On a pump

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Abstract. We examine the Navier-Stokes equations in a two space dimensional time dependent domain. The system is considered with nonhomogeneous slip boundary conditions. The main result shows that under a certain geometrical constraint on deformations of the domain, it is possible to prove existence of solutions globally in time for arbitrary flows across the boundary with a new time independent bound on the vorticity. The geometrical restriction does not imply simply connectedness of the domain. Our model may be treated as a simple model of a pump.

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1 Introduction

The aim of this paper is to analyze the issue of existence for a model of two dimensional motion of an incompressible viscous fluid in a domain with moving boundaries governed by the Navier-Stokes equations with the slip boundary conditions admitting flows across the boundary.

The subject is connected to problems for the Navier-Stokes with nonhomogeneous boundary data. For the Dirichlet boundary conditions the current theory proves general results only for flows in simply connected domains [2, 3-Chap. VIII, 7]. In this case we apply the energy method which requires a construction of an extension of the boundary data. It follows that the modification of the system has homogenous boundary conditions, however the structure of equations loses the conservative character.

In the presented paper we investigate the slip boundary conditions. Their properties enable to reformulate the original system to the coupled system on the vorticity and velocity. It can be done, since the slip boundary conditions define the value of the vorticity at the boundary as a function of the velocity. The obtained form of the equations allows to apply the maximum principal. This way we show a bound of the vorticity uniformly in time without any restrictions on the flow through the boundary. Nevertheless, to realize this technique we need to assume a geometrical constraint on the shape of the time dependent domain. However this condition is independent of the magnitude of the boundary data. The obtained a priori bound has a linear time independent character as the energy estimate for the homogeneous case.

The method of the proof is an adaptation of the approach from [1,6].

The system is described by the evolutionary Navier-Stokes equations

$$\begin{array}{lll} v_t + v\cdot\nabla v - \mathrm{div}\,\mathbf{T}(v,p) = F & \text{in} & P(T), \\ \mathrm{div}\, v = 0 & \text{in} & P(T), \\ n\cdot\mathbf{T}(v,p)\cdot\tau + fv\cdot\tau = 0 & \text{on} & \partial P(T), \\ n\cdot v = V & \text{on} & \partial P(T), \\ v|_{t=0} = v_0 & \text{on} & p(0), \end{array} \tag{1.1}$$

where $v = (v^1, v^2)$ is the velocity of the fluid, p is the pressure, F - the external force, $\mathbf{T}(\cdot,\cdot)$ is the stress tensor for the Newtonian fluids

$$\mathbf{T}(v,p) = \nu\mathbf{D}(v) - p\,Id = \{\nu(v^i_{,j} + v^j_{,i}) - p\delta_{ij}\}_{i,j=1,2}, \tag{1.2}$$

$\mathbf{D}(\cdot)$ is the deformation tensor, ν - the constant viscous coefficient, f - the nonnegative friction coefficient which may be in general non constant, n and τ are the normal and tangent vectors to the boundary and v_0 is the initial datum.

To describe the time dependent domain we define

$$P(T) = \bigcup_{0\le t\le T} p(t)\times\{t\}, \quad \partial P(T) = \bigcup_{0\le t\le T} \partial p(t)\times\{t\}, \tag{1.3}$$

where $p(t)\subset \mathbf{R}^2$ denotes our domain in the moment of time t. The boundary $\partial p(t)$ we divide into two parts

$$\partial p(t) = \Gamma_{in/out}(t)\cup\Gamma_b(t). \tag{1.4}$$

We assume that part $\Gamma_{in/out}(t) = \Gamma_{in/out}$ is rigid, and datum V from $(1.1)_4$ on this part is denoted as follows

$$V_{in/out} = V|_{\Gamma_{in/out}(t)} \tag{1.5}$$

which describes the inflow and outflow into our pipe. The second part defines the moving part of the boundary, moreover datum V on $\Gamma_b(t)$

$$V_b = V|_{\Gamma_b(t)} = \frac{\partial}{\partial t}\varphi(x,t)\cdot n|_{\Gamma_b(t)} \tag{1.6}$$

describes the motion of the boundary, where $\varphi(x,t):(-L,L)\times(0,T)\to\mathbf{R}^2$ defines $\Gamma_b(t)$ - see the picture.

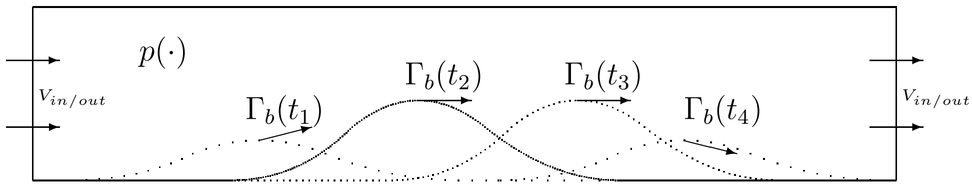
The compatibility conditions require the following constraints on the data

$$\int_{\partial p(t)} V(\cdot,t)d\sigma = 0 \quad \text{for} \quad t\in[0,T) \tag{1.7}$$

and

$$\text{div } v_0 = 0, \quad n\cdot v_0|_{\partial p(0)} = V(\cdot,0). \tag{1.8}$$

We assume that the boundary is sufficiently smooth at least $C^{2,1}$-piecewise and interior angles are $\pi/2$ - see the picture.



$$t_1 < t_2 < t_3 < t_4$$

$$p(t) = \{\bar{x}\in\mathbf{R}^2 : -L < x_1 < L, \;\; \varphi(x_1,t) < x_2 < 1\}$$

$$\Gamma_b(t) = \{(x_1,\varphi(x_1,t)) : -L < x_1 < L\}$$

The picture above shows a perturbation of the domain which describes a model of our titled pump - we may assume that the evolution of the boundary is periodic in time.

