Flux problem for a certain class of 2D domains

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Abstract. We consider the steady Navier-Stokes equations in a two dimensional bounded domain with multiply connected boundary with nonhomogeneous slip boundary conditions. Under a geometrical constraint, restricting the shape of the domain, we construct an a priori bound of solutions for arbitrary large data, including fluxes through each component of the boundary. Our result is an exception to the general theory of the flux problem.

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In this note we study the steady Navier-Stokes equations in two dimensional domains with multiply connected boundaries. Boundary problems for such system are still a source of open questions. Difficulties are related to fluxes described by boundary data on each connected component of the boundary. For the Dirichlet problem, if fluxes are zero (or sufficiently small compared them to the viscosity) or the domain is simply connected, using the Hopf extension [6], we are able to prove existence of solutions [2,5,7,9]. However for the general case the question is still open. In some special cases, if we assume a symmetry of the whole system (including data and domain) the problem is solvable for arbitrary data [1]. However this symmetry reduces the system to the case of simply connected domains. Other interesting examples were studied in [3,10,11], where special forms of Dirichlet boundary data were chosen.

Here we investigate the Navier-Stokes equations with the slip boundary conditions (known also as the Navier relations). For a certain class of domains we find a priori bounds for solutions with no restrictions on flow data. In particular, fluxes through each components of the boundary can be arbitrary large. The class of domains is characterized by a geometrical constraint which depends on the curvature of the boundary and the constant in the

Poincare inequality. It is worthwhile to note that the direct application of Hopf's technique for the system with the slip boundary conditions does not work even for simply connected domains.

The studied system reads:

$$\begin{array}{lll} v\cdot\nabla v-\nu\Delta v+\nabla p=F & \text{in} & \Omega, \\ \operatorname{div} v=0 & \text{in} & \Omega, \\ n\cdot\mathbf{T}(v,p)\cdot\tau+fv\cdot\tau=0 & \text{on} & \partial\Omega, \\ n\cdot v=d & \text{on} & \partial\Omega, \end{array} \tag{1}$$

where $v=(v^1,v^2)$ is the velocity, p - the pressure, F - the external force, f - the friction coefficient, n and τ - the normal and tangent vectors to the boundary $\partial\Omega$, d - the inflow datum, $\mathbf{T}(v,p)$ - the stress tensor, $\mathbf{T}(v,p)=\{\nu(v^i_{,j}+v^j_{,i})-p\delta_{ij}\}_{i,j=1,2}$, where $\nu>0$ is the viscosity coefficient.

Our system is investigated in a domain with non trivial geometry. We assume that the first group of the homotopy of domain Ω is $\mathbf{Z}$. We admit one "hole".

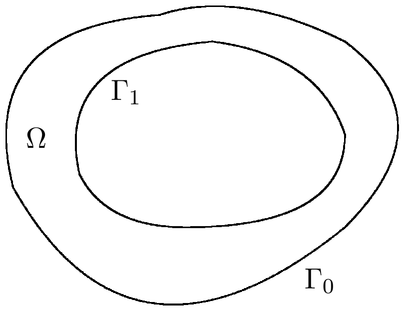

Figure 1.

Under this assumption, boundary $\partial\Omega$ consists of two connected parts Γ_0 and Γ_1 - see Fig. 1. We skip other cases (when the domain has more than one hole), since it is not obvious if a geometrical constraint given by the main result admits richer geometry. To describe properties of domain Ω, let us introduce the following quantity

$$A_0=||\nabla\phi||_{C(\overline{\Omega})}, \tag{2}$$

where function ϕ is a solution to the following elliptic problem

$$\Delta\phi=1, \quad \phi|_{\partial\Omega}=0. \tag{3}$$

By the scaling method we see that

$$A_0\sim\sup_{x\in\Gamma_0}\operatorname{dist}(x,\Gamma_1), \tag{4}$$

where the last quantity is proportional to the constant in the Poincare inequality. Relation (4) says that A_0 is small if the r.h.s. of (4) is sufficiently small, too.

By $(1)_2$ it is required to assume a compatibility condition $\int_{\partial\Omega} d\, d\sigma = 0$. Since boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ - see Fig. 1., we have

$$\int_{\Gamma_0} d\, d\sigma = -\int_{\Gamma_1} d\, d\sigma = \mathcal{D}. \tag{5}$$

The main result describes the class of domains for which we are able to construct an a priori bound for the solutions with no restrictions on $\mathcal{D}$. This bound guarantees the existence, too.

Theorem. *Let $\partial\Omega \in C^2$, $F \in L_2(\Omega)$, $d \in W^1_\infty(\partial\Omega)$ and $\int_{\partial\Omega} d\, d\sigma = 0$. If*

$$A_0 ||2\chi - f/\nu||_{L_\infty(\partial\Omega)} < 1, \tag{6}$$

where χ is the curvature of boundary $\partial\Omega$ and A_0 is given by (2), then there exists at least one weak solution to problem (1) such that

$$rot\, v \in L_\infty(\Omega) + H^1_0(\Omega) \qquad and \qquad v \in C^a(\overline{\Omega})$$

for any $0 < a < 1$. Moreover the following bound is valid

$$||rot\, v||_{L_\infty(\Omega)+H^1_0(\Omega)} + ||v||_{C^a(\overline{\Omega})} \leq B(||d||_{W^1_\infty(\partial\Omega)} + \frac{1}{\nu}||F||_{L_2(\Omega)}), \tag{7}$$

where B is independent of the viscosity coefficient.

