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STEFAN PROBLEM IN A 2D CASE

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Abstract. The aim of this paper is to analyze the well posedness of the one-phase quasi-stationary Stefan problem with the Gibbs–Thomson correction in a two-dimensional domain which is a perturbation of the half plane. We show the existence of a unique regular solution for an arbitrary time interval, under suitable smallness assumptions on initial data. The existence is shown in the Besov–Slobodetskiĭ class with sharp regularity in the L_2-framework.

1. Introduction. The one-phase Stefan problem models phenomena of phase transitions between liquid and solid. The Gibbs–Thomson correction adds the influence of the shape of the free surface to the model. We will investigate the mathematical aspects of this system. We concentrate on the quasistationary case in a two-dimensional domain. We want to investigate the existence of solutions to the system.

This subject has been studied by many authors, e.g. in [2, 3, 4, 6]. These results provide only partial answers. It was not clear which approach gives the best information about the system. An important achievement is paper [5]. The authors noted that the system can be treated as a nonlocal nonlinear parabolic equation of order three. Using the theory of semigroups for abstract parabolic systems, they showed the existence of unique classical solutions locally in time.

In our paper, we construct regular solutions to the system in a domain which is a perturbation of the half plane. The key element of our technique is a Schauder-type estimate for a linearization of the full system. It turns out that the linearized equations are a local version of the following nonlocal parabolic equation of order three:

$$\partial_t \phi + (-\Delta)^{3/2}\phi = m. \tag{1.1}$$

Having this interesting property we are able to prove existence of solutions to the system such that the graph function (describing the free boundary)

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ϕ belongs to $W_2^{7/2,7/6}$ for any $T > 0$. However this can be done only for small data.

This new approach improves the results obtained in [5] as regards regularity and also clarifies the parabolic character of the model. We consider here only the two-dimensional case and the L_2-approach to show precisely the main idea of the technique. The resulting regularity of solutions is optimal in the L_2-framework (regularity of p cannot be decreased if we want to control regular solutions to system (1.2)).

The Stefan problem with the Gibbs–Thomson correction, also known as the Hele–Shaw system, reads:

$$\begin{aligned} &\Delta p = 0 && \text{in } \Omega_t,\\ &p = a\kappa && \text{on } \partial\Omega_t,\\ &\frac{\partial p}{\partial n} = -V_n && \text{on } \partial\Omega_t,\\ &\partial\Omega_t|_{t=0} = \partial\Omega_0, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} \partial\Omega_0 &= \{(x_1,\phi_0(x_1)) : x_1 \in \mathbb{R}\},\\ \partial\Omega_t &= \{(x_1,\phi(x_1,t)) : x_1 \in \mathbb{R}\}, \qquad t\in[0,T),\\ \Omega_t &= \{(x_1,x_2) : x_1 \in \mathbb{R} \text{ and } x_2 > \phi(x_1,t)\}. \end{aligned} \tag{1.3}$$

We are looking for the evolution of the domain Ω_t described by the free boundary $\partial\Omega_t$ and the existence of p.

Here $a > 0$ is a constant and κ denotes the curvature of the boundary $\partial\Omega_t$ and is given by the function ϕ as follows:

$$\kappa = \frac{1}{\sqrt{1+|\phi_{,x_1}|^2}}\,\partial_{x_1}\left(\frac{\phi_{,x_1}}{\sqrt{1+|\phi_{,x_1}|^2}}\right) \tag{1.4}$$

where the comma denotes differentiation. The quantity V_n is the normal velocity of the evolution of the boundary,

$$V_n = -\frac{\partial_t\phi}{\sqrt{1+|\phi_{,x_1}|^2}}. \tag{1.5}$$

The statement of problem (1.2)–(1.5) restricts our attention to the case where the boundary is the graph of a function. This assumption requires suitable smallness of norms of ϕ to avoid difficulties with the description of $\partial\Omega_t$. The function ϕ_0 describes the initial boundary $\partial\Omega_0$.

The main result is the following.

THEOREM 1.1. *Let $T > 0$ and $\phi_0 \in W_2^2(\mathbb{R})$. Then there exists $\varepsilon_0 = \varepsilon_0(T) > 0$ such that if*

$$\|\phi_0\|_{W_2^2(\mathbb{R})} \le \varepsilon_0, \tag{1.6}$$

then there exists a unique regular solution to problem (1.2) on the time interval $[0,T]$ *such that*

$$\begin{aligned}\nabla p &\in L_2(0,T;W_2^1(\Omega_t))\cap W_2^{1/3}(0,T;L_2(\Omega_t)),\\ \phi &\in W_2^{7/2,7/6}(\mathbb{R}\times[0,T]).\end{aligned} \tag{1.7}$$

The key element to prove Theorem 1.1 is a Schauder-type estimate for a linearization of (1.2). We will investigate the following system:

$$\begin{aligned}
&\Delta p = f && \text{in } \mathbb{R}^2_+\times(0,T),\\
&p|_{x_2=0} = a\phi_{,x_1x_1}+g && \text{on } \mathbb{R}\times(0,T),\\
&p_{,x_2}|_{x_2=0} = -\partial_t\phi + h && \text{on } \mathbb{R}\times(0,T),\\
&\phi|_{t=0} = \phi && \text{on } \mathbb{R},\\
&p\to 0 && \text{as } |x|\to\infty.
\end{aligned} \tag{1.8}$$

It will turn out that system (1.8) can be reduced to the parabolic equation (1.1).

The kernel of the paper is the following result.

THEOREM 1.2. *Let* $T>0$ *and assume that*

$$\begin{aligned}
&h, g_{x_1}\in W_2^{1/2,1/6}(\mathbb{R}\times(0,T)),\ \ g\in W_2^{0,1/2}(\mathbb{R}\times(0,T)),\\
&f\in L_2(\mathbb{R}^2_+\times(0,T))\cap W_2^{1/3}(0,T;W_2^{-1}(\mathbb{R}^2_+)),\\
&\phi_0\in W_2^2(\mathbb{R}).
\end{aligned} \tag{1.9}$$

Then there exists a unique solution to problem (1.8) *such that*

$$\begin{aligned}
&\|\phi\|_{W_2^{7/2,7/6}(\mathbb{R}\times(0,T))}+\|\nabla p\|_{W_2^{1,1/3}(\mathbb{R}^2_+\times(0,T))}\\
&\quad\le c(T)(\|h,g_{,x_1}\|_{W_2^{1/2,1/6}(\mathbb{R}\times(0,T))}+\|g\|_{W_2^{0,1/2}(\mathbb{R}\times(0,T))}+\|\phi_0\|_{W_2^2(\mathbb{R})}\\
&\qquad+\|f\|_{L_2(\mathbb{R}^2_+\times(0,T))\cap W_2^{1/3}(0,T;W_2^{-1}(\mathbb{R}^2_+))}),
\end{aligned} \tag{1.10}$$

where $c(T)$ *is an increasing function of* T.