The main result is the following.

Theorem 1.1. *Let $0 < a < 1$, $\partial P(\infty)\in C^{2,1}$, $F\in L_\infty(P(\infty))$, $v_0\in C^a(p(0))$, $rot\, v_0\in L_\infty(p(0))$, $V\in C^{1,a/2}(\partial P(\infty))$; moreover let*

$$||A_0(p(\cdot))||_{C(0,\infty)}\,||2\chi(\cdot,\cdot) - f/\nu||_{C(\partial P(\infty))} < 1, \tag{1.9}$$

where $\chi(\cdot,t)$ is the curvature of curve $\partial p(t)$ and $A_0(p(t))$ is a constant depends only on features of domain $p(t)$ - see (2.11).

Then there exists unique global in time solution to problem (1.1) such that

$$v \in C^{a,a/2}(P(\infty)), \qquad rot\, v \in S_1 + S_2$$

with

$$S_1 = B(0,\infty; L_\infty(p(t))) \ \text{ and } \ S_2 = L_{2(loc)}(0,\infty; H_0^1(p(t))) \cap B(0,\infty; L_N(p(t)))$$

for any $N > 2/(1-a)$ and the following bound is valid

$$||v||_{C^{a,0}(P(\infty))} + ||rot\, v||_{B(0,\infty;L_\infty(p(t)))+L_{2(loc)}(0,\infty;H_0^1(p(t)))\cap B(0,\infty;L_N(p(t)))}$$

$$\leq c(\Omega)\left(\frac{1}{\nu}||F||_{L_\infty(P(\infty))} + ||rot\, v_0||_{L_\infty(p(0))} + ||V||_{C^{1,a/2}(\partial P(\infty))}\right), \qquad (1.10)$$

where $c(\Omega)$ is a constant independent of ν and velocity data.

The condition (1.9) determines the shape of possible deformations of the domain. Since effective friction f/ν is nonnegative we obtain a restriction on the negative part of the curvature of the boundary. Keeping this constraint we are able to prove the existence of solutions to our model with estimate (1.10) for any data $V_{in/out}$ and F for time interval $(0,\infty)$. Of course, V_b defines the evolution of the boundary, hence it defines the curvature, too. However (1.9) implies no restriction on the evolutions of deformations in time.

As we mentioned at the beginning, system (1.1) may be treated as a simple model of a pump. Taking the notation from the picture the effectiveness of the pump is measured by the following quantity

$$\mathcal{E} = \sup_{T\geq 0} \int_T^{T+1} \int_{\Gamma_b(t)} V_b n \cdot e_1 d\sigma dt. \qquad (1.11)$$

Constraint (1.9) gives no bound on the magnitude of the above quantity, since we may add to V_b term $g(t)e_1$ (with a smooth large function $g(t)$) which does not change the curvature, but will change $\mathcal{E}$.

To find an example of a domain with a multiple connected boundary fulfilling (1.9) it is enough to consider an annulus

$$\left\{x \in \mathbf{R}^2 : R_1 < |x| < R_2\right\},$$

then if $R_2 - R_1$ is sufficiently small then condition (1.9) is fulfilled. Small perturbations of this domain still preserve the condition, too.

The structure of system (1.1) requires to introduce a reformulation of the problem. A key role will play the vorticity of the velocity

$$\alpha = v^2_{,1} - v^1_{,2}. \tag{1.12}$$

From $(1.1)_{1,2}$ we obtain the following parabolic equation

$$\alpha_t + v \cdot \nabla\alpha - \nu\Delta\alpha = \text{rot } F \quad \text{in} \quad P(T). \tag{1.13}$$

The Dirichlet datum for equation (1.13) is given by

$$\alpha = (2\chi - f/\nu)v \cdot \tau + 2V_{,s} \quad \text{on} \quad \partial P(T). \tag{1.14}$$

This follows from slip conditions $(1.1)_{3,4}$ by differentiation $(1.1)_4$ with respect to the length parameter s. And the initial datum is taken from $(1.1)_5$

$$\alpha|_{t=0} = \text{rot } v_0 = \alpha_0 \quad \text{on} \quad p(0). \tag{1.15}$$

Next, we introduce the following elliptic system

$$\begin{array}{lll} \text{rot } v = \alpha & \text{in} & p(t), \\ \text{div } v = 0 & \text{in} & p(t), \\ n \cdot v = V & \text{on} & \partial p(t) \end{array} \tag{1.16}$$

for $t \in (0, t)$ to obtain information about the velocity.

We will investigate the coupled system consists of (1.13), (1.14), (1.15) and (1.16), instead of the original problem (1.1).

The paper is organized as follows. First we introduce basic notations. In section 3 a priori estimate is shown, for sufficiently regular solutions. Next, we prove the existence and increase the regularity of the solutions.

2 Notation

Throughout the paper we try to use the standard notation.

By $L_p(\Omega)$ we denote the Banach space with the following norm

$$||f||_{L_p(\Omega)} = \left(\int_\Omega |f|^p dx \right)^{1/p}. \tag{2.1}$$

For simplicity, we denote $L_p(\Omega;\mathbf{R})$ and $L_p(\Omega;\mathbf{R}^2)$ the same - $L_p(\Omega)$. Moreover

$$||f||_{L_\infty(\Omega)} = \text{ess} \sup_{x\in\Omega} |f(x)|, \quad ||f||_{B(\Omega)} = \sup_{x\in\Omega} |f(x)|. \tag{2.2}$$

Since we examine the domain given by (1.3) is time dependent, we precise the definition of spaces defined on this set.

