Long time behavior of a flow
in infinite pipe conforming to slip boundary conditions

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Abstract. The paper analyzes long time behavior of solutions of the Navier-Stokes equations in a two-dimensional pipe-like domain. The system is studied with perfect slip boundary conditions with arbitrary inflow conditions at infinity. The main results show the existence of global in time solutions and of an attractor for the dynamical system generated by the model. The paper also establishes an upper bound for the Hausdorff dimension of the attractor.

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1 Introduction

The theory of global attractors for infinite dimensional dynamical systems has been rapidly expanding for the last three decades. Variety of applications in hydrodynamics [Rob], theory of elasticity [Gao], physics [Pa], biology [Gop], engineering [Tem] and other disciplines has drawn attention of many researchers and a lot of important problems in applied mathematics have been studied from that perspective.

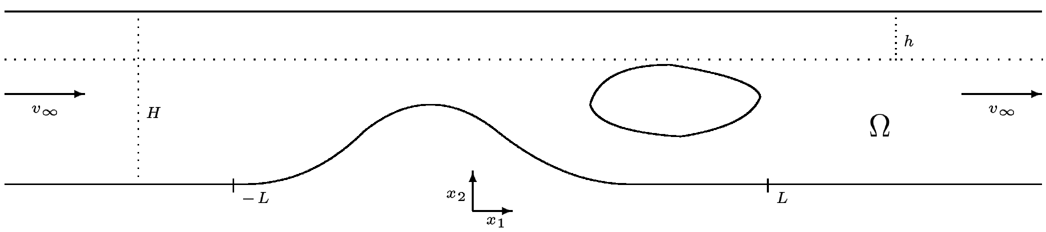
One of the most important and interesting among them with profound consequences to physics, biology or chemistry is the detailed description of the flows of incompressible fluids in unbounded channels. The motion of blood in aorta [Gop] and flows of melted polymers, irons or crystals [Fu, Li-Sh] are closely related to problems in pipe-like domains which can be also viewed as flows in infinite channels. Thus models in such domains have been carefully studied. Long time behavior of solutions to the Navier-Stokes equations (which are the most commonly used to describe the motion of fluids), in particular existence of a global attractor for flows in a two-dimensional case of unbounded domain has been investigated in [Ab] (in weighted spaces) and in [Ro, Ju] (in isotropic spaces). However, these results related to no-slip boundary condition with trivial flux of the flow what, from the physical point of view, cannot model a real nontrivial flow.

In our paper we investigate two-dimensional flow governed by the Navier-Stokes equations with arbitrary flux and data in an infinite pipe. We are interested in data which cause large flux of the flow, which is a natural approach in the context of the applications of this model. A similar problem has already been examined in [Mo] with inhomogeneous Dirichlet conditions. In our paper we examine the system with perfect slip boundary conditions which in contrary to the boundary relations in [Mo] do not require to prescribe velocity at the boundary. Thus they seem more suitable for conditions at infinity and they are much more appropriate for physical analysis. Another novelty of our paper is an estimation of an upper bound for the Hausdorff dimension of the global attractor, which differs from the standard bounds obtained for the flow with Dirichlet boundary conditions [Tem], [Ro].

We consider the following problem:

$$\begin{array}{lll} v_t + v\cdot\nabla v - \text{div }\mathbf{T}(v,p) = f & \text{in} & \Omega\times(0,T), \\ \text{div } v = 0 & \text{in} & \Omega\times(0,T), \\ n\cdot\mathbf{T}(v,p)\cdot\tau = 0, \quad n\cdot v = 0 & \text{on} & \partial\Omega\times(0,T), \\ v\to(v_\infty,0) \text{ as } |x_1|\to\infty & \text{for} & t\in(0,T), \\ v|_{t=0} = v_0 & \text{on} & \Omega, \end{array} \tag{1.1}$$

where $v=(v^1,v^2)$ is the velocity of the fluid, p is the pressure, ν - a constant viscous coefficient, f - an external force and a stress tensor $\mathbf{T}(v,p)=\nu\mathbf{D}(v)-pId$, where $\mathbf{D}(v)=\{v^i_{,j}+v^j_{,i}\}_{i,j=1,2}$. By Ω we denote the interior of the pipe, and by $\partial\Omega$ its boundary. Vectors n and τ describe the normal and tangent vectors to $\partial\Omega$, respectively. The positive constant v_∞ defines the flow at infinity and prescribe the flux of the flow.



To start our investigation we need to make our assumptions on the domain Ω more precise.

H1. $\partial\Omega$ is smooth - at least C^2;
H2. $\Omega \subset (-\infty, +\infty) \times (0, H)$;
H3. $\Omega \setminus ([-L, L] \times [0, H]) = ((-\infty, +\infty) \times (0, H)) \setminus ([-L, L] \times [0, H])$ - the perturbation is local;
H4. $(-\infty, +\infty) \times (H - h, H) \subset \Omega$.

