On new approach to the issue of existence and regularity for the steady compressible Navier–Stokes equations

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Abstract

We prove existence of weak solutions to the steady compressible Navier–Stokes equations in isentropic regime. We consider the two-dimensional problem with the slip boundary conditions. The proof is based on a new idea of the construction of approximative solutions which guarantees that the sought density belongs to the L_∞ (up to the boundary). Our approach improves the known methods reducing number of technical tricks.

1 Introduction

We consider steady flow of a Newtonian fluid in a bounded domain $\Omega \subset \mathbb{R}^2$ in the isentropic regime, i.e.

$$\begin{aligned} \operatorname{div}(\varrho \boldsymbol{v}) &= 0 \quad \text{in } \Omega, \\ \varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \mu \Delta \boldsymbol{v} - (\nu+\mu)\nabla \operatorname{div} \boldsymbol{v} + \nabla \pi(\varrho) &= \varrho \boldsymbol{F} \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\boldsymbol{v} : \Omega \to \mathbb{R}^2$ represents the velocity field, $\pi : \mathbb{R}_0^+ \to \mathbb{R}$, a given function of the density $\varrho : \Omega \to \mathbb{R}_0^+$, represents the pressure and $\boldsymbol{F} : \Omega \to \mathbb{R}^2$ is the volume force. For the sake of simplicity, we consider only the case

$$\pi(\varrho) = \varrho^\gamma, \qquad \gamma > 1. \tag{1.2}$$

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The flow is considered in a fixed domain with rigid walls, i.e.

$$\boldsymbol{v}\cdot\boldsymbol{n}=0 \quad \text{at } \partial\Omega. \tag{1.3}$$

However, we do not consider the often used no slip boundary condition $\boldsymbol{v}\cdot\boldsymbol{\tau}=0$ at $\partial\Omega$ which leads to the homogeneous Dirichlet boundary conditions for the velocity; we take the slip boundary condition

$$\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{v},\pi)\cdot\boldsymbol{\tau}+f\boldsymbol{v}\cdot\boldsymbol{\tau}=0 \quad \text{at } \partial\Omega \tag{1.4}$$

with $f\geq 0$; the stress tensor $\boldsymbol{T}(\boldsymbol{v},\pi)=2\mu\boldsymbol{D}(\boldsymbol{v})+(\nu\,\mathrm{div}\,\boldsymbol{v}-\pi)\boldsymbol{I}$ and $T_{ij}(\boldsymbol{v},\pi)=\mu(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})+\nu\mathrm{div}\boldsymbol{v}\delta_{ij}-\pi\delta_{ij}$; $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the normal and tangent vectors to $\partial\Omega$. Function f describes the friction effects at the boundary. If $f\to\infty$ then (1.3) and (1.4) become the zero Dirichlet conditions and if $f=0$ we get the so-called perfect slip boundary conditions. Relations (1.3)–(1.4) are known as the Navier or friction relations, too. The properties of them one can find in [4], [6].

We will work with weak solutions to (1.1)–(1.4).

Definition 1.1 *We call the pair $(\varrho,\boldsymbol{v})\in L_\gamma(\Omega)\times W^1_2(\Omega)$, $\boldsymbol{v}\cdot\boldsymbol{n}=0$ at $\partial\Omega$ in the trace sense, a weak solution to (1.1)–(1.4) provided*

$$\int_\Omega \varrho\boldsymbol{v}\cdot\nabla\eta dx=0 \qquad \forall\eta\in C^\infty(\overline{\Omega})$$

and

$$\begin{gathered}-\int_\Omega\varrho\boldsymbol{v}\otimes\boldsymbol{v}:\nabla\boldsymbol{\varphi}dx+2\mu\int_\Omega\boldsymbol{D}(\boldsymbol{v}):\boldsymbol{D}(\boldsymbol{\varphi})dx+\nu\int_\Omega\mathrm{div}\,\boldsymbol{v}\,\mathrm{div}\,\boldsymbol{\varphi}dx\\+\int_{\partial\Omega}f(\boldsymbol{v}\cdot\boldsymbol{\tau})(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})dx-\int_\Omega\pi(\varrho)\,\mathrm{div}\,\boldsymbol{\varphi}dx=\int_\Omega\varrho\boldsymbol{F}\cdot\boldsymbol{\varphi}dx\\ \forall\boldsymbol{\varphi}\in C^\infty(\overline{\Omega});\boldsymbol{\varphi}\cdot\boldsymbol{n}=0 \text{ at } \partial\Omega.\end{gathered}$$

The first existence result for the steady compressible Navier–Stokes equations without any assumption on smallness of the data is due to P.L.Lions, see [5]. He considered the no-slip boundary conditions, but the method works in other cases, too. The proof was based on the properties of the effective viscous flux: $-(2\mu+\nu)\mathrm{div}\,\boldsymbol{v}+\pi$. The importance of the effective viscous flux has already been observed earlier, see e.g. [10] or [9]. Another approach, based on the Feiresl's idea in the non-steady case (see [3]), can be found in [7]; see also the monography [11]. The theory enables to increase regularity of the solutions to gain the density in the L_∞-space (proofs concern only the interior regularity, but in the case of the slip boundary conditions method from [5] should work). As follows from the counterexample due to P.L. Lions, in the presence of vacuum states, one cannot expect higher regularity, cf. [5] or [11].

The goal of our paper is to present a new approach to the subject. We illustrate our technique on an example in the two dimensional case. The main result reads as follows

Theorem 1.1 *Let $\Omega\in C^2$ be a bounded domain, the constants $\mu>0$, $2\mu+3\nu>0$, $M>0$, $\gamma>1$, $f\geq 0$ and $\boldsymbol{F}\in L_\infty(\Omega)$. Then there exists a weak solution to (1.1)–(1.4)* *such that*

$$\begin{aligned} \varrho &\in L_\infty(\Omega) \quad and \quad \varrho \geq 0, \\ \boldsymbol{v} &\in W_p^1(\Omega) \quad for \quad \forall p < \infty, \\ \int_\Omega \varrho dx &= M. \end{aligned}$$

To prove Theorem 1.1 we introduce a new construction of approximative sequence of the sought solution to the original problem. Our approach enables to obtain the density directly in the L_∞-space. The known methods from [5] or [11] give weak solutions with the density in the L_p-class with $p = p(\gamma) < \infty$. Next, using very technical methods it is possible to increase the regularity to L_∞. The advantage of our technique is the simplifiction of the whole procedure of the proof. The L_∞-bound reduces the number of technical tricks which are required in the standard approach, e.g. the renormalized sense of the continuity equation is not needed. Our analysis is based on special properties of the approximative sequence of the density. From the construction, the sequence of densities ϱ_ϵ is uniformly bounded in the L_∞-norm, however, to get the desired solution the measure of sets $\{x \in \Omega : \varrho_\epsilon(x) > m\}$ has to be controlled. In other case the limit would not be the sought solution. In particular, the following information

$$\lim_{\epsilon \to 0^+} |\{x \in \Omega : \varrho_\epsilon(x) > m\}| = 0 \tag{1.5}$$

is required for sufficiently large number m, greater than the a priori bound of the density. Then strong convergence $\varrho_\epsilon \to \varrho$ can be shown in quite an easy way.

Here we present only the two dimensional case for the isentropic fluid, however with all details. We hope the technique will work effectively in more complex cases, too.

The next section is devoted to formal a priori estimates showing the L_∞-bound for the density and the W_p^1-bound for the velocity provided the solution is sufficiently smooth. This section should emphasize the main idea and it should help the reader to understand better the next sections, where the rigorous proof for only weak solutions is given. The influence of the slip boundary conditions is explained, too.

Section 3 defines the new approximation scheme for the steady compressible Navier–Stokes equations. Its solvability, together with the main a priori estimates can be found here, too. The method of the proof is standard, however the type of nonlinearity is new in such a scheme, so we present almost the whole proof. The last section contains the most difficult step: the fact that for sufficiently large m we have (1.5), which on one side quite easily implies the strong convergence of the density, on the other side it ensures that for suitably chosen parameters of the approximation, the limit function is indeed the solution to the steady compressible Navier–Stokes equations, not only to its modified version.

We use the standard notation for Lebesgue spaces $L_p(\Omega)$ endowed with the norm $\|\cdot\|_p$ and the Sobolev spaces $W_p^k(\Omega)$ endowed with the norm $\|\cdot\|_{k,p}$. Vector-valued functions are printed bold-faced. We do not distinguish between function spaces for scalar- and vector-valued functions. Generic constants are denoted by C; their value may vary in the same formula or in the same line.

2 A priori estimate

In this part we want to illustrate the main idea of the proof of Theorem 1.1 which is based on the a priori bound. We assume here the existence of sufficiently smooth solutions to the original system (1.1). The obtained bound will play an important role in the existence proof since it is described by $||\boldsymbol{F}||_\infty$ and M, and this constant will be a parameter in the construction of the approximative scheme.

Theorem 2.1 *Let $\boldsymbol{F} \in L_\infty(\Omega)$, $M > 0$ and there exist a sufficiently smooth solution to (1.1)–(1.4). Then the following bound is valid*

$$||\varrho||_\infty \leq c(||\boldsymbol{F}||_\infty, M) =: ||G||_\infty^{1/\gamma}, \tag{2.1}$$

where G is the same as in (2.10).

The proof of Theorem 2.1 will be split into several steps described by separate lemmae. First we compute basic bounds which follow from the energy approach to system (1.1)–(1.4).

Lemma 2.1 *We have*

$$||\boldsymbol{v}||_{1,2} + ||\varrho||_{2\gamma} \leq c(||\boldsymbol{F}||_\infty, M). \tag{2.2}$$

Proof. Multiply $(1.1)_2$ by $\boldsymbol{v}$ and integrate over Ω, then we obtain

$$\int_\Omega (2\mu|\boldsymbol{D}(\boldsymbol{v})|^2 + \nu \mathrm{div}^2\boldsymbol{v})dx + \int_{\partial\Omega} f(\boldsymbol{v}\cdot\boldsymbol{\tau})^2 dS = \int_\Omega \varrho \boldsymbol{F}\cdot\boldsymbol{v}dx. \tag{2.3}$$

To control the right hand-side of (2.3), there is a need of information about the density. We introduce field $\boldsymbol{\Phi}$ satisfying the following relations

$$\begin{array}{lll} \mathrm{div}\ \boldsymbol{\Phi} = \varrho^\gamma - I_\gamma & \text{in} & \Omega, \\ \boldsymbol{\Phi} = 0 & \text{on} & \partial\Omega, \end{array} \tag{2.4}$$

where $I_\gamma = \frac{1}{|\Omega|}\int_\Omega \varrho^\gamma dx$. Next, applying $\boldsymbol{\Phi}$ as the test function, remembering that we control the L_1-norm of the density — the total mass is given — we conclude (2.2). For more details, see the proof of Lemma 3.4. □

In the next step we improve the information given by Lemma 2.1. The slip boundary condition gives a possibility to state the vorticity problem for system (1.1)–(1.4).

Let $\omega = \mathrm{curl}\ \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$ be the vorticity of velocity field $\boldsymbol{v}$, then ω satisfies the following system

$$\begin{array}{lll} -\mu\Delta\omega = -\mathrm{curl}\ (\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \mathrm{curl}\ (\varrho\boldsymbol{F}) & \text{in} & \Omega, \\ \omega = \Big(2\chi - \dfrac{f}{\mu}\Big)\boldsymbol{v}\cdot\boldsymbol{\tau} & \text{at} & \partial\Omega, \end{array} \tag{2.5}$$

where χ is the curvature of curve $\partial\Omega$. Boundary condition $(2.5)_2$ is obtained from (1.3) and (1.4) by differentiation of (1.3) with respect to the length parameter — see [6].

Lemma 2.2 *We have*

$$||\omega||_{1,p_1} \le c(||\boldsymbol{F}||_\infty, M), \tag{2.6}$$

where $1 < p_1 < p_1^*$ *and* $p_1^* = \frac{2\gamma}{\gamma+1}$.

