On lifespan of solutions to the Einstein equations

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Abstract. We investigate the issue of existence of maximal solutions to the vacuum Einstein solutions for asymptotically flat spacetimes. Solutions are established globally in time outside a domain of influence of suitable large compact set, where can appear singularities. Our approach enables to show that metric coefficients obey the following behavior $g_{\alpha\beta} = \eta_{\alpha\beta} + O(r^{-1/2})$ at infinity (where $\eta_{\alpha\beta}$ is the Minkowski metric). The system is studied in the harmonic (wavelike) gauge.

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1 Introduction

The analysis of the issue of existence of solutions to the Einstein equations is the subject of this paper. We want to examine an area of spacetime where we are able to control a metric which describes a searched pseudo-Riemannian manifold. We consider

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \qquad \alpha, \beta = 0, 1, 2, 3 \tag{1.1}$$

with the signature $(-+++)$, where the summing convention is used. Points in the spacetime are denoted by $x = (x^0, x^1, x^2, x^3)$.

We investigate the Cauchy problem for an initial submanifold which is required to be asymptotically flat. The solutions will be searched by the following initial problem

$$\begin{array}{c} G_{\mu\nu} = T_{\mu\nu}, \\ g_{\mu\nu}|_{x^0=0} = g^0_{\mu\nu}, \\ \frac{\partial g_{\mu\nu}}{\partial x^0}|_{x^0=0} = g^1_{\mu\nu}, \end{array} \tag{1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ - the energy-momentum tensor describing influence of external forces. As examples of tensor $T_{\mu\nu}$ we can consider models of a collisionless gas given by the Vlasov equation [1,3,11,14], or the relativistic Maxwell system taking into account influence of the electromagnetic field [2], or others [12,13].

Conditions $(1.2)_3$ can be replaced by assumptions on the curvature tensor of the initial submanifold and the relations between them can be described via the Gauss-Codazzi equations [5].

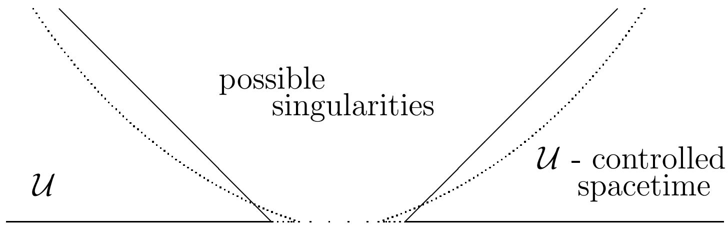
The geometric structure of equations implies the Bianchi identity

$$G^{\mu\nu}_{;\nu} = 0, \qquad T^{\mu\nu}_{;\nu} = 0, \tag{1.3}$$

where ; denotes the covariant differentiation. It follows that we may look at system (1.2) of ten equations, but under constraints (1.3). And it is related to the fact that the same geometry can be described by different metrics.

From the analytical point of view the geometrical invariance of the system causes serious difficulties. It is not obvious which type of coordinates are the best (or the most suitable) to investigate the issue of existence. In the only result [5] about the global in time existence and stability of solutions for the vacuum Einstein equations, the authors consider the so-called traceless coordinates. This form of the metric leads to a good structure of nonlinearity terms which is related to the null condition property from the theory of the nonlinear wave equations [4,9]. This approach enables to control solutions for all times under suitable smallness of initial data.

In our paper we want to consider a more general question about the existence of solutions. We search for maximal solutions to system (1.2) for a suitable large class of initial data, what is related to analysis of the system outside of the cone of influence of possible singularities - see Pict. 1. As an answer we will obtain information about the asymptotic behavior of metric coefficients and about a domain $\mathcal{U}$, where the solution to (1.2) will be well defined.



Picture 1.

The initial data are defined on an initial submanifold Σ_0 as follows

$$\begin{aligned} g_{\mu\nu}|_{x^0=0} &= g^0_{\mu\nu} \qquad \text{on} \quad \Sigma_0, \\ \tfrac{\partial g_{\mu\nu}}{\partial x^0}|_{x^0=0} &= g^1_{\mu\nu} \qquad \text{on} \quad \Sigma_0, \end{aligned} \tag{1.4}$$

where

$$\Sigma_0 = \mathbf{R}^3 \setminus B(0,R). \tag{1.5}$$

Requirements of the asymptotic flatness can be described by the following relations

$$g^0_{\mu\nu} - \eta_{\mu\nu} \to 0 \qquad \text{as} \quad r \to \infty, \tag{1.6}$$

where $\eta_{\mu\nu}$ is the Minkowski metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1.7}$$

and $r = |\bar{x}| = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$.

Looking for some reasonable types of behavior of solutions at infinity we consider an important example, it is the Schwarzschild metric given by the following formula (in the spherical coordinates)

$$ds^2 = -(1 - 2M/r)(dx^0)^2 + (1 - 2M/r)^{-1}(dr)^2 + r^2(d\Omega)^2. \tag{1.8}$$

The above example describes a universe where all mass is localized, and influence of it implies the following asymptotic behavior of metric coefficients

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1}). \tag{1.9}$$

From that reason we should not take into account faster vanishing at infinity. More rigoristic restrictions would have no good physical interpretation.

To concentrate our investigation on dependence from initial data we assume that energy-momentum tensor $T_{\mu\nu}$ is also localized in the spacetime (i.e. we require to the support of $T_{\mu\nu}$ be in the cone of influence of possible singularities - see Pict.1.), hence we reduce problem (1.2) to the case

$$T_{\mu\nu} \equiv 0. \tag{1.10}$$

We want to consider initial data which fulfill at most relation (1.9). However, we are interested not only in globally regular data, we can admit singularities, but located only in ball $B(0, R)$, on submanifold Σ_0 metric coefficients $g^0_{\mu\nu}$, $g^1_{\mu\nu}$ are required to be sufficiently regular. Asymptotic structure allows to examine our issue with no restrictions on largeness of initial data.