Regularity of the boundary and condition (1.9) guarantee that there exists a map $\Phi : P(\infty) \to p(0) \times (0,\infty)$ such that Φ is a $C^{2,1}$-diffeomorfism. Then we define $C^{a,b}(P(T))$ - Hölder space $(0 < a, b < 1)$ with the following norm

$$||f||_{C^{a,b}(P(T))} = ||f(\Phi^{-1}(\cdot))||_{C^{a,b}(p(0)\times(0,T))}. \tag{2.3}$$

The same definition holds for the boundary quantities

$$||f||_{C^{a,b}(\partial P(T))} = ||f(\Phi^{-1}(\cdot))||_{C^{a,b}(\partial p(0)\times(0,T))}, \tag{2.4}$$

where $C^{a,b}$ on the r.h.s. of (2.3)-(2.4) are standard spaces.

The boundary of domain $P(T)$ is irregular, but always it is piecewise smooth at least $C^{2,1}$ and interior angles are $\pi/2$ to control regularity of the solutions. As we introduced in (1.3), $\partial P(T)$ may be split into smooth elements $\Gamma_{in/out}$ and $\bigcup_{0\le t\le T} \Gamma_b(t)$. Our considerations will be done on the level of weak formulations of the solutions, thus we need a weaker information for quantities defined at the boundary. Hence for any function space B defined on the boundary we denote

$$B(\partial P(T)) = B(V_{in/out}) \cap B(\bigcup_{0\le t\le T} V_b(t)) \tag{2.5}$$

and

$$B(\partial p(t)) = B(V_{in/out}) \cap B(V_b(t)). \tag{2.6}$$

Main parts of our study will be based on results of the following elliptic system

$$\begin{array}{lll} \text{rot } v = d & \text{in} & \Omega, \\ \text{div } v = 0 & \text{in} & \Omega, \\ n \cdot v = b & \text{on} & \partial\Omega, \end{array} \tag{2.7}$$

where Ω is a bounded domain in $\mathbf{R}^2$ with properties as domains $p(t)$.

For simply connected domains by the Poincare Lemma and $(2.7)_2$ we find a scalar function φ - the so-called stream function - such that

$$v = \nabla^{\perp}\varphi = (-\partial_{x_2}\varphi, \partial_{x_1}\varphi), \tag{2.8}$$

then (2.7) takes the following form

$$\begin{aligned} \Delta\varphi = d \quad & \text{in} \quad \Omega, \\ \varphi = \tilde{b} \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{2.9}$$

where

$$\tilde{b}(s) = \int_{s_0}^{s} b(s')ds'. \tag{2.10}$$

and s is the length parameter of curve $\partial\Omega$. Function φ is defined up to a constant.

If domain is non simply connected the kernel of rot-div problem (2.7) is non trivial and to keep the uniqueness to problem (2.7) we divide the space of solutions by the kernel of the operator. Next using an extension of the boundary datum we are able to reduce the problem to system (2.9).

The standard theory of the elliptic equations [4-Chap. IV] delivers the following results. Since interior angles are $\pi/2$ there appears no singularities connected to irregularity of the boundary.

Lemma 2.1. *Consider problem (2.7). The following estimates hold:*

(i) if $b = 0$ *and* $d \in L_\infty(\Omega)$*, then*

$$||v||_{C^a(\Omega)} \leq A_a(\Omega)||d||_{L_\infty(\Omega)}; \tag{2.11}$$

(ii) if $b = 0$ *and* $d \in L_N(\Omega)$*, then*

$$||v||_{C^a(\Omega)} \leq A^N(\Omega)||d||_{L_N(\Omega)} \tag{2.12}$$

for $0 \leq a < 1 - 2/N$*;*

(iii) if $d = 0$ *and* $b \in C^a(\partial\Omega)$*, then*

$$||v||_{C^a(\Omega)} \leq c||b||_{C^a(\partial\Omega)}. \tag{2.13}$$

The crucial quantity is the constant in estimate (2.11) for $a = 0$. It appears in the main condition (1.9) in statement of Theorem 1.1.

All generic constants are denoted by letter c. Constants $A_a(\cdot)$ and $A_N(\cdot)$ are given by estimates (2.11) and (2.12). By $DATA$ we denote constants depend on known (obtained) quantities.

3 A priori estimate

In this part we want to show a priori bound. We assume that the solutions exist and they are sufficiently smooth.

Theorem 3.1. *Let v be a sufficiently smooth solution to problem (1.1) and condition (1.9) be fulfilled. Then for* $0 < a < 1 - 2/N$

$$||v||_{B(0,\infty;C^a(p(t)))} + ||rot\, v||_{B(0,\infty;L_\infty(p(t)))+L_{2(loc)}(0,\infty;H_0^1(p(t)))\cap B(0,\infty;L_N(p(t)))} \leq X, \tag{3.1}$$

where

$$X = c(\Omega)\left(\frac{1}{\nu}||F||_{L_\infty(P(\infty))} + ||V||_{B(0,\infty;W^1_\infty(\partial p(t)))} + ||\alpha_0||_{L_\infty(p(0))}\right) \tag{3.2}$$

and

$$c(\Omega) = c\left(1 - ||A_0(p(\cdot))||_{C(0,\infty)}||2\chi - f/\nu||_{C(\partial P(\infty))}\right)^{-1} < 1.$$

The obtained estimate gives slightly weaker regularity of solutions than in Theorem 1.1. Regularity $C^{a,a/2}$ will be a consequence of (3.1) and the weak formulation of the solutions presented in section 4. It is worthwhile to point out that bound (3.1) is linear. The nonlinear term is neglected by features of the maximum principle.