The idea of the proof of Theorem 1.2 is based on the approach used for parabolic-elliptic systems as in [8]. The Fourier transform and the theory of Besov spaces will be basic tools to obtain the bound (1.10).

Theorem 1.1 is a consequence of Theorem 1.2 and the Banach fixed point theorem together with some technical lemmas proved in the Appendix.

The result can be extended to more general cases (n-dimensional for a general initial domain in the L_p-framework), but then more advanced techniques are required (see [7]).

The paper is organized as follows. In Section 2 we introduce the basic notation and some auxiliary results. Next, we prove Theorem 1.2. In Section 4, we prove Theorem 1.1. The last section contains the proofs of the lemmas from Section 2.

2. Notation. Throughout the paper we try to use standard notations.

The main considerations will be carried out in the anisotropic Besov–Slobodetskiĭ spaces $W_p^{m,n}$ for $m,n \geq 0$ and $p \geq 1$ (see [1]) with the norm

$$\|f\|_{W_p^{m,n}(\Omega\times(0,T))} = \|f\|_{L_p(\Omega\times(0,T))} + \langle f\rangle_{W_p^{m,n}(\Omega\times(0,T))}, \tag{2.1}$$

where $\langle f\rangle_{W_p^{m,n}}$ is the *main seminorm* of $\|f\|_{W_p^{m,n}}$ defined by

$$\begin{aligned}\langle f\rangle^p_{W_p^{m,n}(\Omega\times(0,T))} &= \int_0^T \langle f\rangle^p_{W_p^m(\Omega)}\,dt + \int_\Omega \langle f\rangle^p_{W_p^n(0,T)}\,dx \\ &= \sum_{|\alpha|=[m]} \int_0^T dt \int_\Omega dx \int_\Omega dx' \,\frac{|\partial_x^\alpha f(x,t) - \partial_x^\alpha f(x',t)|^p}{|x-x'|^{d+p(m-[m])}} \\ &\quad + \int_\Omega dx \int_0^T dt \int_0^T dt' \,\frac{|\partial_t^{[n]} f(x,t) - \partial_t^{[n]} f(x,t')|^p}{|t-t'|^{1+p(n-[n])}},\end{aligned} \tag{2.2}$$

where $d = \dim \Omega$ and $[\cdot]$ denotes the integer part of a real number.

By $W_2^{-1}(\mathbb{R}^2_+)$ we denote the dual space to

$$V = \{\varphi \in W^1_{2(\mathrm{loc})}(\mathbb{R}^2_+) : \nabla\varphi \in L_2(\mathbb{R}^2_+) \text{ and } \varphi|_{x_2=0} = 0\},$$

and

$$\|f\|_{W_2^{-1}(\mathbb{R}^2_+)} = \sup_\varphi \langle f,\varphi\rangle_{L_2(\mathbb{R}^2_+)}, \tag{2.3}$$

where the sup is taken over $\varphi \in V$ such that $\|\nabla\varphi\|_{L_2} \leq 1$.

The theory of Slobodetskiĭ spaces [1, Chap. XVIII] gives the following imbedding theorem.

Lemma 2.1. *Let $1 \leq p \leq q < \infty$, $m_1 > m_2 \geq 0$, $n_1 > n_2 \geq 0$ and $d = \dim\Omega$. Then*

$$W_p^{m_1,n_1}(\Omega\times(0,T)) \subset W_q^{m_2,n_2}(\Omega\times(0,T)), \tag{2.4}$$

provided

$$\frac{d}{m_1}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{n_1}\left(\frac{1}{p}-\frac{1}{q}\right) \leq \min\{1-(m_1-m_2)/m_1, 1-(n_1-n_2)/n_1\}. \tag{2.5}$$

The spaces $W_2^{3m,m}(\mathbb{R}\times(0,T))$ for $m \in \mathbb{R}_+$ will play a crucial role. The following trace theorem holds for this class.

Lemma 2.2. *Let $3m - 3/2 > 0$ and $u \in W_2^{3m,m}(\Omega\times(0,T))$ for some $T > 0$. Then*

$$\overline{u} = u|_{t=0} \in W_2^{3m-3/2}(\Omega). \tag{2.6}$$

LEMMA 2.3. *Let* $f \in W_2^{5/2,5/6}(\mathbb{R}\times(0,T))$ *and* $g \in W_2^{1/2,1/6}(\mathbb{R}\times(0,T))$. *Then*

$$\|fg\|_{W_2^{1/2,1/6}(\mathbb{R}\times(0,T))} \le c\|f\|_{W_2^{5/2,5/6}(\mathbb{R}\times(0,T))}\|g\|_{W_2^{1/2,1/6}(\mathbb{R}\times(0,T))}. \tag{2.7}$$

LEMMA 2.4. *Let* $f \in W_2^{5/2,5/6}(\mathbb{R}\times(0,T))$ *and* $g,h \in W_2^{3/2,1/2}(\mathbb{R}\times(0,T))$. *Then*

$$\begin{aligned}\|fgh\|_{W_2^{1/2,1/6}(\mathbb{R}\times(0,T))} \\ \le c\|f\|_{W_2^{5/2,5/6}(\mathbb{R}\times(0,T))}\|g\|_{W_2^{3/2,1/2}(\mathbb{R}\times(0,T))}\|h\|_{W_2^{3/2,1/2}(\mathbb{R}\times(0,T))},\end{aligned} \tag{2.8}$$

$$\|fg\|_{W_2^{0,1/2}(\mathbb{R}\times(0,T))} \le c\|f\|_{W_2^{5/2,5/6}(\mathbb{R}\times(0,T))}\|g\|_{W_2^{3/2,1/2}(\mathbb{R}\times(0,T))}. \tag{2.9}$$

Lemmas 2.3 and 2.4 are shown in the Appendix.

For simplicity we introduce the following notation for norms of several variables in the same Banach space, say B:

$$\|a_1,\dots,a_n\|_B = \max\{\|a_1\|_B,\dots,\|a_n\|_B\} \tag{2.10}$$

for $a_1,\dots,a_n \in B$.

3. Model problem in the half space. The goal of this part is the analysis of a linearization of the system with frozen coefficients in $\mathbb{R}^2_+ \times (0,\infty)$. The results are stated in Theorem 1.2.

We consider (1.8) with $T=\infty$,

$$\begin{aligned}
&\Delta p = f && \text{in } \mathbb{R}^2_+\times(0,\infty),\\
&p|_{x_2=0} = a\phi_{,x_1x_1} + g && \text{on } \mathbb{R}\times(0,\infty),\\
&p_{,x_2}|_{x_2=0} = -\partial_t\phi + h && \text{on } \mathbb{R}\times(0,\infty),\\
&\phi|_{t=0} = \phi_0 && \text{on } \mathbb{R},\\
&p\to 0 && \text{as } |x|\to\infty.
\end{aligned} \tag{3.1}$$

The main result of this section is the following.

