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LIMIT OF KINETIC TERM FOR A STEFAN PROBLEM

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Abstract. We investigate solutions to the one-phase quasi-stationary Stefan problem with the surface tension and kinetic term. Main results show existence of unique regular solutions with a suitable bound which enables to obtain the limit as the kinetic term is vanishing. Our problem is considered in anisotropic Besov spaces locally in time.

1. **Introduction.** Phase transition phenomena in liquid-solid systems are most often modeled by Stefan type problems [2],[5]. Nonlocal properties of these equations allow to describe melting and freezing processes by controlling the sought phase surface between liquid and solid states of the same material. To account for the diversity of the studied phenomena different sets of PDE's are required. Here we concentrate on the analysis of the influence of the kinetic term which extends the Gibbs-Thomson correction. This rule describes undercooling effects appearing near the transition surface. Also it is a generalization of the classical system where the temperature is set to zero at the boundary. We will study the behavior of the model for the case of vanishing kinetic effects.

In this note we restrict our attention to investigating solutions to the one-phase quasi-stationary Stefan problem with Gibbs-Thomson correction and kinetic term:

$$\begin{array}{lll} \Delta p^\beta = 0 & \text{in} & \Omega_t^\beta, \\ p^\beta = \kappa^\beta - \beta V_{\vec{n}}^\beta & \text{on} & \partial\Omega_t^\beta, \\ \frac{\partial p^\beta}{\partial \vec{n}^\beta} = -V_{\vec{n}}^\beta & \text{on} & \partial\Omega_t^\beta, \\ \Omega_t^\beta|_{t=0} = \Omega_0. & & \end{array} \tag{1}$$

In the above system we look for the evolution of the unknown free surface $\partial\Omega_t^\beta$ and the function p^β which in the classical interpretation is the temperature. The surface $\partial\Omega_t^\beta$ is the boundary of the unknown domain $\Omega_t^\beta \subset \mathbf{R}^{n+1}$. The quantity κ^β denotes the mean curvature of boundary $\partial\Omega_t^\beta$ (we choose the convention: the curvature of a convex domain is negative), $\vec{n}^\beta$ is the normal outward vector to $\partial\Omega_t^\beta$ and $V_{\vec{n}}^\beta$ is the normal velocity describing the evolution of the free surface. The set Ω_0 is the initial domain, bounded in $\mathbf{R}^{n+1}$. An important quantity is the constant parameter $\beta \geq 0$ which controls the influence of the kinetic term. The limit case $\beta = 0$ will be of special interest.

We will omit in the notation "β" in places where this parameter is fixed.

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The mean curvature κ can be stated in local coordinates $z \in \mathbf{R}^n$ as

$$\kappa \sim \Delta_{\partial\Omega_t} \psi(z) \tag{2}$$

up to lower terms of ψ, where $x(z) = (z, \psi(z))$ describes locally the boundary $\partial\Omega_t$ as the graph of ψ and $\Delta_{\partial\Omega_t}$ is the Laplace-Beltrami operator defined in local coordinates by the following explicite formula:

$$\Delta_{\partial\Omega_t} = (det\{g_{ij}\})^{-1/2} \frac{\partial}{\partial z^i}((det\{g_{ij}\})^{1/2} g^{ij} \frac{\partial}{\partial z^j}), \tag{3}$$

where the metric $\{g_{ij}\}$ is induced by the coordinate system $x(z)$.

The main goal of our paper is to analyze the limit of solutions to the above system with $\beta \to 0^+$. The limit case neglects the kinetic term in the considered model. As we will see, from the mathematical point of view this change causes that the studied equations become a parabolic system of the third order. The considerations will be restricted to local in time solutions.

Let us state the main result in regularity described in Besov spaces – see definitions (7)-(9) below and [1],[11].

Theorem 1. *Let $s > 3$, $p > \frac{n+3}{s-2}$ and $0 < \beta < 1$. If $\partial\Omega_0 \in B^{s-2/p}_{pp}$, then there exists a number $T > 0$ such that for any $\beta : 0 < \beta < 1$ there exist unique solutions to problem (1) such that*

$$\bigcup_{0<t<T} \partial\Omega^\beta_t \in B^{s,s/3}_{pp} \qquad \textit{as submanifold in } \mathbf{R}^{n+1} \times (0,T), \tag{4}$$

where $\partial\Omega^\beta_t$ is the solution to system (1) with parameter β and initial domain Ω_0.

Moreover, the properties of the obtained solutions guarantee existence of the limit for $\beta \to 0^+$ such that

$$\partial\Omega^\beta_t \to \partial\Omega^0_t \quad \textit{in} \;\; C^2 \quad \textit{as} \;\; \beta \to 0^+ \;\; \textit{for}\; t \in (0,T). \tag{5}$$

The well posedness to system (1) for $\beta > 0$ has been established in [12] for the two-phase problem and in [3] – for the evolutionary version of (1). The authors noted that boundary condition $(1)_2$ "contains" a parabolic equation of the second order defined on the sought phase surface. The case $\beta = 0$ is more difficult. First results delivering unique classical solutions are obtained in [4],[6]. The main idea is to use the fact that formally the system with $\beta = 0$ is of parabolic type but of third order. Then the abstract theory of semi-groups enables us to prove local in time existence results. In [10] the author considers the limit of the classical solutions to the evolutionary version of system (1) to solutions of the classical Stefan problem (i.e. the boundary condition $(1)_2$ tends to the form: $p = 0$).