On the basis of the above assumptions we define the domain as follows

$$\Omega = \Omega^{(0)} \cup \Omega^{(1)} \tag{1.2}$$

where $\Omega^{(0)}$ is bounded and

$$\Omega^{(0)} \supset (-L, L) \times (0, H) \tag{1.3}$$

and

$$\Omega^{(1)} \subset (-\infty, \infty) \times (0, H) \setminus (-L, L) \times (0, H). \tag{1.4}$$

We look for solutions to the system (1.1) in the parabolic function space $V^{1,0}(\Omega \times (0, T))$ given by the following norm

$$||v||_{V^{1,0}(\Omega\times(0,T))} = \sup_{0\leq t\leq T} ||v(\cdot, t)||_{L_2(\Omega)} + \left(\int_0^T ||\nabla v(\cdot, t)||^2_{L_2(\Omega)} dt\right)^{1/2}. \tag{1.5}$$

We consider external forces in the space dual to the Hilbert space

$$V = \{u \in H^1(\Omega; \mathbf{R}^2) : \text{div } u = 0 \text{ and } n \cdot u|_{\partial\Omega} = 0\}, \tag{1.6}$$

This space is related to another one

$$H = \{u \in L_2(\Omega) : \text{div } u = 0 \text{ in } \mathcal{D}'(\Omega)\}, \tag{1.7}$$

where $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. We will construct our solutions in the space H.

The first result which we will prove is the existence theorem for weak solutions to the system (1.1).

Theorem 1.1. *Let* $v_\infty \in \mathbf{R}$,

$$v_0|_{\Omega^{(0)}} \in L_2(\Omega^{(0)}), \quad v_0 - (v_\infty, 0)|_{\Omega^{(1)}} \in L_2(\Omega^{(1)}) \tag{1.8}$$

and div $v_0 = 0$. *Moreover, let* $f \in L_\infty(0, +\infty; V^*)$. *Then there exists unique global in time weak solution to the system (1.1) such that*

$$\begin{aligned} v|_{\Omega^{(0)}} &\in V^{1,0}(\Omega^{(0)} \times (k, k+1)), \\ v - (v_\infty, 0)|_{\Omega^{(1)}} &\in V^{1,0}(\Omega^{(1)} \times (k, k+1)) \end{aligned} \tag{1.9}$$

for any $k \in \mathbf{N}$.

The above result establishes existence of global in time solutions for arbitrary large data including the flux of the flow described by v_∞. Since the the domain Ω is two-dimensional the regularity described by (1.7) guarantees not only uniqueness, but also possibility to increase smoothness of solutions.

The proof of Theorem 1.1 is based on a technique applied in [Mu] to the steady version of system (1.1). Mathematical difficulties related to investigations of this type of problems are due to the Dirichlet integral $\int_\Omega \nabla v : \nabla v dx$. If this quantity is infinite we are not able to show similar results for the Navier-Stokes equations in pipe-like domains without restrictions on the magnitude of the flux [Gal, Pil]. However, in our case thanks to the slip boundary conditions the Dirichlet integral is finite and using a modification of the Hopf method we are able to prove existence of solutions for arbitrary fluxes.

Theorem 1.1 is a consequence of Theorem 2.1 proved in the next section.

If we assume that f is time independent, then it follows from Theorem 1.1 that there exists a semigroup

$$\bar{S}(t) : \bar{H} \to \bar{H} \tag{1.10}$$

acting on the space

$$\bar{H} = \{v \in L_{2(loc)}(\Omega) : \text{div } v = 0 \text{ in } \mathcal{D}'(\Omega),$$

$$v|_{\Omega^{(0)}} \in L_2(\Omega^{(0)}), \;\; v - (v_\infty, 0)|_{\Omega^{(1)}} \in L_2(\Omega^{(1)})\} \tag{1.11}$$

such that

$$v(t) = \bar{S}(t) v_0 \tag{1.12}$$

where $v(t)$ is the solution of the system (1.1) with initial condition v_0.

The long time behavior of the dynamical system generated by the semigroup $\bar{S}(t)$ is described by the following theorem.

Theorem 1.2. *Let $f \in V^*$. Then there exists a global attractor $\mathcal{A}$ for the semigroup $\bar{S}(\cdot)$ acting on the space $\bar{H}$. The Hausdorff dimension, $d(\mathcal{A})$, of the attractor $\mathcal{A}$ is finite and satisfies the following estimate*

$$d\mathcal{A} \le c(\Omega) \frac{1+\nu}{\nu^3} (e^{\frac{2v_\infty}{\nu}} + ||f||_{V^*}),$$

where $c(\Omega)$ depends only on the domain Ω.

From this result it follows that the system considered in the paper is another example of an infinite dimensional system which - in some sense

- exhibits similarity to finite dimensional model. However, finite number of degrees of freedom for the flow in unbounded domain is not easily interpreted in terms of standard heuristic considerations [Rob]. It is also worth noticing that the upper bound for the Hausdorff dimension of the attractor which we find in our paper reflects both infiniteness of the pipe and strong dependence on the injection of the fluid at infinity, and is much different from the well known estimations of this dimension in case of the flows in bounded domains and Dirichlet boundary conditions [Tem, Theorem 3.1, part VI].

The paper is organized as follows. In its first part we prove global in time existence of solutions with suitable a priori estimates. Subsequently, we proceed to find the attractor for the system. Eventually, we show finiteness of the dimension of the attractor.