THEOREM 3.1. *The solutions to* (3.1) *exist and satisfy*

$$\begin{aligned}&\langle\phi\rangle_{W_2^{7/2,7/6}(\mathbb{R}\times(0,\infty))} + \langle\nabla p\rangle_{W_2^{1,1/3}(\mathbb{R}^2_+\times(0,\infty))} \\ &\quad\le c(\langle h, g_{,x_1}\rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))} + \langle g\rangle_{W_2^{0,1/2}(\mathbb{R}\times(0,\infty))} + \langle\phi_0\rangle_{W_2^2(\mathbb{R})} \\ &\qquad + \langle f\rangle_{L_2(\mathbb{R}^2_+\times(0,\infty))\cap W_2^{1/3}(0,\infty;W_2^{-1}(\mathbb{R}^2_+))}).\end{aligned} \tag{3.2}$$

Proof. The first step is connected with the function f. We consider the elliptic problem

$$\begin{aligned}
&\Delta p = f && \text{in } \mathbb{R}^2_+,\\
&p|_{x_2=0} = 0 && \text{on } \mathbb{R},
\end{aligned} \tag{3.3}$$

for $t\in(0,\infty)$. We prove

LEMMA 3.1. *Let*

$$f \in L_2(\mathbb{R}^2_+ \times (0,\infty)) \cap W_2^{1/3}(0,\infty; W_2^{-1}(\mathbb{R}^2_+)). \tag{3.4}$$

Then the solution of (3.3) *satisfies*

$$\langle \nabla p \rangle_{W_2^{1,1/3}(\mathbb{R}^2_+ \times (0,\infty))} + \langle p_{,x_2}|_{x_2=0} \rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))} \le c\|f\|_{L_2(\mathbb{R}^2_+\times(0,\infty))\cap W_2^{1/3}(0,\infty;W_2^{-1}(\mathbb{R}^2_+))}. \tag{3.5}$$

Proof. By the weak formulation of problem (3.3), we obtain existence in the space $W^1_{2(\mathrm{loc})}(\mathbb{R}^2_+) \cap \{\psi|_{x_2=0} = 0\}$,

$$(\nabla p, \nabla \psi)_{L_2(\mathbb{R}^2_+)} = -(f, \psi)_{L_2(\mathbb{R}^2_+)} \tag{3.6}$$

for any $\psi \in W^1_2(\mathbb{R}^2_+) \cap \{\psi|_{x_2=0} = 0\}$. Using the definition of $W_2^{1/3}$ (see (2.2)), we easily conclude that

$$\langle \nabla p \rangle_{W_2^{1/3}(0,\infty;L_2(\mathbb{R}^2_+))} \le c\langle f \rangle_{W_2^{1/3}(0,\infty;W^{-1}(\mathbb{R}^2_+))}. \tag{3.7}$$

Moreover, the Parseval identity leads to

$$\langle p \rangle_{W_2^{2,0}(\mathbb{R}^2_+\times(0,\infty))} \le c\|f\|_{L_2(\mathbb{R}^2_+\times(0,\infty))}. \tag{3.8}$$

From the trace theorem, we deduce the estimate for $p_{,x_2}|_{x_2=0}$. Lemma 3.1 is proved.

The above result makes it possible to omit the influence of the function f. Introduce the following form of solutions to problem (3.1):

$$p_{\mathrm{old}} = p_{\mathrm{new}} + p_{L1}, \tag{3.9}$$

where p_{L1} is the solution of (3.3) given by Lemma 3.1 and p_{old} is a solution of (3.1), while p_{new} is a solution of

$$\begin{aligned} &\Delta p = 0 && \text{in } \mathbb{R}^2_+ \times (0,\infty),\\ &p|_{x_2=0} = a\phi_{,x_1x_1} + g && \text{on } \mathbb{R}\times(0,\infty),\\ &p_{,x_2}|_{x_2=0} = -\partial_t\phi + h && \text{on } \mathbb{R}\times(0,\infty),\\ &\phi|_{t=0} = \phi_0 && \text{on } \mathbb{R},\\ &p \to 0 && \text{as } |x| \to \infty, \end{aligned} \tag{3.10}$$

where

$$g_{\mathrm{new}} = g_{\mathrm{old}} - p_{L1}|_{x_2=0}, \qquad h_{\mathrm{new}} = h_{\mathrm{old}} - p_{L1,x_2}|_{x_2=0}, \tag{3.11}$$

and by the estimates from Lemma 3.1 we have

$$\langle h_{\mathrm{new}}, g_{\mathrm{new},x_1} \rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty)} \le c\big(\langle h_{\mathrm{old}}, g_{\mathrm{old},x_1} \rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))} + \langle f \rangle_{L_2(\mathbb{R}^2_+\times(0,\infty))\cap W_2^{1/3}(0,\infty;W_2^{-1}(\mathbb{R}^2_+))}\big). \tag{3.12}$$

Now, we investigate system (3.10):

LEMMA 3.2. *Let $h, g_{,x_1} \in W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))$, $g \in W_2^{0,1/2}(\mathbb{R}\times(0,\infty))$ and $\phi_0 \in W_2^2(\mathbb{R})$. Then solutions to* (3.10) *exist and satisfy*

$$\begin{aligned}(3.13)\qquad &\langle\phi\rangle_{W_2^{7/2,7/6}(\mathbb{R}\times(0,\infty))} + \langle\nabla p\rangle_{W_2^{1,1/3}(\mathbb{R}^2_+\times(0,\infty))}\\ &\le c(\langle h, g_{,x_1}\rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))} + \langle g\rangle_{W_2^{0,1/2}(\mathbb{R}\times(0,\infty))} + \langle\phi_0\rangle_{W_2^2(\mathbb{R})}).\end{aligned}$$

Proof. We apply the standard approach to parabolic-elliptic systems, using the Fourier transform

$$\widehat{\cdot} = \mathcal{F}_{x_1}[\cdot] = \int_{\mathbb{R}} e^{-i\xi x_1}\cdot dx_1. \tag{3.14}$$

Then system (3.10) takes the following form:

$$\begin{aligned}&(-|\xi|^2+\partial_{x_2}^2)\widehat{p} = 0 && \text{in } \mathbb{R}^2_+\times(0,\infty),\\ &\widehat{p}|_{x_2=0} = -a|\xi|^2\widehat{\phi}+\widehat{g} && \text{on } \mathbb{R}\times(0,\infty),\\ &\widehat{p}_{,x_2}|_{x_2=0} = -\partial_t\widehat{\phi}+\widehat{h} && \text{on } \mathbb{R}\times(0,\infty),\\ &\widehat{\phi}|_{t=0} = \widehat{\phi}_0 && \text{on } \mathbb{R},\\ &\widehat{p}\to 0 && \text{as } x_2\to\infty.\end{aligned} \tag{3.15}$$