Condition (6) defines a class of admissible domains. Since f/ν is always nonnegative and f-friction may be described independently, thus condition (6) implies the following geometrical constraint

$$A_0 \inf_{x\in\partial\Omega} 2\chi > -1. \tag{8}$$

(We set for $\Omega = B(0,1)$ curvature $\chi = 1$.) To point an example of a domain satisfying (8) it is enough to consider an annulus $\{x \in \mathbf{R}^2 : R_1 < |x| < R_2\}$, then we see that the l.h.s. of (8) is proportional to $-\frac{1}{R_1}(R_2 - R_1)$, hence taking suitable R_1 and R_2 we obtain (8). Some straightforward calculations enable to show explicite formula on A_0 for this set ($A_0 = 1/4(R_2 - R_1)[1 + (1 + R_1/R_2)((lnR_1/R_2)^{-1} + (1 - R_1/R_2)^{-1})]$). Of course, a perturbation of this domain fulfills condition (8), too. It is not clear if condition (8) can be fulfilled by a domain with more than one “hole”.

Estimate (7) has a linear character similar to the standard energy estimate for the problem with homogeneous boundary data. It is a consequence of the fact that we are able to apply the maximum principle and neglect influence of the nonlinear term and get a bound on the vorticity. Hence an analogical estimate for the Stokes system is exactly the same. Of course, for the linear problem one can show the result with no restriction on the shape of domain using the standard approach. However the geometrical constraint enables application of the maximum principle, which does not hold for the linear system in general case. Another interesting feature of bound (7) is independence from the viscosity coefficient if $F \equiv 0$. This property is a motivation to study in [12] the inviscid limits for solutions to problem (1).

A similar result for a symmetric three dimensional case of the system has been considered in [14].

The sense of obtained solutions is given by the following formulation.

Weak formulation. *We say that v is a weak solution to problem (1) iff $n \cdot v|_{\partial\Omega} = d$, $v \in C(\overline{\Omega})$, $\alpha = rot\, v \in L_\infty(\Omega) + H_0^1(\Omega)$ and the following identity*

$$\nu \int_\Omega \alpha \Delta\phi dx - \nu \int_{\partial\Omega} ((2\chi - f/\nu) v \cdot \tau - 2d_{,s}) \frac{\partial \phi}{\partial n} d\sigma + \int_\Omega \alpha v \cdot \nabla \phi dx = \int_\Omega F \cdot \nabla^\perp \phi dx$$

holds for any $\phi \in C^2(\overline{\Omega}) \cap \{\phi|_{\partial\Omega} = 0\}$.

The regularity of the solutions guarantees us the following result.

Corollary. *Let the assumptions of Theorem be fulfilled, moreover additionally $\partial\Omega \in C^\infty$, $F \in C^\infty(\overline{\Omega})$ and $d \in C^\infty(\partial\Omega)$, then $v \in C^\infty(\overline{\Omega})$.*

To obtain *Corollary* it is enough to increase the regularity of the solutions fulfilling the weak formulation using standard techniques [4,8].

The proof of *Theorem* is based on a reformulation of the problem. Let us consider the vorticity

$$\alpha = \text{rot } v = v^2_{,1} - v^1_{,2} \tag{9}$$

which in the two dimensional case is a scalar function. From (1) we deduce the following system

$$\begin{array}{lcl} v \cdot \nabla\alpha - \nu\Delta\alpha = \text{rot } F & \text{in} & \Omega, \\ \alpha = (2\chi - f/\nu) v \cdot \tau - 2d_{,s} & \text{on} & \partial\Omega, \end{array} \tag{10}$$

where s is the length parameter of curve $\partial\Omega$. To obtain $(10)_1$ we take the rotation of $(1)_1$ and apply $(1)_2$. To get $(10)_2$ we differentiate $(1)_4$ with respect to s and combine it with $(1)_3$ using also the Frenet formula $n_{,s} = \chi\tau$ - see [3, Lemma 2.1].

To obtain information about the velocity we consider the following elliptic problem

$$\begin{array}{lcl} \text{rot } v = \alpha & \text{in} & \Omega, \\ \text{div } v = 0 & \text{in} & \Omega, \\ n \cdot v = d & \text{on} & \partial\Omega. \end{array} \tag{11}$$

Since Ω is not simply connected, the kernel of the elliptic operator from (11) in not trivial and all solutions to the l.h.s. of (11) can be represented as follows (see [8])

$$v = V_d + V_r + tV_k \tag{12}$$

for arbitrary parameter $t \in \mathbf{R}$, where $V_d = \nabla\phi_d$ and ϕ_d satisfies

$$\Delta\phi_d = 0, \quad \left.\frac{\partial\phi_d}{\partial n}\right|_{\partial\Omega} = d, \quad \int_\Omega \phi_d dx = 0; \tag{13}$$

vector $V_r = \nabla^\perp\phi_r$ and ϕ_r is a solutions to the following problem

$$\Delta\phi_r = \alpha, \quad \phi_r|_{\partial\Omega} = 0; \tag{14}$$

and $V_k = \nabla^\perp\phi_k$ and ϕ_k satisfies

$$\Delta\phi_k = 0 \quad \phi_k|_{\Gamma_1} = 1, \quad \phi_k|_{\Gamma_0} = 0. \tag{15}$$

In general, for more complex domains the kernel is of higher order.