Formally system (1) is equivalent to the following linear equation:

$$\partial_t \phi + \frac{(-\Delta)^{3/2}}{1 + \beta(-\Delta)^{1/2}} \phi = \frac{m}{1 + \beta(-\Delta)^{1/2}} \qquad \text{on} \quad \mathbf{R} \times (0,T), \tag{6}$$

where ϕ represents the sought surface and m the external data. This model problem shows how the parameter $\beta = 0$ or $\beta > 0$ determines the type of the system.

The result in Theorem 1 allows us to approximate the system with $\beta = 0$ by a system with $\beta > 0$ with a sufficiently small parameter. Given this fact we will be certain that approximative solutions are in the vicinity of the original solution. On the other side having $\beta > 0$ we gain better regularity of the system. In other words, Theorem 1 shows that system (1) is stable with respect to perturbations of β, even for the limit case $\beta = 0$.

It is worthwhile to underline that the presented results follow from techniques applied for the case $\beta = 0$ for regular solutions in the L_p-framework – see [7] and [9]. This approach “reduces” the problem to the examination of equation (6). Nevertheless, sufficiently high regularity is required to control nonlinearities, traces and nonlocality appearing in the system as well two different types of system (1).

2. **Preliminaries.** To clarify the statement of Theorem 1 we recall the definition of Besov spaces. In the isotropic case for a subdomain $Q \subset \mathbf{R}^d$ the norm describing the space reads:

$$||u||_{B^s_{pp}(Q)} = ||u||_{L_p(Q)} + < u >_{B^s_{pp}(Q)}, \tag{7}$$

where the main semi norm has the form:

$$< u >^p_{B^s_{pp}(Q)} = \sum_{|\alpha|=[s]} \int_Q dx \int_Q dx' \frac{|\partial^\alpha u(x) - \partial^\alpha u(x')|^p}{|x-x'|^{d+p(s-[s])}}. \tag{8}$$

In the anisotropic case for domains of type $Q \times (0,T)$ it reads:

$$\begin{gathered} ||u||_{B^{s,r}_{pp}(Q\times(0,T))} = ||u||_{L_p(Q\times(0,T))} + \\ (\int_0^T < u(\cdot,t) >^p_{B^s_{pp}(Q)} dt)^{1/p} + (\int_Q < u(x,\cdot) >^p_{B^r_{pp}(0,T)} dt)^{1/p}. \end{gathered} \tag{9}$$

For information and properties about these spaces such as imbedding or trace theorems we refer to the books [1] and [11]. Within our considerations we try to use standard notations. Generic constants are denoted by C and dependence on the parameter β is distinguish in crucial parts.

To analyze system (1) an appropriate description of $\partial\Omega_t$ is needed. Let us choose a fixed domain $D \subset \mathbf{R}^{n+1}$ with a smooth (C^∞) boundary such that

$$||\partial D - \partial \Omega_0||_{B^{s-2/p}_{pp}} < \epsilon, \tag{10}$$

where the choice (smallness) of $\epsilon > 0$ is not restricted. Then the sought surface can be represented by a “graph” of a scalar function $\psi : \partial D \to \mathbf{R}$ such that for each $t \in [0,T)$ we have

$$\begin{gathered} \partial\Omega_t = \{x \in \mathbf{R}^{n+1} : x(y,t) = y + \psi(y,t)\vec{n}(y), \\ \text{where } y \in \partial D \text{ and } \vec{n} \text{ is the normal vector to } \partial D \text{ at point } y\}. \end{gathered} \tag{11}$$

A key advantage of the above setting is that the regularity of $\partial\Omega_t$ is represented by the regularity of ψ, since the field of normal vectors $\vec{n}$ is smooth ($\partial D \in C^\infty$). The second advantage is the possibility to construct a diffeomorphism $\Phi_t : D \to \Omega_t$, having the same smoothness as the function ψ and fulfilling the following estimates

$$||Id - \nabla\Phi_t||_{L_\infty} \le C||\psi||_{W^1_\infty} \quad \text{and} \quad ||\nabla\Phi_t||_{B^{s-1,(s-1)/3}_{pp}} \le C(1 + ||\phi||_{B^{s,s/3}_{pp}}). \tag{12}$$

