A new look at equilibria in Stefan type problems in the plane

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Abstract. We study steady states of Stefan-type problems in the plane with the Gibbs-Thomson correction involving general anisotropic energy density function. By a local analysis we prove the global result showing that the solution is the Wulff shape. The key element is a stability result which enables us to approximate singular models by regulars ones.

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1 Introduction

An important aspect of studying dynamical systems is determining its steady states and their stability properties. We have in mind a class of models of phase transitions involving Gibbs-Thomson correction on the interface. They include the Stefan (see [Lu], [AW], [CR] [Ra], [FR], [GR1]) and Hele-Shaw problems (see [DE], [A]). In special cases the modeling system takes the following one phase quasi stationary form

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \Omega(t), \\ p = \kappa_\gamma & \text{on} & \partial\Omega(t), \\ \frac{\partial p}{\partial \mathbf{n}} = -V & \text{on} & \partial\Omega(t). \end{array} \tag{1.1}$$

This is a free boundary problem where we seek the time evolution of an open set $\Omega(t)$ in $\mathbb{R}^n$, $\mathbf{n}$ is the normal to the boundary and V is the normal velocity of the interface $\partial\Omega(t)$. Here κ_γ is the weighted mean curvature, which is the most important object for us, it will be explained below and in more detail in Section 3.

This system is augmented with initial data for Ω. The interpretation of p depends upon the phenomenon we wish to model with (1.1). In the Stefan problem this is the temperature, in the Hele-Shaw problem p is the fluid pressure, in the tumor growth model, see [FR], p is the internal pressure of the proliferating tissue, in the crystal growth from vapor (see [GR1]) p is the supersaturation of the diffusing water vapor.

We want to stress an important fact, that κ_γ in (1.1) is the weighted (anisotropic) mean curvature, i.e.

$$\kappa_\gamma = \mathrm{div}_{\partial\Omega}(\nabla_\xi \bar\gamma|_{\xi=\mathbf{n}}). \tag{1.2}$$

This formula is correct for sufficiently smooth $\bar\gamma$. On the other hand, the case of low smoothness $\bar\gamma$ and singular $\partial\Omega$ will require special considerations.

Function $\bar\gamma$ is related to the anisotropy of the modeled process. Here we assume that function $\bar\gamma : \mathbb{R}^n \to \mathbb{R}$ is one-homogeneous, $\bar\gamma(x) > 0$ for $x \neq 0$, i.e.

$$\bar\gamma(x) = |x|\gamma(\frac{x}{|x|}) \quad \text{if } x \neq 0, \qquad \bar\gamma(0) = 0, \tag{1.3}$$

while function γ depends essentially on the orientation, i.e. $\gamma : S^{n-1} \to \mathbb{R}$. In the terminology of [Gu, Chapter 7] function γ is called the interfacial energy while $\bar\gamma$ is the extended energy.

Our goal is to study static solutions to (1.1), i.e.

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \Omega, \\ p = \mathrm{div}_{\partial\Omega}(\nabla_\xi \bar\gamma|_{\xi=\mathbf{n}}) & \text{on} & \partial\Omega, \\ \frac{\partial p}{\partial \mathbf{n}} = 0 & \text{on} & \partial\Omega. \end{array} \qquad |\Omega| \text{ is given,} \tag{1.4}$$

Momentarily, we are going to explain why the measure of Ω has to be fixed. By $(1.1)_{1,3}$ we conclude that $\int_{\partial\Omega(t)} V\, d\mathcal{H}^{n-1} = 0$, hence the measure of $\Omega(t)$ is constant. Additionally from $(1.4)_{1,3}$ we conclude that $p \equiv const.$, thus, we get

$$\mathrm{div}_{\partial\Omega}(\nabla_\xi \bar\gamma|_{\xi=\mathbf{n}}) = const. \qquad \text{and} \qquad |\Omega| \text{ is given.}$$

By a simple rescaling we arrive at a geometric problem expressed as a differential equation

$$\mathrm{div}_{\partial\Omega}(\nabla_\xi \bar\gamma|_{\xi=\mathbf{n}}) = 1. \tag{1.5}$$

At this moment we have to specify the assumptions on $\bar\gamma$. A minimal hypothesis, besides (1.3), which we impose is *convexity* of $\bar\gamma$. Thus, we immediately conclude that $\bar\gamma$ is Lipschitz continuous.

Under such broad assumptions on $\bar\gamma$ making sense out of (1.5) requires some work. For example, in [BNP] and [GR2] a separate variational problem was considered for the definition of κ_γ. It is our desire to consider quite general, as far as the smoothness is concerned, surface energy density function γ.

In our paper we consider a two dimensional case of (1.5). That is we look for a curve Γ which locally has a constant weighted mean curvature κ_γ. We show existence of such a curve, which turns out to be the boundary of a region, (in particular Γ is closed).

It is possible to adopt an energy point of view and interpret (1.5) as a critical point of surface energy functional $E(S) = \int_S \gamma(\mathbf{n}(x))\, d\mathcal{H}^{n-1}(x)$ and study its minimizers under the volume constraint. In fact this approach has been carried out, see [T], [FM], [Pa], [Mo] and references therein. In [T] Taylor gives the first rigorous proof of Wulff's theorem, stating that the only minimizer of E under the volume constraint is the Wulff shape. Later, various proofs of this result were found, see [FM] and references therein. Palmer, see [Pa], shows that the only stable smooth critical point of E is the Wulff shape. Morgan, [Mo], studied equilibria of E, he dropped the stability and smoothness assumptions. He showed that the only equilibrium is the Wulff shape.

An important assumption in the above papers is that they deal only with manifolds without boundary (e.g. closed curves). Here we adopt a different view, which we may call a local one. Namely, we can regard (1.5) as a *locally* defined differential equation. By this we mean that if (1.5) is given some appropriate data, then there exists a solution which is defined in a neighborhood of the data. Our goal, however, is *global*, to show that one can glue up those local solutions and despite the apparent freedom in choosing the data for the equation one can obtain a uniquely defined geometric object: a closed manifold, in this case a closed curve.

We stress that we set a restricted goal to consider only the two dimensional case. This leads to a significant simplification of (1.5) and the notion of its solution. However thanks to this simplification we will be able to show interesting qualitative properties of obtained solutions. A solution to this equation will be a curve of class W^1_p, whose curvature satisfies (1.5) in an appropriate sense explained later, see Section 3.

We may now state our main result.

Theorem 1.1. *Let us suppose that* $\bar\gamma : \mathbb{R}^2 \to \mathbb{R}$ *is a one-homogeneous, convex function. Then, there exists a solution to equation (1.5). Moreover, the solution defines a closed curve* Γ *and it is unique up to a translation, namely it is the Wulff shape of* $\bar\gamma$.