To reach our aims we will examine system (1.1) in the harmonic (wavelike) coordinates. It follows that the Einstein equations are of hyperbolic type, more precisely, we obtain a set of nonlinear wave equations on metic $g_{\alpha\beta}$ [2,3,6]. As we know this gauge is well defined, however in global analysis we have some difficulties with stability of solutions. Nevertheless, to answer on our question this approach will simplify the structure of the equations, and enables to concentrate the attention on the analytical difficulties.

The presented technique will give the metric with the following asymptotic behavior

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1/2}) \quad \text{in} \quad \mathcal{U}, \tag{1.11}$$

having initial data such that

$$\begin{aligned} g^0_{\mu\nu} &= \eta_{\mu\nu} + O(r^{-1/2-\delta}) \qquad \text{on} \quad \Sigma_0, \\ g^1_{\mu\nu} &= O(r^{-3/2-\delta}) \qquad \text{on} \quad \Sigma_0 \end{aligned} \tag{1.12}$$

for $\delta > 0$ with suitable regularity which will be stated precisely in the next section of the paper. We will improve conditions (1.12), however to define them we need to introduce some notations (see Theorem 2.1). Additionally we assume that (1.12) generates the metic in the harmonic gauge, (so we will require to conditions (2.13) be fulfilled).

The main result of our paper is the following.

Theorem 1.1. *Let $g^0_{\mu\nu}, g^1_{\mu\nu}$ be defined on the initial submanifold Σ_0 and be sufficiently smooth, moreover let*

$$g^0_{\mu\nu}|_{\Sigma_0}, \; g^1_{\mu\nu}|_{\Sigma_0} \; \textit{satisfy condition (1.12)}. \tag{1.13}$$

Then there exists the maximal solution defined on domain $\mathcal{U}$ *- see Pict. 1 such that metric coefficients* $g_{\mu\nu}$ *fulfill (1.11) and set* $\mathcal{U}$ *satisfies the following inclusion*

$$\mathcal{U} \supset \left((0,\infty) \times \mathbf{R}^3\right) \setminus \{x : r \leq (1+\epsilon)x^0 + M\} \tag{1.14}$$

for any $\epsilon > 0$*, if* M *is sufficiently large.*

The result of Theorem 1.1 characterizes the spacetime generated by the initial data satisfying asymptotic behavior (1.12). The method enables to prove inclusion (1.14), which says that outside of a domain of influence of "large data" our spacetime is, by relation (1.11), a perturbation of the Minkowski manifold. The magnitude of M and conditions (1.12) will reduce any large initial data to the case of small solutions.

A similar result has been stated in [10], where authors got a better characterization of set $\mathcal{U}$, however assuming stronger restriction of asymptotic behavior, i.e. there was considered a perturbation of the Schwarzschild metric, hence the solution had a structure described by (1.19). In the presented paper we require only conditions (1.12) to obtain solutions fulfilling (1.11).

Our technique has an analytical character and does not deal with geometrical aspects of the subject. We will concentrate our analysis on examination of a hyperbolic system by a reduction of the problem to a case of "small solutions".

Theorem 1.1 will be a conclusion of Theorem 2.1 stated in the next section. A key element of the proof will be an analysis of the behavior of the speed of propagation for system (1.2).

2 Analytical statement

We want to study a spacetime manifold generated by an initial submanifold which is required to be asymptotically flat. Since we assume that the support of $T_{\mu\nu}$ is localized (see (1.10)), we concentrate our attention on the vacuum equations, i.e. the Einstein system takes the following form

$$G_{\mu\nu} = 0. \tag{2.1}$$

Since the Einstein tensor is given by the following definition

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor and $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, assuming that the metric is not degenerated, we obtain that system (2.1) is equivalent to the following one

$$R_{\mu\nu} = 0. \tag{2.3}$$

Our spacetime is required to be a pseudo-Riemannian manifold with a metric $g_{\alpha\beta}$ with the signature $(- + ++)$. It follows that at least locally we have

$$-g_{00} > a, \quad b|X|^2 \leq g_{kl}X^k X^l \leq c|X|^2 \quad \text{for} \quad X \in \mathbf{R}^3, \tag{2.4}$$

where $a, b, c > 0$ and $k, l = 1, 2, 3$.

Let us recall, the Ricci tensor is given by the Christoffel symbols

$$R_{\alpha\beta} = \partial_\sigma \Gamma^\sigma_{\alpha\beta} - \partial_\alpha \Gamma^\sigma_{\beta\sigma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\rho\sigma} - \Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\beta\rho}, \tag{2.5}$$

where Christoffel symbols are given by the metric coefficients as follows

$$\Gamma^\gamma_{\beta\mu} = \frac{1}{2} g^{\alpha\nu}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}). \tag{2.6}$$

The general form of the Ricci tensor $R_{\alpha\beta}$ is complex and to see a hyperbolic character of equations (2.3) it is better to look on them in a special coordinate system. The easiest way is to take the harmonic coordinates, sometimes because of their properties they called the wavelike ones, too. To define them we need contracted Christoffel symbols

$$\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}. \tag{2.7}$$

And we say that the metic is harmonic if and only if

$$\Gamma^\mu = 0 \qquad \text{for} \;\; \mu = 0, 1, 2, 3. \tag{2.8}$$

Classical results guarantee that the extention of this type of the coordinates [2] exists as far as the metric exists.