The proof of Theorem 3.1 is split into two lemmas.

The first one follows from the maximum principle applied to the vorticity problem (1.13), (1.14) and (1.15).

Lemma 3.1. *Let* $N > 2$,

$$V \in B(0,\infty;W^1_\infty(\partial p(t))) \quad \text{and} \quad v \in B(0,\infty;C(\infty)),$$

then solutions to problem (1.13)-(1.15) satisfy the following bound

$$\begin{aligned}&||\alpha||_{B(0,\infty;L_\infty(p(t)))+L_{2(loc)}(0,\infty;H^1_0(p(t)))\cap B(0,\infty;L_N(p(t)))}\\ &\quad \leq ||2\chi - f/\nu||_{C(\partial P(\infty)))}||v||_{B(0,\infty;C(\partial p(t)))} + X.\end{aligned} \tag{3.3}$$

Proof. Define

$$k^* = \max\{\sup_{(x,t)\in\partial P(\infty)}(2\chi - f/\nu)v\cdot\tau + 2V_{,s}, \sup_{x\in p(0)}\alpha_0\}. \tag{3.4}$$

We introduce

$$(\alpha - k^*)_+ = \max\{\alpha - k^*, 0\}. \tag{3.5}$$

Multiply (1.13) by $(\alpha - k^*)_+$, integrate over $p(t)$, getting

$$\frac{1}{2}\frac{d}{dt}\int_{p(t)}(\alpha-k^*)_+^2 dx+\nu\int_{p(t)}|\nabla(\alpha-k^*)_+|^2 dx = \int_{p(t)} F\cdot\nabla^{\perp}(\alpha-k^*)_+ dx. \tag{3.6}$$

Boundary terms vanishes by properties of k^* - see (3.4). Using the Schwarz inequality we obtain

$$\frac{d}{dt}\int_{p(t)}(\alpha - k^*)_+^2 dx + c\int_{p(t)}|\nabla(\alpha - k^*)_+|^2 dx \le c\int_{p(t)} F^2 dx. \tag{3.7}$$

By the uniform boundedness of domains $p(t)$ function F belongs to $L_\infty(0,\infty; L_2(p(t)))$ and by the Poincare inequality together with the Gronwall inequality, inequality (3.7) implies

$$\sup_{t\in(0,\infty)} ||(\alpha - k^*)_+||_{L_2(p(t))} \le c||F||_{L_\infty(0,\infty;L_2(p(t)))}, \tag{3.8}$$

remembering that $(\alpha - k^*)_+|_{t=0} = 0$.

Applying this information to (3.7) we get

$$\sup_{k\in\mathbf{N}} ||\nabla(\alpha - k^*)_+||_{L_2(k,k+1;L_2(p(t)))} \le c||F||_{L_\infty(0,\infty;L_2(p(t)))}. \tag{3.9}$$

However bound (3.9) is not sufficient and we need to show higher regularity of term $(\alpha-k^*)_+$. We return to equation (1.13) and multiply it by $(\alpha-k^*)_+^{N-1}$; integrating over $p(t)$ we obtain

$$\frac{1}{N}\frac{d}{dt}\int_{p(t)}(\alpha - k^*)_+^N dx + \nu(N-1)\int_{p(t)}|\nabla(\alpha - k^*)_+|^2(\alpha - k^*)_+^{N-2}dx$$

$$= (N-1)\int_{p(t)} F\cdot\nabla^{\perp}(\alpha - k^*)_+(\alpha - k^*)_+^{N-2}dx. \tag{3.10}$$

Again, by the Schwarz inequality, (3.10) yields

$$\frac{d}{dt}\int_{p(t)}(\alpha - k^*)_+^N dx + c\int_{p(t)}|\nabla(\alpha - k^*)_+^{N/2}|^2 dx \le c\int_{p(t)} F^2(\alpha - k^*)_+^{N-2}dx, \tag{3.11}$$

where constants in (3.11) depend on constant N.

Applying the Hölder inequality to the r.h.s. and the Poincare inequality to the second term of the l.h.s., from (3.11) we obtain

$$\frac{d}{dt}\int_{p(t)}(\alpha-k^*)_+^N dx + c\int_{p(t)}|(\alpha-k^*)_+|^N dx \le c\int_{p(t)} F^N dx. \tag{3.12}$$

The above differential inequality guarantees the following bound

$$\sup_{t\in(0,\infty)} ||(\alpha-k^*)_+||_{L_N(p(t))} \le c||F||_{L_\infty(0,\infty;L_N(p(t)))}. \tag{3.13}$$

Similarly we obtain the bound for

$$(\alpha-k_*)_- = \min\{\alpha-k_*,0\} \tag{3.14}$$

with

$$k_* = \min\{\inf_{(x,t)\in\partial P(\infty)}(2\chi-f/\nu)v\cdot\tau+2V_{,s},\ \inf_{x\in p(0)}\alpha_0\}, \tag{3.15}$$

i.e. we are able to show the following estimates

$$\begin{gathered} ||(\alpha-k^*)_-||_{L_\infty(0,\infty;L_2(p(t)))} \le c||F||_{L_\infty(0,\infty;L_2(p(t)))},\\ \sup_{k\in\mathbf{N}}||\nabla(\alpha-k^*)_-||_{L_2(k,k+1;L_2(p(t)))} \le c||F||_{L_\infty(0,\infty;L_2(p(t)))},\\ \sup_{t\in(0,\infty)}||(\alpha-k^*)_-||_{L_N(p(t))} \le c||F||_{L_\infty(0,\infty;L_N(p(t)))}. \end{gathered} \tag{3.16}$$