Transforming system (1) to the rigid domain D we get

$$\begin{array}{lll} \Delta q = (\Delta - \Delta^t)q & \text{in} & D, \\ q = \tilde{\kappa} - \beta v & \text{on} & \partial D, \\ \frac{\partial q}{\partial \vec{n}} = -v + (\frac{\partial}{\partial \vec{n}} - \frac{\partial}{\partial \vec{n}^t})q & \text{on} & \partial D, \\ \psi|_{t=0} = \psi_0 & \text{on} & \partial D, \end{array} \tag{13}$$

where $y = \Phi_t^{-1}(x)$, $(\nabla^t)^{(i)} = \sum_{j=1}^{n+1} \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$, $q(y,t) = p(\Phi_t(y),t)$, $\tilde{\kappa} = \kappa(\Phi_t(y),t)$ and $v(y,t) = V_{\vec{n}}(\Phi_t(y),t)$. The vector $\vec{n}^t(y,t)$ is the normal vector to $\partial\Omega_t$ in the (y,t) coordinates and ψ_0 describes the initial surface.

To analyze the above system a suitable linearization is required. The main mathematical properties of these equations are hidden in the first terms of the r.h.s. of the boundary conditions $(13)_{2,3}$. We propose the following linearization of (13):

$$\begin{array}{lll} \Delta p = F & \text{in} & D \times (0,T), \\ p = \Lambda\phi - \beta\partial_t\phi + G & \text{on} & \partial D \times (0,T), \\ \frac{\partial p}{\partial n} = -\partial_t\phi + H & \text{on} & \partial D \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & \partial D, \end{array} \tag{14}$$

where Λ is an elliptic operator defined on the surface ∂D as follows

$$\Lambda\psi = \sum_k Z_k^*[\Delta' Z_k^{-1^*}(\eta_k\psi)], \tag{15}$$

where (Z_k, U_k) is an atlas for the submanifold ∂D (i.e. $U_k \subset \mathbf{R}^n, Z_k : U_k \to \mathbf{R}^{n+1}$), the functions η_k are a partition of unity for ∂D and Δ' is the Laplace operator on $\mathbf{R}^n$. Additionally, we assume that

$$\sup_k \text{ diam supp } \eta_k \leq \lambda, \tag{16}$$

where λ is a sufficiently small number which will be specified letter. The covering number of $\{\text{supp } \eta_k\}$ depends only on the regularity of ∂D and it is independent of smallness of λ.

System (13) can be viewed in the form of equations (14) as we choose functions α, ζ, ξ in the following way:

$$\begin{aligned} F =&(\Delta - \Delta^t)q = div(\nabla - \nabla^t)q + (div - div^t)\nabla^t q =: \alpha(\psi, q), \\ H =&\partial_t\psi\vec{n} \cdot (\vec{n}^t - \vec{n}) + (\frac{\partial}{\partial\vec{n}} - \frac{\partial}{\partial\vec{n}^t})q =: \xi(\psi, q). \end{aligned} \tag{17}$$

Because of the structure of the r.h.s. of $(13)_2$ is easier to consider $G =: \zeta(\psi)$ in each U^k, separately. Then

$$Z_k^{-1^*}(\eta_k G) = Z_k^{-1^*}(\eta_k\tilde{\kappa}) - \Delta' Z_k^{-1^*}(\eta_k\psi) + \beta Z_k^{-1^*}(\partial_t\psi\vec{n} \cdot (\vec{n}^t - \vec{n})). \tag{18}$$

Taking $p = q$ and $\phi = \psi$, F, G and H as in (17) and (18) we obtain system (13) – which is equivalent to the original system (1). The statement of Theorem 1 in the language of system (13) and the function ψ takes the following form.

Theorem 2. *Let the assumptions to Theorem 1 be fulfilled, $\psi_0 \in B_{pp}^{s-2/p}(\partial D)$. Then there exists $T > 0$ such that there exist unique regular solutions to system (13) such that $\psi^\beta \in B_{pp}^{s,s/3}(\partial D \times (0,T))$ and*

$$||\psi^\beta||_{B_{pp}^{s,s/3}} \leq C(T) \qquad \textit{uniformally in } \beta \ (0 < \beta < 1). \tag{19}$$

Moreover there exists a unique limit ψ^0 for $\beta \to 0^+$ such that

$$||\psi^\beta - \psi^0||_{C^{2,2/3}(\partial D \times (0,T))} \to 0 \quad \textit{as} \quad \beta \to 0^+, \tag{20}$$

where ψ^β with $\beta \geq 0$ describes the solutions to system (13) with parameter β.

The above result is equivalent to Theorem 1, hence we concentrate our attention on Theorem 2. Relation (11) determines uniquely the evolution of the sought surface $\partial\Omega_t^\beta$. A key tool in our analysis is a special estimate of solutions to the linearization (14). The main information, which is required, is a bound on the norms of solutions independent of the magnitude of the parameter β. It will allow to pass to the limit with the kinetic term.

