times(0,T))$ *given by the following formula*

$$R(H,\phi_0) = \sum_k \pi^k\phi^k \quad \textit{we call a } \underline{\textit{regularizer}}. \tag{5.8}$$

The properties of operator $R$ will be shown only for short time, and they will be connected with the following quantity

$$\beta = \epsilon + \lambda + c(\epsilon)T^a \tag{5.8}$$

which will be required to be sufficiently small (constant will be chosen such that: $\lambda$ is given by (2.9), $c(\epsilon)\to\infty$ as $\epsilon\to 0$ and $a>0$). To show Theorem 5.1, two lemmas are required.

**Lemma 5.1.** *We have* $LRH = H + \mathcal{T}H$ *and* $||\mathcal{T}|| \leq c\beta$.

**Proof.** First, let us note that

$$LRH = \sum_k (L(\pi^k\phi^k) - \pi^k L\phi^k) + \sum_k \pi^k L\phi^k. \tag{5.9}$$

The definition of operator $L$ gives the following relation $L\phi^k = \partial_t\phi^k + \frac{\partial p}{\partial\vec n}$, where $p$ satisfies the following problem:

$$\begin{array}{lll} \Delta p = 0 & \text{in} & D, \\ p = a\sum_{l\in\mathcal{N}} Z_l^*[\Delta' Z_l^{-1*}(\pi^l\zeta^l\phi^k)] & \text{on} & \partial D. \end{array} \tag{5.10}$$

The localization and transform into the halfspace lead to the following system for $l$ such that supp $\zeta^l\cap$ supp $\pi^k \neq \emptyset$

$$\begin{array}{lll} \Delta(\bar\pi^k\bar\pi^l\bar\zeta^l\bar p) = F_1^{kl} & \text{in} & \mathbf{R}_+^{n+1}\times(0,T), \\ \bar\pi^k\bar\pi^l\bar\zeta^l\bar p = a\Delta'(\bar\pi^k\bar\pi^l\bar\zeta^l\bar\phi^k) + G_1^{kl} & \text{on} & \mathbf{R}^n\times(0,T), \\ (\bar\pi^k\bar\pi^l\bar\zeta^l\bar p)_{,z_{n+1}} + \partial_t(\bar\pi^k\bar\pi^l\bar\zeta^l\phi^k) = \bar\pi^k(\Delta'\bar\pi^l\bar\zeta^l\bar\phi^k) & \text{on} & \mathbf{R}^n\times(0,T), \\ \pi^k\bar\pi^l\bar\zeta^l\phi^k|_{t=0} = \bar\pi^k\bar\pi^l\bar\zeta^l\zeta^k\bar\phi_0 & \text{on} & \mathbf{R}^n. \end{array} \tag{5.11}$$

But functions $\phi^k$ is given by (5.6), so the localization of (5.6) reads:

$$\begin{array}{lll} \Delta(\bar\pi^k\bar\pi^l\bar\zeta^l\bar q) = F_2^{kl} & \text{in} & \mathbf{R}^{n+1}_+\times(0,T),\\ \bar\pi^k\bar\pi^l\bar\zeta^l\bar q = a\Delta'(\bar\pi^l\bar\zeta^l\bar\phi^k)+G_2^{kl} & \text{on} & \mathbf{R}^n\times(0,T),\\ (\bar\pi^k\bar\pi^l\bar\zeta^l\bar q)_{,z_{n+1}} + \partial_t(\bar\pi^k\bar\pi^l\bar\zeta^l\bar\phi^k) = \bar\pi^k\bar\pi^l\bar\zeta^l Z_k^*[\zeta^k H] & \text{on} & \mathbf{R}^n\times(0,T),\\ \bar\pi^k\bar\pi^l\bar\zeta^l\bar\phi^k|_{t=0} = \bar\pi^k\bar\pi^l\bar\zeta^l Z_k^*[\zeta^k\phi_0] & \text{on} & \mathbf{R}^n. \end{array} \tag{5.12}$$