Norms of functions $(\alpha-k^*)_+$ and $(\alpha-k_*)_-$ are independent of unknown quantities and their bounds depend only on norms of F. That is the reason we split the vorticity into two terms as follows

$$\alpha = \mathcal{A}_1+\mathcal{A}_2, \tag{3.17}$$

where

$$\mathcal{A}_1 = \alpha-(\alpha-k^*)_+-(\alpha-k_*)_-, \qquad \mathcal{A}_2 = (\alpha-k^*)_+ + (\alpha-k_*)_-. \tag{3.18}$$

Estimates (3.9), (3.13) and (3.16) imply the following identity

$$\mathcal{A}_1 \in B(0,\infty;L_\infty(p(t)))$$

and

$$\mathcal{A}_2 \in L_{2(loc)}(0,\infty;H_0^1(p(t)))\cap B(0,\infty;L_N(p(t))).$$

By the definition of $\mathcal{A}_1$ we have

$$||\mathcal{A}_1||_{B(0,\infty;L_\infty(p(t)))} \le \max\{k^*, k_*\}, \tag{3.19}$$

hence by (3.4) and (3.5) we obtain

$$||\mathcal{A}_1||_{B(0,\infty;L_\infty(p(t))} \le ||2\chi - f/\nu||_{C(\partial P(\infty)))}||v||_{B(0,\infty;C(\partial p(t)))} + X; \tag{3.20}$$

and by (3.9), (3.13) and (3.16) we control $\mathcal{A}_2$ as follows

$$||\mathcal{A}_2||_{B(0,\infty;L_N(p(t)))\cap L_{2(loc)}(0,\infty;H^1_0(p(t)))} \le c||F||_{L_\infty(0,\infty;L_N(p(t)))}. \tag{3.21}$$

Constants in the above proof depend on the magnitude of N, but this quantity is connected with constant a and it is fixed ($N > 2/(1-a)$). Lemma 3.1 is proved.

To end the proof of Theorem 3.1 there is a need of the next result concerning the following elliptic system

$$\begin{array}{lll} \text{rot } v = \alpha & \text{in} & p(t), \\ \text{div } v = 0 & \text{in} & p(t), \\ n \cdot v = V & \text{on} & \partial p(t) \end{array} \tag{3.22}$$

for $t \in (0, \infty)$.

Lemma 3.2. *Let $0 \le a < 1$ and α be given by Lemma 3.1 and $V \in C^{a,a/2}(P(\infty))$, then v given as a solution to problem (3.22) belongs to $B(0,\infty;C^a(\partial p(t)))$ and the following bound is valid*

$$||v||_{B(0,\infty;C^a(p(t)))} \le ||A_a(p(\cdot))||_{C(0,\infty)}||\mathcal{A}_1||_{B(0,\infty;L_\infty(p(t)))} +$$

$$||A^N(p(\cdot))||_{C(0,\infty)}||\mathcal{A}_2||_{B(0,\infty;L_N(p(t)))} + B||V||_{B(0,\infty;C^a(p(t)))}, \tag{3.23}$$

where constants $A_a(\cdot)$ and $A^N(\cdot)$ are given by Lemma 2.1.

Proof. To prove Lemma 3.2 we apply Lemmas 2.1. First, we split system (3.22) in the following way

$$\begin{array}{llllll} \text{rot } v_a = \mathcal{A}_1, & \text{rot } v_b = \mathcal{A}_2, & \text{rot } v_c = 0 & \text{in} & p(t), \\ \text{div } v_a = 0, & \text{div } v_b = 0, & \text{div } v_c = 0 & \text{in} & p(t), \\ n \cdot v_a = 0, & n \cdot v_b = 0, & n \cdot v_c = V & \text{on} & \partial p(t), \end{array} \tag{3.24}$$

where

$$v = v_a + v_b + v_c. \tag{3.25}$$

Lemma 2.1 delivers estimates for solutions to (3.24). For the first system, estimate (2.11) gives

$$||v_a||_{C^a(p(t))} \leq A_a(p(t))||\mathcal{A}_1||_{L_\infty(p(t))}. \tag{3.26}$$

Applying (2.12) to the second one, it yields

$$||v_b||_{C^a(\Omega)} \leq A^N(p(t))||\mathcal{A}_2||_{L_N(p(t))}, \tag{3.27}$$

since $a < 1 - 2/N$. And for the last one - (2.13) we have

$$||v_c||_{C^a(p(t))} \leq B||V||_{C^a(p(t))}. \tag{3.28}$$

Estimates (3.26)-(3.28) together with (3.25) implies (3.23). Lemma 3.2 is shown.

Combining Lemmas 3.1 and 3.2 we obtain the following a priori bound on the vorticity

$$\begin{gathered} ||\alpha||_{B(0,\infty;L_\infty(p(t)))+L_{2(loc)}(0,\infty;H_0^1(p(t)))\cap B(0,\infty;L_N(p(t)))} \leq \\ ||2\chi - f/\nu||_{C(\partial P(\infty))}\Big(||A_0(p(\cdot))||_{C(0,\infty)}||\mathcal{A}_1||_{B(0,\infty;L_\infty(p(t)))}+ \\ ||A^N(p(\cdot))||_{C(0,\infty)}||\mathcal{A}_2||_{B(0,\infty;L_N(p(t)))} + B||V||_{B(0,\infty;C^a(p(t)))}\Big) + X. \end{gathered} \tag{3.29}$$

Recalling estimates (3.20) and (3.21), we conclude

$$\begin{gathered} ||\alpha||_{B(0,\infty;L_\infty(p(t)))+L_{2(loc)}(0,\infty;H_0^1(p(t)))\cap B(0,\infty;L_N(p(t)))} \leq \\ (1 - ||A_0(p(\cdot))||_{C(0,\infty)}||2\chi - f/\nu||_{C(\partial P(\infty))})^{-1} DATA. \end{gathered} \tag{3.30}$$

Next, bound (3.23) for $a < 1$ completes the proof of Theorem 3.1. The estimation is closed if the condition (1.9) is fulfilled.