3. **Linear problem.** In this section we deal with the linear system (14). To simplify the notation we introduce the following function space

$$\mathcal{P}(D\times(0,T)) = \Big\{p : p \in L_p(0,T; B_{pp}^{s-2-1/p}(D)) \cap B_{pp}^{(s-3)/3}(0,T; B_{pp}^{1+1/p}(D))$$
$$\text{and } \tfrac{\partial p}{\partial \vec{n}}|_{\partial D\times(0,T)} \in L_p(0,T; B_{pp}^{s-3}(\partial D)) \cap B_{pp}^{(s-3)/3}(0,T; L_p(\partial D)) \text{ with}$$
$$||p||_{\mathcal{P}} = ||p||_{L_pB_{pp}^{s-2-1/p}\cap B_{pp}^{(s-3)/3}B_{pp}^{1+1/p}} + ||\tfrac{\partial p}{\partial \vec{n}}|_{\partial D\times(0,T)}||_{L_pB_{pp}^{s-3}\cap B_{pp}^{(s-3)/3}L_p}\Big\}. \tag{21}$$

The main result of this part, giving us the desired bound, is the following.

Theorem 3. *Let $s>3$, $p>\frac{n+3}{s-2}$, $\phi_0 \in B_{pp}^{s-2/p}(\partial D)$, $G \in B_{pp}^{s-2,(s-2)/3}(\partial D \times (0,T))$, $H \in B_{pp}^{s-3,(s-3)/3}(\partial D\times(0,T))$, $F = A\nabla(B\nabla q)$ with matrices $A, B \in B_{pp}^{s-1,(s-1)/3}(D\times(0,T))$ and $q \in \mathcal{P}(D\times(0,T))$. Then there exists a unique solution to problem (14) such that $\phi \in B_{pp}^{s,s/3}(\partial D\times(0,T))$ and $p \in \mathcal{P}(D\times(0,T))$ and*

$$||\phi||_{B_{pp}^{s,s/3}} + \beta||\nabla\partial_t\phi||_{B_{pp}^{s-3,(s-3)/3}} + ||p||_{\mathcal{P}} \le C(T)DATA, \tag{22}$$

where the constant is independent of β, and $DATA$ denotes the sum of norms of the data in assumed regularity.

The main achievement of the above result is the estimate (22). In particular, we obtain information about the term $\beta\partial_t\nabla\phi$, which plays the crucial role in the considerations for the nonlinear system.

To find a suitable approach to system (14) we remove the inhomogeneity from the r.h.s. of $(14)_1$. For this purpose the following elliptic problem is considered

$$\begin{array}{lll} \Delta p = F & \text{in} & D, \\ p = 0 & \text{on} & \partial D. \end{array} \tag{23}$$

The goal is that the function p should belong to a class of regularity guaranteeing that the trace $\frac{\partial p}{\partial n}|_{\partial D} \in B_{pp}^{s-3,(s-3)/3}$. Thus the trace theorem implies a restriction: $s-3>0$, which has been assumed. To find suitable information about solutions to (23) we restrict our attention only to a special form of F as in Theorem 3.

We have

Lemma 1. *Let F satisfy the assumptions from Theorem 3. Then the solutions to (23) belong to $\mathcal{P}(D\times(0,T))$ and the following estimate is valid*

$$||p||_{\mathcal{P}} \le C||A||_{B_{pp}^{s-1,(s-1)/3}}||B||_{B_{pp}^{s-1,(s-1)/3}}||q||_{\mathcal{P}}.$$

We skip the proof of the above lemma. A similar result can be found in [9]. The main difficulty here is the regularity with respect to time for traces at the boundary. However, the form of the r.h.s. of (23) helps to show it by standard methods.

The next step is to examine the system in the half space. This point is the most important one in the proof of Theorem 3. Having already Lemma 1 we consider a model version of problem (14):

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \mathbf{R}_+^{n+1}, \\ p = \Lambda\phi - \beta\partial_t\phi + G & \text{on} & \mathbf{R}^n, \\ p_{,z_{n+1}} = -\partial_t\phi + H & \text{on} & \mathbf{R}^n, \\ \phi|_{t=0} = \phi_0 & \text{on} & \mathbf{R}^n, \\ \phi, p \to 0 & \text{as} & |x| \to \infty, \end{array} \tag{24}$$

Lemma 2. *Let $G \in B_{pp}^{s-2,(s-2)/3}(\mathbf{R}^n \times (0,T))$, $H \in B_{pp}^{s-3,(s-3)/3}(\mathbf{R}^n \times (0,T))$ and $\phi_0 \in B_{pp}^{s-2/p}(\mathbf{R}^n)$. Then there exists a unique solution to problem (24) such that*

$$\phi \in B_{pp}^{s,s/3}(\mathbf{R}^n \times (0,T)) \quad \text{and} \quad p \in \mathcal{P}(\mathbf{R}_+^{n+1} \times (0,T))$$

and

$$\begin{aligned} &||\phi||_{B_{pp}^{s,s/3}} + \beta||\nabla\partial_t\phi||_{B_{pp}^{s-3,(s-3)/3}} + ||p||_{\mathcal{P}} \\ &\leq C(||\nabla G||_{B_{pp}^{s-3,(s-3)/3}} + ||H||_{B_{pp}^{s-3,(s-3)/3}} + ||\phi_0||_{B_{pp}^{s-2/p}}). \end{aligned} \tag{25}$$

