# Convergence of Rothe's scheme for the Navier-Stokes equations with slip conditions in 2D domains

**Piotr B. Mucha**[1,*] **and Reimund Rautmann**[2,**]

[1] Institute of Applied Mathematics and Mechanics, Warsaw University, ul. Banacha 2, 02097 Warszawa, Poland
[2] Department of Mathematics, Paderborn University, Warburgerstr. 100, 33095 Paderborn, Germany

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We investigate Rothe's scheme for the Navier-Stokes equations in two dimensional bounded domains with slip boundary conditions admitting flow across the boundary. The structure of the model enables to reformulate it into the coupled system of the vorticity and velocity problems. The obtained convergence of our first order approximation is the same as for case in domains without boundaries although we admit inhomogeneous boundary data. The method is based on the maximum principle for the vorticity equation which delivers a new bound for the solutions. However our study can be realized only under a geometrical constraint which restricts the shape of the domain.



## 1 Introduction

We analyze an approximation of the following evolutionary system

$$\begin{aligned} v_t + v\cdot\nabla v - \nu\Delta v + \nabla p = 0 \quad &\text{in} \quad \Omega\times(0,T)\,, \\ \operatorname{div} v = 0 \quad &\text{in} \quad \Omega\times(0,T)\,, \\ v|_{t=0} = v_0 \quad &\text{on} \quad \Omega \end{aligned} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbf{R}^2$ with the following slip boundary conditions

$$\begin{aligned} n\cdot\mathbf{T}(v,p)\cdot\tau + \nu f_0 v\cdot\tau = 0 \quad &\text{on} \quad \partial\Omega\times(0,T)\,, \\ n\cdot v = d \quad &\text{on} \quad \partial\Omega\times(0,T)\,, \end{aligned} \tag{1.2}$$

where $n$ and $\tau$ are the normal and tangent vectors to the boundary, $\nu > 0$ is the constant viscosity coefficient and $f_0 \geq 0$ is the effective friction coefficient. Problem (1.1)–(1.2) describes motion of incompressible viscous fluid governed by the Navier-Stokes equations $(1.1)_{1,2}$ with boundary relations involving friction effects at the boundary. Vector $v = (v^1, v^2)$ describes the velocity of the fluid, $p$ – the pressure, and $\mathbf{T}$ the stress tensor

$$\mathbf{T}(v,p) = \nu\mathbf{D}(v) - p\,Id = \left\{\nu\left(v^i_{,j} + v^j_{,i}\right) - p\delta_{ij}\right\}_{i,j=1,2}\,, \tag{1.3}$$

where the comma denotes the differentiation.

In general we admit a flow across the boundary, thus datum $d$ may be nontrivial. We require to domain $\Omega$ be simply connected and boundary $\partial\Omega$ be smooth at least $C^{4+a}$ for a number $a > 0$.

By $(1.1)_2$ and $(1.2)_2$ the system needs to satisfy the following compatibility conditions on the data

$$v_0\cdot n|_{\partial\Omega} = d|_{t=0} \tag{1.4}$$

and

$$\int_{\partial\Omega} d(x,t)d\sigma_x = 0 \quad \text{for} \quad t\in(0,T)\,. \tag{1.5}$$

* Corresponding author, e-mail: p.mucha@mimuw.edu.pl
** e-mail: rautmann@math.uni-paderborn.de

The model described by (1.1)–(1.5) has slightly different properties than systems given by the standard approach, where we use the Dirichlet boundary data [8,16]. The slip conditions (1.2) give implicate constraints on the velocity at the boundary describing the friction effects between the surface and fluid, instead of describing the velocity which in general is difficult to measure. If the friction is going to infinity and $d \equiv 0$ then from (1.2) we obtain the Dirichlet no slip condition $u \equiv 0$. In fact, relations (1.2) are suitable to model flows in pipe-like domains of fluids with different structures than water. As good examples we may consider flows of polymers or blood [2,4], where Dirichlet data are contradictory to the experiment. Since in our case the magnitude of boundary data (1.2) will not be restricted, the system can model turbulent flows, so our approach seems to be suitable for other systems describing such phenomena [9]. Looking on the mathematical analysis, system (1.1)–(1.2) is a good and natural approximation of the motion of the perfect fluid [2,11]. That is the reason we want to investigate properties of this system from the numerical point of view. We will see in Sect. 2 that some important features of the presented scheme are independent of the viscosity coefficient.

Our goal in this paper is to study a discretization of system (1.1)–(1.2) with respect to time. However because of inhomogeneous boundary data we will not investigate the original problem, but its reformulation.

Taking the rotation of $(1.1)_{1,3}$ we get

$$\begin{aligned} \alpha_t + v \cdot \nabla\alpha - \nu\Delta\alpha = 0 \quad &\text{in} \quad \Omega \times (0,T)\,, \\ \alpha|_{t=0} = \operatorname{rot} v_0 = \alpha_0 \quad &\text{on} \quad \Omega\,, \end{aligned} \tag{1.6}$$

where

$$\alpha = \operatorname{rot} v = v^2_{,1} - v^1_{,2} \tag{1.7}$$

is the vorticity of the velocity. Boundary data for (1.6) are obtained from (1.2) as follows

$$\alpha = (2\chi - f_0) v \cdot \tau - 2d_{,s} \quad \text{on} \quad \partial\Omega \times (0,T)\,, \tag{1.8}$$

where $\chi$ is the curvature of $\partial\Omega$ and $s$ is the unit length parameter of curve $\partial\Omega$. To compute (1.8) it is enough to differentiate $(1.2)_2$ with respect to $s$ and combine it with $(1.2)_1$ - see Appendix.