Let us stress that compared to the literature we have already mentioned, closedeness of curve Γ a *conclusion* of our analysis, not its assumption. Moreover, we consider the weakest possible regularity assumptions on γ, i.e. only convexity of $\bar\gamma$. A general notion of solution is introduced in the Definition of Section 3 by (3.13).

In order to prove this theorem, we first consider the regular case (we will make clear what this means for us). Careful analysis gives us a hint how to proceed for general $\bar\gamma$. We will see that it is easier to show existence of local solutions to (1.5) than to show that they can be glued up to form a closed curve. The last task is simpler for regular curves, because we can use the power of the classical differential geometry. Our goal is reached through an appropriate stability theorem, which is our second important result.

Theorem 1.2. *Let us assume that* $1 \le p < \infty$ *and* $\bar\gamma, \bar\gamma_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ *are convex, one-homogeneous and* γ_ϵ *are defined by (1.3) with* $\bar\gamma$ *replaced by* $\bar\gamma_\epsilon$. *We also assume that* Γ_ϵ *is a solution to*

$$\mathit{div}_{\partial\Omega}(\nabla_\xi \bar\gamma_\epsilon|_{\xi=\mathbf{n}}) = 1$$

and Γ is a solution to (1.5). If $\gamma_\epsilon \to \gamma$ in $W^1_p(S^1)$ as $\epsilon \to 0$, then possibly after a translation of Γ_ϵ it is true that $\Gamma_\epsilon \to \Gamma$ in W^1_p as submanifolds in $\mathbb{R}^2$.

Let us first explain that convergence of Γ_ϵ means convergence of curves locally treated as graphs. The $W^1_p(S^1)$-space is the standard Sobolev space of functions, defined on the unit circle $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$, integrable with its derivative with the p-th power.

The last result enables us to approximate any low regularity problem by its suitable regularization. A crucial point here is the choice of a proper notion of convergence. At first it seems that the measures are the best spaces, because the second derivative of a convex function is in this space or equivalently, the first derivative belongs to the TV-space. However, from our point of view this setting does not appear appropriate, because we are not able to find any smooth approximations for the general case. Hence, we investigate the L_p-approach which implies a weaker topology, but it provides sufficient information. The method developed here gives us a tool to analyze the behavior of possible singularities appearing in our problem.

Let us draw a corollary from Theorem 1.2. If $\|x\|_p$ is the usual p-norm in $\mathbb{R}^2$, i.e. $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ for $p < \infty$, and $\|x\|_\infty = \max\{|x_1|, |x_2|\}$, then we define γ_p by means of (1.3) as $\gamma_p(x) = \|x\|_p$ for $x \in S^1$. It is a well-know fact that Γ_p, which is a solution to (1.5) with γ_p replacing γ, is given as a Wulff shape of $\|\cdot\|_p$, i.e. Γ_p is the unit ball of the norm $\|\cdot\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus, $\Gamma_p = \{\|x\|_q = 1\}$. We can easily see that for any $r < \infty$, if $\gamma_p \to \gamma_\infty$ in W^1_r as $p \to \infty$, then $\Gamma_p \to \Gamma_\infty$ in W^1_r as submanifolds in $\mathbb{R}^2$.

We briefly describe the content of the paper. In Section 2, we recall some well-known facts from the classical differential geometry and explain the meaning of (1.5). Moreover, we prove there Theorem 1.1 in the case of regular γ. This proof will be the starting point for further considerations. Next, we reformulate the problem as a differential inclusion in the case of general γ. In section 4 the main results: Theorems 1.1 and 1.2 are proved. Finally, we analyze qualitative properties of obtained solutions.

2 Preliminary analysis

In this section we recall some facts and introduce further notation. Namely, we shall start writing

$$\bar{\gamma}(x) = r\gamma(\varphi),$$

where $r = |x|$ and φ are defined by the relation $x = r(\cos\varphi, \sin\varphi)$.

Let us suppose that $s \mapsto \mathbf{x}(s)$ is an arc-length parameterization of a given smooth curve Γ, then $\mathbf{t}(s) = \frac{d\mathbf{x}}{ds}(s)$ is a unit vector tangent to Γ. If Γ is the boundary of Ω, then we rewrite the LHS of (1.5) to obtain

$$\mathrm{div}_{\partial\Omega}(\nabla_\xi \bar{\gamma}|_{\xi=\mathbf{n}}) = \mathbf{t} \cdot \frac{\partial}{\partial \mathbf{t}}(\nabla\bar{\gamma}(\mathbf{n})) \equiv \mathbf{t}(s) \cdot \frac{d}{ds}[\nabla\bar{\gamma}(\mathbf{n}(s))]. \tag{2.1}$$

While keeping in mind (2.1), we want to re-write system (1.5) in a suitable coordinate system. For this purpose, we use the normal angle and subsequently the angle parameterization of Γ. Let us suppose first that $\mathbf{n}$ is a unit vector normal to Γ such that moving frame $\{\mathbf{n}, \mathbf{t}\}$ is a positively oriented basis of $\mathbb{R}^2$. Then, the *normal angle* $\varphi(s)$ is defined as a smooth function of s through

$$\begin{aligned} \mathbf{n} &= \mathbf{n}(s) = (\cos\varphi(s), \sin\varphi(s)), \\ \mathbf{t} &= \mathbf{t}(s) = (-\sin\varphi(s), \cos\varphi(s)). \end{aligned} \tag{2.2}$$

We will refer to the range of the function $s \mapsto \varphi(s)$ as the *angle-set.* With the help of this function we define a parameterization of our curve by formula

$$\mathbf{x}(s) = \int_{s_0}^{s} \mathbf{t}(\varphi(t))dt + \mathbf{v_0} = \int_{s_0}^{s} (-\sin\varphi(t), \cos\varphi(t))\, dt + \mathbf{v_0}, \tag{2.3}$$

where $\mathbf{v_0}$ is a fixed point in $\mathbb{R}^2$. Let us write $\Gamma = \mathbf{x}(\mathbb{R})$. This formula is a direct result of the fact that s is an arc-length parameter.