2 Existence of weak solutions

The main difficulty of the problem (1.1) is hidden in the integrability of velocity. As we can see the L_2-norm, by condition $(1.1)_4$, is infinite, so we need to modify the function we search for. A trick used here is the same as in the stationary case - see [Mu]. We are looking for function v in the following form

$$v = u + a, \tag{2.1}$$

where vector field $a : \Omega \to \mathbf{R}^2$ satisfies the relations

$$\begin{array}{lll} \text{div } a = 0 & \text{in} & \Omega, \\ n \cdot \mathbf{D}(a) \cdot \tau = 0, \;\; n \cdot \tau = 0 & \text{on} & \partial\Omega, \\ a \to (v_\infty, 0) \;\text{ as }\; |x_1| \to \infty. & & \end{array} \tag{2.2}$$

We will apply the following result from [Mu].

Lemma 2.1. *For any positive constant α there exists a vector field $a : \Omega \to \mathbf{R}^2$ such that conditions (2.2) are satisfied and the inequality*

$$\left| \int_\Omega u \cdot \nabla a u dx \right| \leq \alpha ||u||^2_{H^1(\Omega)} \tag{2.3}$$

holds for any $u \in V$. Moreover, $|a| \leq c \exp \frac{v_\infty}{\nu}$, where the constant c depends only on Ω.

The construction of the vector field a has been done in [Mu, Theorem B].

We examine now the following modification of problem (1.1)

$$\begin{array}{lll}
u_t + u\cdot\nabla u - \mathrm{div}\ \mathbf{T}(u,p) = & & \\
-u\cdot\nabla a - a\cdot\nabla u - a\cdot\nabla a + \nu\Delta a + f & \text{in} & \Omega\times(0,T), \\
\mathrm{div}\ u = 0 & \text{in} & \Omega\times(0,T), \\
n\cdot u = 0, \quad n\cdot\mathbf{T}(u,p)\cdot\tau = 0 & \text{on} & \partial\Omega\times(0,T), \\
u\to(0,0) \quad \text{as} \quad |x_1|\to\infty & \text{for} & t\in(0,T), \\
u|_{t=0} = v_0 - a =: u_0 & \text{on} & \Omega.
\end{array} \tag{2.4}$$

We define the following weak formulation of the problem (2.4):

Definition 2.1(weak formulation). *We say that u is a weak solution to problem (2.4) if and only if* $u\in L_2(0,T;V)$ *and*

$$\int_\Omega u_t\varphi dx + \int_\Omega u\cdot\nabla u\varphi dx + \nu\int_\Omega \mathbf{D}(u):\nabla\varphi dx = -\int_\Omega u\cdot\nabla a\varphi dx$$

$$-\int_\Omega a\cdot\nabla u\varphi dx - \int_\Omega a\cdot\nabla a\varphi dx + \nu\int_\Omega \Delta a\varphi dx + f\varphi dx \tag{2.5}$$

Holds in the sense of distribution on interval $[0,T)$ *for every* $\varphi\in H^{1,1}(\Omega\times(0,T))$.

The existence of solutions satisfying the above definition follows from

Theorem 2.1. *Let* $f\in L_\infty(0,\infty;V^*)$, $u_0\in L_2(\Omega)$ *and div* $u_0=0$ *(in the distributional sense). Then there exists a unique global in time weak solution to problem (2.4) in the sense of Definition 2.1 such that* $u\in L_{2(loc)}(0,\infty;V)$ *and*

$$||u(\cdot,t)||_{L_2(\Omega)} \le e^{-\tilde{\nu}t}||u_0||_{L_2(\Omega)} + cM, \tag{2.6}$$

$$\sup_{k\in\mathbf{N}}||u||_{L_2(k,k+1;H^1(\Omega))} \le B(M + ||u_0||_{L_2(\Omega)}), \tag{2.7}$$

where

$$M = ||a\nabla a||_{L_2(\Omega)} + ||\nabla a||_{L_2(\Omega)} + ||f||_{L_\infty(0,\infty;V^*)} \tag{2.8}$$

with $B = c\frac{1+\nu}{\nu}$, *where the constant c depends only on* Ω.

Proof. Theorem 2.1 will be proved by the Galerkin method. As we know, the space

$$V = \{u\in H^1(\Omega;\mathbf{R}^2) : \mathrm{div}\ u = 0 \text{ and } n\cdot u|_{\partial\Omega} = 0\} \tag{2.9}$$

is separable Hilbert space and there exists a vector base $\{w_i\}_{i=1}^\infty$ such that the space V is the completion of span$\{w_1,...,w_n,....\}$ in the norm of $H^1(\Omega)$ and

$$\int_\Omega w_i\cdot w_j dx = \delta_{ij}, \tag{2.10}$$

where δ_{ij} is the Kronecker delta. We introduce notation

$$V^N = \text{span}\{w_1, ..., w_N\}. \tag{2.11}$$

It is worthwhile to notice that we are able to choose the base such that vectors w_i have compact supports. It is possible since by definition (2.9) we can prescribe any function from V as $\nabla^\perp \phi = (-\partial_{x_2}\phi, \partial_{x_1}\phi)$ for a scalar function ϕ such that $\phi|_{\partial\Omega} = 0$.

We look for approximate solutions

$$u^N(x,t) = \sum_{j=1}^{N} c_i^N(t) w_i(x), \tag{2.12}$$

where functions $\{c_i^N(t)\}_{i=1,...,N}$ are functions we search for. To find them we solve the following system of ordinary differential equations obtained from the weak formulation. By (2.5) we have

$$\begin{gathered} \tfrac{d}{dt}c_k^N(t) + \textstyle\int_\Omega u^N \cdot \nabla u^N w_k dx + \nu \int_\Omega \mathbf{D}(u^N) : \nabla w_k dx = -\int_\Omega u^N \cdot \nabla a w_k dx \\ -\textstyle\int_\Omega a \cdot \nabla u^N w_k dx - \int_\Omega a \cdot \nabla a w_k + \nu \int_\Omega \Delta a w_k + \int_\Omega f w_k dx, \\ c_k^N(0) = \textstyle\int_\Omega u_0 w_k dx \end{gathered} \tag{2.13}$$

for $k = 1, ..., N$.