Proof. By the basic estimate and the trace theorem we conclude from Lemma 2.1 that $\boldsymbol{v}\cdot\boldsymbol{\tau}|_{\partial\Omega} \in W_2^{1/2}(\partial\Omega)$. This inclusion controls the regularity of the vorticity at the boundary having only the bound from the energy estimate. Additionally we control $\varrho\boldsymbol{F}$ in $L_{2\gamma}(\Omega)$, $\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ in $L_{p_1}(\Omega)$, with p_1 as in (2.6) ($W_2^1(\Omega) \subset BMO(\Omega)$ only). Then by Lemma 2.1 we obtain $\omega \in W_{p_1}^1(\Omega)$ with bound (2.6), since $W_2^{1/2}(\partial\Omega) \subset W_{p_1}^{1-1/p_1}(\partial\Omega)$. Lemma 2.2 is proved. □

The next step concerns the Helmholtz decomposition decomposition of the velocity — see [12]. Having field $\boldsymbol{v}$, we define

$$\boldsymbol{v} = \nabla^\perp A + \nabla\phi, \tag{2.7}$$

where $\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$ and the scalar function ϕ is described by the elliptic problem

$$\begin{array}{lll} \Delta\phi = \operatorname{div}\boldsymbol{v} & \text{in} & \Omega, \\ \frac{\partial\phi}{\partial\boldsymbol{n}} = 0 & \text{on} & \partial\Omega; \end{array} \tag{2.8}$$

the compatibility condition is fulfilled since $\boldsymbol{n}\cdot\boldsymbol{v}|_{\partial\Omega} = 0$. The field A is given by

$$\begin{array}{lll} \Delta A = \operatorname{curl}\boldsymbol{v} & \text{in} & \Omega, \\ \boldsymbol{n}\cdot\nabla^\perp A = 0 & \text{on} & \partial\Omega. \end{array}$$

By Lemma 2.2 we conclude higher regularity of the above part of the velocity, i.e. $\nabla^\perp A \in W_{p_1}^2(\Omega)$.

Let us concentrate on the potential part of the velocity. By features of the decomposition from $(1.1)_2$ we have

$$\nabla(-(2\mu+\nu)\Delta\phi + \varrho^\gamma) = \nabla G, \tag{2.9}$$

where $\nabla G = -\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \mu\Delta\nabla^\perp A + \varrho\boldsymbol{F}$. Hence we are able to choose such a G to obtain

$$-(2\mu+\nu)\Delta\phi + \varrho^\gamma = G. \tag{2.10}$$

To specify the function G, we need to control $\int_\Omega G dx$, but $\frac{\partial\phi}{\partial\boldsymbol{n}}|_{\partial\Omega} = 0$ and $\varrho^\gamma \in L_2(\Omega)$, so G is well-defined. From Lemmae 2.1 and 2.2 we conclude

$$||G||_{1,p_1} \le c(||\boldsymbol{F}||_\infty, M), \tag{2.11}$$

moreover the imbedding theorem implies $G \in W_{p_1}^1(\Omega) \subset L_{q_1}(\Omega)$ with $2 < q_1 < 2\gamma$.

Next, we show the main integral identity which will be the basic tool to increase the regularity of the density.

Lemma 2.3 *Let G by given by (2.10), then for any $l > 0$ the following identity is valid*

$$\int_\Omega \varrho^{\gamma+l+1} dx = \int_\Omega \varrho^{l+1} G dx. \tag{2.12}$$

Proof. We consider the system

$$\begin{array}{ll} -(2\mu+\nu)\Delta\phi + \varrho^\gamma = G & \text{in } \Omega, \\ \text{div } (\varrho \nabla \phi) = -\text{div } (\varrho\, \nabla^\perp A) & \text{in } \Omega, \end{array} \tag{2.13}$$

following from (2.10) and $(1.1)_1$ with (2.7).

Multiplying $(2.13)_1$ by ϱ and combining it with $(2.13)_2$ we obtain

$$(2\mu+\nu)\nabla\phi\nabla\varrho + \varrho^{\gamma+1} = \varrho G - (2\mu+\nu)\text{div } (\varrho\, \nabla^\perp A). \tag{2.14}$$

Next, we multiply (2.14) by ϱ^l and integrate over Ω, getting

$$\int_\Omega \frac{2\mu+\nu}{l+1} \nabla\phi\nabla\varrho^{l+1} dx + \int_\Omega \varrho^{\gamma+l+1} dx = \int_\Omega \varrho^{l+1} G dx - \int_\Omega (2\mu+\nu)\varrho^l \text{div } (\varrho\, \nabla^\perp A). \tag{2.15}$$

The last integral of the right hand-side of (2.15) vanishes since $\text{div}\, \nabla^\perp A = 0$ and on the boundary $\boldsymbol{n}\cdot\ \nabla^\perp A|_{\partial\Omega} = 0$. The first term of the left hand-side of (2.15) is computed by equations $(2.13)_1$ as follows

$$\begin{aligned} &\int_\Omega \frac{2\mu+\nu}{l+1} \nabla\phi\nabla\varrho^{l+1} dx \\ &= -\int_\Omega \frac{2\mu+\nu}{l+1} \Delta\phi\varrho^{l+1} dx = -\frac{1}{l+1}\int_\Omega \varrho^{\gamma+l+1} dx + \frac{1}{l+1}\int_\Omega \varrho^{l+1} G dx, \end{aligned} \tag{2.16}$$

and again the boundary term vanishes since $\frac{\partial\phi}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0$. Hence (2.15) takes the following form

$$(1 - \frac{1}{l+1}) \int_\Omega \varrho^{\gamma+l+1} dx = (1 - \frac{1}{l+1}) \int_\Omega \varrho^{l+1} G dx.$$

Our investigation may deliver interesting information only for $l > 0$, so we obtain (2.12). Lemma 2.3 is proved. □

Lemma 2.4 *Let G be given by (2.10) and fulfill (2.11), then*

$$||\varrho^\gamma||_{2+\epsilon} \leq c(||\boldsymbol{F}||_\infty, M) \tag{2.17}$$

for sufficiently small $\epsilon > 0$.

Proof. To estimate the right hand-side of (2.12) we apply the Hölder inequality

$$\Big|\int_\Omega \varrho^{l+1} G dx\Big| \leq ||\varrho^{l+1}||_{q_1'} ||G||_{q_1},$$

where $q_1' = q_1/(q_1 - 1)$ and $2\gamma/(2\gamma-1) < q_1' < 2$. But we remember that we control ϱ in $L_{2\gamma}(\Omega)$, hence we put $(l+1)q_1' = 2\gamma$, so $l + 1 < 2\gamma - 1$. Taking l satisfying these conditions, we obtain

$$\int_\Omega \varrho^{\gamma+l+1} dx \leq c(||\boldsymbol{F}||_\infty, M).$$

We shown that $\varrho \in L_{3\gamma-1-\epsilon}(\Omega)$ for any $\epsilon : 0 < \epsilon < \gamma - 1$. And since $2\gamma < 3\gamma - 1 - \epsilon$, the regularity of the density is improved. It follows that the pressure satisfies the inclusion $\varrho^\gamma \in L_{(3\gamma-1)/\gamma-\epsilon}(\Omega)$ with $0 < \epsilon < (\gamma-1)/\gamma$. Thus $(3\gamma-1)/\gamma - \epsilon > 2$ which gives (2.17). Lemma 2.4 is proved. □

Now we want to improve the regularity of the velocity. System (1.1)–(1.4) can be restated in the following form

$$\begin{array}{lll} -\mu\Delta \boldsymbol{v} - (\nu+\mu)\nabla \mathrm{div}\, \boldsymbol{v} = \varrho \boldsymbol{F} - \mathrm{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) - \nabla \varrho^\gamma & \text{in} & \Omega, \\ \boldsymbol{n} \cdot 2\mu \mathbf{D}(\boldsymbol{v}) \cdot \boldsymbol{\tau} + f \boldsymbol{v} \cdot \boldsymbol{\tau} = 0 & \text{at} & \partial\Omega, \\ \boldsymbol{n} \cdot \boldsymbol{v} = 0 & \text{at} & \partial\Omega. \end{array}$$

By Lemmae 2.1 and 2.4 we have already shown that for sufficiently small $\epsilon > 0$ we have $\varrho \boldsymbol{v} \otimes \boldsymbol{v}$, $\varrho^\gamma \in L_{2+\epsilon}(\Omega)$, hence the classical theory of the elliptic operators implies $\boldsymbol{v} \in W^1_{2+\epsilon}(\Omega)$ which yields that

$$\boldsymbol{v} \in C(\overline{\Omega}) \quad \text{with} \quad ||\boldsymbol{v}||_{C(\overline{\Omega})} \leq c(||\boldsymbol{F}||_\infty, M). \tag{2.18}$$

The above information is sufficient to show the following result.

Lemma 2.5 *Let $\boldsymbol{v}$ fulfill (2.18) then for any $m \in \mathbb{R}$ such that $2\gamma < m < \infty$ we have*

$$||\varrho||_m \leq c(m, ||\boldsymbol{F}||_\infty, M). \tag{2.19}$$

Proof. Assume, ϱ is sufficiently smooth, i.e. $\varrho \in L_m(\Omega)$. Then as in Lemma 2.2, remembering (2.19), we find $\varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L_{p_m}(\Omega)$ with $p_m = \frac{2m}{m+2} < 2$, so also we have $W_2^{1/2}(\partial\Omega) \subset W_{p_m}^{1-1/p_m}(\partial\Omega)$, hence from the analysis of system (2.5) we obtain $\omega \in W^1_{p_m}(\Omega)$ and $\nabla^\perp A \in W^2_{p_m}(\Omega)$.

By the considerations for (2.11) we conclude $\nabla G \in L_{p_m}(\Omega)$. Hence the imbedding theorem yields $G \in L_m(\Omega)$, because $m = \frac{2p_m}{2-p_m}$. Thus we conclude

$$||G||_m \leq c(m, \|\boldsymbol{F}\|_\infty, M)||\varrho||_m.$$

Let us return to Lemma 2.3, we estimate the right hand-side of (2.12) as follows

$$\left| \int_\Omega \varrho^{l+1} G dx \right| \leq ||\varrho^{l+1}||_{m'} ||G||_m \leq c||\varrho||_m ||\varrho||^{l+1}_{(l+1)m'},$$

where $m' = m/(m-1)$. To find a priori estimate we need to assume $(l+1)m' \leq m$, so $l+1 \leq m-1$. Thus we obtain the following estimate

$$||\varrho||^{l+1}_{(l+1)m'} \leq c(m, \|\boldsymbol{F}\|_\infty, M)(||\varrho||^{l+1}_{2\gamma} + ||\varrho||^{l+1}_m).$$

We put in (2.12) $\gamma + l + 1 = m$ and we find the following relation

$$||\varrho||^m_m \leq c(m, \|\boldsymbol{F}\|_\infty, M)(||\varrho||^{m-\gamma}_{2\gamma} + ||\varrho||^{m-\gamma}_m)||\varrho||_m.$$

Because of $m - \gamma < m - 1$, by the Hölder inequality we are able to find the a priori bound (2.19) for any $2\gamma < m < +\infty$. Lemma 2.5 is proved. □

Proof of Theorem 2.1. A corollary of the above lemmae may be written by the following bounds $\varrho^\gamma \in L_m(\Omega)$ for $m < \infty$ and $\varrho \boldsymbol{v} \otimes \boldsymbol{v} \in L_m(\Omega)$ for $m < \infty$. From Lemma 2.5, the same as for (2.19) we get $\nabla \boldsymbol{v} \in L_m(\Omega)$ for $m < \infty$. Taking sufficiently large m we obtain $\varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v} \in L_3(\Omega)$ and trace $\boldsymbol{v}|_{\partial\Omega} \in W_3^{2/3}(\partial\Omega)$, hence from (2.5) we obtain the following inclusions

$$\omega \in W_3^1(\Omega) \quad \text{and} \quad \nabla^\perp A \in W_3^2(\Omega). \tag{2.20}$$