The function

$$\frac{d\varphi}{ds}(s) = \kappa(s) \tag{2.4}$$

is the Euclidean curvature of Γ. It obeys the Frenet formulas: $\frac{d\mathbf{n}}{ds} = \kappa\mathbf{t}$, $\frac{d\mathbf{t}}{ds} = -\kappa\mathbf{n}$. In order to proceed, we assume for the moment that γ is at least C^2-smooth. Then equation (1.5) becomes

$$[\frac{d}{ds}\varphi][\mathbf{t}\cdot K(\mathbf{n})\cdot\mathbf{t}] = 1, \tag{2.5}$$

where

$$K = \begin{bmatrix} \bar{\gamma}_{11} & \bar{\gamma}_{12} \\ \bar{\gamma}_{21} & \bar{\gamma}_{22} \end{bmatrix} = \frac{1}{r}\left(\frac{d^2}{d\varphi^2}\gamma(\varphi) + \gamma(\varphi)\right) e_\varphi \otimes e_\varphi = d^2\bar{\gamma}(r\mathbf{n}) \tag{2.6}$$

and $e_\varphi(\varphi(s)) = \mathbf{t}(\varphi(s))$. Convexity of $\bar{\gamma}$ is equivalent to

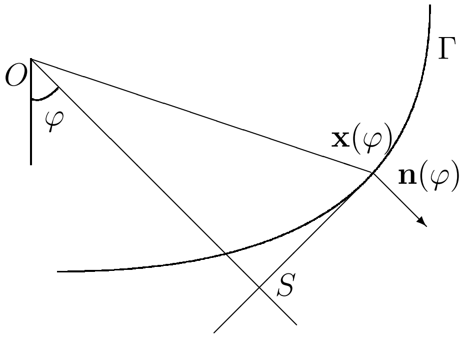
$$\frac{d^2}{d\varphi^2}\gamma(\varphi) + \gamma(\varphi) \geq 0. \tag{2.7}$$

In the simplest case we will rule out the possibility of equality above. Let us stress that, because of one-homogeneity of $\bar{\gamma}$, this function cannot be strictly convex in the usual sense.

We shall call a smooth curve *strictly convex*, if its curvature never vanishes, (in the literature such curves are called often convex). Let us notice that for strictly convex curves an *angle-parameterization,* $\varphi \mapsto \mathbf{x}(\varphi)$ is possible, because the function $s \mapsto \varphi(s)$ is strictly increasing.

Finally, we shall call a surface energy density function γ *strictly stable at* φ_0 if $\frac{d^2\gamma}{d\varphi^2}(\varphi_0) + \gamma(\varphi_0) > 0$ and γ will be called *strictly stable* if $\frac{d^2\gamma}{d\varphi^2}(\varphi) + \gamma(\varphi) > 0$ holds for all φ. The last condition is a substitute for strict convexity of $\bar{\gamma}$. We shall call a surface energy function γ *regular* if it is smooth and strictly stable.

We expect that (2.5) yields a well-posed differential equation having local solutions. The main difficulty will be associated with proving that solutions can be glued up to form a closed curve. Our tool in solving this problem will be the classical concept of the support function. Let us supposes that Θ is the angle set of a curve Γ for a point O. The *support function* $P_0 : \Theta \to \mathbb{R}$ of Γ is defined by the formula (see Fig.1.),



$$P_0(\varphi) = \mathbf{x}(\varphi) \cdot \mathbf{n}(\varphi) = |OS|.$$

Fig. 1

The meaning of $P_0(\varphi)$ is the distance from the tangent to Γ at $\mathbf{x}(\varphi)$ to the origin. One can easily check using the Frenet formula, (2.4) and $\frac{d\mathbf{n}}{d\varphi} = \mathbf{t}$ that P_0 satisfies the following equations,

$$\mathbf{x}(\varphi) = P_0(\varphi)\mathbf{n}(\varphi) + \frac{dP_0}{d\varphi}(\varphi)\mathbf{t}(\varphi), \qquad \frac{d^2 P_0}{d\varphi^2}(\varphi) + P_0(\varphi) = \frac{1}{\kappa}, \tag{2.8}$$

(see also [Bl, §94], [Gu, eq. (1.10)]).

Lemma 2.1. *([Gu, Lemma 1A (b)]) If Γ is a convex curve with curvature $\kappa(\varphi)$, and if P_0 is any solution to (2.8)$_2$ on the angle set of Γ, then (2.8)$_1$ is the angle-parameterization of a curve which differs from Γ by at most a translation.*

Proof. Identity $(2.8)_1$ defines us a curve uniquely up to a translation. Hence assuming that the curves – mentioned in the Lemma – are different, we conclude that they have to intersect at least in one point. Let us consider the difference, u, between P_0 and the support function of Γ. It satisfies $\frac{d^2u}{d\varphi^2} + u = 0$. The general solution to this equation is $u(\varphi) = a_1 \cos\varphi + a_2 \sin\varphi \equiv \mathbf{a} \cdot \mathbf{n}(\varphi)$. However $(2.8)_1$ determined two parameters for the intersection, hence $a_1 = a_2 = 0$. Our claim follows. □

We are now ready for our first local result.

Lemma 2.2. *Let us suppose that γ is smooth and strictly stable. Then, there exists a unique local solution to (2.5) augmented with the initial condition $\varphi(s_0) = \varphi_0$.*

Proof. The assumption of the strict stability of γ implies that there exist positive numbers a_1 and a_2 such that for each point $p = (x(s), y(s))$ of curve Γ defined by (2.3), matrix K satisfies

$$a_1|\xi|^2 \leq \xi \cdot K(p) \cdot \xi \leq a_2|\xi|^2 \quad \text{for any } \xi \in T\Gamma_p, \tag{2.9}$$

where $T\Gamma_p$ denotes the tangent space to Γ at point p.

In this case the existence as well as the uniqueness of solutions are obvious since we reduce the problem to the following elementary ordinary differential equation

$$\frac{d}{ds}\varphi(s) = (\mathbf{t} \cdot K(\mathbf{n}(\varphi(s))) \cdot \mathbf{t})^{-1}. \tag{2.10}$$

with initial data $\varphi(s_0) = \varphi_0$.

By the strict convexity of γ constants a_1 and a_2 in (2.9) are prescribed globally, the solution to (2.10) exists for all $s \in \mathbb{R}$.

□

Subsequently, we will use the above lemma a number of times. It is the necessary step to establishing our first goal, stated below.

Proposition 2.1. *Let us suppose that γ is regular, i.e. smooth and globally strictly stable. Then solutions to (2.5) yield a unique closed C^2-curve Γ being the boundary of a domain Ω.*

Proof. By Lemma 2.2 we have local existence as well as the uniqueness of solutions to (2.5) with initial data $\varphi(s_0) = \varphi_0$. Now, we are going to show that the solution to (2.10) yield a closed curve Γ. First, let us note that by (2.6) and (2.10) we conclude that

$$\frac{d^2}{d\varphi^2}\gamma(\varphi) + \gamma(\varphi) = \frac{1}{\kappa(\varphi)}. \tag{2.11}$$

Because of the global strict stability of γ, function φ is an angle parameterization of a strictly convex curve. Moreover, due to (2.9) there is an interval $[s_0, s_0 + d)$ such that φ is onto $[\varphi_0, \varphi_0 + 2\pi)$, i.e. the angle set of Γ is $[0, 2\pi)$. Hence, by Lemma 2.1 we conclude that P_0 and γ differ by $a_1 \cos\varphi + a_2 \sin\varphi$. Since γ is periodic we deduce that P_0 is periodic as well. Thus, we infer that the curve, whose support function is periodic, must be closed, see $(2.8)_1$ and Lemma 2.1. Now, we need to exclude self intersections of the sought curve Γ. This follows immediately form the fact that $\mathbf{x}(\varphi)$ given by (2.8) is 2π-periodic.