This problem has a unique solution, but because of nonlinearity of the problem the existence of solutions may hold only for a time interval $[0, T_N]$. We will prolong these solutions on $[0, \infty)$, applying the energy estimates which will guarantee a bound on functions $c_k^N(\cdot)$ for all time. To obtain this result, we substitute u by u^N and φ by u^N in (2.5), and after some elementary calculations, using Lemma 2.1, we get the following inequality

$$\frac{1}{2}\frac{d}{dt}||u^N||^2_{L_2(\Omega)} + \nu||\mathbf{D}(u^N)||^2_{L_2(\Omega)} \le \frac{\tilde{\nu}}{4}||u^N||^2_{H^1(\Omega)} + \left|\int_\Omega u^N \cdot \nabla a u^N dx\right| + \frac{c}{2}M^2, \tag{2.14}$$

where M is defined by (2.8), c is a positive constant depending on Ω and $\tilde{\nu}$ is the constant from the Korn inequality [Mu]:

Lemma 2.2 (Korn inequality) *Let Ω fulfills assumptions H1-H4, then*

$$\tilde{\nu}||u||^2_{H^1(\Omega)} \le \nu||\mathbf{D}(u)||^2_{L_2(\Omega)} \tag{2.15}$$

holds for any $u \in V$.

Then we get

$$\frac{1}{2}\frac{d}{dt}||u^N||^2_{L_2(\Omega)} + \tilde{\nu}||u^N||^2_{H^1(\Omega)} \le \frac{\tilde{\nu}}{4}||u^N||^2_{H^1(\Omega)} + \left|\int_\Omega u^N \cdot \nabla a u^N dx\right| + \frac{c}{2}M^2, \tag{2.16}$$

From inequality (2.3) it follows that

$$\frac{d}{dt}||u^N||^2_{L_2(\Omega)} + \tilde{\nu}||u^N||^2_{H^1(\Omega)} \le cM^2. \tag{2.17}$$

Differential inequality (2.17) gives immediately two estimates on norms of u^N. The first one is a consequence of the Gronwall inequality

$$||u^N(\cdot,t)||^2_{L_2(\Omega)} \le e^{-\tilde{\nu}t}||u_0||^2_{L_2(\Omega)} + ce^{-\tilde{\nu}t}\int_0^t e^{\tilde{\nu}s}M^2 ds, \tag{2.18}$$

where M is defined as in (2.8) and from (2.18) we immediately get (2.6). The next estimate is

$$\sup_{k\in\mathbf{N}} ||u^N||^2_{L_2(k,k+1;H^1(\Omega))} \le c(M^2 + ||u_0||^2_{L_2(\Omega)}) \tag{2.19}$$

that follows from (2.18) and (2.17).

Therefore by estimate (2.19) we conclude that, for any $T > 0$, functions $u^N \in L_2(0,T;V)$ and the sequence $\{u^N\}_{N=1}^\infty$ is bounded in this space. Thus (see [Te]) there exists a subsequence $\{u^{N_k}\}_{k=1}^\infty$ which has a weak limit u_* in the space $L_2(0,T;V)$

$$\begin{aligned} u^{N_k} &\rightharpoonup u_* \quad \text{weakly in} \quad L_2(0,T;V) \ \text{ as } \ k\to\infty, \\ u^{N_k} &\rightharpoonup u_* \quad \text{weakly-}* \text{ in} \quad L_\infty(0,T;H) \ \text{ as } \ k\to\infty, \\ u^{N_k}\cdot\nabla u^{N_k} &\rightharpoonup u_*\cdot\nabla u_* \quad \text{weakly in} \quad L_{6/5}(\Omega\times(0,T)) \ \text{ as } \ k\to\infty. \end{aligned} \tag{2.20}$$

This convergence allows us to pass to the limit in the equations satisfied by approximate solutions and conclude that u_* is a solution we look for. By the standard theory this solution is unique (the imbedding theorem in two-dimensional domains) and satisfies estimates (2.6) and (2.7).

Lemma 2.3. *The solution u given by Theorem 2.1 fulfills the energy equality*

$$\int_\Omega u_t\varphi dx + \int_\Omega u\cdot\nabla u\varphi dx + \nu\int_\Omega \mathbf{D}(u) : \nabla\varphi dx = -\int_\Omega u\cdot\nabla a\varphi dx$$

$$-\int_\Omega a\cdot\nabla u\varphi dx - \int_\Omega a\cdot\nabla a\varphi dx + \nu\int_\Omega \Delta a\varphi dx + f\varphi dx \tag{2.21}$$

for any $\varphi \in V^{1,0}(\Omega \times (0,T))$. In particular, the solution belongs to the class of test functions $V^{1,0}$ and

$$u_t \in L_{2(loc)}(0,\infty; V^*). \tag{2.22}$$

Proof. By Definition 2.1 we can see that

$$\int_0^T \int_\Omega u_t \varphi dxdt = -\int_0^T \int_\Omega u \cdot \nabla u \varphi dxdt - \nu \int_0^T \int_\Omega \mathbf{D}(u) : \nabla \varphi dxdt$$
$$-\int_0^T \int_\Omega u \cdot \nabla a \varphi dxdt - \int_0^T \int_\Omega a \cdot \nabla u \varphi dxdt - \int_0^T \int_\Omega a \cdot \nabla a \varphi dxdt$$
$$+\nu \int_0^T \int_\Omega \Delta a \varphi dxdt - \int_0^T \int_\Omega f \varphi dxdt. \tag{2.23}$$