4 Existence

The aim of this section is to establish the existence of solutions which satisfy the a priori bound from section 3. First, we restate the examined system as follows

$$\begin{array}{lll} \alpha_t + v\cdot\nabla\alpha - \nu\Delta\alpha = \text{rot } F & \text{in} & P(T), \\ \alpha = (2\chi - f/\nu)v\cdot\tau + 2V_{,s} & \text{on} & \partial P(T), \\ \alpha|_{t=0} = \alpha_0 & \text{on} & p(0), \\ \text{rot } v = \alpha & \text{in} & P(T), \\ \text{div } v = 0 & \text{in} & P(T), \\ n\cdot v = V & \text{on} & \partial P(T). \end{array} \tag{4.1}$$

We want to show a basic existence by applying the Galerkin method. There is a need to reformulate the system to remove the nonhomogenity from the boundary data. We construct a function b which is defined as a solution to the following parabolic problem

$$\begin{array}{lll} b_t - \nu\Delta b = \text{rot } F & \text{in} & P(T), \\ b = 2V_{,s} & \text{on} & \partial P(T), \\ b|_{t=0} = \alpha_0 & \text{on} & p(0). \end{array} \tag{4.2}$$

Since boundary $\partial P(T)$ is sufficiently smooth we apply the standard theory to prove the following result.

Lemma 4.1. *Let*

$$F \in L_2(P(T)), \quad V_{,s} \in B(0,T;L_\infty(\partial p(t))), \quad \alpha_0 \in L_\infty(p(0))).$$

Then there exists unique weak solution to problem (4.2) such that

$$b \in B(0,\infty;L_\infty(p(t))) + L_2(0,T;H^1_0(p(t))) \tag{4.3}$$

and the following bound is valid

$$||b||_{B(0,T;L_\infty(p(t)))+L_2(0,T;H^1_0(p(t)))} \le X. \tag{4.4}$$

Moreover $b_t \in L_p(0,T;W^{-1}_{q'}(\Omega))$ *with the following bound*

$$||b_t||_{L_p(0,T;W^{-1}_{q'}(\Omega))} \le c(T)X \tag{4.5}$$

for any $1 < p,q < \infty$*, where* X *is defined by (3.2) and* $W^{-1}_{q'}(p(t))$ *is the dual space to* $W^1_q(p(t)) \cap \{\varphi|_{\partial p(t)} = 0\}$ *with* $1/q + 1/q' = 1$.

The solution given by Lemma 4.1 fulfills system (4.2) in the following sense

$$\int_{p(t)} b_t\psi dx + \nu \int_{p(t)} \nabla b^1 \cdot \nabla\psi dx - \nu \int_{p(t)} b^2 \Delta\psi d\sigma$$
$$-\nu \int_{\partial p(t)} 2V_{,s}\frac{\partial\psi}{\partial n} d\sigma = -\int_{p(t)} F \cdot \nabla^{\perp}\psi dx \tag{4.6}$$

for any $\psi \in W_2^{2,1}(p(T)) \cap \{\varphi|_{\partial p(t)} = 0\}$, where $b^1 \in L_2(0,T;H_0^1(p(t)))$ and $b^2 \in B(0,T;L_\infty(p(t)))$ such that $b = b^1 + b^2$. Applying bounds (4.4) into formulation (4.6) we get information about b_t with estimate (4.5).

Function b is an auxiliary tool to investigate the vorticity problem. For the velocity we define a vector function u given as a solution to the following elliptic problem

$$\begin{array}{lll} \text{rot } u = 0 & \text{in} & p(t), \\ \text{div } u = 0 & \text{in} & p(t), \\ n \cdot u = V & \text{on} & \partial p(t) \end{array} \tag{4.7}$$

for $t \in (0,T)$.

The elementary theory delivers us the following result (Lemma 2.1).

Lemma 4.2. *Let $V \in W_\infty^1(\partial p(t))$, then the solutions of problem (4.7) belong to $C^a(p(t))$ for any $0 \le a < 1$ with the following bound*

$$||u(\cdot,t)||_{C^a(p(t))} \le A_a(p(t))||V(\cdot,t)||_{W_\infty^1(\partial p(t))}. \tag{4.8}$$

Now, we reconsider system (4.1) using Lemmas 4.1 and 4.2. By properties of function u and since div $v = 0$ we introduce a scalar function φ such that

$$v = u + \nabla^{\perp}\varphi \quad \text{and} \quad \alpha = \Delta\varphi + b, \tag{4.9}$$

then system (4.1) can be stated as follows

$$\begin{array}{lll} \Delta\varphi_t + (\nabla^{\perp}\varphi + u)\cdot\nabla(\Delta\varphi + b) - \nu\Delta^2\varphi = 0 & \text{in} & P(T), \\ \Delta\varphi + (f/\nu - 2\chi)\frac{\partial\varphi}{\partial n} = 0 & \text{on} & \partial P(T), \\ \varphi = 0 & \text{on} & \partial P(T), \\ \varphi|_{t=0} = 0 & \text{on} & p(0). \end{array} \tag{4.10}$$

The issue of existence for problem (4.1) reduces to finding a solution to problem (4.10). Define a weak solution to problem (4.10).