The velocity may be obtained back from the following elliptic system

$$\begin{aligned} \operatorname{rot} v = \alpha \quad &\text{in} \quad \Omega\,, \\ \operatorname{div} v = 0 \quad &\text{in} \quad \Omega\,, \\ n \cdot v = d \quad &\text{on} \quad \partial\Omega\,. \end{aligned} \tag{1.9}$$

The structure of system (1.6)–(1.9) is more suitable to our study, hence we will investigate it instead of (1.1)–(1.2). The parabolic type of (1.6)–(1.8) will allow to apply the maximum principle and (1.9) is a linear elliptic system which will be simplified to the Laplace equation.

For the coupled system (1.6)–(1.9) we define Rothe's scheme to construct a first order semi-discretization with respect to time in the following way

$$\frac{\alpha^h_k - \alpha^h_{k-1}}{h} + v^h_{k-1} \cdot \nabla\alpha^h_k - \nu\Delta\alpha^h_k = 0 \quad \text{in} \quad \Omega \tag{1.10}$$

with boundary data

$$\alpha^h_k = (2\chi - f_0) v^h_{k-1} \cdot \tau - d^h_{k,\tau} \quad \text{on} \quad \partial\Omega\,, \tag{1.11}$$

where $d^h_k = d(x, kh)$, $k \in \mathbf{N}$, and $h > 0$.

The part of the reformulation describing the velocity reads

$$\begin{aligned} \operatorname{rot} v^h_k = \alpha^h_k \quad &\text{in} \quad \Omega\,, \\ \operatorname{div} v^h_k = 0 \quad &\text{in} \quad \Omega\,, \\ n \cdot v^h_k = d^h_k \quad &\text{on} \quad \partial\Omega\,. \end{aligned} \tag{1.12}$$

Vector $v^h_0$ is taken as the initial data $v_0$ and $\alpha^h_0 = \operatorname{rot} v_0$.

Functions $\alpha_k^h(x)$ are connected with the approximation of the solutions to the problem as follows

$$\alpha_k^h(x) \sim \alpha(x, kh)\,. \tag{1.13}$$

To begin our considerations we need an existence result for solutions to system (1.1)–(1.2) on the considered time interval $[0,T]$. In a special case we will investigate $T = +\infty$. Taking a modification of results from [11, Theorem 1.1] (see also [12, Theorem B]) we conclude the following theorem.

**Theorem 1.1.** *Let* $0 < a < 1$, $f_0 \in L_\infty(\partial\Omega)$, $v_0 \in C^a(\Omega)$, $\operatorname{rot} v_0 \in L_\infty(\Omega)$, *and* $div\, v_0 = 0$, *moreover* $d \in C^{1,1/2}(\partial\Omega \times (0,T))$.
*If*

$$||\gamma_\infty(2\chi - f_0)||_{L_\infty(\partial\Omega)} < 1\,, \tag{1.14}$$

*where* $\gamma_\infty = ||W||_{C(\Omega)}$ *and vector* $W$ *is the solution to the following problem*

$$\begin{aligned} \operatorname{rot} W &= 1 \quad \textit{in} \quad \Omega\,,\\ div\, W &= 0 \quad \textit{in} \quad \Omega\,,\\ n \cdot W &= 0 \quad \textit{on} \quad \partial\Omega\,. \end{aligned} \tag{1.15}$$

*Then there exists unique solution to problem* (1.1)–(1.2) *such that*

$$v \in C^{a,a/2}(\Omega \times (0,T))\,, \qquad \operatorname{rot} v \in L_\infty(\Omega \times (0,T))\,. \tag{1.16}$$

*Additionally the following bound is valid*

$$\sup_{t \in (0,T)} ||\operatorname{rot} v(\cdot,t)||_{L_\infty(\Omega)} \le B(||d||_{C^{1,1/2}(\partial\Omega\times(0,T))} + ||\operatorname{rot} v_0||_{L_\infty(\Omega)}) = S_0\,, \tag{1.17}$$

*where* $B$ *is independent of* $\nu$ *and* $T$. *Hence in particular* $T$ *can be equal* $+\infty$.
*Moreover, recalling that boundary* $\partial\Omega$ *is* $C^{4+a}$*-smooth; if data are sufficiently regular and fulfill additional compatibility conditions, i.e.* $\alpha_0 \in C^{4+a}(\Omega)$, $d_{,s} \in C^{4+a,2+a/2}(\partial\Omega \times (0,T))$, *and* $f \in C^{4+a}(\partial\Omega)$, *then we are able to show existence of solutions such that* $\alpha_t \in C^{2+a,1+a/2}(\Omega \times (0,T))$ *and the following bound is valid*

$$||\Delta\alpha_t||_{L_\infty(\Omega\times(0,T))} \le S_1\,, \tag{1.18}$$

*where* $S_1$ *depends on* $S_0$ *and viscosity* $\nu$, *and is independent of* $T$.