□

Remarks. Let us stress that Lemma 2.2 yields angle parameterizations which are solutions to a differential equation with arbitrary initial data. We showed, that the corresponding geometric object i.e. a closed curve, whose angle parameterization we have constructed, exists and it is unique. The uniqueness is a consequence of the fact that the support functions coincide (up to a periodic function) with γ. Moreover, if γ is strictly stable, then the locus of the vector function given by (2.8), i.e.

$$\mathbf{x}(\varphi) = \gamma(\varphi)\mathbf{n}(\varphi) + \gamma'(\varphi)\mathbf{t}(\varphi) \tag{2.12}$$

is the boundary of the Wulff shape, W_γ, (see [Gu, Theorem 7P]). We recall that by definition,

$$W_\gamma = \{p \in \mathbb{R}^2 : \; p \cdot \mathbf{n}(\varphi) \le \gamma(\varphi)\}. \tag{2.13}$$

In other words, we have proved Theorem 1.1 for smooth, globally stable γ's.

Inequality (2.7) has a geometric interpretation. We recall that the *Frank diagram*, F_γ, of a one-homogeneous function $\bar{\gamma}$ is defined by

$$F_\gamma = \{p \in \mathbb{R}^2 : \; \bar{\gamma}(p) = 1\} = \{(r, \varphi) : \; r = \gamma^{-1}(\varphi)\}. \tag{2.14}$$

It is well-known fact that convexity of F_γ is equivalent to convexity of $\bar{\gamma}$, (e.g. see [Gu, Theorem 7B]). Moreover, one can see that the strict stability of γ is equivalent to smoothness and convexity of F_γ.

We are now ready to discuss the weighted mean curvature (WMC for short) equation for a general anisotropy.

3 The general case of WMC equation for curves

The goal of this section is to restate problem (1.5) for general γ and define a generalization of the solution – see (3.13).

The approach presented in section 2 is not suitable for γ, which is not of class C^2. For this purpose we return to equation (2.1) and re-write it again. Namely, we notice that for smooth γ (1.5) is equivalent to

$$\frac{d}{ds}[(\nabla_\xi\bar{\gamma}|_{\xi=\mathbf{n}})\cdot\mathbf{t}] - (\nabla_\xi\bar{\gamma}|_{\xi=\mathbf{n}})\frac{d}{ds}\mathbf{t} = 1, \tag{3.1}$$

but due to (2.2) we see that $\frac{d}{ds}\mathbf{t} = -(\frac{d}{ds}\varphi)\mathbf{n}$, hence the second term of the LHS of (3.1) reads

$$-(\nabla_\xi\bar{\gamma}|_{\xi=\mathbf{n}})\frac{d}{ds}\mathbf{t} = (\frac{d}{ds}\varphi)(\nabla_\xi\bar{\gamma}|_{\xi=\mathbf{n}})\cdot\mathbf{n}, \tag{3.2}$$

however by one-homogeneity of $\bar{\gamma}$ – see (1.3) – we have $(\nabla_\xi\bar{\gamma}|_{\xi=\mathbf{n}})\cdot\mathbf{n} = \bar{\gamma}(\mathbf{n})$. Hence we get

$$-(\nabla_\xi\bar{\gamma}|_{\xi=\mathbf{n}})(\frac{d}{ds}\mathbf{t}) = (\frac{d}{ds}\varphi)\bar{\gamma}(\mathbf{n}(\varphi)) = \frac{d}{ds}\int_\vartheta^{\varphi(s)}\bar{\gamma}(\mathbf{n}(t))dt, \tag{3.3}$$

where ϑ will be fixed later. Thus, equation (3.1) takes the following form

$$\frac{d}{ds}\left[(\nabla_\xi\bar{\gamma}|_{\xi=\mathbf{n}})\cdot\mathbf{t} + \int_\vartheta^{\varphi(s)}\bar{\gamma}(\mathbf{n}(t))dt\right] = 1. \tag{3.4}$$

We notice that the definition of the normal vector (2.2) implies that $(\nabla_\xi\bar{\gamma}|_{\xi=\mathbf{n}})\cdot\mathbf{t} = \frac{d}{d\varphi}\bar{\gamma}(\mathbf{n}(\varphi))$. Keeping this in mind we rewrite (3.4) as follows,

$$\frac{d}{ds}\frac{d}{d\varphi}\left[\bar{\gamma}(\mathbf{n}(\varphi)) + \int_\vartheta^{\varphi}d\psi\int_\vartheta^{\psi}\bar{\gamma}(\mathbf{n}(t))dt\right] = 1 \tag{3.5}$$

or in a concise form

$$\frac{d}{ds}\frac{d}{d\varphi}I_\vartheta^\gamma(\varphi) = 1, \tag{3.6}$$

where

$$I_\vartheta^\gamma(\varphi) = \bar{\gamma}(\mathbf{n}(\varphi)) + \int_\vartheta^{\varphi}d\psi\int_\vartheta^{\psi}\bar{\gamma}(\mathbf{n}(t))dt. \tag{3.7}$$

We will drop the superscript for fixed γ.

Equation (3.6) does not make sense for $\bar{\gamma}$ which are only Lipschitz continuous. However, we know that $\bar{\gamma}$ is convex and this property of $\bar{\gamma}$ is equivalent to the following distributional relation

$$\frac{d^2}{d\varphi^2}\gamma(\varphi) + \gamma(\varphi) \geq 0 \quad \text{in } \mathcal{D}'(S^1). \tag{3.8}$$

Let us make the following observation

Proposition 3.1. *If one-homogeneous function $\bar{\gamma} : \mathbb{R}^2 \to \mathbb{R}$ is convex, then the function*

$$\mathbb{R} \ni \varphi \to I_\vartheta(\varphi) \in \mathbb{R}$$

is also convex function for fixed ϑ.

Proof. If $\bar{\gamma}$ is as above i.e. $\bar{\gamma}(x) = |x|\gamma(\varphi)$, where $x = |x|(\cos\varphi, \sin\varphi)$, then by (2.6) convexity of $\bar{\gamma}$ is equivalent to inequality $\gamma'' + \gamma \geq 0$ in the sense of distributions (see (3.8)). Once we notice that $\frac{\partial^2}{\partial\varphi^2} I_\vartheta(\varphi) = \gamma'' + \gamma$, our claim follows. □

Thus, the subdifferential $\partial_\varphi I_\vartheta$ of a convex function I_ϑ is always well-defined. Hence, our task consists of finding a suitable section of $\partial_\varphi I_\vartheta$ and we can formally write (3.5) as the following inclusion

$$\frac{d}{ds}\partial_\varphi \left[\bar{\gamma}(\mathbf{n}(\varphi)) + \int_\vartheta^\varphi d\psi \int_\vartheta^\psi \bar{\gamma}(\mathbf{n}(t))dt \right] \ni 1. \tag{3.9}$$