The comparison of (5.11) and (5.12) leads to the following conclusion

$$\sum_k \pi^k L\phi^k = H + \sum_k Z_k^{-1*}[\bar\pi^k(\bar q-\bar p)_{,z_{n+1}}], \tag{5.13}$$

where

$$\begin{array}{lll} \Delta[\bar\pi^k(\bar q-\bar p)] = F_2 - F_1 & \text{in} & \mathbf{R}^{n+1},\\ \bar\pi^k(\bar q-\bar p) = 0 & \text{on} & \mathbf{R}^n. \end{array} \tag{5.14}$$

Direct calculations show that

$$F_2^{kl}-F_1^{kl} \sim (\Delta-\Delta_x)(\bar\pi^k\bar\pi^l\bar\zeta^l\bar p) - Z_k^{-1*}[2\nabla(\pi^k\bar\pi^l\bar\zeta^l)\nabla p + p\Delta\pi^k]. \tag{5.15}$$

Then applying Theorem 3.1 we get (recall $L_pW_p^{-1}\subset W_p^{-1,-1/3}$)

$$||\bar\pi^k\bar\pi^l\bar\zeta^l(\bar q-\bar p)_{,z_{n+1}}||_{W_p^{-1,-1/3}(\mathbf{R}^n\times(0,T))} \le c\lambda||\pi^k p||_{B^{1/p}_{p1}(D)} + c||p||_{L_p(D\cap supp\,\pi^k)}. \tag{5.16}$$

We need to find a bound on the second term of the r.h.s. of (5.16). The easiest approach is to apply the interpolation inequality, i.e.:

$$||p||_{L_p(D)} \le \epsilon||p||_{B^{1/p}_{p1}(D)} + c(\epsilon)||p||_{L_2(D)} \tag{5.17}$$

and examine the last term in (5.17). To get a good estimate for solution to (5.3) we multiply $(5.1)_1$ by $r:\Delta r = p$ and $r=0$ on $\partial D$, then integrating over $D$ we conclude

$$\int_D p^2dx \le \left|\int_{\partial D} a\Lambda\phi^k\frac{\partial r}{\partial\vec n}d\sigma\right|. \tag{5.18}$$

Since $r\in W_2^2(D)$, so $\frac{\partial r}{\partial n}\in W_2^{1/2}(\partial D)$, we deduce

$$RHS(5.18) \le a||\Lambda\phi^k||_{H^{-1/2}(\partial D)}||\frac{\partial r}{\partial\vec n}||_{H^{1/2}(\partial D)}. \tag{5.19}$$

However $\phi^k\in W_p^{2,2/3}(\partial D\times(0,T))$, so

$$||\Lambda\phi^k||_{W_2^{1/2}(\partial D)} \le c||\phi^k||_{W_2^{3/2}(\partial D)} \le c||\phi^k||_{W_p^{3/2}(\partial D)} \le \epsilon||\phi^k||_{W_p^2(\partial D)} + c(\epsilon)||\phi^k||_{L_p(\partial D)}, \tag{5.20}$$

but from the trace theorem we have

$$||\phi^k||_{L_\infty(0,T;L_p(\partial D))} \le c(<\phi^k>_{W_p^{2,2/3}(\partial D\times(0,T))} + ||\phi_0||_{W_p^{2-3/p}(\partial D)}), \tag{5.21}$$

where the constant in (5.21) is independent of $T$. Summing up we get

$$||\bar\pi^k(\bar q-\bar p)_{,z_{n+1}}||_{W_p^{-1,-1/3}(\partial D\times(0,T))} \le (\epsilon + c(\epsilon)T^a)(||H||_{W_p^{-1,-1/3}(\partial D\times(0,T))} + ||\phi_0||_{W_p^{2-3/p}(\partial D)}) \tag{5.22}$$

for a number $a > 0$. Similarly we conclude that

$$||\sum_k (L(\pi^k\phi^k) - \pi^k L\phi^k)||_{W_p^{-1,-1/3}(\partial D\times(0,T))} \le c\beta(||H||_{W_p^{-1,-1/3}(\partial D\times(0,T))} + ||\phi_0||_{W_p^{2-3/p}(\partial D)}). \tag{5.23}$$

Relations (5.13), (5.2) and (5.23) finish the proof of Lemma 5.1.

Using similar argumentation one can prove the second lemma.