The above result is valid only under geometrical condition (1.14). Then we obtain bound (1.16) which by standard techniques guarantees (1.18), also for $T = \infty$ (even if datum $d$ does not vanish for large times). An interesting and important feature of bound (1.17) is a linear character, the same as for the standard energy estimate for problems with homogenous boundary data. Also the independence from the viscosity is worthwhile to underline. The technique of the proof is a generalization of the method for the two dimensional problem in the whole space. The slip conditions give a possibility to apply the maximum principle to the vorticity equation from which it follows that the influence in the main estimate of the nonlinear term is neglected.

Constraint (1.14) restricts shape of considered domains. The curvature is bounded by the magnitude of constant $\gamma_\infty$ given by system (1.15). This constant is proportional to the constant in the Poincaré inequality. It seems that the class of admissible domains is narrow, however all convex sets belong to this class, since a suitable choice of friction function $f_0$ makes quantity $|2\chi - f_0|$ arbitrary small.

Let us explain estimate (1.18), because it has not been shown in [11]. Since the boundary of our domain is sufficiently smooth and condition (1.8) enables to apply the standard bootstrap method to increase regularity of solutions, we are able to use the theory of parabolic equations from [10] and obtain the solution in such a class that $\alpha_t \in C^{2+a,1+a/2}(\Omega \times (0,T))$.

A weak side of the result of Theorem 1.1 is that we are not able to obtain any analog of it in the full three dimensional case, because of the special structure of the vorticity problem which in the three dimensional case is not valid.

Now we state our result concerning scheme (1.10)–(1.12). In this paper we prove the following theorem.

**Theorem 1.2.** *Let assumptions of Theorem 1.1 be fulfilled, in particular we assume condition* (1.14) *and solutions to system* (1.1)–(1.2) *fulfill bound* (1.18).
*Then for any* $T > 0$ *and* $0 < h \le 1$

$$\sup_{k \in \{0,\dots,N\}} ||\operatorname{rot} v(\cdot,kh) - \operatorname{rot} v_k^h||_{L_\infty(\Omega)} \le S_2 e^{S_2 T} h\,, \tag{1.19}$$

*where $N = [T/h]$ and $S_2$ depends on $S_1$ from* (1.18).

*Moreover if $S_1$ is sufficiently small, then the scheme is asymptotically stable and for a certain $p > 2$*

$$\sup_{k\in\mathbf{N}} ||\operatorname{rot} v(\cdot, kh) - \operatorname{rot} v_k^h||_{L_p(\Omega)} \le S_2 h\,. \tag{1.20}$$

*And again constant $S_2$ depends on $S_1$.*

Theorem 1.2 delivers two cases. The first one gives the rate of convergence of scheme (1.10)–(1.12) to the solution of (1.6)–(1.9) for arbitrary large data for any finite $T > 0$. The error estimate in (1.19) depends on $T$. It is connected with a nontrivial dynamics generated by the system, since for large data the attractor of the system is not trivial. That is the reason we are not able to control the approximation uniformly in time. The second part of Theorem 1.2 consider the case for small data and shows bound (1.20) asymptotically in time. It is possible to get it, because in this case the dynamics is trivial – the global in time stability for (1.1)–(1.2) is valid. The smallness of data can be represented also by geometrical properties of the domain. If $\Omega$ is sufficiently thin the restriction on $S_1$ is weaker. It will be pointed out in the proof of the second part of Theorem 1.2 in Sect. 3.

The result obtained for our first order scheme (1.10)–(1.12) is analogical to that for problems in the whole space and in bounded domains with homogeneous boundary data [13–15]. The rates of convergence of Rothe's scheme are the same as in mentioned cases. An achievement is bound (1.19), for the standard approach to the Navier-Stokes equations (1.1) the bound in the $L_\infty$-norm is not possible to obtain in a direct way. In our case it is a simple consequence of estimates for the parabolic equation of type (1.6). Analysis of higher order schemes for the Navier-Stokes problems are discussed in [1, 7].

An important point of our approach is that the reformulation (1.6)–(1.9) simplifies our analysis by the reduction of the original Navier-Stokes equations to a simple parabolic equation and a linear elliptic system. The numerical implementation seems to be more effective than the direct analysis of (1.1)–(1.2). In the next section – Theorem 2.1 – we prove that the regularity of the approximation sequence $\{v_k^h, \alpha_k^h\}$ is independent of the viscosity in the class of regularity described by (1.16). Here the semidiscretization in time is studied, only, so our result can be viewed as a method to reduce the original problem to a system of elliptic problems (1.10), (1.11) and (1.12). Then these elliptic equations can be solved numerically (in space) by the finite element method or as in [1].

Stability of the scheme with respect to data $d$ and $v_0$ follows from stability of the continuous problem and this issue is a consequence of results of Theorem 1.1, too.

In our study we excluded the external force in the r.h.s. of $(1.1)_1$. In the most interesting case this force is potential and can be omitted by putting it into the pressure. In our case the external data are represented by the inflow/outflow condition $(1.2)_2$ which is physically reasonable and more natural for flows in bounded domains. Also from the mathematical point of view inhomogeneity at the boundary is welcomed, because standard techniques require to keep homogenous conditions. It is possible to investigate this system with the external force, but then the structure of the system does not admit the standard maximum principle and there appears a need to apply more technical analysis.

Throughout the paper we are trying to use the standard notation [10, 17]. Generic constant $S$ in proofs depends only on $S_0$ and $S_1$ and do not depend on $T$.