To prove (2.22) it is enough to note that the r.h.s of (2.23) is well defined for any $\varphi \in V^{1,0}$, one point which requires explanation is the estimate for the first term of the r.h.s. of (2.23). However, by the parabolic imbedding (dim $\Omega = 2$) we have

$$V^{1,0}(\Omega \times (0,T)) \subset L_4(\Omega \times (0,T)), \tag{2.24}$$

hence

$$\left| \int_0^T \int_\Omega u \cdot \nabla u \varphi dxdt \right| \leq ||u||_{L_4(\Omega\times(0,T))} ||\nabla u||_{L_2(\Omega\times(0,T))} ||\varphi||_{L_4(\Omega\times(0,T))}. \tag{2.25}$$

Hence we proved (2.22) which implies that the solution fulfills the energy e-quality. Moreover, estimates (2.6) and (2.7) imply that weak solutions belong to $V^{1,0}_{(loc)}(\Omega \times (0,\infty))$.

3 Existence of a global attractor

Now we will focus on a long time behavior of the flow in the infinite pipe. We assume from now on that f is time independent.

To examine the attractor for system (2.4), we introduce the following notation. The problem (2.4) can be treated as a functional equation

$$\frac{d}{dt}u + A(u) + R(u) + B(u,u) = F \quad \text{in} \quad V^*, \tag{3.1}$$

where $A : V \to V^*$, $R : V \to V^*$, $F : V \to V^*$, $B : V \times V \to V^*$ and

$$\begin{array}{rcl} < A(u), v > & = & \nu \int_\Omega \mathbf{D}(u) : \nabla\varphi dx, \\ < R(u), v > & = & (u \cdot \nabla a, v)_H + (a \cdot \nabla u, v)_H, \\ < F, v > & = & (\nu\Delta a - a \cdot \nabla a + f(\cdot), v)_H, \\ < B(u, u), v > & = & (u \cdot \nabla u, v)_H. \end{array} \tag{3.2}$$

By Theorem 2.1 the problem (2.4) defines a semigroup acting on the space H

$$S(t)u_0 = u(t) \tag{3.3}$$

where $u(\cdot)$ is the solution of (2.5) with initial data u_0.

Let us remind now two definitions [Tem].

Definition 3.1. *The set $\mathcal{B} \subset H$ is called an absorbing set for the semigroup $S(t)$ acting on the metric space H iff for any bounded set $B \subset H$ there exists time T such that $S(t)B \subset \mathcal{B}$ for all $t \geq T$.*

Definition 3.2. *The compact set $\mathcal{A} \in H$ is called a global attractor for the semigroup $S(t)$ acting on the metric space H iff $\mathcal{A}$ is invariant ($S(t)\mathcal{A} = \mathcal{A}$) and attracts every bounded set (for any bounded set $B \subset H$ and any $\varepsilon > 0$ there exists time T such that for any $t \geq T$ set $S(t)B$ is included in ε-neighborhood of $\mathcal{A}$).*

One of the main theorems on the existence of a global attractor [Tem] for a semigroup acting on some metric space is based on three conditions on a semigroup $S(t)$:
- existence of a bounded absorbing set;
- continuity of a semigroup;
- some kind of compactness of a semigroup.

The first two conditions are consequences of the results proved in the previous section: continuity of the semigroup follows from Lemma 2.3 and existence of a bounded absorbing set follows from estimate (2.6) since it implies that the set

$$\mathcal{B} = \{v \in H : ||v||_H \leq cM\} \tag{3.4}$$

is absorbing.

However, the compactness condition needs much more attention. Since the domain of the flow is unbounded the operators $S(t)$ are not compact for any $t > 0$. Thus, we will prove another form of compactness, which is asymptotical compactness of the semigroup. We remind that a semigroup is asymptotically compact iff for any bounded sequence u_n and any sequence

$t_n \to \infty$ the set $S(t_n)u_n$ is precompact. To prove this property we need two following lemmas.

Lemma 3.1. *Operator $A+R$ defined by (3.1) is coercive.*

Proof. From Lemma 2.1, setting $\delta = \tilde{\nu}/2$ we obtain

$$|\int_\Omega u \cdot \nabla a u dx| \leq \frac{\tilde{\nu}}{2}||u||^2_{H^1(\Omega)} \tag{3.5}$$

So from (2.15) we have

$$(Au + Ru, u) \geq \tilde{\nu}||u||^2_{H^1(\Omega)} - \frac{\tilde{\nu}}{2}||u||^2_{H^1(\Omega)} = \frac{\tilde{\nu}}{2}||u||^2_{H^1(\Omega)}. \tag{3.6}$$