Solving the first equation, in view of $(3.15)_5$ (the 5th equation in (3.15)) we obtain

$$\widehat{p}(\xi,x_2,t) = \widehat{q}(\xi,t)e^{-|\xi|x_2} \tag{3.16}$$

for a function $q(\cdot,\cdot)$. Then the boundary conditions $(3.15)_{2,3}$ read

$$\begin{aligned}\widehat{q} &= -a|\xi|^2\widehat{\phi}+\widehat{g} && \text{on } \mathbb{R},\\ -|\xi|\widehat{q} &= -\partial_t\widehat{\phi}+\widehat{h} && \text{on } \mathbb{R}.\end{aligned} \tag{3.17}$$

Inserting the first equation into the second one we obtain

$$(\partial_t + a|\xi|^3)\widehat{\phi} = \widehat{h} + |\xi|\widehat{g} = \widehat{m} \quad \text{in } \mathbb{R}\times(0,\infty). \tag{3.18}$$

The above equation contains the main information carried by the system. It is a parabolic equation with an elliptic operator of order three which determines the type of regularity of solutions and also determines the whole procedure. As we saw, the above form of the equation is equivalent to (1.1).

To solve (3.18), we first construct an extension of the problem to $t<0$. By assumption $m \in W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))$. Introduce

$$\widetilde{m}(x,t) = \begin{cases} m(x,t) & \text{for } t\le 0,\\ m(x,-t) & \text{for } t<0.\end{cases} \tag{3.19}$$

By the definition of the Slobodetskiĭ spaces, $\widetilde{m} \in W_2^{1/2,1/6}(\mathbb{R}^2)$ and

$$\|\widetilde{m}\|_{W_2^{1/2,1/6}(\mathbb{R}\times\mathbb{R})} \le c\|m\|_{W^{1/2,1/6}(\mathbb{R}\times(0,\infty))}. \tag{3.20}$$

Now we modify $\widetilde{m}$ by defining

$$M(x,t)=\zeta(t)\widetilde{m}(x,t), \tag{3.21}$$

where $\zeta:\mathbb{R}\to[0,1]$ is a smooth function such that

$$\zeta(t)=\begin{cases}1 & \text{for } t\ge 0,\\ \in[0,1] & \text{for } -1<t<0,\\ 0 & \text{for } t\le -1.\end{cases} \tag{3.22}$$

Note that still $M\in W_2^{1/2,1/6}(\mathbb{R}^2)$ and

$$\|M\|_{W_2^{1/2,1/6}(\mathbb{R}\times\mathbb{R})}\le c\|m\|_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))} \tag{3.23}$$

with $M|_{t\ge 0}=m$.

Consider the initial value problem

$$\begin{aligned}(\partial_t+a|\xi|^3)\widehat{\phi}_1&=\widehat{M} \quad \text{in } \mathbb{R}\times(-1,\infty),\\ \widehat{\phi}_1|_{t=-1}&=0 \quad \text{on } \mathbb{R}.\end{aligned} \tag{3.24}$$

By the uniqueness in time and properties of M, we extend the system to $t<-1$ by zero and apply the Fourier transform with respect to time $(\tau\leftrightarrow t)$. Thus (3.24) reads

$$(i\tau+a|\xi|^3)\widehat{\phi}_1=\widehat{M} \quad \text{in } \mathbb{R}\times\mathbb{R}. \tag{3.25}$$

From (3.25), we get

$$\widehat{\phi}_1=\frac{\widehat{M}}{i\tau+a|\xi|^3}. \tag{3.26}$$

Applying the Parseval identity and the definition of the Besov–Slobodetskiĭ spaces (see Section 2) we obtain the bounds

$$\langle\phi_1\rangle_{W_2^{7/2,7/6}(\mathbb{R}\times\mathbb{R})}\le c\langle h,\partial_{x_1}g\rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))}, \tag{3.27}$$

provided that $h,\partial_{x_1}g\in W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))$.

Since $\phi_1=0$ for $t=-1$, we control the L_2-norm of the solution for finite time, hence in particular we have

$$\phi_1|_{t=0}\in W_2^2(\mathbb{R}) \quad\text{and}\quad \|\phi_1|_{t=0}\|_{W_2^2(\mathbb{R})}\le c\langle m\rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))}. \tag{3.28}$$

Putting

$$\phi=\phi_1+\phi_2, \tag{3.29}$$

by (3.18) and (3.24), we get

$$\begin{aligned}(\partial_t+a|\xi|^3)\widehat{\phi}_2&=0 \quad \text{in } \mathbb{R}\times(0,\infty),\\ \widehat{\phi}_2|_{t=0}&=\widehat{\phi}_0-\widehat{\phi}_1|_{t=0} \quad \text{on } \mathbb{R}.\end{aligned} \tag{3.30}$$

To solve (3.30) we prove the following result.

LEMMA 3.3. *Let* $\phi_0 \in W_2^2(\mathbb{R})$. *Then there exists a unique solution to the parabolic problem*

$$\begin{aligned} &\partial_t \phi + a\mathcal{F}_x^{-1}[|\xi|^3 \widehat{\phi}] = 0 \qquad && \text{in } \mathbb{R} \times (0,\infty), \\ &\phi|_{t=0} = \phi_0 && \text{on } \mathbb{R}, \end{aligned} \tag{3.31}$$

such that $\phi \in W_2^{7/2,7/6}(\mathbb{R} \times (0,\infty))$ *and*

$$\langle \phi \rangle_{W_2^{7/2,7/6}(\mathbb{R}\times(0,\infty))} \le c\|\phi_0\|_{W_2^2(\mathbb{R})}. \tag{3.32}$$

Proof. After the application of the Fourier transform system (3.31) reads

$$\begin{aligned} &\partial_t \widehat{\phi} + a|\xi|^3 \widehat{\phi} = 0 \qquad && \text{on } \mathbb{R} \times (0,\infty), \\ &\widehat{\phi} = \widehat{\phi}_0 && \text{in } \mathbb{R}. \end{aligned} \tag{3.33}$$

Thus we obtain the explicit formula

$$\widehat{\phi}(\xi,t) = \widehat{\phi}_0(\xi) e^{-a|\xi|^3 t}. \tag{3.34}$$

Let us estimate the seminorm of the solution given by (3.34). If the domain is $\mathbb{R}$, we can apply an equivalent definition of norms in W_2^s:

$$\langle f \rangle_{W_2^s(\mathbb{R})} = \int_{\mathbb{R}} d\xi\, |\xi|^{2s} |\widehat{f}|^2. \tag{3.35}$$