## 2 Basic properties of the approximation

The main aim of this part is to show uniform bounds for the approximation sequence $\{\alpha_k^h, v_k^h\}$. The properties of them are described by the following result.

**Theorem 2.1.** *Let assumptions of Theorem 1.1 be fulfilled. Then approximation sequence $\{\alpha_k^h, v_k^h\}$ prescribed by scheme* (1.10)–(1.12) *is defined uniquely and the following bound is valid*

$$\sup_{k=0,\ldots,[T/h]} \Big( ||\alpha_k^k||_{L_\infty(\Omega)} + ||v_k^h||_{C^a(\Omega)} \Big) \le c \Big( ||d||_{C^{1,1/2}(\partial\Omega\times(0,T))} + ||\alpha_0||_{L_\infty(\Omega)} \Big) \tag{2.1}$$

*for any $h > 0$, where $c$ depends on geometrical condition* (1.14) *and $c$ is independent of $\nu$, $h$, and $T$ (in particular we can put $T = \infty$).*

An interesting feature of our approach is independence of (2.1) from the viscosity. It is a consequence of an application of the maximum principle which is possible to show under condition (1.14). Unfortunately this property will not be valid in the investigation of the convergence of the scheme, since the higher regularity of the solutions depend on $\nu$ – see $S_1$ from (1.18).

Proof. We want to show that

$$||\alpha_k^h||_{L_\infty(\Omega)} \le K\,, \tag{2.2}$$

 

where $K$ depends on $S_0$. The theory of the elliptic systems implies the following bound of solutions of problem (1.12)

$$||v_k^h||_{C(\Omega)} \leq \gamma_\infty K + B||d||_{W^1_\infty(\partial\Omega)} \,. \tag{2.3}$$

Estimate (2.3) can be obtained as a consequence of a result for a modification of system (1.12). Since domain $\Omega$ is simply connected, by $(1.12)_2$ we are able to transform system (1.12) into the following one

$$\begin{aligned} \Delta\phi_k^h &= \alpha_k^h \quad \text{in} \quad \Omega\,, \\ \phi_k^h &= b_k^h \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{2.4}$$

where

$$b_k^h = \int_0^s d(x(s'), kh)ds' \,, \tag{2.5}$$

where $s$ is the length parameter of curve $\partial\Omega$ and $x(0)$ is a fixed point on the boundary. The searched function is described by

$$v_k^h = (\partial_{x_2}\phi_k^h, -\partial_{x_1}\phi_k^h) \,. \tag{2.6}$$

The theory of elliptic operator [6, 18] gives the following bound for the solutions to (2.4)

$$||\phi_k^h||_{W^2_p(\Omega)} \leq c\left(||\alpha_k^h||_{L_p(\Omega)} + ||b_k^h||_{W_p^{2-2/p}(\partial\Omega)}\right) \,. \tag{2.7}$$

Since we assumed (2.2) and domain $\Omega$ is bounded

$$||\alpha_k^h||_{L_p(\Omega)} \leq K \tag{2.8}$$

and definition of $b_k^h$ – see (2.5) – guarantees that for any $p < \infty$

$$||b_k^h||_{W_p^{2-2/p}(\partial\Omega)} \leq c||d_k^h||_{W^1_\infty(\partial\Omega)} \,. \tag{2.9}$$

To prove (2.3) it is enough to note that (2.7) holds for any $2 < p < \infty$, hence by the imbedding theorem $(\dim \Omega = 2)$

$$W_p^2(\Omega) \subset C^1(\Omega) \,. \tag{2.10}$$

By imbedding (2.10) we are able to obtain the Hölder continuity for any $a < 1$, too, because

$$W_p^2(\Omega) \subset C^{1+a}(\Omega) \quad \text{for} \quad p > 2/(1-a) \,. \tag{2.11}$$

Thus, for any $0 < a < 1$ we find so large $p$ that (2.11) is valid, hence (2.7) implies the following bound

$$||v_k^h||_{C^a(\Omega)} \leq c(K + ||d||_{W^1_\infty(\partial\Omega)}) \,, \tag{2.12}$$

too.

Our technique requires the following technical result.

**Lemma 2.1.** *Let $d \in W^1_\infty(\partial\Omega), \alpha_0 \in L_\infty(\Omega)$, moreover let*

$$\gamma_\infty||2\chi - f_0||_{L_\infty(\partial\Omega)} < 1 \,, \tag{2.13}$$

*then there exists such a number $K > 0$ that if*

$$||\alpha_0||_{L_\infty(\Omega)} \leq K \,, \tag{2.14}$$

*then*

$$||2\chi - f_0||_{L_\infty(\partial\Omega)}||v_0||_{C(\Omega)} + B||d||_{W^1_\infty(\partial\Omega)} \leq K \,, \tag{2.15}$$

*where vector $v_0$ fulfills the following problem*

$$\begin{aligned} \operatorname{rot} v_0 &= \alpha_0 \quad \textit{in} \quad \Omega\,, \\ \operatorname{div} v_0 &= 0 \quad \textit{in} \quad \Omega\,, \\ n \cdot v_0 &= d \quad \textit{on} \quad \partial\Omega\,. \end{aligned} \tag{2.16}$$