Proof. We want to concentrate our analysis on the boundary of the half space. Taking the Fourier transform with respect to tangent directions

$$\mathcal{F}_x[u](\xi) = \hat{u}(\xi) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} e^{-ix\xi}u(x)dx, \tag{26}$$

we get from $(24)_{1,5}$ the form of the function p: $\hat{p} = \hat{p}(\xi,0,t)e^{-|\xi|z_{n+1}}$. It allows to consider the boundary relations $(24)_{2,3}$ as follows

$$\hat{p} = -|\xi|^2\hat{\phi} - \beta\partial_t\hat{\phi} + \hat{G} \quad \text{and} \quad |\xi|\hat{p} = \partial_t\hat{\phi} + \hat{H} \quad \text{on} \ \mathbf{R}^n \times (0,T). \tag{27}$$

From (27) we get

$$(1+\beta|\xi|)\partial_t\hat{\phi} + |\xi|^3\hat{\phi} = |\xi|\hat{G} + \hat{H} \qquad \text{on} \ \mathbf{R}^n \times (0,T). \tag{28}$$

To see the type of the above equation we rewrite (28) in the following form

$$\begin{gathered} \partial_t\hat{\phi} + \tfrac{|\xi|^3}{1+\beta|\xi|}\hat{\phi} = \tfrac{|\xi|\hat{G}+\hat{F}}{1+\beta|\xi|} \quad \text{on } \mathbf{R}^n \times (0,T), \\ \phi|_{t=0} = \phi_0 \quad \text{on } \mathbf{R}^n. \end{gathered} \tag{29}$$

Here we obtain equation (6) which has been mentioned at the beginning. System (29) determines our whole analysis of the problem (1).

For a fixed $\beta > 0$ equation (29) is parabolic with an elliptic operator of order two. The standard L^2-energy methods imply the uniqueness in time.

It will allow to extend the system with respect to this direction on the whole $\mathbf{R}$. Extending the initial data such that:

$$\begin{gathered} \tilde{\phi}|_{t=0} = \phi_0, \qquad \tilde{\phi}_t|_{t=0} = \mathcal{F}_x^{-1}\left[\tfrac{|\xi|\hat{G}+\hat{F}}{1+\beta|\xi|}|_{t=0} - \tfrac{|\xi|^3}{1+\beta|\xi|}\hat{\phi}_0\right], ..., \\ \partial_t^{[s]}\tilde{\phi}|_{t=0} = \mathcal{F}_x^{-1}\left[\partial_t^{[s]}[\tfrac{|\xi|\hat{G}+\hat{F}}{1+\beta|\xi|}]|_{t=0} - \tfrac{|\xi|^3}{1+\beta|\xi|}\partial_t^{[s]-1}\tilde{\phi}|_{t=0}\right], \end{gathered} \tag{30}$$

by the theory from [11] we are able to preserve the norm. However here we use the fact that $\phi_0 \in B_{pp}^{s-2/p}$, hence we choose such that extension $\tilde{\phi} \in B_{pp}^{s,s/2}$. Then the imbedding theorem [1, Chap. XVIII] implies $\nabla\partial_t\tilde{\phi} \in B_{pp}^{s-3,(s-3)/2}$. In particular, we get the following bound

$$||\partial_t\tilde{\phi}||_{B_{pp}^{0,s/3}} + ||\nabla^3\tilde{\phi}||_{B_{pp}^{s,0}} + ||\nabla\partial_t\tilde{\phi}||_{B_{pp}^{s-3,(s-3)/3}} \leq C||\phi_0||_{B_{pp}^{s-2/p}}. \tag{31}$$

In this place the higher regularity of the initial data is required. We can not take only $\phi_0 \in B_{pp}^{s-3/p}$, since the control of the last term of the l.h.s. would be lost.

The extension (30) modifies the solution to (29) as follows

$$\phi_1 = \phi - \tilde{\phi}, \tag{32}$$

then the special form of the chosen extension $\tilde{\phi}$ implies that ϕ_1 satisfies

$$\partial_t\hat{\phi}_1 + \frac{|\xi|^3}{1+\beta|\xi|}\hat{\phi}_1 = \frac{\hat{m}}{1+\beta|\xi|} \qquad \text{on} \ \mathbf{R}^n \times (-\infty,T) \tag{33}$$

with a function $m \in B_{pp}^{s-3,(s-3)/3}(\mathbf{R}^n \times (-\infty, T))$ such that supp $m \in \mathbf{R}^n \times [0,T]$. Given this fact we conclude that $\phi_1 \equiv 0$ for $t < 0$.