**Lemma 5.2.** *We have $RL\phi = \phi + \mathcal{W}\phi$ and $||\mathcal{W}|| \le c\beta$.*

We skip the proof since it is based on the same analysis as for Lemma 5.1.

**Proof of Theorem 5.1.** Lemmas 5.1 and 5.2 guarantee that for sufficiently small $T$ operator $(1+\mathcal{T})^{-1}$ and $(1+\mathcal{W})^{-1}$ are well defined and their norms are bounded. Hence we have

$$LR(1+\mathcal{T})^{-1}H = H \quad \text{and} \quad (1+\mathcal{W})^{-1}RL\Phi = \Phi, \tag{5.24}$$

which implies that $L^{-1} = R(1+\mathcal{T})^{-1} = (1+\mathcal{W})^{-1}R = L^{-1}$. Boundedness of $L^{-1}$ follows from properties of operator $R$. Hence $\phi = L^{-1}(H,\phi_0)$ is the solution (unique) to problem (5.3), but only for short time $T$. However constants in lemmas are fixed (does not depend on data), so we can continue the solutions on any finite time interval step by step. As a result the constant in (5.5) depends on $T$. Theorem 5.1 is proved.

Theorem 2.1 is a consequence of Theorem 5.1 and considerations for system (5.1).

# 6 Proof of Theorem 1.1.

To prove the result for the nonlinear system we apply the Banach fixed point theorem to system (2.7). First, we remove inhomogenuity from the initial condition. Take the solution to the following problem

$$\begin{array}{lll} \Delta\bar{q} = 0 & \text{in} & D\times(0,T), \\ \bar{q} = a\Lambda\bar{\psi} & \text{on} & \partial D\times(0,T), \\ \frac{\partial\bar{q}}{\partial\vec{n}} = -\partial_t\bar{\psi} & \text{on} & \partial D\times(0,T), \\ \bar{\psi}|_{t=0} = \psi_0 & \text{on} & \partial D. \end{array} \tag{6.1}$$

Theorem 2.1 implies that $\bar{q} \in L_p(0,T;B^{1/p}_{p1}(D))$, $\frac{\partial\bar{q}}{\partial\vec{n}} \in L_p(0,T;W_p^{-1}(\partial D))$ and $\bar{\psi} \in W_p^{2,2/3}(\partial D\times(0,T))$. Properties of $L_p$-spaces allow to find $T_1$ so small that for a given $\delta > 0$ we have

$$||\bar{q}||_{L_p(0,T_1;B^{1/p}_{p1}(D))} + ||\frac{\partial\bar{q}}{\partial\vec{n}}||_{L_p(0,T_1;W_p^{-1}(\partial D))} + <\bar{\psi}>_{W_p^{2,2/3}(\partial D\times(0,T_1))} \le \delta_1. \tag{6.2}$$

Therefore in the next considerations we will investigate only $T < T_1$.

The sought functions will be searched in the following form

$$q = u + \bar{q} \quad \text{and} \quad \psi = f + \bar{\psi}. \tag{6.3}$$

Then system (2.7) reads

$$\begin{array}{lll} \Delta u = F_1 & \text{in} & D\times(0,T), \\ u = a\Lambda f + G_1 & \text{on} & \partial D\times(0,T), \\ \frac{\partial u}{\partial\vec{n}} = -\partial_t f + H_1 & \text{on} & \partial D\times(0,T), \\ f|_{t=0} = 0 & \text{on} & \partial D, \end{array} \tag{6.4}$$

where

$$F_1 = \text{div }(Id - A_t)u + \text{div }(Id - A_t)\bar{q}, \quad G_1 = (\Delta_{\partial\Omega_t} - \Lambda)f + (\Delta_{\partial\Omega_t} - \Lambda)\bar{\psi},$$
$$H_1 = \vec{n}(\vec{n}_t - \vec{n})\partial_t f + \vec{n}(\vec{n}_t - \vec{n})\partial_t\bar{\psi} + (\vec{n}_t - \vec{n})\nabla u + (\vec{n}_t - \vec{n})\nabla\bar{q}. \tag{6.5}$$