A good side of the chosen setting is the form of the Ricci tensor. In the harmonic coordinates it reads

$$R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} g_{\mu\nu} + H_{\mu\nu}, \tag{2.9}$$

where $H_{\mu\nu} = H_{\mu\nu}(g_{\alpha\beta}, \frac{\partial g_{\alpha\beta}}{\partial x^\gamma})$ and term H_μ is a bilinear operator with respect to the first derivative of the metric coefficients, i.e. symbolically we have

$$H \sim \tilde{H}(g) \cdot Dg \cdot Dg, \tag{2.10}$$

where D denotes the whole gradient operator, i.e. $D = (\partial_0, \partial_1, \partial_2, \partial_3)$.

Hence by choosing the harmonic gauge the Einstein equations read in our case as follows

$$-\frac{1}{2} g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} g_{\mu\nu} + H_{\mu\nu} = 0. \tag{2.11}$$

To start the investigation we describe the initial data. At the beginning we require

$$\begin{array}{c} g_{\mu\nu}|_{x^0=0} = \eta_{\mu\nu} + o(r^{-1/2}), \\ \frac{\partial g_{\mu\nu}}{\partial x^0}|_{x^0=0} = o(r^{-3/2}), \end{array} \tag{2.12}$$

moreover due to our harmonic gauge we assume additionally the following initial conditions

$$\Gamma^\mu|_{x^0=0} = 0, \quad \frac{\partial \Gamma^\mu}{\partial x^0}|_{x^0=0} = 0 \tag{2.13}$$

which will guarantee the metric in the harmonic coordinates. In general, these conditions are required to be satisfied for

$$\bar{x} = (x^1, x^2, x^3) \in \mathbf{R}^3 \setminus B(0, R) \tag{2.14}$$

for sufficiently large R.

Our aim is to establish an existence result for a domain

$$\mathcal{M} = \big((0, +\infty) \times \mathbf{R}^3\big) \setminus S, \tag{2.15}$$

where

$$S = \{x \in (0, \infty) \times \mathbf{R}^3 : r \leq \kappa x^0 + M\}, \tag{2.16}$$

where M will be chosen letter, $r = |\bar{x}| = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$ and κ is a bound of the maximal speed of the propagation to system (2.11), i.e.

$$\kappa > s^* = \max\{\text{speed of propagation in } \mathcal{M} \text{ to (2.11)}\}. \tag{2.17}$$

This type of results it is possible to prove to hyperbolic systems, since we are able to control the speed of propagation of information. It follows that can to modify equations (2.11) in sector S and solutions will not change in domain $\mathcal{M}$.

Let us introduce the following auxiliary (cut off) function.

$$w(x) = \begin{cases} 1 & \text{for} \quad r \geq \kappa x^0 + M \\ 0 & \text{for} \quad r \leq \kappa_* x^0 + M_0 \\ \in [0,1] & \text{for} \quad \kappa_* x^0 + M_0 < r < \kappa x^0 + M \end{cases} \tag{2.18}$$

where κ is given by (2.17) and numbers κ_*, M_0 and M satisfies the below relations

$$1 < \kappa_* < \kappa \quad \text{and} \quad M_0 < M. \tag{2.19}$$

Moreover $w(\cdot)$ is sufficiently smooth and $|Dw| \leq c((\kappa - \kappa_*)x^0)^{-1}$. Also to simplify the examination we set $\kappa_* < 2$.

Applying function $w(\cdot)$ we modify searched functions $g_{\alpha\beta}$ as follows

$$d_{\alpha\beta} = \eta_{\alpha\beta} + w h_{\alpha\beta}, \tag{2.20}$$

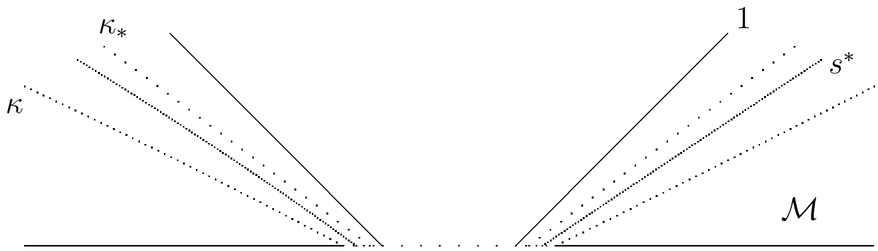
where $h_{\alpha\beta}$ are given as solutions to the following system being a modification of equations (2.11)-(2.12)

$$\begin{array}{lll} -\frac{1}{2} d^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} h_{\mu\nu} + w H_{\mu\nu}(d_{\alpha\beta}, D d_{\alpha\beta}) = 0 & \text{in} & \mathbf{R}^3 \times (0,T), \\ h_{\mu\nu}|_{x^0=0} = h^0_{\mu\nu} = w g^0_{\mu\nu} & \text{on} & \mathbf{R}^3, \\ h_{\mu\nu,0}|_{x^0=0} = h^1_{\mu\nu} = w g^1_{\mu\nu} & \text{on} & \mathbf{R}^3. \end{array} \tag{2.21}$$

The analysis of the above system will give information for problem (2.11)-(2.12) in set $\mathcal{M}$. A key idea is that smallness of solutions $h_{\mu\nu}$ will imply that we are able to control the maximal speed of propagation in set $\mathcal{M}$. It is possible, since choosing suitable large M_0 we obtain smallness of data $h^0_{\mu\nu}$ and $h^1_{\mu\nu}$, although (2.12) can be arbitrary large. As a consequence of these considerations we get the following relations between solutions to systems (2.11)-(2.12) and (2.21)

$$\begin{array}{ll} h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} & \text{in} \quad \mathcal{M}, \\ d_{\alpha\beta} = g_{\alpha\beta} & \text{in} \quad \mathcal{M}, \end{array} \tag{2.22}$$

where $g_{\mu\nu}$ is the solution to problem (2.11)-(2.12) in the harmonic gauge - see Pict. 2..