This form is more convenient to estimate spatial regularity. We have

$$\begin{aligned} \langle \phi \rangle^2_{W_2^{7/2,0}(\mathbb{R}\times(0,\infty))} &= \int_0^\infty \int_{\mathbb{R}} d\xi\, |\widehat{\phi}_0|^2 |\xi|^7 e^{-2a|\xi|^3 t}\, dt \\ &= \int_{\mathbb{R}} d\xi\, |\xi|^4 |\widehat{\phi}_0|^2 |\xi|^3 \frac{1}{2a|\xi|^3}\, d\xi = \frac{1}{2a} \langle \phi_0 \rangle^2_{W_2^2(\mathbb{R})}. \end{aligned} \tag{3.36}$$

And for the time regularity we apply the definition given by (2.2):

$$\begin{aligned} &\langle \partial_t \phi \rangle_{W_2^{0,1/6}(\mathbb{R}\times(0,\infty))} \\ &\qquad = \int_{\mathbb{R}} d\xi \int_0^\infty dt \int_0^\infty dt' \, \frac{\left| a|\xi|^3 \widehat{\phi}_0 e^{-a|\xi|^3 t} - a|\xi|^3 \widehat{\phi}_0 e^{-a|\xi|^3 t'} \right|^2}{|t-t'|^{1+1/3}}; \end{aligned} \tag{3.37}$$

introducing new coordinates $s = |\xi|^3 t$ and $s' = |\xi|^3 t'$, we see that this equals

$$\langle \phi_0 \rangle^2_{W_2^2(\mathbb{R})} \int_0^\infty ds \int_0^\infty ds' \, \frac{|e^{-as} - e^{-as'}|^2}{|s-s'|^{4/3}}. \tag{3.38}$$

The last integral is finite. Hence (3.36) and (3.38) imply (3.32). Lemma 3.3 is proved.

To finish the proof of Theorem 3.1, we need to find estimates on p. By the boundary equations (3.17), we deduce that

$$\langle q\rangle_{W_2^{3/2,1/2}(\mathbb{R}\times(0,\infty))}+\langle q_{,x_1}\rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))} \le c(\langle h,g_{,x_1}\rangle_{W_2^{1/2,1/6}(\mathbb{R}\times(0,\infty))}+\langle g\rangle_{W_2^{0,1/2}(\mathbb{R}\times(0,\infty))}). \tag{3.39}$$

By (3.16) and (3.35) we also obtain

$$\langle \mathcal{F}_{x_1}^{-1}[|\xi|^{1/2}\widehat{p}]\rangle_{W_2^{3/2,1/2}(\mathbb{R}^2_+\times(0,\infty))}+\langle \mathcal{F}_{x_1}^{-1}[|\xi|^{3/2}\widehat{p}]\rangle_{W_2^{1/2,1/6}(\mathbb{R}^2_+\times(0,\infty))} \le c(\langle h,\partial_{x_1}g\rangle_{W_2^{1/2,1/6}(\mathbb{R}^2_+\times(0,\infty))}+\langle g\rangle_{W_2^{0,1/2}(\mathbb{R}\times(0,\infty))}). \tag{3.40}$$

To get regularity with respect to x_2 it is enough to use arguments similar to (3.37)–(3.38) applied to (3.16). To control the norm of $p_{,x_2x_2}$ we apply $(3.10)_1$. Estimate (3.2) follows from (3.5), (3.12), (3.13), (3.32) and (3.40). Theorem 3.1 is proved.

Note that Theorem 3.1 only gives information on the main seminorms of solutions. To prove Theorem 1.2 we need information for finite time, which follows from the next lemma and estimate (3.28) for the whole norm.

LEMMA 3.4. *Let $f\equiv 0$, $g=h=0$ and $\phi_0=0$. Then system* (3.1) *admits only one regular solution.*

Proof. We want to show that all possible solutions in this case are trivial ($p=0$ and $\phi=0$). Multiply $(3.1)_1$ by p and integrate over $\mathbb{R}^2_+$, to get

$$0=-\int_{\mathbb{R}^2_+}\Delta p\cdot p\,dx=\int_{\mathbb{R}^2_+}|\nabla p|^2\,dx-\int_{\mathbb{R}}\frac{\partial p}{\partial n}p\,dx_1; \tag{3.41}$$

from the boundary conditions we obtain

$$\begin{aligned}\int_{\mathbb{R}^2_+}|\nabla p|^2\,dx+\int_{\mathbb{R}}p_{,x_2}p\,dx_1&=\int_{\mathbb{R}^2_+}|\nabla p|^2\,dx+a\int_{\mathbb{R}}(-\partial_t\phi)\phi_{,x_1x_1}\,dx_1\\&=\int_{\mathbb{R}^2_+}|\nabla p|^2\,dx+\frac{a}{2}\frac{d}{dt}\int_{\mathbb{R}}|\phi_{x_1}|^2\,dx_1.\end{aligned} \tag{3.42}$$

Since $\phi_0\equiv 0$, also $\int|\phi_{0,x_1}|^2dx_1=0$. Hence $\phi\equiv 0$ and $p\equiv 0$. Lemma 3.4 is shown.

The proof of Theorem 1.2 is complete.

4. Proof of Theorem 1.1. To analyze problem (1.2) we need to control the influence of the free boundary. The easiest solution is to introduce a transformation of Ω_t onto $\mathbb{R}^2_+$ as follows:

$$\Phi_t(\widetilde{x}_1,\widetilde{x}_2)=(\widetilde{x}_1,\widetilde{x}_2-\phi(\widetilde{x}_1,t)). \tag{4.1}$$

Note that the regularity of the transformation is equivalent to the smoothness of ϕ.

After the transformation (4.1) into the half space problem (1.2) reads

$$
\begin{aligned}
&\Delta p = (\Delta - \Delta_\phi)p && \text{in } \mathbb{R}^2_+ \times (0,T),\\
&p|_{x_2=0} = a\phi_{,x_1x_1} + b_1(\phi_{,x_1})\phi_{,x_1x_1} && \text{on } \mathbb{R} \times (0,T),\\
&p_{,x_2}|_{x_2=0} = -\partial_t\phi + b_2(\phi_{,x_1})\cdot\nabla p + b_3(\phi_{,x_1})\partial_t\phi && \text{on } \mathbb{R} \times (0,T),\\
&\phi|_{t=0} = \phi_0 && \text{on } \mathbb{R},
\end{aligned}
\tag{4.2}
$$

where

$$
\Delta_\phi = \Big(\sum_{l,k=1,2} \frac{\partial x_k}{\partial \widetilde{x}_l}\,\partial_{x_k}\Big)^2,
$$

$$
\begin{aligned}
b_1(\phi_{,x_1}) &= \frac{1}{1+|\phi_{,x_1}|^2} - 1,\\
b_2(\phi_{,x_1}) &= \bigg(\frac{\phi_{,x_1}}{\sqrt{1+|\phi_{,x_1}|^2}}, 1 - \frac{1}{\sqrt{1+|\phi_{,x_1}|^2}}\bigg),\\
b_3(\phi_{,x_1}) &= 1 - \frac{1}{\sqrt{1+|\phi_{,x_1}|^2}}
\end{aligned}
\tag{4.3}
$$

and $x_1 = \widetilde{x}_1$, $x_2 = \widetilde{x}_2 - \phi(\widetilde{x}_1, t)$.