Proof. By the elementary estimate for solutions to (2.16) - the same as for (2.3) - we find the following bound

$$||v_0||_{C(\Omega)} \leq \gamma_\infty ||\alpha_0||_{L_\infty(\Omega)} + B||d||_{W^1_\infty(\partial\Omega)}\,, \tag{2.17}$$

thus

$$||v_0||_{C(\Omega)} \leq \gamma_\infty K + B||d||_{W^1_\infty(\partial\Omega)}\,. \tag{2.18}$$

Hence by (2.18)

$$||2\chi - f_0||_{L_\infty(\partial\Omega)}||v_0||_{C(\Omega)} + B||d||_{W^1_\infty(\partial\Omega)} \leq \gamma_\infty ||2\chi - f_0||_{L_\infty(\partial\Omega)} K + B||d||_{W^1_\infty(\partial\Omega)}\,. \tag{2.19}$$

Taking

$$K \geq B||d||_{W^1_\infty(\partial\Omega)}(1 - \gamma_\infty ||2\chi - f_0||_{L_\infty(\partial\Omega)})^{-1}\,, \tag{2.20}$$

we get (2.15). □

**Lemma 2.2.** *Let $K$ be given by Lemma 2.1. Then*

$$||\alpha_k^h||_{L_\infty(\Omega)} \leq K \tag{2.21}$$

*for any $k \in \mathbf{N}$ independently of $h$.*

Proof. Take $K$ given by Lemma 2.1 with suitable data. By this result we see that $\alpha_0$ fulfills (2.21).
We examine step $k$ having step $k-1$.
By the hypothesis for $k-1$ we see that for almost all points

$$\alpha_{k-1}^h - K \leq 0 \quad \text{in} \quad \Omega\,. \tag{2.22}$$

Moreover

$$||v_{k-1}^h||_{C(\Omega)} \leq \gamma_\infty K + B||d||_{W^1_\infty(\partial\Omega)}\,. \tag{2.23}$$

Thus, by Lemma 2.1 and boundary condition (1.11) we conclude that

$$\alpha_k^h \leq K \quad \text{on} \quad \partial\Omega\,. \tag{2.24}$$

Introduce

$$(\alpha_k^h - K)_+ = \max\{\alpha_k^h - K, 0\}\,. \tag{2.25}$$

Multiplying (1.10) by the above function, integrating over $\Omega$, we get

$$\int_\Omega \frac{(\alpha_k^h - K)_+^2}{h}\,dx + \nu \int_\Omega |\nabla(\alpha_k^h - K)_+|^2\,dx \leq \frac{1}{h}\int_\Omega (\alpha_{k-1}^h - K)(\alpha_k^h - K)_+ dx \leq 0\,. \tag{2.26}$$

Thus $(\alpha_k^h - K)_+ = 0$a.e., i.e.

$$\alpha_k^h \leq K \quad \text{in the } L_\infty\text{-sense}\,. \tag{2.27}$$

The same we get for $(\alpha_k^h + K)_- = \min\{\alpha_k^h + K, 0\}$, getting

$$\alpha_k^h \geq -K \quad \text{in the } L_\infty\text{-sense}\,. \tag{2.28}$$

Relations (2.27) and (2.28) imply (2.21). Lemma 2.2 has been proved by the induction. □

Thus, we showed (2.21) and estimate (2.3) finishes the proof of (2.1). Theorem 2.1 is proved. □

 

## 3 Rate of convergence

In this part of the paper we prove our main result – Theorem 1.2. We want to analyze the behavior of the approximation $\{v_k^h, \alpha_k^h\}_{k\in\mathbf{N}}$ with respect to the original solution.

Denote

$$\alpha(k) = \alpha(x, kh) \quad \text{and} \quad v(k) = v(x, kh) \tag{3.1}$$

for fixed $h > 0$, the same as in scheme (1.10)–(1.12), functions $\alpha(\cdot,\cdot)$ and $v(\cdot,\cdot)$ are solutions to problem (1.6)–(1.9). We want to reformulate the original evolutionary problem into a form suitable for the approximation system.

Integrate (1.6) over $((k-1)h, hk)$ with respect to time and divide by constant $h$, then we get

$$\frac{\alpha(k)-\alpha(k-1)}{h} + v(k-1)\cdot\nabla\alpha(k) - \nu\Delta\alpha(k) = X_1 \quad \text{in } \Omega, \tag{3.2}$$

where

$$X_1 = -v(k-1)\cdot\nabla\left[\frac{1}{h}\int_{(k-1)h}^{kh}[\alpha(x,t)-\alpha(k)]dt\right]dt - \nu\left[\frac{1}{h}\int_{(k-1)h}^{kh}\Delta(\alpha(x,t)-\alpha(k))dt\right].$$

Theorem 1.1 guarantees that $\Delta\alpha_t \in L_\infty(\Omega\times(0,T))$, hence we are able to control term $X_1$ and get the following bound

$$||X_1||_{L_\infty(\Omega)} \le Sh\,. \tag{3.3}$$

Boundary condition (1.8) takes the form

$$\alpha(k) = (2\chi - f/\nu)v(k-1)\cdot\tau - 2d_{,s}(k) + X_2 \quad \text{on} \quad \partial\Omega\,, \tag{3.4}$$

where

$$X_2 = (2\chi - f_0)(v(k-1) - v(k))\cdot\,\tau\,.$$

The same as for (3.3) we conclude that

$$||X_2||_{C(\partial\Omega)} \le Sh\,. \tag{3.5}$$

Introduce quantities which measure the error of the scheme

$$A_k = \alpha(k) - \alpha_k \quad \text{and} \quad V_k = v(k) - v_k\,. \tag{3.6}$$