Picture 2.

To begin the statement of our results we introduce some notations and relations to precise our mathematical background.

First, let us define vector fields related to the Minkowski spacetime. We introduce

$$T_\mu = \partial_\mu \qquad T = \{T_\mu \,:\, \mu = 0,1,2,3\}, \tag{2.23}$$

$$L_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \qquad L = \{L_{\mu\nu} \,:\, 0 \le \mu < \nu \le 3\}, \tag{2.24}$$

$$S = x^\mu \partial_\mu. \tag{2.25}$$

The whole set of the above vectors fields are denoted by A and

$$A = (T, L, S) = \{\Gamma_a, a \in I\}, \tag{2.26}$$

where I is a finite set of appropriate indeces.

Next, we define Banach spaces $G^m(\mathbf{R}^3; 0, T)$ by the following norm

$$||u||_{G^m(\mathbf{R}^3;0,T)} = \sup_{0\le t\le T} ||u(t,\cdot)||_{\bar{G}^m(\mathbf{R}^3)}, \tag{2.27}$$

where

$$||u||_{\bar{G}^m(\mathbf{R}^3)} = \sum_{0\le l\le m}\sum_{\Gamma\in A}\left(\int_{\mathbf{R}^3} |\Gamma_1 ... \Gamma_l u(x)|^2 dx\right)^{1/2}, \tag{2.28}$$

where the sum extended over all vector fields $\Gamma_1, ..., \Gamma_l$ belonging to set A.

The kernel of the paper is the following result.

Theorem 2.1. *Let κ, κ_* fulfill (2.19), and initial data $(2.21)_{2,3}$ satisfy the following regularity conditions*

$$h^0_{\alpha\beta}(1+r)^{1/2} \in L_\infty(\mathbf{R}^3) \quad \textit{and} \quad \nabla h^0_{\alpha\beta},\ h^1_{\alpha\beta} \ \textit{s.t.} \ Dh_{\alpha\beta}|_{x^0=0} \in \bar{G}^2(\mathbf{R}^3). \tag{2.29}$$

If

$$||h^0_{\alpha\beta}(1+r)^{1/2}||_{L_\infty(\mathbf{R}^3)} + ||Dh_{\alpha\beta}|_{x^0=0}||_{\bar{G}^2(\mathbf{R}^3)} \leq X_0 \qquad (2.30)$$

and X_0 is sufficiently small, then there exists global in time solution to problem (2.21) such that

$$Dh_{\alpha\beta} \in G^2(\mathbf{R}^3; 0, \infty) \qquad (2.31)$$

and

$$h_{\alpha\beta}(1+r)^{1/2} \in L_\infty(\mathbf{R}^3 \times (0, \infty)), \qquad (2.32)$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$ and $D = (\partial_0, \partial_1, \partial_2, \partial_3)$.

Although, the general theory for nonlinear wave equations with nonlinearity of the second order as (2.10) does not imply global in time existence of solutions [7], we are able to obtain a priori estimates (2.31) and (2.32) using standard techniques. Our approach is effective, since the character of nonlinear terms reduces our consideration only to analysis on the support of function w. This modification allows to apply the whole information which can be obtained from analysis in spaces G^m (see Propositions 3.1-3.3 in the next section).

The required regularity of initial data is not optimal. However in our approach it is better to work with integer order of derivatives, because definitions (2.27) and (2.28) would be more complex and the energy method could be less effective. The sharp result in the L_2-framework has been obtained in [6].

Proof of Theorem 1.1. Having already proved Theorem 2.1, we show Theorem 1.1. Choosing $\epsilon > 0$, we prescribe $\kappa = 1 + \epsilon$, it follows that we need to choose so large M_0 and M in (2.18) that X_0 is so small that the solutions given by Theorem 2.1 generate a spacetime with maximal speed of propagation less than $1 + \epsilon$. It can be possible as we have assumptions (2.12) with extra restrictions on regularity of initial data. Then the basic features of hyperbolic systems imply that on domain $\mathcal{M}$ we have (2.22). Let us note that there is no un-uniqueness effects, since the regularity of solutions is sufficiently large. Theorem 1.1 has been proved.

In the next section we introduce some definition and auxiliary results for spaces G^m. Section 4 proves the a priori bound, which via local existence results will show Theorem 2.1.

3 Preliminaries

In this section we introduce some auxiliary results and notations necessary to prove Theorem 2.1.

First, let us recall some results for spaces G^m from [8].

Proposition 3.1. *Let* $u \in G^2(\mathbf{R}^3; 0, \infty)$, *then for any* $x \in (0, \infty) \times \mathbf{R}^3$ *holds*

$$|u(x)| \leq c(1 + |r - x^0|)^{-1/2}(1 + |r + x^0|)^{-1}||u||_{G^2(\mathbf{R}^3;0,\infty)}. \tag{3.1}$$

Proposition 3.2. *Using notation (2.23)-(2.25), we have* $\partial_\mu = (x^\nu L_{\mu\nu} + x_\mu S) < S, S >^{-1}$, *where* $< S, S >= -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$. *It follows that for sufficiently smooth functions u the following estimate is valid*

$$|Du(x)| \leq c|x^0 - r|^{-1}(|Su| + \sum_{0 \leq \mu < \nu \leq 3} |L_{\mu\nu} u|) = c|x^0 - r|^{-1}|\Lambda u|. \tag{3.2}$$