Hence by (2.20) we obtain $G \in W_3^1(\Omega)$. Since $N = 2$, the imbedding theorem implies

$$G \in L_\infty(\Omega) \quad \text{and} \quad ||G||_\infty \le c(||\boldsymbol{F}||_\infty, M).$$

We again return to Lemma 2.3 and estimate the right hand-side of (2.12) as follows

$$\Big| \int_\Omega \varrho^{l+1} G dx \Big| \le ||G||_\infty |\Omega|^{\gamma/(\gamma+l+1)} ||\varrho||_{\gamma+l+1}^{l+1}.$$

Then from (2.12) we obtain

$$||\varrho||_{\gamma+l+1} \le ||G||_\infty^{1/\gamma} |\Omega|^{1/(\gamma+l+1)}.$$

Passing with $l \to \infty$ we obtain $||\varrho||_\infty \le ||G||_\infty^{1/\gamma}$. The a priori bound and Theorem 2.1 are proved. □

3 Approximation

Let α, ε and h be positive numbers. We consider the following approximative system

$$\left.\begin{array}{c} \alpha h \boldsymbol{v} + \alpha \varrho \boldsymbol{v} + \dfrac{1}{2} \operatorname{div}(K(\varrho)\varrho \boldsymbol{v} \otimes \boldsymbol{v}) + \dfrac{1}{2} K(\varrho) \varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v} \\ -\mu \Delta \boldsymbol{v} - (\mu+\nu) \nabla \operatorname{div} \boldsymbol{v} + \nabla P(\varrho) = K(\varrho) \varrho \boldsymbol{F} \end{array}\right\} \text{ in } \Omega, \tag{3.1}$$

$$\alpha \varrho + \operatorname{div}(K(\varrho)\varrho \boldsymbol{v}) - \varepsilon \Delta \varrho = \alpha h K(\varrho) \quad \text{in } \Omega, \tag{3.2}$$

$$\frac{\partial \varrho}{\partial \boldsymbol{n}} = 0 \quad \text{at } \partial\Omega, \tag{3.3}$$

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{at } \partial\Omega, \tag{3.4}$$

$$\boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{v}, P(\varrho)) \cdot \boldsymbol{\tau} + f \boldsymbol{v} \cdot \boldsymbol{\tau} = 0 \quad \text{at } \partial\Omega, \tag{3.5}$$

where

$$K(\varrho) = \begin{cases} 1 & \varrho \le m_1 \\ 0 & \varrho \ge m_2 \\ \in (0,1) & \varrho \in (m_1, m_2), \end{cases}$$

$K'(\varrho) < 0$ in (m_1, m_2), $K(\cdot) \in C^1(\mathbb{R})$ and $P(\varrho) = \gamma \int_0^\varrho t^{\gamma-1} K(t)dt$. We fix the difference $m_2 - m_1$. As its value does not play now any important role, we set, say, $m_2 - m_1 = 1$. To make our construction effective, the choice of m_1 and m_2 should guarantee that

$$||G||_\infty^{1/\gamma} < m_1 < m_2.$$

Constant $h = M/|\Omega|$ is the average of the original density. For the sake of simplicity, for solutions to (3.1)–(3.5), we will write only $\boldsymbol{v}$ and ϱ and we will not use indeces ε and α, throughout this and the following section.

This approximation is a modification to the standard approximation used for the construction of weak solutions to the Navier–Stokes equation, cf. [11]. The main difference is the presence of $K(\cdot)$ at several places. As will be seen below, this fact has rather important consequences for the properties of the solution to (3.1)–(3.5). It will imply that ϱ is bounded by m_2, thus independently on the approximation parameters α and ε. On the other hand, in order to conclude that passing with the parameters to zero we get really a solution to the original system (1.1)–(1.4), we will have to show that $K(\cdot)$ is actually identically one, i.e. $\varrho \le m_1 < m_2$. Thus improved L_∞ estimates for the density will be necessary to show Theorem 1.1.

In the section we will prove the existence of a solution for the approximative system:

Theorem 3.1 *Let $\Omega \in C^2$, α, ε, $h > 0$, $f \ge 0$, $K(\cdot)$ and $P(\cdot)$ be defined as above. Then there exists a strong solution $(\varrho, \boldsymbol{v})$ to (3.1)–(3.5), $\varrho \in W^2_p(\Omega)$, $\boldsymbol{v} \in W^2_p(\Omega)$ for all $p < \infty$. Moreover*

$$0 \le \varrho \le m_2 \quad \text{in } \Omega, \tag{3.6}$$

$$\int_\Omega \varrho dx \le h|\Omega|. \tag{3.7}$$

Before starting the existence part of the proof, let us show estimates (3.6) and (3.7).

Step 1: (Proof of (3.7)) Integrate (3.2) over Ω:

$$\alpha \int_\Omega \varrho dx + \int_{\partial\Omega} K(\varrho)\varrho \boldsymbol{v} \cdot \boldsymbol{n} dS - \varepsilon \int_{\partial\Omega} \frac{\partial \varrho}{\partial \boldsymbol{n}} dS = \alpha h \int_\Omega K(\varrho) dx.$$

Thanks to the regularity of $\boldsymbol{v}$ and ϱ, the boundary integrals vanish due to the boundary conditions and thus

$$\int_\Omega \varrho dx = h \int_\Omega K(\varrho) dx$$

which, due to the definition of $K(\cdot)$, leads to (3.7).

Step 2: (Non-negativity of the density) We integrate equation (3.2) over $\Omega^- = \{x \in \Omega : \varrho(x) < 0\}$.* As the density is sufficiently smooth, we have

$$\alpha \int_{\Omega^-} \varrho dx + \int_{\partial\Omega^-} K(\varrho)\varrho \boldsymbol{v} \cdot \boldsymbol{n} dS - \varepsilon \int_{\partial\Omega^-} \frac{\partial \varrho}{\partial \boldsymbol{n}} dS = \alpha h \int_{\Omega^-} K(\varrho) dx.$$

*If Ω^- is not a regular domain, we integrate over $\Omega^-_{\varepsilon_n} = \{x \in \Omega; \varrho(x) < \varepsilon_n\}$ and pass with $\varepsilon_n \to 0^+$.

As $\frac{\partial\varrho}{\partial\boldsymbol{n}} \geq 0$ at $\partial\Omega^-$ and either $\varrho = 0$ or $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ at $\partial\Omega^-$, we get

$$\alpha\int_{\Omega^-}\varrho dx \geq \alpha h\int_{\Omega^-}K(\varrho)dx$$

which implies $|\Omega^-| = 0$ and thus (recall that $\varrho \in W^2_p(\Omega)$ for all $p < \infty$) $\varrho \geq 0$ in Ω.
Step 3: (Upper bound for the density) Finally we integrate (3.2) over $\Omega^+ = \{x \in \Omega : \varrho(x) \geq m_2\}$. We have

$$\alpha\int_{\Omega^+}\varrho dx + \int_{\partial\Omega^+}K(\varrho)\varrho\boldsymbol{v}\cdot\boldsymbol{n}dS - \varepsilon\int_{\partial\Omega^+}\frac{\partial\varrho}{\partial\boldsymbol{n}}dS = \alpha h\int_{\Omega^+}K(\varrho)dx.$$

As $\frac{\partial\varrho}{\partial\boldsymbol{n}} \leq 0$ at $\partial\Omega^+$ and either $K(\varrho) = 0$ or $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ at $\partial\Omega^+$, we get $\int_{\Omega^+}\varrho dx \leq 0$ which implies similarly as above $\varrho \leq m_2$ in Ω.

We prove the existence of a solution to (3.1)–(3.5). The idea is similar to the "standard" situation — i.e. with $K(\cdot) \equiv 1$. We define for $p \in [1,\infty]$

$$M_p = \{\boldsymbol{w} \in W^1_p(\Omega); \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ at } \partial\Omega\}$$

and

$$\mathcal{T} : M_\infty \to M_\infty, \qquad S : M_\infty \to W^2_p(\Omega), \quad 1 \leq p < \infty, \tag{3.8}$$

where

$$\begin{gathered} S(\boldsymbol{v}) = \varrho, \\ -\varepsilon\Delta\varrho + \alpha\varrho + \operatorname{div}(K(\varrho)\varrho\boldsymbol{v}) = \alpha h K(\varrho) \qquad \text{in } \Omega, \\ \frac{\partial\varrho}{\partial\boldsymbol{n}} = 0 \;\text{ at } \partial\Omega \end{gathered} \tag{3.9}$$

and

$$\begin{gathered} \mathcal{T}(\boldsymbol{v}) = \boldsymbol{w}, \\ -\mu\Delta\boldsymbol{w} - (\mu+\nu)\nabla\operatorname{div}\boldsymbol{w} = -\alpha h\boldsymbol{v} - \alpha\varrho\boldsymbol{v} - \frac{1}{2}\operatorname{div}(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}) \\ -\frac{1}{2}K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla P(\varrho) + K(\varrho)\varrho\boldsymbol{F}, \qquad \varrho = S(\boldsymbol{v}) \;\text{ in } \Omega, \\ \boldsymbol{w}\cdot\boldsymbol{n} = 0 \;\text{ at } \partial\Omega, \\ \boldsymbol{n}\cdot(2\mu\boldsymbol{D}(\boldsymbol{w}) + \nu\operatorname{div}\boldsymbol{w}\boldsymbol{I})\cdot\boldsymbol{\tau} + f\boldsymbol{v}\cdot\boldsymbol{\tau} = 0 \;\text{ at } \partial\Omega. \end{gathered} \tag{3.10}$$

The proof of the existence part of Theorem 3.1 will be done using the Leray–Schauder fixed point theorem for the operator $\mathcal{T}$. First, let us show that S is well defined.

Proposition 3.1 *Let assumptions of Theorem 3.1 be satisfied. Then S is well defined as the operator from M_∞ to $W^2_p(\Omega)$ for any $p < \infty$. Moreover, if $\varrho = S(\boldsymbol{v})$, then $\varrho \geq 0$ in Ω,*

$$\int_\Omega \varrho dx \leq h|\Omega|$$

and if $\|\boldsymbol{v}\|_{1,\infty} \leq L$, $L > 0$, then

$$\|\varrho\|_{2,p} \leq C(\varepsilon,p,\Omega)(1+L)h, \quad 1 \leq p < \infty.$$

Its proof will be based on several lemmae given below with basic hints of their proof.

Lemma 3.1 *Let $t \in [0,1]$, $\varrho_t = T_t(\xi)$, $\xi \in W^2_p(\Omega)$, where*

$$(3.11)\qquad \begin{aligned} -\varepsilon\Delta\varrho_t &= -t\,\mathrm{div}(K(\xi)\xi\boldsymbol{v}) + \alpha t\xi - \alpha t h K(\xi) \qquad \text{in } \Omega,\\ \frac{\partial\varrho_t}{\partial\boldsymbol{n}} &= 0 \qquad \text{at } \partial\Omega. \end{aligned}$$

Let $1 < p < \infty$, $L > 0$, $\boldsymbol{v} \in M_\infty$, $\|\boldsymbol{v}\|_{1,\infty} \le L$, $\Omega \in C^2$, $f \ge 0$, α, ε and $h > 0$. Let $\varrho \in W^{1,p}(\Omega)$ be such that $\varrho = T_t(\varrho)$ and $\int_\Omega \varrho dx \le h|\Omega|$.

Then there exists $C = C(p, L, \varepsilon, \Omega, h)$ such that

$$\|\varrho\|_{1,p} \le C, \quad \text{i.e.} \quad 0 \notin (I - T_t)(\partial B_C),$$

where

$$B_C = \Big\{\varrho \in W^1_p(\Omega); \|\varrho\|_{1,p} \le C, \int_\Omega \varrho dx = h\int_\Omega K(\varrho)dx\Big\}$$

and I is the identity operator.