To show existence we apply the standard Banach procedure. We will look for the solution as a fixed point of the map

$$
\Xi(q,\psi) = (p,\phi), \tag{4.4}
$$

where (p,ϕ) solves the system

$$
\begin{aligned}
&\Delta p = (\Delta - \Delta_\psi)q && \text{in } \mathbb{R}^2_+ \times (0,T),\\
&p|_{x_2=0} = a\phi_{,x_1x_1} + b_1(\psi_{,x_1})\psi_{,x_1x_1} && \text{on } \mathbb{R} \times (0,T),\\
&p_{,x_2}|_{x_2=0} = -\partial_t\phi + b_2(\psi_{,x_1})\cdot\nabla q + b_3(\psi_{,x_1})\partial_t\psi && \text{on } \mathbb{R} \times (0,T),\\
&\phi|_{t=0} = \phi_0 && \text{on } \mathbb{R}.
\end{aligned}
\tag{4.5}
$$

The solutions to problem (4.5) will be searched for in the space

$$
\begin{aligned}
\Pi = {}&(L_2(\mathbb{R}^2_+ \times (0,T)) \cap W_2^{1/3}(0,T; W_2^{-1}(\mathbb{R}^2_+)))\\
&\times (W_2^{7/2,7/6}(\mathbb{R}\times(0,T)) \cap \{\psi|_{t=0} = \phi_0\}).
\end{aligned}
\tag{4.6}
$$

First, we wish to find $\delta_0 > 0$, describing a set in Π, so small that

$$
\text{if} \quad \|(q,\psi)\|_\Pi \le \delta_0, \quad \text{then} \quad \|(p,\phi)\|_\Pi \le \delta_0. \tag{4.7}
$$

Next we show that map Ξ is a contraction on this set.

LEMMA 4.1. *If $\|\phi_0\|_{W_2^2(\mathbb{R})}$ is sufficiently small then the map Ξ is a contraction.*

Proof. From Theorem 1.2 and the form of system (4.5) we have

$$\begin{aligned}(4.8)\quad &\|(p,\phi)\|_{\Pi}\\ &\le c(\|\partial_{x_1}(b_1(\psi_{,x_1})\psi_{,x_1x_1}), b_2(\psi_{,x_1})\nabla q, b_3(\psi_{,x_1})\partial_t\psi\|_{W_2^{1/2,1/6}(\mathbb{R}\times(0,T))}\\ &\quad+\|b_1(\psi_{,x_1})\psi_{,x_1x_1}\|_{W_2^{0,1/2}(\mathbb{R}\times(0,T))}\\ &\quad+\|(\Delta-\Delta_\psi)q\|_{L_2(\mathbb{R}^2_+\times(0,T))\cap W_2^{1/3}(0,T;W_2^{-1}(\mathbb{R}^2_+))}+\|\phi_0\|_{W_2^2(\mathbb{R})}).\end{aligned}$$

Applying Lemmas 2.3 and 2.4 we conclude that

$$(4.9)\qquad \|(p,\phi)\|_{\Pi}\le a_1\|(q,\psi)\|_{\Pi}^2+a_2\|\phi_0\|_{W_2^2(\mathbb{R})}.$$

Taking (q,ψ) such that

$$(4.10)\qquad \|(q,\psi)\|\le\min\{2a_2\|\phi_0\|_{W_2^2(\mathbb{R})},1/2a_1\}=\delta_0,$$

we get (4.7).

We want to show

$$(4.11)\qquad \|\Xi(q,\psi)-\Xi(\widetilde{q},\widetilde{\psi})\|_{\Pi}\le(1-\varepsilon)\|(q,\psi)-(\widetilde{q},\widetilde{\psi})\|_{\Pi},$$

provided that

$$(4.12)\qquad \|(q,\psi),(\widetilde{q},\widetilde{\psi})\|_{\Pi}\le\delta_1$$

for sufficiently small δ_1 such that $\delta_0>\delta_1>0$.

To prove (4.11), we examine the following system which comes from (4.5) and the definition of Ξ:

$$(4.13)\qquad \begin{aligned}&\Delta(p-\widetilde{p})=F && \text{in } \mathbb{R}^2_+\times(0,T),\\ &(p-\widetilde{p})|_{x_2=0}=a(\phi-\widetilde{\phi})_{,x_1x_1}+G && \text{on } \mathbb{R}\times(0,T),\\ &(p-\widetilde{p})_{,x_2}|_{x_2=0}=-\partial_t(\phi-\widetilde{\phi})+H && \text{on } \mathbb{R}\times(0,T),\\ &(\phi-\widetilde{\phi})|_{t=0}=0 && \text{on } \mathbb{R}\times(0,T),\end{aligned}$$

where

$$(4.14)\qquad \begin{aligned}&F=(\Delta-\Delta_\psi)q-(\Delta-\Delta_{\widetilde{\psi}})\widetilde{q},\\ &G=b_1(\psi_{,x_1})\psi_{,x_1x_1}-b_1(\widetilde{\psi}_{,x_1})\widetilde{\psi}_{,x_1x_1},\\ &H=b_2(\psi_{,x_1})\cdot\nabla q-b_2(\widetilde{\psi}_{,x_1})\cdot\nabla\widetilde{q}+b_3(\psi_{,x_1})\partial_t\psi-b_3(\widetilde{\psi}_{,x_1})\partial_t\widetilde{\psi}.\end{aligned}$$

To find suitable estimates for solutions to (4.13) it is enough to use Lemma 2.3, since the terms of $(4.14)_2$ are products of functions from $W_2^{5/2,5/6}$ and $W_2^{1/2,1/6}$, hence

$$(4.15)\qquad \|H\|_{W_2^{1/2,1/6}}\le c\|(q,\psi),(\widetilde{q},\widetilde{\psi})\|_{\Pi}\|(q-\widetilde{q},\psi-\widetilde{\psi})\|_{\Pi}.$$