Then from (1.10), (3.2), (1.11), and (3.4) we conclude

$$\begin{aligned} \frac{A_k - A_{k-1}}{h} + v_{k-1}\cdot\nabla A_k - \nu\Delta A_k &= X_1 - V_{k-1}\nabla\alpha(k) && \text{in} \quad \Omega\,, \\ A_k &= (2\chi - f_0)V_{k-1}\cdot\tau + X_2 && \text{on} \quad \partial\Omega\,. \end{aligned} \tag{3.7}$$

To obtain the desired estimate we apply a certain modification of the maximum principle. Introduce

$$(A_k - K^*)_+ = \max\{A_k - K^*, 0\}\,, \tag{3.8}$$

where

$$K_k^* = \sup_{l\in\{1,\dots,k\}}\sup_{x\in\partial\Omega}\{(2\chi - f/\nu)V_{l-1}\cdot\tau + X_2\}\,. \tag{3.9}$$

Take $\epsilon > 0$. Multiply $(3.7)_1$ by $(A_k - K_k^*)_+^{1+2\epsilon}$ and integrate over $\Omega$, then we obtain

$$\begin{aligned} &\int_\Omega \frac{(A_k - K_k^*)_+^{2+2\epsilon}}{h}dx + \nu\int_\Omega \frac{1+2\epsilon}{(1+\epsilon)^2}|\nabla(A_k - K_k^*)_+^{1+\epsilon}|^2 dx \\ &= \int_\Omega \frac{(A_{k-1} - K_k^*)_+(A_k - K_k^*)_+^{1+2\epsilon}}{h}dx - \int_\Omega V_{k-1}\nabla\alpha(k)(A_k - K_k^*)_+^{1+2\epsilon}dx + \int_\Omega X_1(A_k - K_k^*)^{1+2\epsilon}dx\,. \end{aligned} \tag{3.10}$$

It follows that

$$||(A_k - K_k^*)_+||^{2+2\epsilon}_{L_{2+2\epsilon}} + a_1 \frac{1+2\epsilon}{(1+\epsilon)^2} ||(A_k - K_k^*)||^{2+2\epsilon}_{L_{2+2\epsilon}} h$$
$$\leq ||(A_{k-1} - K_k^*)_+||_{L_{2+2\epsilon}} ||(A_k - K_k^*)_+)||^{1+2\epsilon}_{L_{2+2\epsilon}} + Sh||V_{k-1}||_C ||(A_k - K_k^*)_+||^{1+2\epsilon}_{L_{2+2\epsilon}} + Sh^2 ||(A_k - K_k^*)_+||^{1+2\epsilon}_{L_{2+2\epsilon}} . \tag{3.11}$$

Since the trace at the boundary of $(A_k - K_k^*)_+$ is zero, to obtain the second term of the l.h.s. of (3.11), we applied the Poincaré inequality ($a_1$ depends on features of domain $\Omega$)

$$a_1 ||(A_k - K_k^*)_+||^{2+2\epsilon}_{L_{2+2\epsilon}} \leq \nu \int_\Omega \frac{1+2\epsilon}{(1+\epsilon)^2} |\nabla (A_k - K_k^*)_+^{1+\epsilon}|^2 dx$$

and for the last one of the r.h.s. of (3.11) we used bound (3.3). Applying the Young inequality to the first term of the r.h.s. of (3.11) we get

$$\left(1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h\right) ||(A_k - K_k^*)_+||_{L_{2+2\epsilon}} \leq ||(A_{k-1} - K_k^*)_+||_{L_{2+2\epsilon}} + Sh||V_{k-1}||_C + Sh^2 . \tag{3.12}$$

The same bound we compute for

$$(A_k - K_{*k})_- = \min\{A_k - K_{*k}, 0\} \tag{3.13}$$

with

$$K_{*k} = \min_{l \in \{1,\dots,k\}} \inf_{x \in \partial\Omega} \{(2\chi - f/\nu) V_{l-1} \cdot \tau + X_2\} , \tag{3.14}$$

i.e.

$$\left(1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h\right) ||(A_k - K_{*k})_-||_{L_{2+2\epsilon}} \leq ||(A_{k-1} - K_{*k})_-||_{L_{2+2\epsilon}} + Sh||V_{k-1}||_C + Sh^2 . \tag{3.15}$$

To simplify the notation we introduce the following quantities

$$\begin{aligned} B_k^+ &= \max\left\{||(A_k - K_k^*)_+||_{L_{2+2\epsilon}(\Omega)}, B_{k-1}^+\right\} , \\ B_k^- &= \max\left\{||(A_k - K_{*k})_-||_{L_{2+2\epsilon}(\Omega)}, B_{k-1}^-\right\} , \\ D_k &= \max\left\{||V_k||_{C(\Omega)}, D_{k-1}\right\} . \end{aligned} \tag{3.16}$$

And

$$\begin{aligned} B_k &= B_k^+ + B_k^- , \\ K_k &= \max\{|K_k^*|, |K_{*k}|, K_{k-1}\} . \end{aligned} \tag{3.17}$$