Lemma 3.2. *The semigroup $S(t)$ is weakly continuous: if $u_n \rightharpoonup u$ weakly in H then for any $T > 0$*

$$S(T)u_n \rightharpoonup S(T)u \quad \text{weakly in } H \tag{3.7}$$

and

$$S(\cdot)u_n \rightharpoonup S(\cdot)u \text{ weakly in } L^2(0,T;V). \tag{3.8}$$

Proof. According to Theorem 2.1 the sequence $S(\cdot)u_n$ is bounded in $L^\infty(0,T;H)$ and in $L^2(0,T;V)$. Thus, some subsequence of $S(\cdot)u_n$ is weakly convergent in these spaces to some function $\tilde{u}$. We will prove that the convergence holds for all sequence and that $\tilde{u}$ is the solution of (2.5) on a time interval $[0,T]$ with initial condition u. Take any smooth test function φ with support contained in a ball B_R, $R \in \mathbf{N}$. By (2.7) and (2.22) from the Aubin-Lions theorem ([Te]) we can choose subsequence of $S(\cdot)u_n$ strongly convergent to $\tilde{u}$ in $L^2(0,T;H(\Omega \cap B_R))$. This strong convergence allows us to pass to the limit in the equation (2.5) with fixed φ. Since this procedure is possible for any φ we obtain that $\tilde{u}$ is the solution of (2.5) with initial condition u. From uniqueness of solutions follows that the whole sequence $S(\cdot)u_n$ is weakly convergent to $\tilde{u} = S(\cdot)u$ in $L^2(0,T;V)$.

Now we will prove that $S(T)u_n \rightharpoonup S(T)u$ in H. We will proceed as in [Ro]. Let v be any smooth function with support in B_R. From the strong convergence implied by the Aubin-Lions theorem we have

$$\int_\Omega S(t)u_n(x)v(x)dx \to \int_\Omega S(t)u(x)v(x)dx$$

for almost all $t \in [0,T]$. We prove that the convergence holds for all t. First from 2.7 we know that the functions $m_n(t) = \int_\Omega S(t)u_n(x)v(x)dx$ are uniformly bounded. From the inequality

$$(S(t+r)u_n(x) - S(t)u_n(x), v(x)) =$$

$$\int_t^{t+r} < u_n'(s), v > ds \leq \sqrt{r}||u'||_{L^2(0,T,V^*)}||v||_V \tag{3.9}$$

we deduce that these functions are also equally continuous on $[0,T]$ [Ro]. From the Arzela-Ascoli theorem and density of smooth functions in H we get (3.7).

Now we are ready to prove the following

Theorem 3.1. *There exists a global attractor $\mathcal{A}$ for the semigroup $S(t)$ acting on the metric space H.*

Proof. The proof of existence of a global attractor follows straightforward from the lemmas proven above and energy method first introduced by Rosa. However, we will sketch the main steps of the proof (following [Ro]) for the convenience of the reader. As the existence of a bounded absorbing set and continuity of the semigroup were proved before, we need only to show that the semigroup $S(t)$ is asymptotically compact [Fo, III, 3.1].

Let u_n be a bounded sequence in H and let $t_n \to \infty$. From the existence of a bounded absorbing set $\mathcal{B}$ follows that for any $T \in \mathbf{N}$ the sequences $S(t_n)u_n$ and $S(t_n - T)u_n$ are bounded in H. Since H is reflexive it follows from Banach-Alaoglu theorem that there exists subsequence (still denoted the same) such that $S(t_n)u_n \rightharpoonup u$ and $S(t_n - T)u_n \rightharpoonup u_T$ weakly in H for some u and u_T in H. We need to show that the convergence $S(t_n)u_n$ to u is strong for some subsequence. Because the weak convergence and convergence of norms imply strong convergence in Hilbert spaces it is enough to prove that

$$\limsup_n ||S(t_n)u_n||_H \leq ||u||_H \tag{3.10}$$

since the inequality

$$\liminf_n ||S(t_n)u_n||_H \geq ||u||_H \tag{3.11}$$

holds for any weakly convergent sequence $S(t_n)u_n$. To that end we rewrite the energy equation (2.21) in the following manner

$$\frac{1}{2}\frac{d}{dt}||u||_H^2 = -\frac{\tilde{\nu}}{4}||u||_H^2 - < (A+R)u, u > + \frac{\tilde{\nu}}{4}||u||_H^2 + < f, u >$$

$$= -\frac{\tilde{\nu}}{4}||u||_H^2 - < Du, u > + < f, u >, \tag{3.12}$$

where - according to Lemma 3.1 - the operator D is coercive

$$< Du, u > = < (A+R)u, u > -\frac{\tilde{\nu}}{4}||u||_H^2 \geq \frac{\tilde{\nu}}{4}||u||^2_{H^1(\Omega)} \geq \frac{\tilde{\nu}}{4}||u||_H^2. \tag{3.13}$$

Now from the energy equation we obtain that

$$||S(t_n)u_n||_H^2 = e^{-\frac{\tilde{\nu}T}{2}}||S(t_n - T)u_n||_H^2$$

$$-\int_0^T e^{-\frac{\tilde{\nu}(T-s)}{2}} < DS(s)S(t_n - T)u_n, S(s)S(t_n - T)u_n > ds$$

$$+\int_0^T e^{-\frac{\tilde{\nu}(T-s)}{2}} < f, S(s)S(t_n - T)u_n > ds. \tag{3.14}$$