We will make it precise below. Let us first make some observations about I_ϑ. If γ is smooth, then $\frac{d}{d\varphi} I_{\vartheta_1}$ and $\frac{d}{d\varphi} I_{\vartheta_2}$ differ by a constant, i.e.

$$\frac{d}{d\varphi} I_{\vartheta_1}(\varphi) = \frac{d}{d\varphi} I_{\vartheta_2}(\varphi) + \int_{\vartheta_1}^{\vartheta_2} \gamma(t)\, dt. \tag{3.10}$$

Let us notice that this identity will remain valid if γ is Lipschitz continuous and we replace $\frac{d}{d\varphi}$ with ∂_φ. Mere continuity of γ yields,

$$I_\vartheta(2\pi + \varphi) = I_\vartheta(\varphi) + I_\vartheta(2\pi + \vartheta) - \gamma(\vartheta) + (\varphi - \vartheta) \int_\vartheta^{2\pi+\vartheta} \gamma(t)\, dt. \tag{3.11}$$

Indeed, for $\varphi \geq \vartheta$ we have

$$I_\vartheta(2\pi + \varphi) - I_\vartheta(2\pi + \vartheta) = \gamma(\varphi) - \gamma(\vartheta) + \int_{2\pi+\vartheta}^{2\pi+\varphi} \int_\vartheta^{2\pi+\vartheta} \gamma(t)\, dt d\psi + \int_{2\pi+\vartheta}^{2\pi+\varphi} \int_{2\pi+\vartheta}^{\psi} \gamma(t)\, dt d\psi.$$

Hence, (3.11) follows.

Suppose again that γ is smooth, then we can infer from (3.10) that by an appropriate choice of ϑ we may achieve $\varphi(\bar{s}) = \bar{\varphi}$ for arbitrary $\bar{\varphi}$, $\bar{s} \in \mathbb{R}$.

Furthermore, let us integrate (3.5) over $[\bar{s}, s]$ where $\bar{s}$ is an arbitrary parameter and $s - \bar{s}$ is less than the length of the curve Γ we are looking for.

After integration we obtain

$$\frac{d}{d\varphi}I_\vartheta(\varphi(s)) - \frac{d}{d\varphi}I_\vartheta(\bar\varphi) = s - \bar s \tag{3.12}$$

where $\bar\varphi = \varphi(\bar s)$ may be chosen at will.

If γ is smooth, then the LHS of (3.12) is well-defined for all $\varphi \in \mathbb{R}$. However, this is no longer the case if γ is merely Lipschitz continuous. For such a γ, the LHS of (3.12) is well-defined only a.e.

Now, we are in a position to define a notion of a generalized solution to (1.5) with minimal assumptions on $\bar\gamma$.

Definition. If γ is one-homogeneous and convex, then by a *solution to (1.5)* we mean a closed curve Γ whose the angle parameterization $\varphi(\cdot)$ is a monotone (increasing) multivalued function, which can be treated locally as an L_1 function and the following differential inclusion is valid

$$\partial_\varphi I_\vartheta(\varphi) \ni s - \bar s + s^\star \qquad a.e., \tag{3.13}$$

with initial data $\varphi|_{s=\bar s} = \bar\varphi$ and $s^\star \in \partial_\varphi I_\vartheta(\bar\varphi)$. The inclusion (3.13) holds for such s that $\varphi(s)$ is singlevalued.

Let us notice that (3.10), (with $\frac{d}{d\varphi}$ replaced by ∂_φ), again yields possibility of choosing ϑ so that $\bar s = s^\star \in \partial_\varphi I_\vartheta(\bar\varphi)$ for any $\bar s$, $\bar\varphi \in \mathbb{R}$. Thus, (3.13) will read

$$\partial_\varphi I_\vartheta(\varphi) \ni s. \tag{3.14}$$

We note an expected result.

Corollary 3.1. *Let us suppose that γ is smooth and a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(\bar s) = \bar\varphi$ is a solution to (3.13), then φ is a solution to (3.12). Moreover, if in addition γ is globally stable, then the solution is unique.*

Proof. This fact is obvious, once we realize that for smooth γ, the function $I_\vartheta(\cdot)$ is also smooth and

$$\partial_\varphi I_\vartheta(\varphi) = \{\frac{d}{d\varphi}I_\vartheta(\varphi)\},$$

where $\frac{d}{d\varphi}I_\vartheta(\varphi)$ is the usual derivative of $I_\vartheta(\varphi)$. Uniqueness of solutions follows from strict monotonicity of $\partial_\varphi I_\vartheta(\varphi)$, provided that γ is strictly stable. □

4 The general existence and stability

In this section we prove Theorems 1.1 and 1.2. We want to proceed in a manner similar to that of Section 2. We first show local existence of solutions to (3.13), which replaces (2.5) for non-smooth γ. However, in order to complete the proof of Theorem 1.1, we have to show that the solutions we have constructed yield closed curves. Indeed, this goal is achieved through an approximation procedure. We show for this purpose Theorem 1.2. It states that solutions to (3.13) depend continuously on $\bar{\gamma}$. Thus, the approximation of any energy density function γ by a sequence of smooth, strictly stable energy density functions γ_ϵ yields the desired result.

We begin with an analog of Lemma 2.2 for non-smooth energy density function .

Proposition 4.1. *Let us suppose that $\bar{\gamma}$ is convex, one-homogeneous and ϑ is fixed. Then,*

(a) the multivalued operator $\partial I_\vartheta : \mathbb{R} \to \mathbb{R}$ is maximal monotone;

(b) for any $s \in \mathbb{R}$ there exists $\varphi \in \mathbb{R}$ a solution to $\partial I_\vartheta(\varphi) \ni s$.

Proof. Part (a) is a conclusion from the general theory, see e.g. [Ba] or [Br]. (b) By (3.11) we notice

$$\partial I_\vartheta(2\pi + \varphi) = \partial I_\vartheta(\varphi) + L(\gamma),$$

where $L(\gamma) = \int_\vartheta^{2\pi+\vartheta} \gamma(t)\,dt > 0$. We recall that γ is 2π-periodic, thus $L(\gamma)$ is independent from θ. Since $R(\partial I_\vartheta)$, the range of ∂I_ϑ is connected and contains a sequence which is neither bounded from below nor from above, we conclude that $R(\partial I_\vartheta)$ is equal to $\mathbb{R}$. □

Once we decide upon $\bar{s}$ and $\bar{\varphi}$ we fix ϑ and we will drop for notational convenience the subscript ϑ. The choice of value $\bar{s}$ which is the beginning of counting the length of the solution curve seems irrelevant, however it is not quite so. We will explain it momentarily. Since the function $\varphi \mapsto I(\varphi)$ is convex, then it has one-sided derivatives everywhere. They are equal, i.e. I is differentiable at all points of $\mathbb{R}$ except at most countably many where $\frac{d^- I}{d\varphi}(\psi) < \frac{d^+ I}{d\varphi}(\psi)$. In other words the derivative $\frac{dI}{d\varphi}$ is well defined a.e. and its graph $\Gamma(\frac{dI}{d\varphi})$ is a subset of ∂I. Since ∂I is maximal monotone, so is its inverse graph, $(\partial I)^{-1}$. If we now consider any function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) \in (\partial I)^{-1}(x)$, i.e. $\Gamma(h) \subset (\partial I)^{-1}$, then it is monotone, hence continuous, except, possibly a countable number of points, where h suffers jumps. For this reason, any two such functions differ on a set of measure zero and define a unique element of $L_{p(loc)}(\mathbb{R})$. This observation permits us to identify (when necessary) ∂I or $(\partial I)^{-1}$ with monotone functions defining unique elements of $L_{p(loc)}(\mathbb{R})$.