Definition 4.1. *We say that* $\varphi \in W_2^{2,1}(P(T)) \cap \{\varphi|_{\partial P(T)} = 0\}$ *is the weak solution to problem (4.10) iff the following identity*

$$\int_{p(t)} \nabla\varphi_t \nabla\psi dx + \int_{p(t)} (\Delta\varphi + b)(\nabla^{\perp}\varphi + u) \cdot \nabla\psi dx$$

$$+\nu \int_{p(t)} \Delta\varphi\Delta\psi dx + \nu \int_{\partial p(t)} (f/\nu - 2\chi) \frac{\partial\varphi}{\partial n}\frac{\partial\psi}{\partial n} d\sigma = 0 \qquad (4.11)$$

for $\psi \in W_2^{2,1}(P(T)) \cap \{\psi|_{\partial P(T)} = 0\}$ *holds in the distributional sense on time interval* $(0, T)$.

Theorem 4.1. *Let assumptions of Theorem 1.1 be fulfilled, then there exists unique weak solution to problem (4.10) in the sense of Definition 4.1.*

Proof. Since $\partial P(T)$ is sufficiently smooth - we can choose a vector sequence $\{w_k\}_{k=1}^{\infty}$ such that $\{w_k(\cdot, t)\}_{k=1}^{\infty}$ is a base in $H^2(p(t)) \cap \{\varphi|_{\partial p(t)} = 0\}$ for each $t \in (0, T)$.

Moreover, let

$$V^N(t) = span\{w_1(\cdot, t), ..., w_N(\cdot, t)\} \qquad (4.12)$$

denotes the finite dimensional approximation of $H^2(p(t))$, additionally

$$\sup_{t\in(0,\infty)} \det[\{(\nabla w_i(\cdot, t), \nabla w_j(\cdot, t))_{L_2(p(t))}\}_{i,j=1,..,N}] \geq \gamma_N > 0. \qquad (4.13)$$

and

$$\sup_{k\in\mathbf{N}} ||\partial_t w_k||_{C(P(\infty))} < \infty. \qquad (4.14)$$

By geometrical condition (1.9) there exists smooth transformation $\Phi(\cdot, t) : p(t) \to p(0)$ which is also smooth with respect to time. This map describes the base $\{w_k\}$ as the range of Φ of the base on domain $p(0)$. Our approach one can find in [8] concerning similar problems in time dependent domains (an alternative in [5]).

We apply the Galerkin method we define the following approximation

$$\varphi^N = \sum_{k=1}^{N} c_k^N(t) w_k(x, t). \qquad (4.15)$$

To compute unknown coefficients $c_k^N(t)$ we solve the following ordinary differential system, which follows from the weak formulation of the solutions.

$$\int_{p(t)} \nabla\varphi_t^N \nabla w_k dx + \int_{p(t)} (\nabla^\perp \varphi^N + u)(\Delta\varphi^N + b)\nabla w_k dx$$

$$+\nu \int_{p(t)} \Delta\varphi^N \Delta w_k dx + \nu \int_{\partial p(t)} (f/\nu - 2\chi)\frac{\partial \varphi^N}{\partial n}\frac{\partial w_k}{\partial n} d\sigma = 0 \tag{4.16}$$

with initial data

$$c_k^N(0) = 0 \quad \text{for} \quad k = 1, 2, .., N. \tag{4.17}$$

The local in time existence for the above system is delivered by the theory of the ordinary differential equations only on interval $[0, T_N]$. Since we are not able to control T_N, we look for a bound on φ^N uniformly in N.

The technique from section 3 can be repeated for the approximation given by the Galerkin method, because to obtain estimates we apply the maximum principal and use the weak formulation for the vorticity problem - see also Definition 4.2 below.

By the a priori estimate our approximation sequence satisfies the following bounds

$$\begin{array}{c} ||u^N + \nabla^\perp \varphi^N||_{B(0,\infty;C^a(p(t)))} \leq DATA, \\ ||b^N + \Delta\varphi^N||_{B(0,T;L_\infty(p(t)))+L_2(0,T;H_0^1(p(t)))} \leq DATA, \end{array} \tag{4.18}$$

where u^N, b^N are projections of u and b on subspace V^N. In particular, estimates (4.18) imply

$$||\nabla^2\varphi^N||_{L_2(P(T))} + ||\nabla\varphi^N||_{B(0,T;L_2(p(t)))} \leq c(T)DATA. \tag{4.19}$$

Additionally, we want to increase the regularity with respect to time. To obtain suitable information we recall the following definition

$$\begin{array}{c} ||\Delta\varphi_t^N||_{L_2(0,T;(H^2(p(t))\cap\{\phi|_{\partial p(t)}=0\})^*)} \\ = \sup\left\{\int_0^T \int_{p(t)} \Delta\varphi_t^N \psi^N dx : \psi^N = \sum_{k=1}^N c_k(t) w_k(x,t), \right. \\ \left. ||\psi^N||_{L_2(0,T;H^2(p(t)))} = 1.\right\}, \end{array} \tag{4.20}$$

The above quantity may be given from the weak formulation (4.16) - (we substitute w_k by ψ^N), then applying (4.19) we conclude

$$\Delta\varphi_t^N \in L_2(0,T;(H^2(p(t)) \cap \{\varphi|_{\partial p(t)} = 0\})^*) \tag{4.21}$$

for any p, q such that $1 < p, q < \infty$. Hence we obtain

$$||\varphi_t^N||_{L_2(P(T))} \leq c(T)DATA. \tag{4.22}$$

The above bounds imply that for a subsequence

$$\varphi^{N_k} \rightharpoonup \varphi_* \quad \text{weakly in } W_2^{2,1}(P(T)) \tag{4.23}$$

and by the Relich theorem

$$\nabla\varphi^{N_k} \to \nabla\varphi_* \text{ strongly in } L_{4-\epsilon}(0,T;W^1_{4-\epsilon}(p(t))) \tag{4.24}$$

for small $\epsilon > 0$. Hence φ_* is a weak solution to problem (4.10) in the sense of Definition 4.1. Theorem 4.1 is proved.