Due to Lemma 3.1 the expression

$$||S(\cdot)u_0||_*^2 = \int_0^T e^{\frac{\tilde{\nu}(T-s)}{2}} < DS(s)u_0, S(s)u_0 > ds \tag{3.15}$$

defines a norm $||\cdot||_*$ on $L^2(0,T;V)$ equivalent to the usual norm. Thus from Lemma 3.2 it follows that

$$-\limsup ||S(\cdot)S(t_n - T)u_n||_* \leq -\liminf ||S(\cdot)S(t_n - T)u_0||_* \leq -||S(\cdot)u_T||_*. \tag{3.16}$$

Moreover, $S(t_n - T)u_n$ is absorbed by an absorbing set $\mathcal{B}$ which we may choose to be a ball of some radius R. Finally, we obtain

$$\limsup_n ||S(t_n)u_n||_H^2 \leq e^{-\frac{\tilde{\nu}T}{2}}R^2 - \int_0^T e^{-\frac{\tilde{\nu}(T-s)}{2}} < DS(s)u_T, S(s)u_T > ds$$

$$+\int_0^T e^{-\frac{\tilde{\nu}(T-s)}{2}} < f, S(s)u_T > ds. \tag{3.17}$$

Substituting into (3.17) the last two terms from equation

$$||u||_H^2 = ||S(T)u_T||_H^2 = e^{-\frac{\tilde{\nu}}{2}T}||u_T||_H^2 - \int_0^T e^{-\frac{\tilde{\nu}(T-s)}{2}} < DS(s)u_T, S(s)u_T > ds$$

$$+\int_0^T e^{-\frac{\tilde{\nu}(T-s)}{2}} < f, S(s)u_T > ds \tag{3.18}$$

and sending T to infinity we get the result.

Remark 1. The crucial tool in the proof above is energy equation which holds due to Lemma 2.3. Energy inequality would not be sufficient which is clearly seen at the end of the proof.

Remark 2. We proved existence of the global attractor for the semigroup acting on the space H. It obviously implies the existence of a global attractor for the semigroup defined by (1.10)-(1.12).

Remark 3. Using the above method it is possible to prove the existence of a global attractor also in a case of external forces dependent on time [Lu-Sa].

4 Dimension of a global attractor

In this section we turn our attention to estimation of Hausdorff dimension of the attractor $\mathcal{A}$.

Theorem 4.1. *The Hausdorff dimension $d_H(\mathcal{A})$ of the global attractor $\mathcal{A}$ is finite and satisfies following inequality*

$$d_H(\mathcal{A}) \leq \frac{c(1+\nu)M\kappa}{\nu\tilde{\nu}^2} \tag{4.1}$$

where c is some positive constant depending on Ω, $\tilde{\nu}$ is the constant from Korn inequality, κ is the constant from Lieb-Thirring inequality (cf [Tem], [Bo]) and

$$M = ||a\nabla a||_{L_2(\Omega)} + ||\nabla a||_{L_2(\Omega)} + ||f||_{V^*}. \tag{4.2}$$

Proof. The proof is based on the standard procedure in which we linearize the flow around the attractor. We consider volumes of m-dimensional infinitesimal parellepipedes and we look for such m that the volume decreases in time. According to the classical results [Tem] this is equivalent to searching for such m that the expression $\mathcal{L}_m$ defined below is negative. Thus let us consider the following formula

$$\mathcal{L}_m = \limsup_{t\to\infty} \sup \frac{1}{t}\int_0^t L_m(\tau)\ d\tau, \tag{4.3}$$

where

$$L_m = \sum_{j=1}^{m} \{(-(A(\varphi_j), \varphi_j) - (B(\varphi_j, v), \varphi_j))$$
$$-(B(v, \varphi_j), \varphi_j)) - (R(\varphi_j), \varphi_j))\}\ \mathrm{d}\tau \tag{4.4}$$

and $\{\varphi_j\}_{j=1}^m$ is a family of functions in V orthonormal in H, v is the solution with initial condition $v_0 \in H$ and supremum is taken over all families orthonormal in H and all initial conditions $v_0 \in \mathcal{A}$. We are going to find such m that $\mathcal{L}_m$ is negative.

First we notice that

$$(B(v, \varphi_j), \varphi_j) = 0. \tag{4.5}$$

Moreover, from Lemma 3.1 we have

$$((A + R)(\varphi_j), \varphi_j) \ \geq \frac{\tilde{\nu}}{2} ||\varphi_j||_{H^1}^2. \tag{4.6}$$

Finally, we estimate the second term of the r.h.s. of (4.4). We have for any positive α

$$|\sum_{j=1}^n (B(\varphi_j, v), \varphi_j)| \leq \int \sum_{j=1}^n |\varphi_j|^2 |\nabla v| dx \leq \frac{\alpha}{2} \int (\sum_{j=1}^n |\varphi_j|^2)^2 dx + \frac{1}{2\alpha} \int |\nabla v|^2 dx. \tag{4.7}$$

Then from Lieb-Thirring inequality [Fo, p.160] we know that there exists a constant κ independent from n such that

$$\int (\sum_{j=1}^n |\varphi_j|^2)^2 dx \leq \kappa \sum_{j=1}^n \int |\nabla \varphi_j|^2 dx. \tag{4.8}$$