Proof. First note that due to the regularity for the Neumann problem for the Laplace equation, $\nabla\varrho \in W^1_p(\Omega)$ for all $p < \infty$. Thus, using as test functions $\eta^\pm = \pm(\varrho^\pm + l)^\beta$, for $\beta \in (0,1)$ we get, exactly as in [11] — Lemma 4.30

$$\|\varrho\|_{\beta+1} \le c(\beta, \Omega)h.$$

Thus we can now consider instead of (3.11) a weak Neumann problem

$$\begin{aligned} -\varepsilon\Delta\varrho_t &= \mathrm{div}\,\boldsymbol{b} \quad \text{in } \Omega,\\ \frac{\partial\varrho}{\partial\boldsymbol{n}} &= \boldsymbol{b}\cdot\boldsymbol{n} \quad \text{at } \partial\Omega, \end{aligned}$$

where $\boldsymbol{b} = \alpha t\mathcal{B}(K(\varrho)h - \varrho) - t\varrho\boldsymbol{v}K(\varrho)$. Here $\mathcal{B} : L_p(\Omega)$ with zero mean $\mapsto W^1_p(\Omega) \cap \{\boldsymbol{u} = 0 \text{ at } \partial\Omega\}$ is the solution operator to $\mathrm{div}\,\boldsymbol{u} = g$ in Ω, $\boldsymbol{u} = 0$ at $\partial\Omega$. Due to the regularity of the Neumann system (see [11], Lemma 4.27 and citations therein)

$$\|\nabla\varrho\|_p \le c(\alpha, \varepsilon, s, \Omega)(1 + L)h, \quad s = \beta + 1.$$

Finally, proceeding as in the proof of Auxiliary Lemma 4.30 in [11]

$$\|\varrho\|_{1,p} \le c(\alpha, \varepsilon, p, \Omega)(1 + L)h \quad \forall p < \infty$$

and $C = 1 + c(1 + L)h$ finishes the proof. □

Lemma 3.2 *(*Cf. Auxiliary Lemma 4.31 in [11]*)*
Let p, t, L, Ω, $\boldsymbol{v}$, h, ε and α be as in Lemma 3.1 and B be a ball of center 0 in $W^{1,p}(\Omega)$ restricted on functions such that $\int_\Omega \xi = h\int_\Omega K(\xi)dx$. Then

(i) *for any $t \in [0,1]$, $T_t(B)$ is a precompact set in $W^1_p(\Omega)$;*

(ii) *for any* $t,\ s \in [0,1]$

$$\|T_t(\xi) - T_s(\xi)\|_{1,p} \le C|t-s|(\|\xi\|_{1,p} + h).$$

Proof. One may copy the proof of Auxiliary Lemma 4.31 from [11] step by step. □

Proof of Proposition 3.1. The only difference with respect to Proposition 4.22 from [11] is to show existence of the unique solution to (3.11) with $t = 0$ in B_C. The solution is given by $\varrho = M = const$, where M is the unique solution to $M = K(M)h$ for $h > 0$. (If $h \le m_1$, then $M = h$, if $h > m_1$, then we need the monotonicity of $K(\cdot)$ to ensure the unique solvability; $M \in (m_1, m_2)$). The rest is the same as in the proof of Proposition 4.22. The existence of a solution is the consequence of the Leray-Schauder fixed point theorem, see 1.4.11.8 in [11]. □

Next we consider the operator $\mathcal{T}$, i.e. we consider the Lamé system (3.11). First we have:

Lemma 3.3 *Let* $1 < p < \infty$, $\Omega \in C^2$, $F \in (M_p)^*$, $1 < p < \infty$, $\mu > 0$, $2\mu + 3\nu > 0$, $f \ge 0$. *Then there exists the unique* $\boldsymbol{w} \in M_p$, *solution to (3.11). Moreover,*

$$\|\boldsymbol{w}\|_{1,p} \le C(p,\Omega)\|\boldsymbol{F}\|_{(M_p)^*}.$$

If $\Omega \in C^{k+2}$, $\boldsymbol{F} \in W^k_p(\Omega)$, $k = 0, 1, \dots,$ *then* $\boldsymbol{w} \in W^{k+2,p}(\Omega)$ *and*

$$\|\boldsymbol{w}\|_{k+2,p} \le C(k,p,\Omega)\|\boldsymbol{F}\|_{k,p}.$$

Proof. The existence of a weak solution for $f > 0$ is an easy consequence of the Lax–Milgram theorem. If $f = 0$ and Ω not "axially" symmetric, we get the same due to Korn's inequality

$$c\|\boldsymbol{u}\|_{1,2} \le \int_\Omega |\boldsymbol{D}(\boldsymbol{u})|^2 + \int_{\partial\Omega} |\boldsymbol{u}\cdot\boldsymbol{n}|^2 dS$$

$\boldsymbol{u} \in W^1_2(\Omega)/R(\Omega)$, $R(\Omega) = \{\boldsymbol{u} \in W^1_2(\Omega); \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ at } \partial\Omega; \boldsymbol{D}(\boldsymbol{u}) = \boldsymbol{0}\}$, see [11], [13]. In the case of "symmetric" domain (as a ball) we restrict our attention to space described by the constraint $\int_\Omega (x_2, -x_1)\cdot\boldsymbol{u}dx = 0$.

The rest is, similarly as in [11], Lemma 4.32, the consequence of the standard elliptic theory, cf. [1]. □

Now, we would like to apply the Leray–Schauder fixed point theorem on the operator $\mathcal{T}$. The only difference with respect to the situation studied in [11] is the presence of the term $K(\varrho)$. Thus the only part which deserves more careful studying is the first a priori estimate. We state it here in a little bit different form which will be useful later on.

Lemma 3.4 *Let* $t \in [0,1]$, $\boldsymbol{v} \in M_\infty$ *be a fixed point* $\boldsymbol{v} = \mathcal{T}_t(\boldsymbol{v})$, *where* $\mathcal{T}_t(\boldsymbol{v}) = \boldsymbol{w}$ *and*

$$\begin{gathered}
-\mu\Delta\boldsymbol{w} - (\mu+\nu)\nabla\operatorname{div}\boldsymbol{w} = t[-\alpha h\boldsymbol{v} - \alpha\varrho\boldsymbol{v} - \frac{1}{2}\operatorname{div}(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v})] \\
+t[-\frac{1}{2}K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla P(\varrho) + K(\varrho)\varrho\boldsymbol{F}], \quad \varrho = S(\boldsymbol{v}) \quad \text{in } \Omega, \\
\boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad \text{at } \partial\Omega, \\
\boldsymbol{n}\cdot(2\mu\boldsymbol{D}(\boldsymbol{w}) + \nu\operatorname{div}\boldsymbol{w}\boldsymbol{I})\boldsymbol{\tau} + f\boldsymbol{v}\cdot\boldsymbol{\tau} = 0 \quad \text{at } \partial\Omega.
\end{gathered}$$

Then

$$\int_\Omega \Big(t\alpha h|\boldsymbol{v}|^2 + t\alpha\varrho|\boldsymbol{v}|^2 + \mu|\boldsymbol{D}(\boldsymbol{v})|^2 + \nu|\operatorname{div}\boldsymbol{v}|^2 + \frac{\alpha\gamma t}{\gamma-1}P(\varrho) + \varepsilon\gamma t\varrho^{\gamma-2}|\nabla\varrho|^2\Big)dx$$
$$+\int_{\partial\Omega} f|\boldsymbol{v}\cdot\boldsymbol{\tau}|^2 dS = \frac{\alpha t\gamma h}{\gamma-1}\int_\Omega K(\varrho)\varrho^{\gamma-1}dx + t\int_\Omega K(\varrho)\varrho\boldsymbol{F}\cdot\boldsymbol{v}dx$$

and

$$\|\boldsymbol{v}\|_{1,2} + \|P(\varrho)\|_2 \leq L$$

with L *independent of* α, ε *and* m_2.

Proof. Take as test functions to (3.11) with $\boldsymbol{v}=\boldsymbol{w}$ the solution itself and recall that $\varrho = S(\boldsymbol{v})$. Thus

$$\alpha th\int_\Omega |\boldsymbol{v}|^2dx + \alpha t\int_\Omega \varrho|\boldsymbol{v}|^2dx + \mu\int_\Omega |\boldsymbol{D}(\boldsymbol{v})|^2dx + \nu\int_\Omega |\operatorname{div}\boldsymbol{v}|^2dx$$
$$+\int_{\partial\Omega} f|\boldsymbol{v}\cdot\boldsymbol{\tau}|^2dS - \frac{1}{2}t\int_\Omega K(\varrho)\varrho(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{v}dx + \frac{1}{2}t\int_\Omega K(\varrho)\varrho(\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\boldsymbol{v}dx$$
$$+t\int_\Omega \boldsymbol{v}\cdot\nabla P(\varrho)dx = t\int_\Omega K(\varrho)\varrho\boldsymbol{F}\cdot\boldsymbol{v}dx.$$

Now

$$\int_\Omega \boldsymbol{v}\cdot\nabla P(\varrho)dx = \gamma\int_\Omega \varrho^{\gamma-1}K(\varrho)\boldsymbol{v}\cdot\nabla\varrho dx = \frac{\gamma}{\gamma-1}\int_\Omega K(\varrho)\varrho\boldsymbol{v}\cdot\nabla(\varrho^{\gamma-1})dx$$
$$= \frac{\gamma}{\gamma-1}\int_\Omega(\alpha\varrho^\gamma - \varepsilon\Delta\varrho\varrho^{\gamma-1} - \alpha hK(\varrho)\varrho^{\gamma-1})dx$$
$$= \frac{\alpha\gamma}{\gamma-1}\int_\Omega \varrho^\gamma dx + \varepsilon\gamma\int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2dx - \frac{\alpha h\gamma}{\gamma-1}\int_\Omega K(\varrho)\varrho^{\gamma-1}dx.$$

Thus

$$\begin{aligned}&\alpha th\int_\Omega |\boldsymbol{v}|^2dx + \alpha t\int_\Omega \varrho|\boldsymbol{v}|^2dx + \mu\int_\Omega |\boldsymbol{D}(\boldsymbol{v})|^2dx + \nu\int_\Omega |\operatorname{div}\boldsymbol{v}|^2dx\\ &+\int_{\partial\Omega} f|\boldsymbol{v}\cdot\boldsymbol{\tau}|^2dS + \frac{t\alpha\gamma}{\gamma-1}\int_\Omega \varrho^\gamma dx + t\varepsilon\gamma\int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2dx\\ &= \frac{t\alpha h\gamma}{\gamma-1}\int_\Omega K(\varrho)\varrho^{\gamma-1}dx + t\int_\Omega K(\varrho)\varrho\boldsymbol{F}\cdot\boldsymbol{v}dx\\ &\leq C_1 + \frac{1}{4}\frac{t\alpha\gamma}{\gamma-1}\int_\Omega \varrho^\gamma dx + Ct\|\boldsymbol{v}\|_{1,2}\|K(\varrho)\varrho\|_{1+\eta}\end{aligned} \tag{3.12}$$

with $\eta > 0$, arbitrarily small. Recall that for $\beta \in (1, 2\gamma)$

$$\int |K(\varrho)\varrho|^\beta dx \leq \|P(\varrho)\|_{\frac{\beta}{\gamma}}^{\frac{\beta}{\gamma}} \leq \|\varrho\|_1^{\frac{2\gamma-\beta}{2\gamma-1}}\|P(\varrho)\|_2^{\frac{\beta-1}{2\gamma-1}}$$

and thus

$$\alpha th\int_\Omega |\boldsymbol{v}|^2dx + \alpha t\int_\Omega \varrho|\boldsymbol{v}|^2dx + \mu\int_\Omega |\boldsymbol{D}(\boldsymbol{v})|^2dx + (\mu+\nu)\int_\Omega |\operatorname{div}\boldsymbol{v}|^2dx$$
$$+\int_{\partial\Omega} f|\boldsymbol{v}\cdot\boldsymbol{\tau}|^2dS + \frac{t\alpha\gamma}{\gamma-1}\int_\Omega \varrho^\gamma dx + t\varepsilon\gamma\int_\Omega \varrho^{\gamma-2}|\nabla\varrho|^2dx$$
$$\leq C(\delta) + C(h)t^2\|P(\varrho)\|_2^\delta$$

with $\delta > 0$, arbitrarily small.