We are now ready for a definition. If we fix $\vartheta \in \mathbb{R}$ and a one-homogeneous convex γ, then we shall call $s^\star \in \mathbb{R}$ *a regular point of* I_ϑ^γ, if it is a continuity point of any function h defined on $\mathbb{R}$, such that $\Gamma(h) \subset (\partial I_\vartheta^\gamma)^{-1}$. This scant continuity of $(\partial I_\vartheta^\gamma)^{-1}$ will play an important role.

We stress that regularity of $s^\star$ depends upon ϑ, i.e. $s^\star$ may cease to be regular if we change ϑ. In particular, we may assume that in equation (3.13) $s_0 = s^\star - \bar{s}$ is a regular point, possibly after adjusting ϑ and $\bar{\varphi}$. In this case we obtain a unique definition of the solution

$$\varphi(\cdot) = (\partial I)^{-1}(\cdot + s_0), \tag{4.1}$$

where φ is treated as a multivalued function, but we can write $\{\varphi_0\} = (\partial I)^{-1}(s_0)$, because $(\partial I)^{-1}(s_0)$ is a singleton.

Part (b) of Proposition 4.1 immediately yields existence of a solution to (3.13), that is a function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(0) = \varphi_0$, which is an angle parameterization of a curve Γ we seek. We recall that this curve is given (up to a translation) by formula (2.3). We also notice that this formula yields the same result for all L^1 equivalent functions $\varphi(\cdot)$. We should comment on the domain of definition of $\mathbf{x}$, i.e. the length of Γ. We will also explain the meaning of $L(\gamma)$ and the role of regular points. We have just constructed a candidate

Proposition 4.2. *Let us suppose that s_0, φ_0 are fixed, then:*
(a) $(\partial I^\gamma_\vartheta)^{-1}[s_0, s_0 + L(\gamma)]$ contains an interval of length 2π;
(b) if $0 < b < L(\gamma)$ and s_0 is a regular point of I^γ_ϑ, then $(\partial I^\gamma_\vartheta)^{-1}[s_0, s_0 + b]$ does not contain any interval of length 2π.

Proof. Part (a) follows immediately from (3.11).

(b) For a regular point s_0 equation (3.11) takes the form

$$\frac{dI}{d\varphi}(\varphi_0 + 2\pi) = \frac{dI}{d\varphi}(\varphi_0) + L(\gamma). \tag{4.2}$$

Hence, $\frac{dI}{d\varphi}(\varphi_0 + 2\pi) > \frac{dI}{d\varphi}(\varphi_0) + b$ and our claim follows. ☐

We can draw some corollaries.

Corollary 4.1. *Let us suppose that s_0 is a regular point of I^γ_ϑ and curve Γ defined by (2.3), where $\varphi(\cdot)$ is given by (4.1).*
(a) If γ is smooth and strictly stable, then the length of Γ, $d(\Gamma)$, is equal to $L(\gamma)$.
(b) If $\{\gamma_\epsilon\}$ is a sequence of smooth, strictly stable functions converging to γ_0 in $W^1_p(S^1)$, then $d(\Gamma_0)$ is equal to $L(\gamma_0)$.

Proof. (a) By (4.2) we notice that $\varphi(s_0) + 2\pi = \varphi(s_0 + L(\gamma))$. Thus, we see that $d(\Gamma) = L(\gamma)$.

(b) If $\gamma_\epsilon \to \gamma_0$ in $W^1_p(S^1)$ and $\vartheta_\epsilon \to \vartheta_0$, then the parameterizations given by (2.3) converge as well, i.e. $\mathbf{x}_\epsilon \to \mathbf{x}_0$ in W^1_p. Thus,

$$L(\gamma_0) = \lim_{\epsilon\to 0} L(\gamma_\epsilon) = \lim_{\epsilon\to 0} d(\Gamma_\epsilon) = d(\Gamma_0),$$

and our claim follows. ☐

The key point is to show that indeed Γ is a closed curve. The main step toward this goal is the following result on continuous dependence of solutions to (3.13) on the anisotropy $\bar\gamma$. For this purpose we will investigate solutions to (2.1), i.e. curves parameterized by $\mathbf{x}(\cdot)$ (see (2.3)), upon γ. A simple lemma below is our basic tool.

Lemma 4.1. *Let $1 \le p < \infty$ and $\gamma_\epsilon(\cdot) \to \gamma_0(\cdot)$ in $W^1_p(S^1)$, then there exists a sequence $\{\vartheta_\epsilon\}_{\epsilon>0}$ converging to ϑ_0 and for $I_\epsilon := I^\epsilon_{\vartheta_\epsilon}$ we have*

$$(\partial I_\epsilon)^{-1}(\cdot) \to (\partial I_0)^{-1}(\cdot) \quad \text{in} \quad L_{p(loc)}(\mathbb{R}). \tag{4.3}$$

Moreover, if s_0 is a regular point of $I^{\gamma_0}_{\vartheta_0}$ (i.e. $\{\varphi_0\} = (\partial I_0)^{-1}(s_0)$), then we can find a sequence $\{s_\epsilon\}$ such that s_ϵ is a regular point of $I^{\gamma_\epsilon}_{\vartheta_\epsilon}$ (i.e. $\{\varphi_\epsilon\} = (\partial I_\epsilon)^{-1}(s_\epsilon)$) and $s_\epsilon \to s_0$.

Proof. Since $I_\epsilon = I^{\gamma_\epsilon}_{\vartheta_\epsilon}$, then for almost-everywhere-convergence of $\frac{d}{d\varphi} I_\epsilon$ to $\frac{d}{d\varphi} I_0$ it is necessary that $\vartheta_\epsilon \to \vartheta_0$. To show (4.3) it is enough to recall the definition (3.7) and remind that all $(\partial I_\epsilon)^{-1}$ are monotone. The last fact implies that our functions are continuous except at most countable sets, so the rest of Lemma 4.1 is clear, too. Once we realize that, the remaining details become quite elementary and they are left to the interested reader.