To find estimates for $V_{k-1}$ we consider system following from (1.6) and (1.9)

$$\begin{aligned} \operatorname{rot} V_{k-1} &= \alpha(k-1) - \alpha^h_{k-1} && \text{in} \quad \Omega , \\ \operatorname{div} V_{k-1} &= 0 && \text{in} \quad \Omega , \\ n \cdot V_{k-1} &= 0 && \text{on} \quad \partial\Omega . \end{aligned} \tag{3.18}$$

Solving problem (3.18) we get

$$||V_{k-1}||_C \leq \gamma_{2+2\epsilon} B_{k-1} + \gamma_\infty K_{k-1} . \tag{3.19}$$

To obtain (3.19) there is a need to keep $\epsilon > 0$. It is a consequence of the dimension of $\Omega$ and the imbedding theorem ($W^1_p(\Omega) \subset C(\Omega)$ if $p > 2$ for dim $\Omega = 2$).

 

By (3.5), (3.9), (3.14), and (3.17) we conclude

$$K_{k-1} \leq ||2\chi - f_0||_{L_\infty} D_{k-1} + Sh\,. \tag{3.20}$$

Hence

$$D_{k-1} \leq (1 - \gamma_\infty ||2\chi - f_0||_{L_\infty})^{-1} (\gamma_{2+2\epsilon} B_{k-1} + Sh)\,. \tag{3.21}$$

From (3.12) and (3.15) by features of (3.16) and (3.17) we obtain

$$B_k \left(1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h\right) \leq B_{k-1} + ShB_{k-1} + Sh^2\,. \tag{3.22}$$

The above relation we restate as follows

$$B_k \leq \kappa B_{k-1} + \frac{S}{1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h} h^2\,, \tag{3.23}$$

where

$$\kappa = \frac{1 + Sh}{1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h}\,. \tag{3.24}$$

The definition of quantity $\kappa$ delivers us two possibilities which splits the thesis of Theorem 1.2.

The first part describes the case for arbitrary large data, then by (3.24)

$$\kappa > 1\,;$$

in this case for simplicity we can consider $S$ to such that $S - \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 \geq 1$.

From (3.23) we obtain

$$\begin{aligned} B_k &\leq \frac{Sh^2}{1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h} + \kappa \left(\kappa B_{k-2} + \frac{Sh^2}{1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h}\right) \\ &\leq \sum_{l=0}^{k-1} \kappa^l \frac{Sh^2}{1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h} = \frac{\kappa^k - 1}{\kappa - 1} \frac{Sh^2}{1 + \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 h} \leq \kappa^k \frac{Sh^2}{(S - \frac{1+2\epsilon}{(1+\epsilon)^2} a_1) h} \leq \kappa^k Sh \leq \kappa^N Sh\,. \end{aligned} \tag{3.25}$$

And recall that $N = [T/h]$. It is important to underline that bound (3.25) is independent of parameter $\epsilon$.

Let us find a suitable bound for the r.h.s. of (3.25). Elementary features of (3.24) with $\kappa > 1$ leads us to the following bound

$$\kappa^N \leq \left(1 + \frac{ST}{N}\right)^N \leq e^{ST}\,. \tag{3.26}$$

Hence from (3.25) we deduce that

$$B_k \leq Se^{ST} h \tag{3.27}$$

and by (3.21) and (3.20) we obtain

$$K_k + D_k \leq Se^{ST} h\,. \tag{3.28}$$

Note that (3.27) as well as (3.28) are independent of $\epsilon$, thus we are able to pass with this quantity to infinity. Hence by (3.16) and (3.17) we obtain from (3.27) and (3.28) the following limit estimate

$$||A_k||_{L_\infty(\Omega)} = \lim_{\epsilon \to \infty} ||A_k||_{L_\infty + L_{2+2\epsilon}(\Omega)} \leq Se^{ST} h\,. \tag{3.29}$$

Bounds (3.27), (3.28), and (3.29) together with the definition of $A_k$ – see (3.6) – prove (1.19). The first part of Theorem 1.2 is proved.

The second part of Theorem 1.2 analyzes the case of small data described by (1.20). In case

$$\kappa < 1$$

we are able to neglect dependence from time. From (3.24) we obtain for a fixed $\epsilon > 0$ the following relation

$$S < \left( \frac{1+2\epsilon}{(1+\epsilon)^2} \right) a_1 \,, \tag{3.30}$$

where $a_1$ is the constant in the Poincaré inequality. The above smallness condition can be fulfilled only by finite $\epsilon$, since the r.h.s. of (3.30) vanishes for $\epsilon$ going to infinity. Also relation (3.30) factor $p$ form (1.20). Restriction (3.30) can be fulfilled if domain is relatively thin comparing to the magnitude of $S$, then the constant from the Poincaré inequality is relatively large.

Repeating estimation as in (3.25), remembering that $\kappa < 1$, we obtain

$$B_k \leq \sum_{l=0}^{k-1} \kappa^l S h^2 \leq \frac{Sh^2}{1-\kappa} \,. \tag{3.31}$$

Hence by (3.24) we conclude that

$$B_k(\epsilon) \leq Sh \,, \tag{3.32}$$

since $0 < h \leq 1$ we have

$$1 - \kappa \geq \left( \frac{1+2\epsilon}{(1+\epsilon)^2} a_1 - S \right) h \,. \tag{3.33}$$

Again estimates (3.21) and (3.20) guarantee

$$K_k + D_k \leq Sh \tag{3.34}$$

independently of $k \in \mathbf{N}$.