Proposition 3.3. *Let* $u \in G^1(\mathbf{R}^3; 0, \infty)$, *then* $u \in L_\infty(0, \infty; L_4(\mathbf{R}^3))$ *and*

$$||Wu(x^0, \cdot)||_{L_4(\mathbf{R}^3)} \leq c(1 + (\kappa_* - 1)x^0)^{-3/4}||u||_{G^1(\mathbf{R}^3,0,\infty)}, \tag{3.3}$$

where

$$W = \begin{cases} 1 & for \quad x \in supp\ w \\ 0 & for \quad x \in \mathbf{R}^3 \times (0, \infty) \setminus supp\ w \end{cases} \tag{3.4}$$

Proof. Since Proposition 3.3 is not proved in [8], we show it here. Imbedding $u \in L_4(\mathbf{R}^3)$ is trivial. To prove estimate (3.3) we apply the Marcinkiewicz interpolation theorem [15]. By Proposition 3.1 we see that

$$||Wu||_{L_\infty(\mathbf{R}^3)} \leq c(1 + (\kappa_* - 1)x^0)^{-3/2}||u||_{G^2}. \tag{3.5}$$

Thus, operator T, being an embedding $u \to u(x^0, \cdot)$, is bounded as a map

$$T : G^2 \to L_\infty, \tag{3.6}$$

where the norm is prescribed by Proposition 3.1. Moreover we consider the trivial embedding

$$T : G^0 \to L_2. \tag{3.7}$$

Applying the Marcinkiewicz interpolation theorem we get boundedness of the following operator

$$T : (G^2, G^0)_{1/2,2} \to (L_\infty, L_2)_{1/2,2} \tag{3.8}$$

with the following norm

$$||T||_{L((G^2,G^0)_{1/2,2};(L_\infty,L_2)_{1/2,2})} = ||T||^{1/2}_{L(G^2,L_\infty)}||T||^{1/2}_{L(G^0,L_2)}. \tag{3.9}$$

Since the norm of (3.7) is equal one, from (3.5) and (3.9) we conclude the following bound of the norm of operator (3.8)

$$||T||_{L((G^2,G^0)_{1/2,2};(L_\infty,L_2)_{1/2,2})} \leq c(1 + (\kappa_* - 1)x^0)^{-3/4}. \tag{3.10}$$

To finish the proof, let us note that

$$(L_\infty, L_2)_{1/2,2} = L_{4,2} \subset L_4, \tag{3.11}$$

where $L_{4,2}$ is the standard Lorentz space, moreover we have

$$(G^2, G^0)_{1/2,2} = G^1. \tag{3.12}$$

Thus we proved (3.4). The proof of Proposition 3.3 is finished.

Let us recall commutator rules for the standard d'Alambert operator (where $\Box = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$)

$$[\Box, T_\mu] = 0 \qquad \text{for} \quad 0 \leq \mu \leq 3, \tag{3.13}$$

$$[\Box, L_{\mu\nu}] = 0 \qquad \text{for} \quad 0 \leq \mu < \nu \leq 3, \tag{3.14}$$

$$[\Box, S] = 2\Box \tag{3.15}$$

and for any $\Gamma_m, \Gamma_n \in A$.

However, the main considerations will be concentrated on system (2.21), hence we introduce the following operator

$$\Box_d = d^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta} \tag{3.16}$$

and for the above operator the following commutator rules hold (which can be stated symbolically as follows)

$$[\Box_d, T_\mu] \sim (Dd)D^2 \qquad \text{for} \quad 0 \leq \mu \leq 3, \tag{3.17}$$

$$[\Box_d, L_{\mu\nu}] \sim (wh)D^2 + (L_{\mu\nu}d)D^2 \qquad \text{for} \quad 0 \le \mu < \nu \le 3, \tag{3.18}$$

$$[\Box_d, S] \sim (wh)D^2 + (Sd)D^2 + 2\Box_d. \tag{3.19}$$

For any function defined on $(0, \infty) \times \mathbf{R}^3$ we introduce

$$E_d[u] = \left(\int_{\mathbf{R}^3} (d^{00}u_{,0}^2 + d^{kl}u_{,k}u_{,l})dx \right)^{1/2}. \tag{3.20}$$

The above quantity defines function spaces where solutions will be looked for. Within our considerations we require to metric coefficients $d^{\alpha\beta}$ fulfill (2.4) but with fixed and controlled constants, i.e.

$$-d^{00} \ge 1/2, \qquad 1/2|X|^2 \le d^{kl}X_kX_l \le 2|X|^2 \quad \text{for } X \in \mathbf{R}^3. \tag{3.21}$$

The above restrictions will be easily satisfied by solutions as we will be able to control smallness of functions $h_{\mu\nu}$. We can replace numbers 1/2 and 2 by $1-\epsilon$ and $1+\epsilon$, but it would not change the final result.

As an elementary corollary of above facts we see that this quantity is equivalent to the following norm

$$E_d[u] \simeq ||Du(t,\cdot)||_{L_2(\mathbf{R}^3)}. \tag{3.22}$$

The main quantity, which controls the norm of solutions, is the following

$$\mathcal{X} = \sup_{0 \le t < \infty} \left(E_d^2[h(t,\cdot)] + E_d^2[\Gamma h(t,\cdot)] + E_d^2[\Gamma\Gamma h(t,\cdot)] \right)^{1/2} + X_0, \tag{3.23}$$

where X_0 described the norm of initial data - see (2.30), and

$$E_d^2[\Gamma h] = \sum_{\Gamma_a \in A} E_d^2[\Gamma_a h], \quad E_d^2[\Gamma\Gamma h] = \sum_{\Gamma_a, \Gamma_b \in A} E_d^2[\Gamma_a \Gamma_b h] \tag{3.24}$$

denote sums over all possible indeces. Finiteness of quantity $\mathcal{X}$ implies that $Dh \in G^2(\mathbf{R}^3; 0, \infty)$.