Next, we extend the function m, denote it by m_2, on the whole $\mathbf{R}$ (preserving the norm), hence we get a problem in the whole space time

$$\partial_t \hat{\phi}_2 + \frac{|\xi|^3}{1+\beta|\xi|}\hat{\phi}_2 = \frac{\hat{m}_2}{1+\beta|\xi|} \quad \text{in } \mathbf{R}^n \times (-\infty, +\infty). \tag{34}$$

Next, we apply the Fourier transform with respect to time ($t \leftrightarrow \xi_0$). For the extended problem we are able to write the explicit formula for the solution to (34)

$$\hat{\phi}_2 = \frac{\hat{m}_2}{i\xi_0(1+\beta|\xi|)+|\xi|^3}. \tag{35}$$

To get our sought bound on the norms of the function ϕ_2 we apply Marcinkiewicz's theorem (see [11]) for multipliers (multiplicators) which says

Lemma 3. *(Marcinkiewicz's theorem) Let $\Phi : \mathbf{R}^n \setminus \{0\} \to \mathbf{C}$ be a smooth function such that $|\partial^\alpha \Phi| \le C_\alpha |\xi|^{-|\alpha|}$ for all multi-indeces $|\alpha| \le [\frac{n}{2}] + 1$, let T be a linear operator such that $Tg = \mathcal{F}_x^{-1}[\Phi \mathcal{F}_x[g]]$, then for $1 < p < \infty$ and $s > 0$ it holds:*

$$||Tg||_{B^s_{pp}} \le M_{p,s}||g||_{B^s_{pp}}.$$

The formula (35) yields

$$\begin{gathered}\partial_t \phi_2 = \mathcal{F}^{-1}_{x,t}\left[\tfrac{i\xi_0}{i\xi_0(1+\beta|\xi|)+|\xi|^3}\hat{m}_2\right], \quad \partial_{x_j}\partial_{x_k}\partial_{x_l}\phi_2 = \mathcal{F}^{-1}_{x,t}\left[\tfrac{-i\xi_j\xi_k\xi_l}{i\xi_0(1+\beta|\xi|)+|\xi|^3}\hat{m}_2\right], \\ \beta\partial_t\partial_{x_j}\phi_2 = \mathcal{F}^{-1}_{x,t}\left[\tfrac{-\beta\xi_0\xi_j}{i\xi_0(1+\beta|\xi|)+|\xi|^3}\hat{m}_2\right].\end{gathered} \tag{36}$$

Multipliers appearing in (36) fulfill the conditions from Lemma 3, thus remembering the definition (9) we easily conclude

$$||\phi_2||_{B^{s,s/3}_{pp}} + \beta||\nabla\partial_t\phi_2||_{B^{s-3,(s-3)/3}_{pp}} \le C||m_2||_{B^{s-3,(s-3)/3}_{pp}}. \tag{37}$$

We should underline that the r.h.s. of (37) does not depend on β, although the l.h.s. does. The form of (37) is a consequence of the explicit formula (35) (and (36)) which characterizes the dependence of solutions to (34) from β.

By the uniqueness we obtain the solution to system (29) on the time interval $(0,T)$ via (32) in the form

$$\phi = \phi_2 + \tilde{\phi}. \tag{38}$$

Next, using standard techniques (see [9]) from $(24)_{1,2}$ we deduce the regularity of the function p described by estimate (25). □

To end the proof of Theorem 3 we need to show a result as in Lemma 2, but for a bounded domain. The classical theory [8] reduces this problem, by the method of the regularizer, to a system in the half space. This technique is based on a suitable localization of the studied problem. Thus, the kernel of the proof of Theorem 3 is Lemma 2 proved here in detail. We skip the rest of the proof of Theorem 3 and we refer to [9], where the whole proof for the case $\beta = 0$ with all details of this technique can be found.

4. **Existence – $\beta > 0$ case.** In this section we prove the first part of Theorem 2 – the existence of solutions to system (13). Having already Theorem 3 we find a solution to problem (13) with $\beta > 0$ by an application of the Banach fixed point theorem to the following problem

$$\begin{array}{lll} \Delta p = \alpha(\psi, q) & \text{in} & D \times (0,T), \\ p = \Lambda\phi - \beta\partial_t\phi + \zeta(\psi) & \text{on} & \partial D \times (0,T), \\ \frac{\partial p}{\partial \vec{n}} = -\partial_t\phi + \xi(\psi, q) & \text{on} & \partial D \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & \partial D. \end{array} \tag{39}$$

The functions α, ζ, ξ are defined by (17) and (18), then the fixed point to system (39) defines a solution to problem (13).

By the choice of D we control the smallness of the initial function ψ_0 – see (10) – in the $B^{s-2/p}_{pp}$-norm. Hence our investigations are restricted to a small data problem. Solutions will be searched in the set

$$\begin{aligned} \Xi = \Big[&(\phi, p) \in B^{s,s/3}_{pp}(\partial D \times (0,T)) \times \mathcal{P}(D \times (0,T)) \ \text{ and} \\ &\partial_t \nabla\phi \in B^{s-3,(s-3)/3}_{pp}(\partial D \times (0,T)), \quad \text{additionally } ||(\phi,p)||_\Xi \leq \delta, \\ &\text{where} \quad ||(\phi,p)||_\Xi = ||\phi||_{B^{s,s/3}_{pp}} + \beta||\partial_t\nabla\phi||_{B^{s-3,(s-3)/3}_{pp}} + ||p||_{\mathcal{P}} \Big]. \end{aligned} \tag{40}$$

We will show that the map $K(\psi, q) = (\phi, p)$, described by system (39), is a contraction on the set Ξ, provided δ and T are sufficiently small.