The solution to problem (6.4) will be found as a fixed point to the following map

$$\mathcal{M}(\bar{u}, \bar{f}) = (u, f), \quad \text{where} \quad (\bar{u}, \bar{f}) \in \Xi \tag{6.6}$$

and set $\Xi$ is defined as follows

$$\Xi = \{(u, f) \in L_p(0, T; B^{1/p}_{p1}(D) \times W^{2,2/3}_p(\partial D \times (0, T)) :$$
$$\text{additionally } \; f|_{t=0} = 0, \qquad \partial_t f, \tfrac{\partial u}{\partial \vec{n}}|_{\partial D} \in L_p(0, T; W^{-1}_p(\partial D)) \text{ and}$$
$$||(u, f)||_\Xi = ||u||_{L_p(0,T;B^{1/p}_{p1}(D))} + ||\partial_t f, \tfrac{\partial u}{\partial \vec{n}}|_{\partial D}||_{L_p(0,T;W^{-1}_p(\partial D))} + < f >_{W^{2,2/3}_p(\partial D\times(0,T))} < \delta\}. \tag{6.7}$$

Pair $(u, f)$ from (6.6) is the solution to the following problem

$$\begin{array}{lll} \Delta u = \overline{F_1} & \text{in} & D \times (0, T), \\ u = a\Lambda f + \overline{G_1} & \text{on} & \partial D \times (0, T), \\ \frac{\partial u}{\partial \vec{n}} = -\partial_t f + \overline{H_1} & \text{on} & \partial D \times (0, T), \\ f|_{t=0} = 0 & \text{on} & \partial D, \end{array} \tag{6.8}$$

where

$$\overline{F_1} = \text{div }(Id - \overline{A_t})\bar{u} + \text{div }(Id - \overline{A_t})\bar{q}, \quad \overline{G_1} = (\Delta_{\overline{\partial\Omega_t}} - \Lambda)\bar{f} + (\Delta_{\overline{\partial\Omega_t}} - \Lambda)\bar{\psi},$$
$$\overline{H_1} = \vec{n}(\overline{n_t} - \vec{n})\partial_t\bar{f} + \vec{n}(\overline{n_t} - \vec{n})\partial_t\bar{\psi} + (\overline{n_t} - \vec{n})\nabla\bar{q} + (\overline{n_t} - \vec{n})\nabla\bar{u}. \tag{6.9}$$

First we want to find so small $\delta$ to prove $\mathcal{M}(\Xi) \subset \Xi$. To find bounds on solutions to (6.8) we apply Theorem 2.1 and estimate the r.h.s. of (6.8) as follows: the form of $\overline{F_1}$, together with properties of $A_t$ and the imbedding theorem implies that

$$||\overline{F_1}||_{L_p(0,T;B^{1/p-2}_{p1}(D))} \le c||\bar{\psi} + \bar{f}||_{W^{2,2/3}_p(\partial D\times(0,T))}(||\bar{u}||_{L_p(0,T;B^{1/p}_{p1}(D))} + ||\bar{q}||_{L_p(0,T;B^{1/p}_{p1}(D))}). \tag{6.10}$$

Locally on each $\Omega^k \cap \partial D$ for $k \in \mathcal{N}$ the following pointwise bound is valid

$$|(\bar{f} + \bar{\psi})| + |\nabla(\bar{f} + \bar{\psi})| \le \epsilon, \tag{6.11}$$

where $\epsilon$ is taken from (2.6), so we can write

$$||(\Delta_{\overline{\partial\Omega_t}} - \Lambda)\bar{f}||_{L_p(\partial D\times(0,T))} \le c(||\nabla(\bar{f} + \bar{\psi})\nabla^2\bar{f}||_{L_p(\partial D\times(0,T))} + ||\nabla^2(\bar{f} + \bar{\psi})\nabla\bar{f}||_{L_p(\partial D\times(0,T))})$$
$$\le \epsilon||\nabla^2\bar{f}||_{L_p(\partial D\times(0,T))} + (\delta_1 + ||\nabla^2\bar{f}||_{L_p(\partial D\times(0,T))})||\nabla\bar{f}||_{L_p(\partial D\times(0,T))} \tag{6.12}$$

and the same we deduce

$$||(\Delta_{\overline{\partial\Omega_t}} - \Lambda)\bar{\psi}||_{L_p(\partial D\times(0,T))} \le \epsilon||\nabla^2\bar{f}||_{L_p(\partial D\times(0,T))} + (\delta_1 + ||\nabla^2\bar{f}||_{L_p(\partial D\times(0,T))})||\nabla\bar{f}||_{L_p(\partial D\times(0,T))}. \tag{6.13}$$