Therefore the choice of the next test function is evident. We use

$$\boldsymbol{\psi} = \mathcal{B}\left(P(\varrho) - \frac{1}{|\Omega|}\int_\Omega P(\varrho)dx\right),$$

where $\mathcal{B}(f)$ is defined above. We recall that (see [2] or also [11])

$$\|\nabla\boldsymbol{\psi}\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

Thus we get

$$(3.13)\quad \begin{aligned} t\int_\Omega |P(\varrho)|^2 dx = \frac{t}{|\Omega|}\Big(\int_\Omega P(\varrho)dx\Big)^2 + t\alpha h\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\psi}dx + \alpha t\int_\Omega \varrho\boldsymbol{v}\cdot\boldsymbol{\psi}dx \\ -\frac{1}{2}t\int_\Omega K(\varrho)\varrho(\boldsymbol{v}\otimes\boldsymbol{v}):\nabla\boldsymbol{\psi}dx + \frac{1}{2}t\int_\Omega K(\varrho)\varrho(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{\psi}dx + \mu\int_\Omega \nabla\boldsymbol{v}:\nabla\boldsymbol{\psi}dx \\ +(\mu+\nu)\int_\Omega \operatorname{div}\boldsymbol{v}\operatorname{div}\boldsymbol{\psi}dx - t\int_\Omega K(\varrho)\varrho\boldsymbol{F}\cdot\boldsymbol{\psi}dx = \sum_{j=1}^{8} I_j. \end{aligned}$$

We estimate each term separately:

$$I_1 = \frac{t}{|\Omega|}\left(\int_\Omega P(\varrho)dx\right)^2 \le C(h)t\|P(\varrho)\|_2^\beta$$

with some $\beta < 2$.

$$I_2 = t\alpha h\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\psi}dx \le C\sqrt{\alpha ht}\sqrt{\alpha ht\|\boldsymbol{v}\|_2^2}\|\boldsymbol{\psi}\|_2 \le C_1(\delta) + Ct\|P(\varrho)\|_2^{1+\delta}.$$

Next

$$\begin{aligned} I_3 = \alpha th\int_\Omega \varrho\boldsymbol{v}\cdot\boldsymbol{\psi}dx &\le C\sqrt{\alpha t\int_\Omega \varrho|\boldsymbol{v}|^2 dx}\|\nabla\boldsymbol{\psi}\|_2\sqrt{\alpha t}\|\varrho\|_\gamma^{\frac{1}{2}} \\ &\le Ct\|P(\varrho)\|_2^{1+\delta} + C_1(\delta). \end{aligned}$$

Further

$$\begin{aligned} I_4 + I_5 &= -\frac{t}{2}\int_\Omega K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}:\nabla\boldsymbol{\psi}dx + \frac{t}{2}\int_\Omega K(\varrho)\varrho(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{\psi}dx \\ &\le Ct\|P(\varrho)\|_2\|\boldsymbol{v}\|_{1,2}^2\|K(\varrho)\varrho\|_{2+\delta} \le Ct\|P(\varrho)\|_2^{1+\delta} + C_1(\delta). \end{aligned}$$

$$\begin{aligned} I_6 + I_7 = \mu\int_\Omega \nabla\boldsymbol{v}:\nabla\boldsymbol{\psi}dx + (\mu+\nu)\int_\Omega \operatorname{div}\boldsymbol{v}\operatorname{div}\boldsymbol{\psi}dx &\le C\|\nabla\boldsymbol{v}\|_2\|P(\varrho)\|_2 \\ &\le Ct\|P(\varrho)\|_2^{1+\delta} + C_1(\delta). \end{aligned}$$

Finally

$$I_8 = -t\int_\Omega K(\varrho)\varrho\boldsymbol{F}\cdot\boldsymbol{\psi}dx \le Ct\|\psi\|_{1,2}\|\boldsymbol{F}\|_\infty\|K(\varrho)\varrho\|_{1+\eta} \le Ct\|P(\varrho)\|_2^{1+\delta}.$$

Thus combining (3.12) and (3.13) with estimates of I_j given above we have

$$\begin{gathered}\alpha th\int_\Omega |\boldsymbol{v}|^2dx+\alpha t\int_\Omega \varrho|\boldsymbol{v}|^2dx+\mu\int_\Omega|\boldsymbol{D}(\boldsymbol{v})|^2dx+(\mu+\nu)\int_\Omega|\operatorname{div}\boldsymbol{v}|^2dx\\ +\int_{\partial\Omega}f|\boldsymbol{v}\cdot\boldsymbol{\tau}|^2dS+\frac{t\alpha\gamma}{\gamma-1}\int_\Omega\varrho^\gamma dx+t\varepsilon\gamma\int_\Omega\varrho^{\gamma-2}|\nabla\varrho|^2dx+t\int_\Omega(P(\varrho))^2dx\\ \le C_1(\delta)+t\|P(\varrho)\|_2^{1+\delta}.\end{gathered}$$

Especially, taking $t=1$ and looking for estimates independent of the parameters ε and α,

$$\|\nabla\boldsymbol{v}\|_2+\|P(\varrho)\|_2\le L$$

where L is also independent of m_2. Further, we also have $\|\boldsymbol{v}\|_{1,2}\le L$, provided $f\ge 0$, see the proof of Lemma 3.3. □

Proof of Theorem 3.1. We may now copy step by step the proofs of Auxiliary Lemmae 4.34 and 4.35 from [11]. This, together with estimates from Proposition 2.12 and Lemma 3.4 finishes the proof of Theorem 3.1. □

Let us summarize our a priori estimates. We have

$$\|\varrho\|_\infty\le m_2,\qquad \|\boldsymbol{v}\|_{1,2}+\|P(\varrho)\|_2\le L, \tag{3.14}$$

where L is independent of ε, α and m_2.

Using the regularity of the Lamé system, we can get also estimates of $\nabla\boldsymbol{v}$ in higher $L_q(\Omega)$. They will be independent of α and ε, however, they will depend on m_2. For the purpose below, we will calculate the dependence on m_2 precisely.

We will apply Lemma 3.3 on system (3.1), written in the form

$$\begin{gathered}-\mu\Delta\boldsymbol{v}-(\mu+\nu)\nabla\operatorname{div}\boldsymbol{v}=-\alpha h\boldsymbol{v}-\alpha\varrho\boldsymbol{v}-\frac12\operatorname{div}(K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v})\\ -\frac12K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v}-\nabla P(\varrho)+K(\varrho)\varrho\boldsymbol{F}\qquad\text{in }\Omega,\\ \boldsymbol{v}\cdot\boldsymbol{n}=0\qquad\text{at }\partial\Omega,\\ \boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{v},P(\varrho))\cdot\boldsymbol{\tau}+f\boldsymbol{v}\cdot\boldsymbol{\tau}=0\quad\text{at }\partial\Omega.\end{gathered}$$

It yields

$$\begin{aligned}\|\nabla\boldsymbol{v}\|_q\ \le\ & C(\alpha\|\boldsymbol{v}\|_{\frac{2q}{2+q}}+\alpha\|\varrho\boldsymbol{v}\|_{\frac{2q}{2+q}}+\|K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}\|_q+\|K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{\frac{2q}{2+q}}\\ &+\|P(\varrho)\|_q+\|K(\varrho)\varrho\boldsymbol{F}\|_{\frac{2q}{2+q}}).\end{aligned}$$

We have

$$\begin{aligned}\alpha\|\boldsymbol{v}\|_{\frac{2q}{2+q}}&\le C\alpha\|\boldsymbol{v}\|_2,\\ \alpha\|\varrho\boldsymbol{v}\|_{\frac{2q}{2+q}}&\le\sqrt{\alpha}\|\varrho|\boldsymbol{v}|^2\|_1^{\frac12}\sqrt{\alpha}\|\varrho\|_\gamma^{\frac{1}{2\gamma}}\|\boldsymbol{v}\|_{1,2},\\ \|K(\varrho)\varrho\boldsymbol{v}\otimes\boldsymbol{v}\|_q+\|K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{\frac{2q}{2+q}}&\le C\|\boldsymbol{v}\|_{1,2}^2\|P(\varrho)\|_{\frac{q}{\gamma}}^{\frac1\gamma},\\ \|P(\varrho)\|_q&\le\|P(\varrho)\|_2^{\frac2q}\|P(\varrho)\|_\infty^{1-\frac2q}\le Cm_2^{(1-\frac2q)\gamma},\\ \|K(\varrho)\varrho\boldsymbol{F}\|_{\frac{2q}{2+q}}&\le C\|P(\varrho)\|_2^{\frac1\gamma}.\end{aligned}$$

Thus

$$\|\nabla \boldsymbol{v}\|_q \le C_1 + C_2 m_2^{(1-\frac{2}{q})\gamma}. \tag{3.15}$$

Moreover, C_1 and C_2 are independent of ε and α.

Next we need an estimate of the gradient of the density which will blow up for $\varepsilon \to 0$, however, with a controlled rate. We multiply (3.9) by ϱ and integrate over Ω. It reads

$$\varepsilon \int_\Omega |\nabla \varrho|^2 dx = \alpha h \int_\Omega K(\varrho)\varrho dx - \alpha \int_\Omega \varrho^2 dx - \int_\Omega K(\varrho)\varrho \boldsymbol{v} \cdot \nabla \varrho dx. \tag{3.16}$$

The last term,

$$\begin{aligned} \int_\Omega K(\varrho)\varrho \boldsymbol{v} \cdot \nabla \varrho dx &= \int_\Omega \boldsymbol{v} \cdot \nabla \left(\int_0^\varrho K(t) t dt \right) dx \\ = - \int_\Omega \operatorname{div} \boldsymbol{v} \left(\int_0^\varrho K(t) t dt \right) dx &\le \int_\Omega |\operatorname{div} \boldsymbol{v}||\varrho|^2 dx \le C(m_2), \end{aligned} \tag{3.17}$$

however, the constant is independent of α and ε. Therefore $\sqrt{\varepsilon}||\nabla \varrho||_2$ is bounded.

To conclude this section, we will compute a priori estimates of the vorticity. This is the key point why we work with slip boundary conditions and where the procedure fails for the Dirichlet boundary conditions. We denote as above

$$\omega = \operatorname{curl} \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

We have, at least in the weak sense

$$\begin{aligned} -\mu \Delta \omega = -\alpha h \omega - \alpha \operatorname{curl}(\varrho \boldsymbol{v}) - \frac{1}{2} \operatorname{curl} \operatorname{div}(K(\varrho)\varrho \boldsymbol{v} \otimes \boldsymbol{v}) \\ -\frac{1}{2} \operatorname{curl}(K(\varrho)\varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v}) + \operatorname{curl}(K(\varrho)\varrho \boldsymbol{F}) \qquad \text{in } \Omega, \end{aligned} \tag{3.18}$$

$$\omega = \Big(2\chi - \frac{f}{\mu}\Big) \boldsymbol{v} \cdot \boldsymbol{\tau} \qquad \text{at } \partial\Omega, \tag{3.19}$$

where χ is the curvature of $\partial\Omega$.

Writing $\omega = \omega_1 + \omega_2$ with

$$\begin{gathered} -\mu \Delta \omega_1 = -\frac{1}{2} \operatorname{curl} \operatorname{div}(K(\varrho)\varrho \boldsymbol{v} \otimes \boldsymbol{v}) \qquad \text{in } \Omega, \\ \omega_1 = 0 \text{ at } \partial\Omega, \\ -\mu \Delta \omega_2 = -\alpha h \omega - \operatorname{curl}(\varrho \boldsymbol{v}) - \frac{1}{2} \operatorname{curl}(K(\varrho)\varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v}) + \operatorname{curl}(K(\varrho)\varrho \boldsymbol{F}) \qquad \text{in } \Omega, \\ \omega_2 = \Big(2\chi - \frac{f}{\mu}\Big) \boldsymbol{v} \cdot \boldsymbol{\tau} \qquad \text{at } \partial\Omega, \end{gathered}$$

we get the following estimates

$$\|\omega_1\|_q \le C \|K(\varrho)\varrho \boldsymbol{v} \otimes \boldsymbol{v}\|_q \le C(m_2, q)$$

with C independent of m_2 for $q < 2\gamma$ and $C = C_0 m_2^{\frac{1}{\gamma}-\frac{2}{q}+\eta}$ for $q \geq 2\gamma$, $\eta > 0$ arbitrarily small.