To estimate G we need to show in particular that

$$b_1'(\psi_{,x_1})\psi_{,x_1x_1}(\psi_{,x_1x_1}-\widetilde{\psi}_{,x_1x_1})\in W_2^{1/2,1/6}(\mathbb{R}\times(0,T)),$$

but $\psi_{x_1x_1} \in W_2^{3/2,1/2}$. By Lemma 2.4 and boundedness of the norms given by (4.7) we obtain

$$\|\partial_{x_1} G\|_{W_2^{1/2,1/6}} + \|G\|_{W_2^{0,1/2}} \le c\|\psi,\widetilde{\psi}\|_{W_2^{7/2,7/6}}\|\psi-\widetilde{\psi}\|_{W_2^{7/2,7/6}}. \tag{4.16}$$

To deal with F, we first study the L_2-norm

$$\|(\Delta-\Delta_\psi)(q-\widetilde{q})\|_{L_2} + \|(\Delta_\psi-\Delta_{\widetilde{\psi}})\widetilde{q}\|_{L_2}. \tag{4.17}$$

Note that, pointwise, we have

$$|(\Delta-\Delta_\psi)(q-\widetilde{q})| \le c(|\nabla\psi|\,|\nabla^2(q-\widetilde{q})| + |\nabla^2\psi|\,|\nabla q|). \tag{4.18}$$

By the regularity of ψ and Lemma 2.1 we see that

$$\begin{aligned} &|\nabla\psi| \in C(\mathbb{R}\times(0,T)),\\ &\nabla^2\psi \in W_2^{3/2,1/2}(\mathbb{R}\times(0,T)) \subset L_5(\mathbb{R}\times(0,T)), \end{aligned} \tag{4.19}$$

and from the properties of q and Lemma 2.1 we have

$$\nabla(q-\widetilde{q}) \in W_2^{1,1/3}(\mathbb{R}\times(0,T)) \subset L_{10/3}(\mathbb{R}\times(0,T)). \tag{4.20}$$

Hence the Hölder inequality yields

$$\|(\Delta-\Delta_\psi)(q-\widetilde{q})\|_{L_2} \le c\|\psi\|_{W_2^{7/2,7/6}}\|\nabla(q-\widetilde{q})\|_{W_2^{1,1/3}}. \tag{4.21}$$

The second term of (4.17) can be handled as follows:

$$|(\Delta_\psi-\Delta_{\widetilde{\psi}})\widetilde{q}| \le c(|\nabla(\psi-\widetilde{\psi})|\,|\nabla^2\widetilde{q}| + |\nabla^2(\psi-\widetilde{\psi})|\,|\nabla\widetilde{q}|). \tag{4.22}$$

By the same reasons as for (4.18), we conclude that

$$\|(\Delta_\psi-\Delta_{\widetilde{\psi}})\widetilde{q}\|_{L_2} \le c\|\psi-\widetilde{\psi}\|_{W_2^{7/2,7/6}}\|\nabla\widetilde{q}\|_{W_2^{1,1/3}}. \tag{4.23}$$

To estimate the next part of the norm, we recall that

$$\|f\|^2_{W_2^{1/3}(0,T;W_2^{-1}(\mathbb{R}^2_+))} = \int_0^T dt\int_0^T dt'\,\frac{|\sup_\phi\langle f(t)-f(t'),\phi\rangle_{L_2}|^2}{|t-t'|^{1+2/3}}, \tag{4.24}$$

where the sup is taken over $\phi \in W_2^1(\mathbb{R}^2_+)\cap\{\phi|_{x_2=0}=0\}$ and $\|\nabla\phi\|_{L_2}\le 1$.

Considering the same F as for the L_2-norm we have

$$\begin{aligned} &\langle(\Delta-\Delta_\psi)(q-\widetilde{q}),\phi\rangle_{L_2}\\ &\quad= \langle\nabla(\widetilde{q}-q),\nabla\phi\rangle_{L_2} + \langle\nabla_\psi(q-\widetilde{q}),\nabla_\psi\phi\rangle_{L_2}\\ &\quad= -\langle(\nabla-\nabla_\psi)(q-\widetilde{q}),\nabla\phi\rangle_{L_2} + \langle\nabla_\psi(q-\widetilde{q}),(\nabla-\nabla_\psi)\phi\rangle_{L_2}. \end{aligned} \tag{4.25}$$

Since

$$\|(\nabla-\nabla_\psi)\phi\|_{L_2} \le c\|\nabla\psi\|_C\|\phi\|_{W_2^1}, \tag{4.26}$$

we conclude that

$$\int_0^T dt \int_0^T dt' \, \frac{|\sup_\phi \langle \nabla_\psi (q-\widetilde{q}), (\nabla - \nabla_\psi)\phi\rangle|^2}{|t-t'|^{1+2/3}} \le c\|\psi\|^2_{W_2^{7/2,7/6}} \|\nabla(q-\widetilde{q})\|^2_{W^{1/3}(0,T;L_2(\mathbb{R}^2_+))}. \tag{4.27}$$

The analogous estimate holds for the first term of the r.h.s. of (4.25).

The term $(\varDelta_\psi - \varDelta_{\widetilde{\psi}})\widetilde{q}$ can be treated similarly. Thus we show that

$$\|F\|_{W_2^{1/3}(0,T;W_2^{-1}(\mathbb{R}^2_+))} \le c\|(q,\psi),(\widetilde{q},\widetilde{\psi})\|_\Pi \|(q-\widetilde{q}),(\psi-\widetilde{\psi})\|_\Pi. \tag{4.28}$$

Summing up we obtain

$$\|(p-\widetilde{p},\phi-\widetilde{\phi})\|_\Pi \le a_3\|(q,\psi),(\widetilde{q},\widetilde{\psi})\|_\Pi \|(q-\widetilde{q},\psi-\widetilde{\psi})\|_\Pi. \tag{4.29}$$

Since we assumed that

$$\|(q,\psi),(\widetilde{q},\widetilde{\psi})\|_\Pi \le \delta_1 \tag{4.30}$$

and δ_1 is so small that $a_3\delta_1 \le 1-\varepsilon$, the map $\varXi$ is a contraction. Lemma 4.1 is proved.

Lemma 4.1 and the choice $\varepsilon \le \delta_1$ complete the proof of Theorem 1.1.

5. Appendix

Proof of Lemma 2.3. We only deal with the seminorms. First,

$$\langle fg\rangle^2_{W_2^{1/2,0}} = \int_0^T dt \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx' \, \frac{|f(x,t)g(x,t) - f(x',t)g(x't)|^2}{|x-x'|^{1+1}}$$

$$\le \int_0^T dt \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx' \left(\frac{|f(x,t)|^2|g(x,t)-g(x',t)|^2}{|x-x'|^2} + \frac{|g(x,t)|^2|f(x,t)-f(x',t)|^2}{|x-x'|^2} \right)$$

$$= I_1 + I_2.$$

By the imbedding theorem, $f \in C(\mathbb{R}\times(0,T))$, hence

$$I_1 \le c\|f\|^2_{L_\infty} \|g\|^2_{W_2^{1/2,0}},$$

$$I_2 = \int_0^T dt \int dx\, |g(x,t)|^2 \int dx' \, \frac{|f(x,t)-f(x',t)|^2}{|x-x'|^2}$$

$$= \int_0^T dt \int dx\, |g(x,t)|^2 \Big(\int_{|x-x'|>1} + \int_{|x-x'|\le 1} \Big) \frac{|f(x,t)-f(x',t)|^2}{|x-x'|^2}$$

$$= I_{21} + I_{22}.$$

By the imbedding theorem we have $g \in W_2^{1/2,1/6} \subset L_{8/3}$, hence $g^2 \in L_{4/3}$.