4 A priori bound

In this section we find the a priori bound on quantity $\mathcal{X}$ to control solutions of (2.21) globally in time. To prove Theorem 2.1 it is enough to have an a priori estimate, since by the local existence results (see [6,11]), controlling norms of solutions we are able to prolong the domain of lifespan. Hence we skip the part of the proof concerning the issue of existence.

For our purpose we apply the energy method, which in our case is split into three steps.

The first energy estimate. Multiplying $(2.21)_1$ by $h_{\mu\nu,0}$, integrating over $\mathbf{R}^3$, next integrating by parts the l.h.s., we obtain the following inequality

$$\frac{1}{4}\frac{d}{dx^0}E_d^2[h_{\mu\nu}] \le \int_{\mathbf{R}^3} \left(|wH_{\mu\nu}(d_{\alpha\beta}, Dh_{\alpha\beta})h_{\mu\nu,0}| + |Dh||Dd|^2\right) dx = I_1. \quad (4.1)$$

To obtain the above inequality we used relation (3.17).

Recalling the form of terms $H_{\mu\nu}$, the both terms can be estimated similarly and the estimation reduces to analyze the following integral

$$\int_{\mathbf{R}^3} W|Dh||Dh|^2 dx, \quad (4.2)$$

where W is given by (3.4). To apply Proposition 3.1 we estimate term (4.2) as follows

$$I_1 \le \sup_{x\in\mathbf{R}^3} |W|Dh|| \int_{\mathbf{R}^3} |Dh|^2 dx. \quad (4.3)$$

Since supp $W = \{r \ge \kappa_* x^0 + M_0\}$ relation (3.1) implies that

$$I_1 \le c(1 + (\kappa_* - 1)x^0)^{-3/2}\mathcal{X}, \quad (4.4)$$

where $\mathcal{X}$ is given by (3.23), which implies that $Dh \in G^2$.

Thus, summing over indeces μ and ν, we get

$$\frac{d}{dx^0}E_d^2[h] \le c(1 + (\kappa_* - 1)x^0))^{-3/2}\mathcal{X}E_d^2[h], \quad (4.5)$$

and by the Gronwall inequality we obtain

$$\sup_{0\le t\le\infty} E_d^2[h(t)] \le E_{d_0}^2[h|_{x^0=0}] \exp\{c\int_0^\infty (1 + (\kappa_* - 1)t)^{-3/2}\mathcal{X}dt\}$$

$$\le E_{d_0}^2[h|_{x^0=0}] \exp\{c\mathcal{X}\}, \quad (4.6)$$

where $d_0 = d|_{x^0=0}$.

The second energy estimate. Let $\Gamma \in A$, then by (3.17)-(3.19) we have

$$-\frac{1}{2}\Box_d(\Gamma h_{\mu\nu}) = \Gamma(wH_{\mu\nu}) + \frac{1}{2}[\Box_d, \Gamma]h_{\mu\nu}. \tag{4.7}$$

Multiplying (4.7) by $(\Gamma h_{\mu\nu})_{,0}$, we obtain

$$\frac{1}{4}\frac{d}{dx^0}E_d^2[\Gamma h_{\mu\nu}] \leq \int_{\mathbf{R}^3}\left(|\Gamma(wH_{\mu\nu}(\Gamma h_{\mu\nu})_{,0}| + \frac{1}{2}|[\Box_d, \Gamma]h_{\mu\nu}(\Gamma h_{\mu\nu})_{,0}|\right)dx$$

$$= I_2 = I_{21} + I_{22}. \tag{4.8}$$

Taking the first term of the r.h.s. of (4.8) we split it into two terms

$$I_{21} \leq \int_{\mathbf{R}^3}|(\Gamma w)H_{\mu\nu}(\Gamma h_{\mu\nu})_{,0}|dx + \int_{\mathbf{R}^3}|w(\Gamma H_{\mu\nu})(\Gamma h_{\mu\nu})_{,0}|dx = I_{211} + I_{212}. \tag{4.9}$$

The second term of (4.9) is treated the same as term I_1 in the first energy estimate, i.e. we have

$$I_{212} \leq c(1 + (\kappa_* - 1)x^0)^{-3/2}\mathcal{X}E_d^2[\Gamma h]. \tag{4.10}$$

To consider the first term we note that by the definition of the cut off function (2.18) we have globally in $(0, \infty) \times \mathbf{R}^3$ for a certain constant the following bound

$$|\Gamma w| \leq c \sim O(1/(\kappa - \kappa_*)), \tag{4.11}$$

which follows from properties of the support of w.

To examine I_{211} we see that by (4.9) and Proposition 3.1 we have

$$I_{211} \leq c\textstyle\int_{\mathbf{R}^3} W|Dh|\,|Dh|\,|(\Gamma h)_{,0}|dx \leq c\sup_{\bar{x}\in\mathbf{R}^3}|W|Dh||\int_{\mathbf{R}^3}|Dh|\,|(\Gamma h)_{,0}|dx$$

$$\leq c(1 + (\kappa_* - 1)x^0)^{-3/2}\mathcal{X}E_d[h]E_d[\Gamma h]. \tag{4.12}$$