For ω_2, we can estimate, independently of α and ε, also its gradient:

$$\|\omega_2\|_{1,q} \leq C(\alpha h\|\boldsymbol{v}\|_{\frac{2q}{2+q}} + \alpha\|\varrho\boldsymbol{v}\|_q + \|K(\varrho)\varrho\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q$$
$$+\|K(\varrho)\varrho\boldsymbol{F}\|_q) + C(\Omega)\|\boldsymbol{v}\cdot\tau\|_{1-\frac{1}{q},q,\partial\Omega} \leq C_1 + C_2\|\nabla\boldsymbol{v}\|_2^2\|K(\varrho)\varrho\|_{\frac{2q}{2-q}+\eta} + C\|K(\varrho)\varrho\|_q,$$

provided $q < 2$. If $q \geq 2$, we must use m_2-dependent estimates of higher gradients of $\boldsymbol{v}$ and thus

$$\|\omega_2\|_{1,q} \leq C$$

with C independent of m_2 provided $q < \frac{2\gamma}{1+\gamma}$, $C = C(m_2)$ if $\frac{2\gamma}{1+\gamma} \leq q < \infty$.

4 Proof of Theorem 1.1

The last section is devoted to the limit passage of the parameters α and ε to zero. In what follows, we set $\alpha = \varepsilon$ and let $\varepsilon \to 0$. From (3.14), (3.15), (3.16) and (3.17) we have the following estimates independent of ε:

$$\|\varrho_\varepsilon\|_\infty + \|\boldsymbol{v}_\varepsilon\|_{1,q} + \varepsilon\|\nabla\varrho_\varepsilon\|_2^2 \leq C(m_2) \quad \text{for} \quad 1 \leq q < \infty, \qquad \|P(\varrho_\varepsilon)\|_2 + \|\boldsymbol{v}_\varepsilon\|_{1,2} \leq C. \tag{4.1}$$

Thus, passing with $\varepsilon \to 0$ we have (actually, we pass with a sequence $\varepsilon_n \to 0^+$; however, we will mostly write $\varepsilon \to 0^+$ instead of the correct $\varepsilon_n \to 0^+$)

$$\boldsymbol{v}_\varepsilon \rightharpoonup \overline{\boldsymbol{v}} = \boldsymbol{v} \ \text{ in } W^1_q(\Omega) \quad \text{and} \quad \varrho_\varepsilon \rightharpoonup^* \overline{\varrho} = \varrho \ \text{ in } L_\infty(\Omega). \tag{4.2}$$

Therefore, there is a subsequence $\boldsymbol{v}_{\varepsilon_n} \to \boldsymbol{v}$ in $L_p(\Omega)$, $1 \leq p \leq \infty$. The limit solves

$$\frac{1}{2}\operatorname{div}(\overline{K(\varrho)\varrho}\boldsymbol{v}\otimes\boldsymbol{v}) + \frac{1}{2}\boldsymbol{v}\overline{K(\varrho)\varrho\nabla\boldsymbol{v}} - \mu\Delta\boldsymbol{v} -(\mu+\nu)\nabla\operatorname{div}\boldsymbol{v} + \nabla\overline{P(\varrho)} = \overline{K(\varrho)\varrho}\boldsymbol{F} \ \text{ in } \Omega, \tag{4.3}$$

$$\operatorname{div}(\overline{K(\varrho)\varrho}\boldsymbol{v}) = 0 \ \text{ in } \Omega, \tag{4.4}$$

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{at } \partial\Omega, \tag{4.5}$$

$$\boldsymbol{n}\cdot\boldsymbol{T}(\boldsymbol{v},P(\varrho))\cdot\boldsymbol{\tau} + f\boldsymbol{v}\cdot\boldsymbol{\tau} = 0 \ \text{ at } \partial\Omega, \tag{4.6}$$

where the bar over the nonlinear quantity denotes its weak limit.

Thus there are several problems:

(1) Is $K(\varrho) = 1$, i.e. is our solution in fact a solution to the compressible Navier–Stokes equations?

(2) Is it true that $\overline{P(\varrho)} = P(\varrho)$? This, as will be seen below, is in fact equivalent to the strong convergence of the density.

(3) In which sense (3.5) is satisfied?

In order to answer **(1)** positively, we will prove that there is a suitable choice of m_1 and m_2 (in fact, sufficiently large) such that for certain $m < m_1$ and for a certain sequence $\varepsilon_n \to 0^+$

$$\lim_{\varepsilon_n \to 0^+} |\{x \in \Omega; \varrho_{\varepsilon_n}(x) > m\}| = 0. \tag{4.7}$$

Having proved (4.7), it will be relatively easy to deduce that $\varrho_\varepsilon \to \varrho$ strongly in $L_p(\Omega)$ for all $p < \infty$. The fact that the corresponding boundary condition is satisfied will be an easy consequence of the weak and strong convergence.

The main tool to show (4.7) is the so-called effective viscous flux. We decompose the velocity field into the divergence-free part and the gradient part, i.e.

$$\boldsymbol{v} = \nabla\varphi + \nabla^\perp A,$$

where $\nabla^\perp$ was defined above and A is the so-called stream function. Recall that

$$\begin{aligned} \Delta A &= \operatorname{curl} \boldsymbol{v} \quad \text{in } \Omega, \\ \nabla^\perp A \cdot \boldsymbol{n} &= 0 \quad \text{at } \partial\Omega \end{aligned}$$

and thus

$$\begin{aligned} \|\nabla^\perp A\|_{1,q} &\leq C\|\operatorname{curl} \boldsymbol{v}\|_q = C\|\omega\|_q, \\ \|\nabla\nabla^\perp A\|_{-1,q} &\leq C\|\operatorname{curl} \boldsymbol{v}\|_{-1,q}. \end{aligned}$$

Inserting this decomposition into the limit equation, remembering that due to (4.4), $\overline{K(\varrho)\varrho\nabla\boldsymbol{v}\boldsymbol{v}} = \overline{K(\varrho)\varrho}\boldsymbol{v} \cdot \nabla\boldsymbol{v}$, we get

$$\nabla(-(2\mu+\nu)\Delta\varphi + \overline{P(\varrho)}) = \mu\Delta\nabla^\perp A + \overline{K(\varrho)\varrho}\boldsymbol{F} - \overline{K(\varrho)\varrho}\boldsymbol{v} \cdot \nabla\boldsymbol{v}. \tag{4.8}$$

We denote

$$G = -(2\mu+\nu)\Delta\varphi + \overline{P(\varrho)} = -(2\mu+\nu)\operatorname{div} \boldsymbol{v} + \overline{P(\varrho)}. \tag{4.9}$$

As $\int_\Omega G dx = \int_\Omega \overline{P(\varrho)} dx$, we get

Lemma 4.1 *We have*

$$\|G\|_2 \leq C_1, \qquad \|G\|_\infty \leq C_2 m_2^{1+\eta},$$

where C_1 and C_2 do not depend on m_1 and m_2, i.e. on the choice of $K(\cdot)$ and $1 < 1+\eta < \gamma$.

Proof. We take $q > 2$ and estimate from (4.9) $\|\nabla G\|_q$. We have

$$\|\nabla G\|_q \leq C(\|\Delta\nabla^\perp A\|_q + \|\overline{K(\varrho)\varrho}\boldsymbol{F}\|_q + \|\overline{K(\varrho)\varrho}\boldsymbol{v} \cdot \nabla\boldsymbol{v}\|_q).$$

Evidently $\|\overline{K(\varrho)\varrho}\boldsymbol{F}\|_q \leq C$ for $q \leq 2\gamma$. Further $q > 2$ by (3.15)

$$\|\overline{K(\varrho)\varrho}\boldsymbol{v} \cdot \nabla\boldsymbol{v}\|_q \leq C\|\nabla\boldsymbol{v}\|_q^2 \|\overline{K(\varrho)\varrho}\|_\infty \leq C m_2^{1+2(1-\frac{2}{q})\gamma}$$

for q sufficiently close to 2 (q is chosen such that $\gamma - 1 > 2\gamma(1-\frac{2}{q})$ and the last quantity describes η). Finally

$$\|\Delta\nabla^{\perp}A\|_q \le C\|\nabla\omega\|_q.$$

We have in the weak sense

$$\begin{aligned} -\mu\Delta\omega &= -\operatorname{curl}(\overline{K(\varrho)\varrho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}) - \operatorname{curl}(\overline{K(\varrho)\varrho}\boldsymbol{F}) \quad \text{in } \Omega,\\ \omega &= \Big(2\chi - \frac{f}{\mu}\Big)\boldsymbol{v}\cdot\boldsymbol{\tau} \quad \text{at } \partial\Omega. \end{aligned} \tag{4.10}$$

(We pass to the limit in (3.18).) Thus, ω is actually a weak solution to (4.10) and

$$\|\omega\|_{1,q} \le C(\Omega)\|\boldsymbol{v}\|_{1,q} + C(\|\overline{K(\varrho)\varrho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_q + \|\overline{K(\varrho)\varrho}\boldsymbol{F}\|_q) \le Cm_2^{1+2\gamma(1-\frac{2}{q})}$$

due to arguments given above. Thus, using also the fact that we control the mean value of G, we get the L_∞ estimates via the Sobolev embedding. The first inequality is trivial, since

$$\|G\|_2 \le C(\|\nabla\boldsymbol{v}\|_2 + \|P(\varrho)\|_2).$$

□

Next we consider the decomposition $\boldsymbol{v}_\varepsilon = \nabla\varphi_\varepsilon + \nabla^{\perp}A_\varepsilon$ for $\varepsilon > 0$. Similarly as above we get

$$\begin{aligned} \nabla\left(-(2\mu+\nu)\Delta\varphi_\varepsilon + P(\varrho_\varepsilon)\right) &= \mu\Delta\nabla^{\perp}A_\varepsilon + K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{F}\\ -\frac{1}{2}K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon &- \frac{1}{2}\operatorname{div}(K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) - \varepsilon h\boldsymbol{v}_\varepsilon - \varepsilon\varrho_\varepsilon\boldsymbol{v}_\varepsilon. \end{aligned}$$

We denote again

$$G_\varepsilon = -(2\mu+\nu)\Delta\varphi_\varepsilon + P(\varrho_\varepsilon). \tag{4.11}$$

Then

Lemma 4.2 *We have $G_\varepsilon \to G$ in $L_2(\Omega)$ (strongly).*

Proof. We have

$$\begin{aligned} \nabla(G_\varepsilon - G) &= (K(\varrho_\varepsilon)\varrho_\varepsilon - \overline{K(\varrho)\varrho})\boldsymbol{F} - \frac{1}{2}K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon\\ -\frac{1}{2}\operatorname{div}(K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) &+ \overline{K(\varrho)\varrho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \varepsilon h\boldsymbol{v}_\varepsilon - \varepsilon\varrho_\varepsilon\boldsymbol{v}_\varepsilon + \mu\Delta\nabla^{\perp}(A^{\varepsilon} - A). \end{aligned} \tag{4.12}$$

Now

$$(K(\varrho_\varepsilon)\varrho_\varepsilon - \overline{K(\varrho)\varrho})\boldsymbol{F} \rightharpoonup 0 \quad \text{in } L^q(\Omega)\ \forall q<\infty$$

and thus, as we consider a general sequence g_ϵ, then we have

$$\nabla(g_\varepsilon - g) \rightharpoonup 0 \ \text{ in } \ L_q(\Omega) \Longrightarrow g_\varepsilon - g \to \textit{const} \text{ in } L_q(\Omega),$$

i.e. the first term gives the strong convergence. Next

$$\begin{aligned} &\frac{1}{2}K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon + \frac{1}{2}\operatorname{div}(K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon) - \overline{K(\varrho)\varrho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\\ &= \frac{1}{2}\operatorname{div}(K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon + K(\varrho_\varepsilon)\varrho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\nabla\boldsymbol{v}_\varepsilon - \overline{K(\varrho)\varrho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}. \end{aligned} \tag{4.13}$$

Now, for the first term we have

$$\frac{1}{2}\operatorname{div}(K(\varrho_\varepsilon)\varrho_\varepsilon \boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon = \frac{1}{2}\varepsilon\Delta\varrho_\varepsilon \boldsymbol{v}_\varepsilon - \frac{1}{2}\varepsilon\varrho_\varepsilon \boldsymbol{v}_\varepsilon + \frac{1}{2}\varepsilon h K(\varrho_\varepsilon)\boldsymbol{v}_\varepsilon$$

and by (4.1) it converges to zero strongly in $W_2^{-1}(\Omega)$. The first term of the right hand-side determines the space of convergence. The other two terms in (4.13) converge to zero weakly in $L_q(\Omega)$ thanks to an argument explained above and to the fact that $\boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon K(\varrho_\varepsilon)\varrho_\varepsilon$ is bounded in any $L_q(\Omega)$, $q<\infty$.