In this case the result is independent of $T$. And the scheme is asymptotically stable. Since domain $\Omega$ is bounded we have

$$L_\infty(\Omega) + L_{2+2\epsilon}(\Omega) \subset L_{2+2\epsilon}(\Omega) \,, \tag{3.35}$$

hence bounds (3.32) and (3.34) shows (1.20) with $p = 2 + 2\epsilon$. The second part of Theorem 1.2 is proved. □

## 4 Appendix

**Boundary relation (1.8).** We show relation (1.8) coming from (1.2). Since in both conditions all quantities are geometrical invariant, without loss of generality, we may assume that in the considered point from the boundary the normal and tangent vector are equal $(1,0)$ and $(0,1)$, respectively. In this setting in the examined point we have

$$v \cdot n = v^1 \quad \text{and} \quad v \cdot \tau = v^2 \,. \tag{4.1}$$

From $(1.2)_1$ by (1.3) we obtain

$$v^1_{,2} + v^2_{,1} + f_0 v^2 = 0 \,. \tag{4.2}$$

Differentiating $(1.2)_2$ with respect to the length parameter we get

$$v^1_{,2} + \chi v^2 = d_{,s} \,, \tag{4.3}$$

where $\chi$ is the curvature of the boundary ($\frac{d}{ds} n = \chi \tau$) and $s$ is the length parameter of $\partial\Omega$.

Combining (4.2) and (4.3) we obtain the following boundary relation

$$v^2_{,1} - v^1_{,2} = (2\chi - f_0) v^2 - 2 d_{,s} \,. \tag{4.4}$$

By (1.7) and (4.1) we conclude (1.8), i.e.

$$\alpha = (2\chi - f_0) v \cdot \tau - 2 d_{,s} \,. \tag{4.5}$$

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# Book Review

*P. E.A. Turchi, A. Gonis, K. Rajan, A. Meike (Eds.) Complex Inorganic Solids. Structural, Stability, and Magnetic Properties of Alloys*, Springer-Verlag Berlin Heidelberg New York. 2005, XVIII, 426 pp. Hardcover, EUR 184.95 (net price), SFR 313.00, £ 142.50, US $ 239.00, ISBN 0-387-24811-0

Bei dem vorliegenden Werk handelt es sich um den Tagungsband der Third International Alloy Conference, die 2002 in Portugal stattfand. Thema der Konferenz waren Legierungen (wörtlich „alloys"), worunter die Herausgeber feste Materialien verstehen, die mehr als eine chemische Species enthalten. Diese weitgefasste Definition umfasst alle Materialien außer den reinsten Elementen (Enthält ein Einkristall eines Elementes mit technischen Verunreinigungen nicht schon mindestens 2 Species?). Analysiert man jedoch die Beiträge, so werden neben den metallischen Legierungen ungeordnete metallische Mischkristalle (Oxide), Silicatgläser, mit Hartpartikeln verstärkte Metalle, keramische Hartstoffe und magnetische Halbleiter behandelt. Ziel der Konferenz war daher, die wissenschaftlichen, vorwiegend rechnerisch theoretischen Möglichkeiten darzustellen, Mischphasen und heterogene Phasensysteme zu beschreiben. Als eine Art roter Faden zieht sich der Begriff strukturelle Ordnung (bzw. Unordnung) durch die Mehrzahl der 31 Beiträge.

Als Methoden werden bevorzugt Computersimulationen und mathematische Ableitungen eingesetzt. Lediglich 5 von den 31 Beiträgen haben einen experimentellen Charakter. Damit hat die Anwendung mathematischer Methoden breiten Raum in dem Konferenzband. Die Methoden sind der theoretischen Physik entliehen (Density Functional Theory, Molekular – Dynamische Simulation, Monte-Carlo-Verfahren u. a.) oder der theoretischen Chemie (CALPHAD: Calculation of phase diagrams).

Die Philosophie des Kongresses wird in dem Beitrag von Gonis und Turchi (zweien der Herausgeber) näher entwickelt. Sie fordern von einer Theorie über Legierungen, dass sie fundamental mathematischen Charakter hat mit der Betonung von ab-initio Berechungen, ohne dass die Theorie durch Fit-Parameter durchbrochen wird. Des Weiteren werden Konsistenz, eindeutige Kommunizierbarkeit, Überprüfbarkeit durch Experimente und ein minimalistisches Konzept (wenige Elemente der Theorie sollen ein breites Feld beschreiben) gefordert. Zum Ende werden die bestehenden Theorien bewertet, mit dem Schluss, dass das Ziel noch nicht erreicht sei, möglicherweise nicht zu erreichen ist.

Die einzelnen Beiträge sind dem Konferenzziel zumeist angepasst und haben in einigen Fällen den Charakter einer Übersicht über Arbeitsrichtungen der beteiligten Arbeitsgruppen. Die Auswahl erscheint jedoch etwas stochastisch und ersetzt kein Literaturstudium, sollte man sich mit einer dieser Arbeitsrichtungen auseinandersetzen wollen.

Halle (Saale) **H. Roggendorf**