To finish the estimation for the second energy estimate we find a bound for the last term of the r.h.s. of (4.8)

$$I_{22} \leq c\int_{\mathbf{R}^3}|[\Box_d, \Gamma]h_{\mu\nu}(\Gamma h)_{,0}|dx$$

$$\leq \int_{\mathbf{R}^3} W(|(\Gamma h)D^2h(\Gamma h)_{,0}| + |(wh)D^2h(\Gamma h)_{,0}| + |\Box_d h(\Gamma h)_{,0}|)dx$$

$$= I_{221} + I_{222} + I_{223}. \tag{4.13}$$

Take I_{221}. By the definition of elements of set A - (see (2.26)), we note that

$$|\Gamma h| \leq c(r + x^0)|Dh|. \tag{4.14}$$

Moreover by Proposition 3.2 we have

$$|D^2 h| \leq c(1 + (r - x^0))^{-1}|\Lambda Dh|. \tag{4.15}$$

By properties of the support of function w we see that for a certain constant the following pointwise estimate holds

$$\left| W \frac{r + x^0}{1 + r - x^0} \right| \leq c. \tag{4.16}$$

Hence we conclude from (4.14), (4.15) and (4.16) the following inequality

$$I_{221} \leq c \int_{\mathbf{R}^3} W|Dh|\,|\Lambda Dh|\,|(\Gamma h)_{,0}|dx. \tag{4.17}$$

Repeating steps for the first energy estimate we get

$$I_{221} \leq c(1 + (\kappa_* - 1)x^0)^{-3/2} \mathcal{X} E_d^2[\Gamma h]. \tag{4.18}$$

Let us take the second term of the r.h.s. of (4.13). The crucial point is to analyze the behavior of the function h with respect to time. By the form of I_{222} we consider only points from the support of function w. For this purpose we examine the following representation of searched functions

$$h_{\alpha\beta} = h^0_{\alpha\beta} + \int_0^{x^0} h_{\alpha\beta,0} dt. \tag{4.19}$$

We are interested only in finding a suitable estimate on $Wh_{\alpha\beta}$, so we may apply here assumption (2.30) and get

$$\sup_{\bar{x} \in \mathbf{R}^3} |Wh^0_{\alpha\beta}| \leq \frac{X_0}{(1 + x^0)^{1/2}}. \tag{4.20}$$

Thus, to find the desired bound on $h_{\alpha\beta}$, we estimate the following integral

$$W \int_0^{x^0} h_{\alpha\beta,0} dt; \tag{4.21}$$

and by Proposition 3.1 we have;

$$\leq cW\int_0^{x^0}(1+(r-t))^{-1/2}(1+r+t)^{-1}\mathcal{X}dt; \tag{4.22}$$

and by properties of the support of W we find;

$$\leq c\mathcal{X}W\int_0^{x^0}(1+r+t)^{-3/2}dt \leq c\mathcal{X}\frac{1}{(1+x^0)^{1/2}}. \tag{4.23}$$

From (4.20) and (4.23) we conclude that

$$\sup_{\bar{x}\in\mathbf{R}^3}|Wh(x)| \leq c(1+x^0)^{-1/2}\mathcal{X}. \tag{4.24}$$

However this bound is not sufficient. To get the full information, we need to repeat arguments from considerations for I_{221}, then we get

$$I_{222} \leq \int_{\mathbf{R}^3}|hD^2h(\Gamma h)_{,0}|dx \leq$$

$$c(1+x^0)^{-1/2}\mathcal{X}\int_{\mathbf{R}^3}W(1+(r-x^0))^{-1}(\Lambda Dh)(\Gamma h)_{,0}dx$$
$$\leq c(1+x^0)^{-1/2}\mathcal{X}(1+(\kappa_*-1)x^0)^{-1}E_d^2(\Gamma h). \tag{4.25}$$

The third term I_{223} is reduced to the first one since $-\frac{1}{2}\Box_d h = wH$.

Summing (4.10), (4.12), (4.18) and (4.25) we obtain the following differential inequality

$$\frac{d}{dx^0}E_d^2[\Gamma h] \leq c(1+(\kappa_*-1)x^0)^{-3/2}\mathcal{X}E_d[\Gamma h](E_d[\Gamma h]+E_d[h]). \tag{4.26}$$

The third energy estimate. Differentiating (4.7) by $\Gamma \in A$ we obtain

$$-\frac{1}{2}\Box_d(\Gamma\Gamma h_{\mu\nu})_{,0} = \Gamma(\Gamma(wH_{\mu\nu})+\frac{1}{2}[\Box_d,\Gamma]h_{\mu\nu})+\frac{1}{2}[\Box_d,\Gamma]\Gamma h_{\mu\nu}. \tag{4.27}$$

Multiplying (4.27) by $(\Gamma\Gamma h_{\mu\nu})_{,0}$, integrating over $\mathbf{R}^3$ we get

$$\frac{1}{4}\frac{d}{dx^0}E_d^2[\Gamma\Gamma h_{\mu\nu}] \leq \int_{\mathbf{R}^3}|\Gamma(\Gamma(wH_{\mu\nu}))(\Gamma\Gamma h_{\mu\nu})_{,0}|dx$$

$$+\int_{\mathbf{R}^3}\frac{1}{2}|\Gamma([\Box_d,\Gamma]h_{\mu\nu})(\Gamma\Gamma h_{\mu\nu})_{,0}|dx+\int_{\mathbf{R}^3}\frac{1}{2}|[\Box_d,\Gamma](\Gamma h_{\mu\nu})(\Gamma\Gamma h_{\mu\nu})_{,0}|dx$$
$$=I_3=I_{31}+I_{32}+I_{33}. \quad (4.28)$$

Take I_{31}. Here, we have

$$I_{31}\le\int_{\mathbf{R}^3}W|\Gamma(\Gamma w)|\,|H_{\mu\nu}|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|dx+2\int_{\mathbf{R}^3}W|\Gamma w|\,|\Gamma H_{\mu\nu}|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|dx$$
$$+\int_{\mathbf{R}^3}W|\Gamma\Gamma H_{\mu\nu}|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|dx=I_{311}+I_{312}+I_{313}. \quad (4.29)$$