The next two terms go to zero even strongly in any $L_q(\Omega)$. To conclude, we have to study the last term in (4.12).

Let us first show that

$$\nabla(\omega_\varepsilon - \omega) = B_\varepsilon^1 + B_\varepsilon^2,$$

where $B_\varepsilon^1 \rightharpoonup 0$ in $L_2(\Omega)$ while $B_2^\varepsilon \to 0$ in $W_2^{-1}(\Omega)$.

We have

$$\Delta(\omega_\varepsilon - \omega) = -\varepsilon h\omega_\varepsilon - \varepsilon \operatorname{curl}(\varrho_\varepsilon \boldsymbol{v}_\varepsilon) - \frac{1}{2}\operatorname{curl}(K(\varrho_\varepsilon)\varrho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon)$$
$$-\frac{1}{2}\operatorname{curl}\operatorname{div}(K(\varrho_\varepsilon)\varrho \boldsymbol{v}_\varepsilon \otimes \boldsymbol{v}_\varepsilon)) + \operatorname{curl}(\overline{K(\varrho)\varrho}\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \operatorname{curl}((K(\varrho_\varepsilon)\varrho_\varepsilon - \overline{K(\varrho)\varrho})\boldsymbol{F}) \quad \text{in } \Omega,$$
$$\omega_\varepsilon - \omega = \Big(2\chi - \frac{f}{\nu}\Big)(\boldsymbol{v}_\varepsilon - \boldsymbol{v})\cdot\boldsymbol{\tau} \qquad \text{at } \partial\Omega.$$

The first two terms on the right hand-side go to zero even strongly in $L_2(\Omega)$. Next, as above

$$K(\varrho_\varepsilon)\varrho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \boldsymbol{v}_\varepsilon - \overline{K(\varrho)\varrho}\boldsymbol{v}\cdot\nabla\boldsymbol{v} = B_\varepsilon^1 + B_\varepsilon^2$$

with $B_\varepsilon^1 \to 0$ strongly in $W_2^{-1}(\Omega)$ and $B_\varepsilon^2 \rightharpoonup 0$ weakly in $L_2(\Omega)$. The convergence to zero for $\operatorname{div}(K(\varrho_\varepsilon)\varrho_\varepsilon \boldsymbol{v}_\varepsilon)\boldsymbol{v}_\varepsilon$ and $(K(\varrho_\varepsilon)\varrho_\varepsilon - \overline{K(\varrho)\varrho})\boldsymbol{F}$ can be shown as above. Now, as

$$\|\Delta\nabla^\perp(A_\varepsilon - A)\|_{-1,2} \le \|\nabla(\omega_\varepsilon - \omega)\|_{-1,2}$$

we have that

$$\Delta\nabla^\perp(A_\varepsilon - A) = B_\varepsilon^1 + B_\varepsilon^2$$

with $B_\varepsilon^1 \to 0$ strongly in $W_2^{-1}(\Omega)$ and $B_\varepsilon^2 \rightharpoonup 0$ weakly in $L_2(\Omega)$. Therefore, we get the same for $\nabla(G_\varepsilon - G)$ and thus

$$G_\varepsilon - G \to const \quad \text{in } L_2(\Omega).$$

But

$$\int_\Omega (G_\varepsilon - G)dx = \int \Delta(\varphi_\varepsilon - \varphi)dx + \int_\Omega (P(\varrho_\varepsilon) - \overline{P(\varrho)})dx \to 0$$

as

$$\int_{\partial\Omega} \frac{\partial\varphi}{\partial\boldsymbol{n}}dS = \int_{\partial\Omega}\frac{\partial\varphi_\varepsilon}{\partial\boldsymbol{n}}dS = 0$$

and thus the constant is zero. The lemma is proved. □

Next we show the crucial information about the convergence of approximative sequence of the density.

Lemma 4.3 *Let $\kappa > 0$ and let m satisfy $||G||_\infty^{1/\gamma} < m < m_1$ and $\frac{m^{\gamma+1}}{m_2} \geq ||G||_\infty + \kappa$, then*

$$\lim_{\varepsilon_n \to 0^+} |\{x \in \Omega : \varrho_{\varepsilon_n}(x) > m\}| = 0.$$

Proof. At the beginning let us note that the choice of m is possible since numbers m_1 and m_2 are sufficiently greater than $||G||_\infty^{1/\gamma}$.

First, let us introduce a function $M(\cdot) \in C^1(\mathbb{R})$ such that

$$M(t) = \begin{cases} 1 & t \leq m \\ \in (0,1) & m < t < m+1 \\ 0 & t+1 \leq \varrho \end{cases}$$

such that $M'(t) < 0$ in $(m, m+1)$ and $m+1 < m_1$. Take $l \in \mathbb{N}$ and multiply the approximative continuity equation by $M^l(\varrho_\varepsilon)$. As

$$\int_\Omega \varepsilon \Delta \varrho_\varepsilon M^l(\varrho_\varepsilon) dx = -\varepsilon l \int_\Omega M'(\varrho_\varepsilon) M^{l-1}(\varrho_\varepsilon) |\nabla \varrho_\varepsilon|^2 dx \geq 0,$$

we have

$$\int_\Omega M^l(\varrho_\varepsilon) \operatorname{div}(K(\varrho_\varepsilon)\varrho_\varepsilon \boldsymbol{v}_\varepsilon) dx = -\varepsilon \int_\Omega \varrho_\varepsilon M^l(\varrho_\varepsilon) dx + \int_\Omega \varepsilon \Delta \varrho_\varepsilon M^l(\varrho_\varepsilon) dx$$
$$+\varepsilon h \int_\Omega K(\varrho_\varepsilon) M^l(\varrho_\varepsilon) dx \geq -\varepsilon \int_\Omega \varrho_\varepsilon M^l(\varrho_\varepsilon) dx + \varepsilon h \int_\Omega K(\varrho_\varepsilon) M^l(\varrho_\varepsilon) dx \equiv B_\varepsilon,$$

where $B_\varepsilon \to 0$ as $\varepsilon \to 0$. Thus, integrating by parts on the left hand-side

$$-l \int_\Omega M^{l-1}(\varrho_\varepsilon) M'(\varrho_\varepsilon) K(\varrho_\varepsilon) \varrho_\varepsilon \boldsymbol{v}_\varepsilon \cdot \nabla \varrho_\varepsilon dx \geq B_\varepsilon.$$

As $M'(\varrho_\varepsilon) = 0$ for $\varrho_\varepsilon \geq m+1$, we have

$$\int_\Omega \left(\int_0^{\varrho_\varepsilon} t M^{l-1}(t) M'(t) dt \right) \operatorname{div} \boldsymbol{v}_\varepsilon dx \geq \frac{1}{l} B_\varepsilon.$$

Next, recall the definition of G_ε. We have

$$-\int_\Omega \left(\int_0^{\varrho_\varepsilon} t M^{l-1}(t) M'(t) dt \right) P(\varrho_\varepsilon) dx \leq -\int_\Omega \left(\int_0^{\varrho_\varepsilon} t M^{l-1}(t) M'(t) dt \right) G_\varepsilon dx - \frac{2\mu + \nu}{l} B_\varepsilon$$

and therefore

$$-m \int_\Omega \left(\int_0^{\varrho_\varepsilon} M^{l-1}(t) M'(t) dt \right) P(\varrho_\varepsilon) dx \leq m_2 \int_\Omega \left| \int_0^{\varrho_\varepsilon} -M^{l-1}(t) M'(t) dt \right| |G_\varepsilon| dx + \frac{1}{l} |B_\varepsilon|.$$

Thus

$$\frac{m}{m_2} \int_{\{\varrho_\varepsilon > m\}} (1 - M^l(\varrho_\varepsilon)) P(\varrho_\varepsilon) dx \leq \int_{\{\varrho_\varepsilon > m\}} (1 - M^l(\varrho_\varepsilon)) |G_\varepsilon| dx + \frac{1}{m_2} |B_\varepsilon|. \tag{4.14}$$

Take $\delta > 0$, sufficiently small. Then by the Lebesgue theorem, noting that $M^l(t) \to 0$ for $t > m$ as $l \to \infty$, we deduce existence of a number $l \in \mathbb{N}$ sufficiently large, $l = l(\delta, \varepsilon)$, such that

$$\|M^l(\varrho_\varepsilon)\|_{L_2(\{\varrho_\varepsilon > m\})} \leq \delta.$$

Inequality (4.14) and the definition of $P(\cdot)$ then yields

$$\frac{m}{m_2} m^\gamma |\{\varrho_\varepsilon > m\}| \leq \frac{m}{m_2} \|P(\varrho_\varepsilon)\|_{L_2(\{\varrho_\varepsilon > m\})} \|M^l(\varrho_\varepsilon)\|_{L_2(\{\varrho_\varepsilon > m\})}$$
$$+\|G\|_\infty |\{\varrho_\varepsilon > m\}| + C(|\Omega|)\|G - G_\varepsilon\|_2 + \frac{1}{m_2}|B_\varepsilon|.$$

Thus

$$\left(\frac{m^{\gamma+1}}{m_2} - \|G\|_\infty\right) |\{\varrho_\varepsilon > m\}| \leq C\delta + C\|G - G_\varepsilon\|_2 + \frac{1}{m_2}|B_\varepsilon|.$$

Recall that $\|G\|_\infty \leq C m_2^{1+\eta}$ with $1 + \eta < \gamma$ and $||G||_\infty$ is described by $||F||_\infty$ and M as in Theorem 2.1. Thus due to the choice of m and m_2,

$$\frac{m^{\gamma+1}}{m_2} - \|G\|_\infty \geq \kappa$$

and therefore

$$|\{\varrho_\varepsilon > m\}| \leq C(\kappa)(\delta + \|G - G_\varepsilon\|_2 + |B_\varepsilon|). \tag{4.15}$$

Passing with $\varepsilon_n \to 0$ (the fact that $l = l(\varepsilon)$ does not matter here), by Lemma 4.2, we conclude

$$\lim_{\varepsilon_n \to 0} |\{x \in \Omega : \varrho_{\varepsilon_n}(x) > m\}| \leq C\delta,$$

at least for a suitably chosen subsequence $\varepsilon_n \to 0$. As $\delta > 0$, arbitrary, we have in fact

$$\lim_{\varepsilon_n \to 0} |\{x \in \Omega : \varrho_{\varepsilon_n}(x) > m\}| = 0.$$

□

To simplify the notation, we will write again $\varepsilon \to 0$ instead of the correct $\varepsilon_n \to 0$. Now

$$\int_\Omega \varrho_\varepsilon K(\varrho_\varepsilon)\varphi dx = \int_\Omega \varrho_\varepsilon \varphi dx + \int_{\{\varrho_\varepsilon > m_1\}} \varrho_\varepsilon (K(\varrho_\varepsilon) - 1)\varphi dx \to \int_\Omega \varrho \varphi dx$$

for any φ smooth and thus problem **(1)** from the beginning of this section is solved.