The next step is to increase the regularity of the solutions.

Definition 4.2. *We say that v is the global in time weak-$*$ solution to problem (4.1) iff*

$$v \in C(0,\infty;C^a(p(t))), \quad \alpha \in S_1 + S_2$$

with

$$S_1 = B(0,\infty;L_\infty(p(t))) \text{ and } S_2 = L_{2(loc)}(0,\infty;H^1_0(p(t)))\cap B(0,\infty;L_N(p(t))),$$

and the following identity

$$-\int_{p(t)} \alpha_t\psi dx + \int_{p(t)} \alpha v\cdot\nabla\psi dx + \nu\int_{p(t)} \mathcal{A}_1\Delta\psi dx - \nu\int_{p(t)} \nabla\mathcal{A}_2\cdot\nabla\psi dx$$
$$+\nu\int_{\partial p(t)} (f/\nu - 2\chi - 2V_{,s})v\cdot\tau\frac{\partial\psi}{\partial n} = \int_{p(t)} F\cdot\nabla^\perp\psi dx \tag{4.25}$$

holds for any $\psi \in W_1^{2,1}(P(\infty)) \cap \{\psi|_{\partial P(\infty)} = 0\}$, where

$$\alpha = rot\, v = \mathcal{A}_1 + \mathcal{A}_2 \tag{4.26}$$

with

$$\mathcal{A}_1 \in B(0,\infty;L_\infty(\Omega)) \tag{4.27}$$

and

$$\mathcal{A}_2 \in L_{2(loc)}(0,\infty;H^1_0(p(t))) \cap B(0,\infty;L_N(p(t))). \tag{4.28}$$

Theorem 4.2. *The weak solution given by Theorem 4.1 is the weak-$*$ solution, moreover for any $1 < p,q < \infty$*

$$\alpha_t \in L_{p(loc)}(0,\infty;(W_q^2(p(t)) \cap \{\varphi|_{\partial p(t)} = 0\})^*). \tag{4.29}$$

Proof. To show the above result we use the Galerkin approximation from the proof of Theorem 4.1. By (4.18) we conclude

$$\begin{gathered} ||v^N||_{B(0,\infty;C^a(p(t)))} \leq DATA, \\ ||\alpha^N||_{B(0,T;L_\infty(p(t)))+L_{2(loc)}(0,T;H^1_0(p(t)))\cap B(0,\infty;L_N(p(t)))} \leq DATA. \end{gathered} \tag{4.30}$$

Hence for a suitable subsequence we have

$$\begin{gathered} v^N = \nabla^\perp \varphi^N + v^N \rightharpoonup \nabla^\perp \varphi_* + u = v, \\ \alpha^N = \Delta \varphi^N + b^N \rightharpoonup \Delta \varphi_* + b = \alpha \end{gathered} \tag{4.31}$$

weakly-$*$ in the spaces as in estimate (3.30). Then functions v and α fulfills (4.25) with $\psi \in C_0^\infty(P(\infty)))$ in the distributional sense. Thus similarly as for (4.21) we prove that

$$||\alpha_t||_{L_p(0,T;(W^2_q(p(t))\cap\{\varphi|_{\partial p(t)}=0\})^*)} \leq c(T)DATA \tag{4.32}$$

for any $T > 0$ and $1 < p, q < \infty$, which follows from the definition of the norm in this space

$$||\alpha_t||_{L_p(0,T;(W^2_q(p(t))\cap\{\varphi|_{\partial p(t)}=0\})^*)} = \sup \int_0^T \int_{p(t)} \alpha_t \psi dx dt, \tag{4.33}$$

where sup is taken over

$$\psi : \psi \in L_{p^*}(0,T; (W^2_q(p(t)) \cap \{\varphi|_{\partial p(t)} = 0\})),$$

$$||\psi||_{L_{p^*}(0,T;(W^2_q(p(t))\cap\{\varphi|_{\partial p(t)}=0\}))} = 1 \quad \text{and} \quad 1/p + 1/p^* = 1.$$

Using (4.25) to control the r.h.s. of (4.33), remembering about bounds (4.30) we conclude (4.32). Theorem 4.2 is proved.

In a similar way we obtain an improvement of (4.33)

$$\sup_{k\in\mathbf{N}} ||\alpha_t||_{L_p(k,k+1;(W^2_q(p(t))\cap\{\varphi|_{\partial p(t)}=0\})^*)} \leq DATA \tag{4.34}$$

which delivers us information independent of time. Since we assumed that $V \in C^{a,a/2}(\partial P(\infty))$, by Lemma 2.1 and bound (4.34), the solution to elliptic problem (1.16) belongs to $C^{a,a/2}$ (if we take sufficiently large p) with the following bound

$$||v||_{C^{a,a/2}(P(\infty))} \leq DATA. \tag{4.35}$$

We applied here Lemma 2.1 (ii) and the imbedding theorem. Estimate (4.34) guarantees that a part of the velocity belongs to $W_p^{1,1/2}$, thus if $p > 4/(1-a)$, then $W_p^{1,1/2} \subset C^{a,a/2}$, which implies (4.35).

Having (4.35) and Theorems 4.1 and 4.2 we finish the proof of Theorem 1.1. The classical theory [9] for the Navier-Stokes equations guarantees us that the regularity of the solutions given by Theorem 1.1 can be increase with no restrictions.

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