The results of Theorem 3 deliver the following bound on solutions to (39)

$$\begin{aligned} ||(\phi,p)||_\Xi \leq &C(T)(||\alpha(\psi,q)||_{.} + ||\nabla\zeta(\psi)||_{B^{s-3,(s-3)/3}_{pp}} \\ &+ ||\xi(\psi,q)||_{B^{s-3,(s-3)/3}_{pp}} + ||\psi_0||_{B^{s-2/p}_{pp}}), \end{aligned} \tag{41}$$

where $||\cdot||_{.}$ denotes the norm appearing in the r.h.s. of the bound in Lemma 1.

Take the term $\alpha(\psi, q)$. Applying (17) and Lemma 1 we get

$$||\alpha(\psi,q)||_{.} \leq ||div(\nabla - \nabla^t)q||_{.} + ||(div - div^t)\nabla^t q||_{.} \tag{42}$$

$$\leq C||Id - \nabla\Phi_t||_{B^{s-1,(s-1)/3}_{pp}} ||\nabla\Phi_t||_{B^{s-1,(s-1)/3}_{pp}} ||q||_{\mathcal{P}}. \tag{43}$$

By the properties of Φ_t given by (12) we obtain

$$||\alpha(\psi,q)||_{.} \leq C(1 + ||\psi||_{B^{s,s/3}_{pp}})||\psi||_{B^{s,s/3}_{pp}}||q||_{\mathcal{P}}. \tag{44}$$

Next, recalling (17)-(18) and (10)-(12) we deduce

$$\begin{aligned} ||\nabla\zeta(\psi)||_{B^{s-3,(s-3)/3}_{pp}} \leq &\sum_k ||Z^{-1*}_k(\eta_k\psi) - \Delta' Z^{-1*}_k(\eta_k\psi)||_{B^{s-2,(s-2)/3}_{pp}} \\ &+ \sum_k ||\beta\nabla(Z^{-1*}_k(\partial_t\psi(\vec{n}(\vec{n}^t - \vec{n}))||_{B^{s-3,(s-3)/3}_{pp}} \\ \leq &C(\epsilon + ||\psi||_{B^{s,s/3}_{pp}} + \beta||\partial_t\nabla\psi||_{B^{s-3,(s-3)/3}_{pp}})||\psi||_{B^{s-3,(s-3)/3}_{pp}}. \end{aligned} \tag{45}$$

From (45) and (10) we have

$$||\nabla\zeta(\psi)||_{B^{s-3,(s-3)/3}_{pp}} \leq C(\epsilon + ||\psi||_{B^{s,s/3}_{pp}})||\psi||_{B^{s,s/3}_{pp}}. \tag{46}$$

Similarly we find the bound for ξ

$$||\xi(\psi,q)||_{B^{s-3,(s-3)/3}_{pp}} \leq C||\psi||_{B^{s,s/3}_{pp}}(||\psi||_{B^{s,s/3}_{pp}} + ||q||_{\mathcal{P}}). \tag{47}$$

Hence from (41)-(47) we get

$$||(\phi,p)||_{\Xi} \leq C(T)[(\epsilon + ||(\psi,q)||_{\Xi})||(\psi,q)||_{\Xi} + ||\psi_0||_{B^{s-2/p}_{pp}}]. \tag{48}$$

If the norm of ψ_0 and T are sufficiently small, we find $\epsilon, \delta > 0$ such that

$$K : \Xi \to \Xi. \tag{49}$$

Next, we prove that the map K is a contraction on the set Ξ by an analysis of the difference

$$K(\psi_1, q_1) - K(\psi_2, q_2) = (\phi_1 - \phi_2, p_1 - p_2).$$

It is equivalent to study the following problem arising from (39)

$$\begin{array}{lll} \Delta(p_1 - p_2) = \alpha(\psi_1, q_1) - \alpha(\psi_2, q_2) & \text{in} & D \times (0,T), \\ p_1 - p_2 = \Lambda(\phi_1 - \phi_2) - \beta\partial_t(\phi_1 - \phi_2) + \zeta(\psi_1) - \zeta(\psi_2) & \text{on} & \partial D \times (0,T), \\ \frac{\partial}{\partial n}(p_1 - p_2) = \partial_t(\phi_1 - \phi_2) + \xi(\psi_1, q_1) - \xi(\psi_2, q_2) & \text{on} & \partial D \times (0,T), \\ \phi_1 - \phi_2 = 0 & \text{on} & \partial D. \end{array} \tag{50}$$

Again repeating the considerations for (41)-(48) for system (39) we easily get the following estimate for solutions to (50)

$$||(\phi_1 - \phi_2, p_1 - p_2)||_{\Xi} \leq \frac{1}{2}||(\psi_1 - \psi_2, q_1 - q_2)||_{\Xi}, \tag{51}$$

provided ϵ, δ, T are sufficiently small.