To analyze I_{311}, we note that by features of function w, we have globally the following bound

$$|\Gamma\Gamma w|\le c\sim O(1/(\kappa-\kappa_*)). \quad (4.30)$$

Hence I_{311} is treated as I_{211} and we get

$$I_{311}\le c(1+(\kappa_*-1)x^0)^{-3/2}\mathcal{X}E_d[h]E_d[\Gamma\Gamma h]. \quad (4.31)$$

The same we have for I_{312}, since

$$I_{312}\le c\int_{\mathbf{R}^3}W|Dh|\,|\Gamma Dh|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|dx, \quad (4.32)$$

thus

$$I_{312}\le c(1+(\kappa_*-1)x^0)^{-3/2}\mathcal{X}E[\Gamma h]E[\Gamma\Gamma h]. \quad (4.33)$$

The last term delivers a new type of nonlinearity. Let us note that I_{313} is bounded as follows

$$I_{313}\le c\int_{\mathbf{R}^3}W|Dh|\,|\Gamma\Gamma Dh|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|dx+c\int_{\mathbf{R}^3}W|\Gamma Dh|\,|\Gamma Dh|\,|(\Gamma\Gamma h_{\mu\nu})_{,0}|dx$$
$$=I_{3131}+I_{3132}. \quad (4.34)$$

The first term can be estimated as follows

$$I_{3131}\le c(1+(\kappa_*-1)x^0)^{-3/2}\mathcal{X}E_d^2[\Gamma\Gamma h]. \quad (4.35)$$

To analyze the second term of the r.h.s. of (4.34) we apply Proposition 3.3. Since we can assume that $\Gamma Dh\in G^1$, applying the Hölder inequality to integral I_{3132}, we deduce the following estimate

$$I_{3132}\le c||\Gamma Dh||^2_{L_4(\mathbf{R}^3)}||\Gamma\Gamma Dh||_{L_2(\mathbf{R}^3)}$$

$$
\begin{aligned}
&\leq c(1+(\kappa_*-1)x^0)^{-3/2}(E_d[\Gamma h]+E_d[\Gamma\Gamma h])^2 E_d[\Gamma\Gamma h] \\
&\leq c(1+(\kappa_*-1)x^0)^{-3/2}\mathcal{X}(E_d[\Gamma h]+E_d[\Gamma\Gamma h])E_d[\Gamma\Gamma h]. \qquad (4.36)
\end{aligned}
$$

Thus, from (4.31), (4.32), (4.35) and (4.36) we conclude the following differential inequality

$$
\frac{d}{dx^0}E_d^2[\Gamma\Gamma h] \leq c(1+(\kappa_*-1)x^0)^{-3/2}\mathcal{X}(E_d^2[h]+E_d^2[\Gamma h]+E_d^2[\Gamma\Gamma h]). \qquad (4.37)
$$

The above estimate finished the third step.

A priori estimate. Summing up three energy inequalities (4.5), (4.26) and (4.37) we obtain

$$
\begin{aligned}
&\frac{d}{dx^0}\left\{E_d^2[h]+E_d^2[\Gamma h]+E_d^2[\Gamma\Gamma h]\right\} \\
&\leq c(1+(\kappa_*-1)x^0)^{-3/2}\mathcal{X}\left\{E_d^2[h]+E_d^2[\Gamma h]+E_d^2[\Gamma\Gamma h]\right\}. \qquad (4.38)
\end{aligned}
$$

Applying to (4.39) the Gronwall inequality we get the following bound

$$
\begin{aligned}
&\sup_{0\leq t<\infty}\left\{E_d^2[h]+E_d^2[\Gamma h]+E_d^2[\Gamma\Gamma h]\right\} \\
&\leq \left\{E_{d^0}^2[h^0]+E_{d^0}^2[\Gamma h^0]+E_{d^0}^2[\Gamma\Gamma h^0]\right\}\exp\{\pi_0\mathcal{X}\}. \qquad (4.39)
\end{aligned}
$$

Since π_0 in (4.40) is an absolute constant, we can require to initial data satisfy the following bound

$$
3\left\{E_{d^0}^2[h^0]+E_{d^0}^2[\Gamma h^0]+E_{d^0}^2[\Gamma\Gamma h^0]\right\}+X_0 \leq 1/\pi_0, \qquad (4.40)
$$

then from (4.39) and (4.40), if we assume suitable smallness of $\mathcal{X}$, we get

$$
\sup_{0\leq t<\infty}\left\{E_d^2[h]+E_d^2[\Gamma h]+E_d^2[\Gamma\Gamma h]\right\} \leq 3\left\{E_{d^0}^2[h^0]+E_{d^0}^2[\Gamma h^0]+E_{d^0}^2[\Gamma\Gamma h^0]\right\}, \qquad (4.41)
$$

what, by (3.24) and (4.40), closes the estimation, so we conclude

$$
\sup_{0\leq t<\infty}\left\{E_d^2[h(t,\cdot)]+E_d^2[\Gamma h(t,\cdot)]+E_d^2[\Gamma\Gamma h(t,\cdot)]\right\}+X_0 \leq 1/\pi_0. \qquad (4.42)
$$

To be more formal, to obtain estimate (4.42), we can examine the l.h.s. of (4.39) only for a finite fixed time instead of the infinity. Then we obtain the same bound which is valid for all times and this estimate implies (by continuity) searched restriction (4.42). Finally by definition of W we obtain

$$\mathcal{X} \leq 4X_0, \tag{4.43}$$

which finishes the proof of the a priori bound. By previous remarks and bound (4.43) we conclude the thesis of Theorem 2.1.

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