Thus setting $\alpha = \frac{\tilde{\nu}}{2\kappa}$ we get

$$|(B(\varphi_j, v), \varphi_j)| \leq \frac{\tilde{\nu}}{4} \sum_{j=1}^m ||\varphi_j||_{H^1(\Omega)}^2 + \frac{\kappa}{\tilde{\nu}} ||v||_{H^1(\Omega)}^2 \tag{4.9}$$

From (4.6) and (4.9) we conclude that

$$\sum_{j=1}^m (-(A(\varphi_j), \varphi_j) - (B(\varphi_j, v), \varphi_j) - (R(\varphi_j), \varphi_j))$$

$$\leq -\frac{\tilde{\nu}}{4} \sum_{j=1}^m ||\varphi_j||_{H^1(\Omega)}^2 + \frac{\kappa}{\tilde{\nu}} ||v||_{H^1(\Omega)}^2 \leq -\frac{\tilde{\nu}}{4} m + \frac{\kappa}{\tilde{\nu}} ||v||_{H^1(\Omega)}^2, \tag{4.10}$$

since $\{\varphi_j\}$ are orthonormal in L^2. Now from (4.3) and (4.10) it follows that

$$\mathcal{L}_m \leq -\frac{\tilde{\nu}}{4} m + \limsup_{t \to \infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{t} \int_0^t \frac{\kappa}{\tilde{\nu}} ||v||_{H^1(\Omega)}^2 \tag{4.11}$$

From estimates (2.7) and the existence of absorbing ball of radius cM it follows that

$$\limsup_{t\to\infty} \sup_{u_0\in\mathcal{A}} \frac{1}{t}\int_0^t ||v||^2_{H^1(\Omega)} \leq B(M + \sup_{u_0\in\mathcal{A}} ||v_0||_H)) \leq BM(1+c), \quad (4.12)$$

where M is defined as in (2.8) and B is defined in Theorem 2.1. Thus, we easily obtain that for $m \geq \frac{4(1+c)BM\kappa}{\tilde{\nu}^2}$ the expression $\mathcal{L}_m$ is negative. Using the fact that $B = c\frac{1+\nu}{\nu}$ we get (4.1).

Remark 1. From theorem 4.1 we get theorem 1.2 since $\tilde{\nu} = c(\Omega)\nu$ (with some constant c depending only on Ω) and $|a| \leq ce^{\frac{v_\infty}{\nu}}$.

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References:

[Ab] F. Abergel *Attractors for a Navier-Stokes flow in an unbounded domain*, RAIRO Model. Math. Anal. Num., 23(3), 359-370, 1989.

[Bo] Boukrouche M., Łukaszewicz *Attractor dimension estimate for plane shear flow of micropolar fluid with free boundary*, to appear in Math. Methods Appl. Sci.

[Fo] C. Foias, O. Manley, R. Rosa, R. Temam *Navier-Stokes Equations and Turbulence*, Cambridge University Press, 2001.

[Fu] H. Fujita *Remarks on the Stokes flow under slip and leak boundary conditions of friction type.*, Topics in mathematical fluid mechanics, 73–94, Quad. Mat. 10, 2002.

[Gal] G.P. Galdi *An introduction to the mathematical theory of the Navier-Stokes equations.*, Springer-Verlag, 1994.

[Gao] H. Gao *Global attractor for the semilinear thermoelastic problem.*, Math. Methods Appl. Sci. 26 (2003), no. 15, 1255–1271.

[Gop] N.P. Gopalan *Pulsatile blood flow in a rigid pulmonary alveolar sheet with porous wall*, Bull. Math. Biol. 43, no. 5, 563-577, 1981.

[Ju] N. Ju *The H^1 compact global attractor for the solutions to the Navier-Stokes equations in two-dimensional unbounded domains*, Nonlinearity 13, 1227-1238, 2000.

[Li-Sh] Ch. Liu, J. Shen *On liquid crystal flows with free-slip boundary conditions.* Discrete Contin. Dynam. Systems 7, no. 2, 307–318, 2001.

[Lu-Sa] G. Łukaszewicz, W. Sadowski *Uniform attractor for 2D magneto-micropolar fluid flow in some unbounded domains*, Z. Angew. M ath. Phys., vol. 55, 2004.

[Mo] I.Moise, R. Rosa, X. Wang *Attractors for non-compact semigroups via energy equations*, Nonlinearity 11, 1369-1393, 1998.

[Mu] P.B. Mucha, On the Navier-Stokes equations with slip boundary conditions in an infinite pipe, *Acta Appl. Math. 76*, No. 1, 1-15 (2003).

[Pa] Pata, Vittorino *Attractors for a damped wave equation on R^3 with linear memory.*, Math. Methods Appl. Sci. 23 (2000), no. 7, 633–653.

[Pil] K. Pileckas *Recent advances in the theory of Navier-Stokes equations in domains with non-compact boundaries.*, Mathematical theory in fluid mechanics (Paseky, 1995), 30-85; Pitman Res. Notes Math. Ser. 354, 1996.

[Rob] J.C. Robinson *Infinite dimensional dynamical systems. An introduction.*, Cambridge 2000.

[Ro] R. Rosa *The global attractor for the 2D Navier-Stokes equations in two-dimensional unbounded domains*, Nonlinear Anal., TMA, Vol. 32,71-85, 1998.

[Tem] R. Temam *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Second Edition, Springer-Verlag, New York, 1997.

[Te] R. Temam *The Navier-Stokes Equations. Theory and Numerical Analysis*, North Holland, Amsterdam, 1977.