□

Theorem 4.1. *If $1 \le p < \infty$ and $\gamma_\epsilon \to \gamma_0$ in $W^1_p(S^1)$, then $\Gamma_\epsilon \to \Gamma_0$ in W^1_p.*

Proof. It is sufficient to show that φ_ϵ (the angle parameterization of Γ_ϵ) converges to φ_0, as long as $\mathbf{v_0}^\epsilon$ converges to $\mathbf{v_0}$, see formula (2.3). These functions are defined on intervals of changing length $d(\Gamma_\epsilon)$. However, due to convergence of $d(\Gamma_\epsilon)$ to $d(\Gamma_0)$, we may restrict our attention to functions φ_ϵ and φ_0 only on interval $[0, d(\Gamma_0)]$, which is sufficient to prove convergence in the L_p-space. By Lemma 4.1 by a proper choice of ϑ_0 and ϑ_ϵ we can make sure s_ϵ, s_0 are regular points, in particular

$$\{\varphi_0(0)\} = (\partial I_0)^{-1}(s_0), \qquad \{\varphi_\epsilon(0)\} = (\partial I_\epsilon)^{-1}(s_\epsilon),$$

moreover $s_\epsilon \to s_0$. Subsequently we will identify $(\partial I_\epsilon)^{-1}$ with its L_p-representative.

Let us note that

$$\begin{aligned} \varphi_\epsilon(s) - \varphi_0(s) &= \partial I_\epsilon^{-1}[s + s_\epsilon] - \partial I_0^{-1}[s + s_0] \\ &= \partial I_\epsilon^{-1}[s + s_\epsilon] - \partial I_0^{-1}[s + s_\epsilon] + \partial I_0^{-1}[s + s_\epsilon] - \partial I_0^{-1}[s + s_0]. \end{aligned} \tag{4.4}$$

Since $\partial I_\epsilon^{-1} \to \partial I_0^{-1}$ in L_p on compact sets, we have

$$||\partial I_\epsilon^{-1}[\,\cdot + s_\epsilon] - \partial I_0^{-1}[\,\cdot + s_\epsilon]||_{L_p(0,d(\Gamma_0))} \le \left(\int_0^{d(\Gamma_0)} |\partial I_\epsilon^{-1}(s + s_\epsilon) - \partial I_0^{-1}(s + s_\epsilon)|^p \, ds \right)^{1/p}$$

$$= \left(\int_{s_\epsilon}^{d(\Gamma_0)+s_\epsilon} |\partial I_\epsilon^{-1}(s) - \partial I_0^{-1}(s)|^p \, ds \right)^{1/p} \le ||\partial I_\epsilon^{-1} - \partial I_0^{-1}||_{L_p(s_0-1, d(\Gamma_0)+s_0+1)} \to 0 \tag{4.5}$$

as $\epsilon \to 0$. Next, we note that

$$||\partial I_0^{-1}(\,\cdot + s_\epsilon) - \partial I_0^{-1}(\,\cdot + s_0)||_{L_p(0,d(\Gamma_0))} \to 0 \tag{4.6}$$

which follows from convergence of s_ϵ to s_0 regularity of s_0 and continuity of the shift operator in the L_p-spaces for $p < \infty$.

As a consequence we get

$$\varphi_\epsilon(\cdot) \to \varphi_0(\cdot) \;\text{ in }\; L_p(0, d(\Gamma_0)). \tag{4.7}$$

Lemma 4.1 implies $\Gamma_\epsilon \to \Gamma_0$ in W^1_p. □

Once the above theorem is at hand we conclude the limiting curve Γ_0 is closed.

Lemma 4.2. *Let us suppose that $\bar{\gamma}_0$ is one-homogeneous, convex and there exists a sequence $\bar{\gamma}_\epsilon$ of one-homogeneous, convex and strictly stable functions such that $\gamma_\epsilon \to \gamma_0$ in $W^1_p(S^1)$. Then any curve Γ whose angle parameterization is a solution to problem (3.13) is a closed curve, moreover it is the Wulff shape of $\bar{\gamma}_0$, up to translation.*

Proof. We have to prove that the curve Γ is closed. For smooth strictly convex γ's the answer is given by Proposition 2.1 in Section 2.

Let us consider an arbitrary $\gamma_0(\cdot)$ and a curve Γ_0 given by solution φ to problem (3.13). Let us assume that Γ_0 is not closed, i.e. there exists $\varphi_0 \in [0, 2\pi)$ such that

$$\mathbf{x}_0(s(\varphi_0)) \neq \mathbf{x}_0(s(\varphi_0 + 2\pi)),$$

in particular

$$|\mathbf{x}_0(s(\varphi_0) - \mathbf{x}_0(s(\varphi_0 + 2\pi))| = T > 0 \tag{4.8}$$

for a number $T > 0$. Since γ_ϵ is a smooth approximation of γ_0 after a proper selection of $\mathbf{v_0}^\epsilon$ in (2.3) we guarantee that

$$||\Gamma_\epsilon - \Gamma||_{C(0,2\pi)} < T/3. \tag{4.9}$$

This is always possible having Theorem 4.1 at hand. But by Proposition 2.1 we know that Γ_ϵ is closed, so by (4.9) we have

$$\begin{aligned} |\mathbf{x}_0(s(\varphi_0)) - \mathbf{x}_0(s(\varphi_0 + 2\pi))| &\leq |\mathbf{x}_0(s(\varphi_0) - \mathbf{x}_\epsilon(s(\varphi_0))| + |\mathbf{x}_\epsilon(s(\varphi_0)) - \mathbf{x}_\epsilon(s(\varphi_0 + 2\pi))| \\ &+ |\mathbf{x}_\epsilon(s(\varphi_0 + 2\pi)) - \mathbf{x}_0(s(\varphi_0 + 2\pi))| < T/3 + 0 + T/3 = 2/3T \end{aligned} \tag{4.10}$$

which contradicts (4.8). Finally, we know that Γ_ϵ is the Wulff shape of γ_ϵ. By (2.13) and the uniform convergence of γ_ϵ to γ_0, we deduce that Γ_0 defined by (4.1) and (2.3) is the Wulff shape of γ_0. □

In order to finish *the proofs of Theorems 1.1 and 1.2*, it is enough to note that finding a sequence approximating general function γ is elementary. For example, for a proper choice of a smooth function ρ, we may take

$$\gamma_\epsilon(\varphi) = \epsilon + \frac{1}{\epsilon} \int_{S^1} \gamma(t) \rho(\frac{\varphi - t}{\epsilon})\, dt. \tag{4.11}$$

Hence Lemma 4.2 implies the first of our main results, stated in Theorem 1.1., i.e. the existence of φ fulfilling the definition of the generalized solution (3.13). Theorem 1.2 follows from Theorem 4.1.

5 Properties of the solutions

Since our case concerns only plane domains, Hessian $K(\mathbf{n}(\cdot))$ – see (2.6) – can be zero on an interval, i.e. $\gamma'' + \gamma = 0$ on (a,b). This in turn is equivalent to F_γ containing a line segment between the angles a and b, (see [Gu, Lemma 7D]). The Euclidean curvature of the Frank diagram is zero there. We want to control precisely the behavior of the system at such points, because singularities may appear there.