Inequality (51) proves that the map K is a contraction on the set Ξ, implying existence of unique solutions (ψ^β, q^β) to problem (13). Moreover the properties of the estimate from definitions (40) and (21) enable us to deduce the following bound

$$||\psi^\beta||_{B^{s,s/3}_{pp}} + \beta||\partial_t\nabla\psi^\beta||_{B^{s-3,(s-3)/3}_{pp}} + ||q^\beta||_{B^{s-2+1/p,(s-2)/3}_{pp}} \leq C(T, ||\psi_0||_{B^{s-2/p}_{pp}}), \tag{52}$$

where the constant in (52) is independent of β.

5. **The limit.** We show the last part of Theorem 2 – we study the limit $\beta \to 0^+$. Thanks to the properties of the estimate (52) we are able to show that

$$\begin{array}{c} \psi^\beta \rightharpoonup \psi^0 \quad \text{weakly in } \ B^{s,s/3}_{pp}(\partial D \times (0,T)) \text{ as } \beta \to 0^+, \\ q^\beta \rightharpoonup q^0 \quad \text{weakly in } \ B^{s-2+1/p,(s-2)/3}_{pp}(D \times (0,T)) \text{ as } \beta \to 0^+. \end{array} \tag{53}$$

The above convergence holds only for a subsequence, however at the end of our analysis we obtain this convergence for the general limit $\beta \to 0^+$.

To show that the limit is a solution to system (13) with $\beta = 0$ we rewrite problem (13) in the following weak sense – recall (17)-(18). A pair (ψ^β, q^β) is a weak solution to (13) iff (ψ^β, q^β) fulfills estimate (52) and the following integral identity is valid

$$\begin{array}{c} \displaystyle\int_{\partial D}\left(\partial_t\psi^\beta v + (\Lambda\psi^\beta - \beta\partial_t\psi^\beta)\frac{\partial v}{\partial \vec{n}}\right)d\sigma - \int_D q^\beta \Delta v dx = \\ \displaystyle\int_{\partial D}\left(\xi(\psi^\beta, q^\beta)v - \zeta(\psi^\beta)\frac{\partial v}{\partial \vec{n}}\right)d\sigma + \int_D \alpha(\psi^\beta, q^\beta)v dx \end{array} \tag{54}$$

for each $v \in C^\infty(\overline{D} \times [0,T])$.

From (53) we conclude that almost all terms in (54) tend to the limit at least in the L^1-norm. The chosen regularity is so high that, by the Rellich theorem, the strong convergence, even for term $\zeta(\psi^\beta) \to \zeta(\psi^0)$, in the L^1-space ($s > 3$

and $p > \frac{n+3}{s-2}$) is obtained. Indeed, for our class of regularity we get the strong convergence being a conclusion from (53). In particular, we have

$$\begin{aligned} \psi^\beta \to \psi^0 \quad \text{strongly} \quad C^{2,2/3}(\partial D \times (0,T)) \text{ as } \beta \to 0^+, \\ q^\beta \rightharpoonup q^0 \quad \text{strongly} \quad B_{pp}^{s-2,(s-2)/4}(D \times (0,T)) \text{ as } \beta \to 0^+. \end{aligned} \tag{55}$$

Considering the last term in (54) with α which is, in general case, a distribution only, we note that the norm of the space $\mathcal{P}$ controls the traces. Hence we can pass to the limit in the boundary terms, too.

Thus, we get that the limit (ψ^0, q^0) fulfills the following identity

$$\begin{aligned} \int_{\partial D} \left(\partial_t \psi^0 v + \Lambda \psi^0 \frac{\partial v}{\partial \vec{n}} \right) d\sigma - \int_D q^0 \Delta v dx = \\ \int_{\partial D} \left(\zeta(\psi^0, q^0) v - \xi(\psi^0) \frac{\partial v}{\partial \vec{n}} \right) d\sigma + \int_D \alpha(\psi^0, q^0) v dx \end{aligned} \tag{56}$$

for each $v \in C^\infty(\overline{D} \times [0,T])$.

The integral identity (56) is the weak form of system (13) with $\beta = 0$. However the regularity implies that the solution, satisfying (56), is regular (we look at the regularity of the surface, the first equation may be satisfied in a weak sense). So we get a regular solution to system (13) with $\beta = 0$. By the results from [6] and [9] the obtained regularity guarantees the uniqueness of solutions, hence there is no need to restrict our considerations to a subsequence, so limits (53) and (55) hold for the general case $\beta \to 0^+$. Theorem 2 (as well Theorem 1) has been proved.

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