Thus, to estimate I_{21} we need to bound the expression

$$\Big(\int\limits_{|x-x'|>1} \frac{|f(x,t)-f(x',t)|^2}{|x-x'|^2}\Big)^3;$$

by the Hölder inequality the above quantity is bounded by

$$\Big(\int\limits_{|x-x'|>1} \frac{dx'}{|x-x'|^{4/3}}\Big)^{3/4}\Big(\int\limits_{|x-x'|>1} \frac{|f(x,t)-f(x',t)|^6}{|x-x'|^{1+6\cdot 1/3}}\Big).$$

The first integral is uniformly bounded and the second is controlled by the norm of f, since we have the imbedding $W_2^{5/2,5/6} \subset W_6^{1/3,0}$. Hence

$$I_{21} \le c\|g\|^2_{W_2^{1/2,1/6}}\|f\|^2_{W_2^{5/2,5/6}}.$$

To estimate I_{22} we estimate

$$\Big(\int\limits_{|x-x'|\le 1} \frac{|f(x,t)-f(x',t)|^2}{|x-x'|^2}\Big)^3$$

by

$$\Big(\int\limits_{|x-x'|\le 1} \frac{dx'}{|x-x'|^{8/9}}\Big)^{3/4}\Big(\int\limits_{|x-x'|>1} \frac{|f(x,t)-f(x',t)|^6}{|x-x'|^{1+6\cdot 1/2}}\Big).$$

The first integral is uniformly bounded and the second is estimated by the norm of f since we have the imbedding

$$W_2^{5/2,5/6}(\mathbb{R}^2_+ \times (0,T)) \subset W_6^{1/2,0}(\mathbb{R}^2_+ \times (0,T)).$$

Thus

$$\|I_{22}\| \le c\|g\|^2_{W_2^{1/2,1/6}}\|f\|^2_{W_2^{5/2,5/6}}.$$

Let us consider the regularity with respect to time:

$$\langle fg\rangle^2_{W_2^{0,1/6}} = \int dx \int\limits_0^T dt \int\limits_0^t dt' \,\frac{|f(x,t)g(x,t)-f(x,t')g(x,t')|^2}{|t-t'|^{1+2\cdot 1/6}}$$

$$\le \int dx \int\limits_0^T dt \int\limits_0^t dt' \Big(\frac{|f(x,t)|^2|g(x,t)-g(x,t')|^2}{|t-t'|^{1+1/3}} + \frac{|g(x,t')|^2|f(x,t)-f(x,t')|^2}{|t-t|^{1+1/3}}\Big)$$

$$= J_1 + J_2.$$

To find the bound for J_1 we use the same argument as for I_1. So

$$J_2 \le \int dx \int\limits_0^T dt\, |g(x,t)|^2 \int\limits_0^T dt' \,\frac{|f(x,t)-f(x,t')|^2}{|t-t'|^{4/3}};$$

but $g^2 \in L_{4/3}$, and

$$\int dx \int_0^T dt \bigg(\int_0^T dt' \frac{|f(x,t)-f(x,t')|^2}{|t-t'|^{4/3}} \bigg)^3 \le C(T) \int dx \int_0^T dt \int_0^T dt' \, \frac{|f(x,t)-f(x,t')|^6}{|t-t'|^{1+6\cdot 1/6}}.$$

We have the imbedding $W_2^{5/2,5/6}(\mathbb{R}\times(0,T)) \subset W_6^{0,1/6}(\mathbb{R}\times(0,T))$. So

$$\langle fg\rangle^2_{W_2^{0,1/6}} \le c\|g\|^2_{W_2^{1/2,1/6}} \|f\|^2_{W_2^{5/2,5/6}}.$$

Lemma 2.3 is proved.

Proof of Lemma 2.4. We will just estimate one term. The others can be considered in a similar way. Take the seminorm connected with the regularity with respect to time,

$$\begin{aligned}\langle fgh\rangle^2_{W_2^{0,1/6}} &= \int_{\mathbb{R}} dx \int_0^T dt \int_0^T dt' \, \frac{|f(x,t)g(x,t)h(x,t)-f(x,t')g(x,t')h(x,t')|^2}{|t-t'|^{1+2\cdot 1/2}} \\ &\le c\int_{\mathbb{R}} dx \int_0^T dt \int_0^T dt' \bigg(\frac{|f(x,t)-f(x,t')|^2 |g(x,t)|^2 |h(x,t)|^2}{|t-t'|^{1+1/3}} \\ &\quad + \frac{|g(x,t)-g(x,t')|^2 |f(x,t)|^2 |h(x,t)|^2}{|t-t'|^{1+1/3}} \\ &\quad + \frac{|h(x,t)-h(x,t')|^2 |f(x,t)|^2 |g(x,t')|^2}{|t-t'|^{1+1/3}} \bigg) \\ &= I_1 + I_2 + I_3.\end{aligned}$$

We only handle I_1. From Lemma 2.1 we deduce

$$g^2h^2 \in L_2(\mathbb{R}\times(0,T)).$$

Hence

$$\begin{aligned} I_1 &\le c\|g\|^2_{W_2^{3/2,1/2}} \|h\|^2_{W_2^{3/2,1/2}} \int_{\mathbb{R}} dx \int_0^T dt \bigg(\int_0^T dt' \, \frac{|f(x,t)-f(x,t')|^2}{|t-t'|^{1+1/3}} \bigg)^2 \\ &\le c\|g\|^2_{W_2^{3/2,1/2}} \|h\|^2_{W_2^{3/2,1/2}} \bigg(\int_0^T \frac{dt'}{|t'|^{2/3}} \bigg) \bigg(\int_{\mathbb{R}} dx \int_0^T dt \int_0^T dt' \, \frac{|f(x,t)-f(x,t')|^4}{|t-t'|^{1+4\cdot 1/4}} \bigg) \\ &\le c(T)\|g\|^2_{W_2^{3/2,1/2}} \|h\|^2_{W_2^{3/2,1/2}} \|f\|^2_{W_2^{5/2,5/6}}, \end{aligned}$$

since $W_2^{5/2,5/6} \subset W_4^{1/4,0}$.

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