For general γ, we concentrate our attention only on the support of K, i.e. on the following set

$$\operatorname{supp} K := \overline{\{\varphi \in S^1 : \mathbf{t}(\varphi) \cdot K(\mathbf{n}(\varphi)) \cdot \mathbf{t}(\varphi) \neq 0\}}. \tag{5.1}$$

It is implicitly understood that the points of non-differentiability of γ belong to $\operatorname{supp} K$. Roughly speaking, $\operatorname{supp} K$ consists of points of strict stability of γ. Certainly, due to our previous considerations $(\partial I)^{-1}$ is single valued and continuous on this set. In order to understand the whole structure of solutions we need to control the behavior of intervals $(a,b) \subset S^1$ such that

$$\mathbf{t}(\varphi) \cdot K(\mathbf{n}(\varphi)) \cdot \mathbf{t}(\varphi) \equiv 0 \qquad \text{for } \varphi \in (a,b). \tag{5.2}$$

In order to make our considerations precise we introduce the following definition. We call an interval (a,b) on which (5.2) holds *maximal* iff for any sufficiently small $\epsilon > 0$ there exist a_* and b^* such that $0 < a - a_* < \epsilon$, $0 < b^* - b < \epsilon$ and for all $w_a \in \partial I(a_*)$ and $w_b \in \partial I(b^*)$ the following relations are satisfied

$$w_a < \partial I(a+\epsilon) \quad \text{and} \quad \partial I(b-\epsilon) < w_b. \tag{5.3}$$

Obviously, all points of interval (a,b) are continuity points of ∂I. However we have no control over behavior of the ends of this interval, which explains the form of (5.3). The above remarks imply that to analyze the behavior on the interval (a,b) it is enough to consider sets $(a-\epsilon, b+\epsilon)$.

The control over these singularities is guaranteed by the following lemma.

Lemma 5.1. *Let us suppose that γ is not strictly stable. If (5.2) holds, then for any solution $\varphi(\cdot)$ of system (2.5) with the range in $\operatorname{supp} K$, for each interval (a,b) satisfying (5.2) there exists a parameter s_0 such that*

$$\lim_{s\to s_0^-} \varphi(s) \le a \quad \textit{and} \quad \lim_{s\to s_0^+} \varphi(s) \ge b. \tag{5.4}$$

Moreover, if (a,b) is maximal (in the meaning of (5.3)) , then

$$\lim_{s\to s_0^-} \varphi(s) = a \quad \textit{and} \quad \lim_{s\to s_0^+} = b. \tag{5.5}$$

Proof. By considerations in the last section we can introduce a strictly stable approximation of the studied γ. We put:

$$\mathbf{t}(\varphi_\epsilon) \cdot K_\epsilon(\varphi_\epsilon) \cdot \mathbf{t}(\varphi_\epsilon) = \epsilon \quad \text{ for } \varphi_\epsilon \in (a,b) \tag{5.6}$$

for $\epsilon > 0$ and

$$\mathbf{t}(\varphi_\epsilon) \cdot K_\epsilon(\varphi_\epsilon) \cdot \mathbf{t}(\varphi_\epsilon) = \mathbf{t}(\varphi_\epsilon) \cdot K(\varphi_\epsilon) \cdot \mathbf{t}(\varphi_\epsilon) + \epsilon \qquad \text{for } S^1 \setminus (a,b). \tag{5.7}$$

By (2.6) we can put $\gamma_\epsilon = \gamma + \epsilon$ getting (5.6) and (5.7). The above definition guarantees us that $\gamma_\epsilon \to \gamma$ in W^1_p at least on a neighborhood of interval (a, b), hence the application of Theorem 1.2 is possible.

Let us suppose that s_a is defined by the following relation $\varphi_\epsilon(s_a) = a$ if (a, b) is maximal. Otherwise, we require $\varphi_\epsilon(s_a) \leq a$ in general one. Then by (2.10) we see that for the approximation satisfying (5.6) we have

$$\frac{d}{ds}\varphi_\epsilon = \frac{1}{\epsilon} \quad \text{as} \quad \varphi \in (a, b), \tag{5.8}$$

hence, we are able to find s_b such that $\varphi_\epsilon(s_b) = b$ in the "maximal" case or $\varphi_\epsilon(s_b) \geq b$ in general one. Formula (5.8) yields $\varphi_\epsilon(s) = \frac{1}{\epsilon}(s - s_a) + a$, so we conclude that $s_b - s_a = \epsilon(b - a)$. Passing with ϵ to 0 we get at the limit $s_a = s_b =: s_0$. Remembering that φ_ϵ and φ are monotone we easily deduce (5.4). From the definition (5.3) we obtain (5.5). We can prove the same result by applying the approximation by regular γ_ϵ's defined by (4.11). However this approach yields additional technical difficulties. Thus, one can conclude that any regular approximation of this case leads to (5.4) or (5.5), respectively.

□

Remark. We are tempted to write formally at point s_0, where function φ has a jump that

$$\left.\frac{d}{ds}\varphi\right|_{s=s_0} = (b - a)\delta(s - s_0).$$

From the geometrical point of view the sought curve Γ will have an interior angle of measure $(b - a)$ at point s_0.

Examples. Here we present examples of solutions which are constructed by our method. We present only the extreme cases: the isotropic and a crystalline $\bar{\gamma}$.

(1) The isotropic energy density is given by $\bar{\gamma}(x) = |x|$, i.e. $\gamma(\varphi) = 1$. Let us take $\vartheta = 0$, $s_0 = 0$, hence $I(\varphi) = \bar{\gamma}(\mathbf{n}(\varphi)) + \int_0^\varphi d\psi \int_0^\psi \bar{\gamma}(\mathbf{n}(t))dt = 1 + \frac{1}{2}\varphi^2$ thus $\partial_\varphi I(\varphi) = \{\frac{d}{d\varphi}I(\varphi)\} = \{\varphi\}$ and $L(\gamma) = 2\pi$. Equation (3.13) takes the form $\varphi(s) = s$. Then we conclude that the solution is a circle, whose length is given by $L(\gamma) = 2\pi$.

(2) This case is sometimes called *crystalline*, because the Wulff shape is a polygon. We may consider the following anisotropy: $\bar{\gamma}(x) = \max\{|x_1|, |x_2|\}$, hence

$$\gamma(\varphi) = \max\{|\cos\varphi|, |\sin\varphi|\}.$$

Elementary calculations, which are left to the interested reader (we may take $s_0 = 0, \vartheta = 0$ in I), and Lemma 5.1 lead us to a conclusion that $d(\Gamma) = 4\sqrt{2}$ and Γ is a square (up to translation) with vertexes at $(\pm 1, 0), (0, \pm 1)$.

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