The next aim is to show that the convergence of the density is in fact strong in any $L_q(\Omega)$, $q < \infty$, namely to solve problem **(2)**. Take $\delta > 0$ and multiply the approximative continuity equation by $\ln m_2 - \ln(\varrho_\varepsilon + \delta)$. First

$$\int_\Omega \varepsilon \Delta \varrho_\varepsilon (\ln m_2 - \ln(\varrho_\varepsilon + \delta))dx = \varepsilon \int_\Omega |\nabla \varrho_\varepsilon|^2 \frac{1}{\varrho_\varepsilon + \delta} dx \geq 0.$$

Thus

$$\int_\Omega [\operatorname{div}(K(\varrho_\varepsilon)\varrho_\varepsilon \boldsymbol{v}_\varepsilon) + \varepsilon \varrho_\varepsilon - \varepsilon h K(\varrho_\varepsilon)][\ln m_2 - \ln(\varrho_\varepsilon + \delta)]dx \geq 0,$$

i.e.
$$\int_\Omega K(\varrho_\varepsilon)\varrho_\varepsilon \frac{\boldsymbol{v}_\varepsilon\cdot\nabla\varrho_\varepsilon}{\varrho_\varepsilon+\delta}dx \geq \int_\Omega [\varepsilon h K(\varrho_\varepsilon)-\varepsilon\varrho_\varepsilon][\ln m_2 - \ln(\varrho_\varepsilon+\delta)]dx.$$
Next we pass with $\delta\to 0^+$. The only difficult term is the first one on the right hand-side. But
$$0\leq \int_\Omega \varepsilon h K(\varrho_\varepsilon)\ln\frac{m_2}{\varrho_\varepsilon+\delta} 1_{\{\varrho_\varepsilon < m_2-\frac{1}{2}(m_2-m_1)\}}dx$$
$$\leq \int_\Omega [K(\varrho_\varepsilon)\varrho_\varepsilon \frac{\boldsymbol{v}_\varepsilon\cdot\nabla\varrho_\varepsilon}{\varrho_\varepsilon+\delta} + \varepsilon\varrho_\varepsilon \ln\frac{m_2}{\varrho_\varepsilon+\delta} - \varepsilon h K(\varrho_\varepsilon)\ln\frac{m_2}{\varrho_\varepsilon+\delta}1_{\{\varrho_\varepsilon\geq m_2-\frac{1}{2}(m_2-m_1)\}}]dx$$
and the Fatou lemma implies
$$\int_\Omega K(\varrho_\varepsilon)\boldsymbol{v}_\varepsilon\cdot\nabla\varrho_\varepsilon dx \geq \int_\Omega [\varepsilon h K(\varrho_\varepsilon)-\varepsilon\varrho_\varepsilon][\ln m_2 - \ln\varrho_\varepsilon]dx.$$
Thus
$$-\int_\Omega \varrho_\varepsilon \operatorname{div}\boldsymbol{v}_\varepsilon dx \geq \int_\Omega (1-K(\varrho_\varepsilon))\boldsymbol{v}_\varepsilon\cdot\nabla\varrho_\varepsilon dx + \varepsilon\int_\Omega hK(\varrho_\varepsilon)\ln\frac{m_2}{\varrho_\varepsilon}dx$$
$$-\varepsilon\int_\Omega \varrho_\varepsilon \ln\frac{m_2}{\varrho_\varepsilon}dx \geq -\int_\Omega\Big(\int_0^{\varrho_\varepsilon}(1-K(t))dt\Big)\operatorname{div}\boldsymbol{v}_\varepsilon dx - \varepsilon\int_\Omega \varrho_\varepsilon\ln\frac{m_2}{\varrho_\varepsilon}dx$$
as the middle term is non-negative. This yields, due to the fact that $\operatorname{div}\boldsymbol{v}_\varepsilon$ is bounded in $L_2(\Omega)$ and $|\{x\in\Omega;\varrho(x)>m_1\}|\to 0$,
$$-\int_\Omega \varrho_\varepsilon\operatorname{div}\boldsymbol{v}_\varepsilon dx \geq B_\varepsilon$$
with $B_\varepsilon\to 0$ as $\varepsilon\to 0$. Thus, recalling the definition of G_ε we have
$$\int_\Omega P(\varrho_\varepsilon)\varrho_\varepsilon dx + B_\varepsilon \leq \int_\Omega G_\varepsilon\varrho_\varepsilon dx.$$
This reads, after passing with $\varepsilon\to 0$,
$$\int_\Omega \overline{P(\varrho)\varrho}dx \leq \int_\Omega G\varrho dx, \tag{4.16}$$
recall Lemma 4.2.

Next we consider the limit equation
$$\begin{aligned} -(2\mu+\nu)\operatorname{div}\boldsymbol{u} + \overline{P(\varrho)} &= G,\\ \operatorname{div}(\varrho\boldsymbol{v}) &= 0.\end{aligned}$$
We define the distribution
$$\boldsymbol{v}\cdot\nabla\varrho = \operatorname{div}(\varrho\boldsymbol{v}) - \varrho\operatorname{div}\boldsymbol{v}.$$
We will use the following lemma. Its proof can be found in [8], Theorem 1.1:

Lemma 4.4 *Let* $\Omega \in C^{0,1}$, $\boldsymbol{v} \in W^1_q(\Omega)$, $1 < q < \infty$, $\varrho \in L_p(\Omega)$, $1 < p < \infty$, $\boldsymbol{v}\cdot\nabla\varrho \in L^s(\Omega)$, $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. *Then there exists* $\varrho_n \in C^\infty(\overline{\Omega})$ *such that*

$$\boldsymbol{v}\cdot\nabla\varrho_n \to \boldsymbol{v}\cdot\nabla\varrho \quad \text{in } L_s(\Omega) \qquad \text{and} \quad \varrho_n \to \varrho \quad \text{in} \quad L_p(\Omega).$$

The assumptions of Lemma 4.4 are fulfilled, recall that $\varrho \in L_\infty(\Omega)$, $\boldsymbol{v} \in W^1_p(\Omega)$ for any $1 \le p < \infty$. First, let us note that we may use as a test function for the continuity equation any smooth function up to the boundary. (We pass to the limit in the weak formulation for the approximative continuity equation.) Take ϱ_n from the lemma above. Then

$$\int_\Omega \operatorname{div}(\varrho_n\boldsymbol{v})dx = \int_{\partial\Omega} \varrho_n\boldsymbol{v}\cdot\boldsymbol{n}dS = 0.$$

Thus

$$\int_\Omega (\varrho_n \operatorname{div}\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\varrho_n)dx = 0$$

and passing with $n\to\infty$ we get

$$\int_\Omega (\varrho \operatorname{div}\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\varrho)dx = 0.$$

Now, let us emphasize that ϱ_n is constructed using the partition of unity and the Friedrichs lemma. As a matter of fact, we have $0 \le \varrho_n \le m_2$. Thus, let $\delta > 0$ and we use as the test function to the continuity equation $-\ln\frac{\delta}{\varrho_n+\delta}$. We get

$$0 = -\int_\Omega \varrho\boldsymbol{v}\cdot\nabla\ln\frac{\delta}{\varrho_n+\delta}dx = \int_\Omega \frac{\varrho\boldsymbol{v}\cdot\nabla\varrho_n}{\varrho_n+\delta}dx.$$

We now pass with $n\to\infty$. As $\boldsymbol{v}\cdot\nabla\varrho_n \to \boldsymbol{v}\cdot\nabla\varrho$ in any $L_p(\Omega)$ and $\varrho_n\to\varrho$ a.e., $\delta \le \varrho_n + \delta < m_2 + \delta$, we have

$$0 = \int_\Omega \frac{\varrho}{\varrho+\delta}\boldsymbol{v}\cdot\nabla\varrho dx.$$

Finally, passing with $\delta\to 0^+$ we have $\int_\Omega \boldsymbol{v}\cdot\nabla\varrho = 0$, i.e. $\int_\Omega \varrho\operatorname{div}\boldsymbol{v}dx = 0$. Thus, multiplying by ϱ the equality

$$G = -(2\mu+\nu)\operatorname{div}\boldsymbol{v} + \overline{P(\varrho)}$$

and integrating over Ω reads

$$\int_\Omega \overline{P(\varrho)}\varrho dx = \int_\Omega G\varrho dx.$$

Thus

$$\int_\Omega \overline{P(\varrho)\varrho}dx \le \int_\Omega G\varrho dx = \int_\Omega \overline{P(\varrho)}\varrho dx.$$

Next, by standard arguments, we show the strong convergence of the approximative density in L_γ which easily implies the strong convergence in $L_p(\Omega)$ for all $p < \infty$. Let $B \subset \Omega$. The inequalities

$$\begin{aligned}\frac{1}{|B|}\int_B \varrho_\varepsilon dx &\le \Big(\frac{1}{|B|}\int_B 1dx\Big)^{1-\frac{1}{\gamma}}\Big(\frac{1}{|B|}\int_B \varrho_\varepsilon^\gamma dx\Big)^{\frac{1}{\gamma}},\\ \frac{1}{|B|}\int_B \varrho_\varepsilon^\gamma dx &\le \Big(\frac{1}{|B|}\int_B \varrho_\varepsilon^{\gamma+1} dx\Big)^{1-\frac{1}{\gamma}}\Big(\frac{1}{|B|}\int_B \varrho_\varepsilon dx\Big)^{\frac{1}{\gamma}}\end{aligned}$$

imply

$$\Big(\frac{1}{|B|}\int_B \varrho_\varepsilon dx \frac{1}{|B|}\int_B \varrho_\varepsilon^\gamma dx\Big)^{1-\frac{1}{\gamma}} \le \Big(\frac{1}{|B|}\int_B 1dx\Big)^{1-\frac{1}{\gamma}}\Big(\frac{1}{|B|}\int_B \varrho_\varepsilon^{\gamma+1} dx\Big)^{1-\frac{1}{\gamma}}.$$

Passing with $\varepsilon \to 0$ and then with $|B| \to 0$, we get

$$\varrho\overline{\varrho^\gamma} \le \overline{\varrho^{\gamma+1}}$$

a.e. in Ω. But, on the other hand, $\int_\Omega \overline{P(\varrho)\varrho} dx \le \int_\Omega \overline{P(\varrho)}\varrho dx$ and thus we have

$$\overline{\varrho^\gamma}\varrho = \overline{\varrho^{\gamma+1}}$$

a.e. in Ω. Similarly as above we may show that for a.a. $x \in \Omega$

$$\overline{\varrho^\gamma}^{\frac{\gamma+1}{\gamma}}(x) \le \overline{\varrho^{\gamma+1}}(x), \qquad \varrho(x) \le \overline{\varrho^\gamma}^{\frac{1}{\gamma}}(x).$$

Thus

$$\varrho(x)\overline{\varrho^\gamma}(x) = \overline{\varrho^{\gamma+1}}(x) \ge \overline{\varrho^\gamma}^{\frac{\gamma+1}{\gamma}}(x) \ge \varrho(x)\overline{\varrho^\gamma}(x).$$

The chain of inequalities is in fact the chain of equalities and we deduce

$$\varrho^\gamma(x) = \overline{\varrho^\gamma}(x)$$

for a.a. $x \in \Omega$ which implies together with (4.2) the strong convergence of the approximative densities to the sought function in the $L_p(\Omega)$-norm for any $p < \infty$.

To conclude, we have to show the validity of the boundary conditions in the weak sense for the momentum equation. But it follows easily passing to the limit in the corresponding weak formulation. Lemma 5.3 guarantees that the total mass $\int_\Omega \varrho dx = M$. Limits (4.2) fulfill Definition 1.1 and problem **(3)** got the answer. Theorem 1.1 is proved.

Remark. Note that the density obtained above is bounded by m from Lemma 4.3. To get the bound given by Theorem 2.1 we need to consider m_1 and m_2 sufficiently close to $||G||_\infty^{1/\gamma}$ and κ (from Lemma 4.3) sufficiently small. Then we obtain the solution from Theorem 1.1 fulfilling the a priori bound from Theorem 2.1.

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