However to estimate $||\nabla\bar{\psi}||_{L_\infty}$ we apply (6.11), so finally we get

$$||\overline{G_1}||_{L_p(\partial D\times(0,T))} \leq (\epsilon + \delta_1 + ||\bar{f}||_{W_p^{2,2/3}(\partial D\times(0,T))})||\bar{f}||_{W_p^{2,2/3}(\partial D\times(0,T))}. \tag{6.14}$$

The last one is $\overline{H_1}$; we consider one term since the rest is similar

$$||\vec{n}(\overline{n_t}-\vec{n})\partial_t\bar{f}||_{L_p(0,T;W_p^{-1}(\partial D))} \leq \sup\left|\int_{\partial D\times(0,T)}\vec{n}(\overline{n_t}-\vec{n})\partial_t\bar{f}\pi dxdt\right|, \tag{6.15}$$

where sup is taken over all $\pi\in W_q^{1,0}(\partial D\times(0,T))$ such that $||\pi||=1$ and $q=p/(p-1)$. Hence it is enough to show that $\vec{n}(\overline{n_t}-\vec{n})\pi\in W_q^{1,0}$, but for $p>n$ we have

$$||\vec{n}(\overline{n_t}-\vec{n})\pi||_{L_p(0,T;W_p^1(\partial D))} \leq c(\epsilon + ||\bar{f}||_{W_p^{2,2/3}(\partial D\times(0,T))}),$$

hence

$$||\vec{n}(\overline{n_t}-\vec{n})\partial_t\bar{f}||_{L_p(0,T;W_p^{-1}(\partial D)} \leq c(\epsilon+||\bar{f}||_{W_p^{2,2/3}(\partial D\times(0,T))})||\partial_t\bar{f}||_{L_p(0,T;W_p^{-1}(\partial D))}. \tag{6.16}$$

The rest of terms from $\overline{H_1}$ can be estimated similarly, so we conclude – see (6.2)

$$||\overline{H_1}||_{L_p(0,T;W_p^{-1}(\partial D))} \leq c(\epsilon+\delta_1+(||(\bar{u},\bar{f})||_\Xi)||(\bar{u},\bar{f})||_\Xi. \tag{6.17}$$

Summing over above inequalities (6.10), (6.14) and (6.17), by Theorem 2.1 we get the estimate for the solution to (6.8) as follows

$$||(u,f)||_\Xi \leq c[(\epsilon+\delta_1)||(\bar{u},\bar{f})||_\Xi + ||(\bar{u},\bar{f})||_\Xi^2]. \tag{6.18}$$

So taking suitable $\delta>0$ in definition (6.7) we obtain $\mathcal{M}(\Xi)\subset\Xi$.

Next, having specified $\delta$ in definition (6.7) we show the map $\mathcal{M}:\Xi\to\Xi$ is the contraction. For this purpose it is enough to prove that

$$||\mathcal{M}(u_1,f_1)-\mathcal{M}(u_2,f_2)||_\Xi \leq \frac{1}{2}||(u_1-u_2,f_1-f_2)||_\Xi. \tag{6.19}$$

To prove (6.19) it is enough to repeat all considerations as in the proof of (6.18), with possible less $\delta$ – sufficiently small – and the application of all tricks as for $\overline{F_1}$, $\overline{G_1}$ and $\overline{H_1}$. Hence we deduce that map $\mathcal{M}$ is the contraction on set $\Xi$, so the unique fixed point $(u^*,f^*)\in\Xi$ exists and describes the sought solution to the original problem in the meaning of definition (1.4). It is a consequence of definition (2.19) and properties of transform (2.1). The obtained time is relatively short. We can repeat this procedure to prolong the solution, but the maximal existence time – in general – will be finite. Theorem 1.1 is proved.

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