To keep uniqueness in definition (12) we put $t = 0$.

Solutions of the coupled system (10)-(11) define solutions to the Navier-Stokes equations (1) in the sense of definition of *Weak formulation.* We neglect the kernel of operator rot-div.

A priori bound. We investigate the vorticity problem using a modification of the maximum principle. Introduce

$$(\alpha - K^*)_+ = \max\{\alpha - K^*, 0\}, \tag{16}$$

where $K^* = \text{ess sup}_{x\in\partial\Omega}(2\chi - f/\nu)v \cdot \tau - 2d_{,s}$. This definition guarantees that $(\alpha - K^*)_+$ has zero trace at the boundary, thus

$$v \cdot \nabla(\alpha - K^*)_+ - \nu\Delta(\alpha - K^*)_+ = \text{rot } F \text{ in the weak sense in } \Omega. \tag{17}$$

Multiplying (17) by $(\alpha - K^*)_+$ we get

$$\nu \int_\Omega |\nabla(\alpha - K^*)_+|^2 dx = \int_\Omega F \cdot \nabla^\perp(\alpha - K^*)_+ dx. \tag{18}$$

From (18) we obtain the following bound

$$||\nabla(\alpha - K^*)_+||_{L_2(\Omega)} \leq \frac{1}{\nu}||F||_{L_2(\Omega)}. \tag{19}$$

The same we have for

$$(\alpha - K_*)_- = \min\{\alpha - K_*, 0\} \tag{20}$$

with $K_* = \text{ess inf}_{x\in\partial\Omega}(2\chi - f/\nu)v\cdot\tau - 2d_{,s}$, i.e.

$$||\nabla(\alpha - K_*)_-||_{L_2(\Omega)} \leq \frac{1}{\nu}||F||_{L_2(\Omega)}. \tag{21}$$

Estimates (19) and (21) give

$$||\Pi_{H^1_0}\alpha||_{H^1_0(\Omega)} \leq \frac{2}{\nu}||F||_{L_2(\Omega)}, \tag{22}$$

where

$$\Pi_{H^1_0}\alpha = (\alpha - K^*)_+ + (\alpha - K_*)_-$$

is the H^1_0-part of $\alpha \in L_\infty(\Omega) + H^1_0(\Omega)$.

The norm of the L_∞-part of α, i.e. $\Pi_{L_\infty}\alpha = \alpha - \Pi_{H^1_0}\alpha$, is bounded by

$$K = \max\{K^*, |K_*|\}. \tag{23}$$

To estimate the above quantity, we find a bound on v given by (12) (with $t = 0$). Since $d \in W^1_\infty(\partial\Omega)$, the theory of the Laplace operator implies that the solutions to (13) satisfies ($0 < a < 1$, constants γ_0 and γ_a depend on features of the domain)

$$||\nabla\phi_d||_{C(\Omega)} \leq \gamma_0||d||_{W^1_\infty(\partial\Omega)}, \quad ||\nabla\phi_d||_{C^a(\Omega)} \leq \gamma_a||d||_{W^1_\infty(\partial\Omega)}. \tag{24}$$

For ϕ_r-solution to (14), the classical theory (for dim $\Omega = 2$) guarantees the following bounds

$$\begin{gathered} ||\nabla\phi_r||_{C(\overline{\Omega})} \leq A_0||\Pi_{L_\infty}\alpha||_{L_\infty(\Omega)} + A^1||\Pi_{H^1_0}\alpha||_{H^1_0(\Omega)}, \\ ||\nabla\phi_r||_{C^a(\overline{\Omega})} \leq A_a||\alpha||_{L_\infty(\Omega)+H^1_0(\Omega)}. \end{gathered} \tag{25}$$

Constant A_0 in $(25)_1$ is defined by (2). Taking the definition of K we obtain

$$K \le ||2\chi - f/\nu||_{L_\infty(\partial\Omega)}||v||_{C(\overline{\Omega})} + 2||d||_{W^1_\infty(\partial\Omega)}. \tag{26}$$

To finish the estimation we note that

$$||\Pi_{H^1_0}\alpha||_{H^1_0(\Omega)} + ||\Pi_{L_\infty}\alpha||_{L_\infty(\Omega)} \le \frac{2}{\nu}||F||_{L_2(\Omega)} + K. \tag{27}$$

Recalling the form of v given by (12), taking into account (22)-(27), we conclude the following inequality

$$||\Pi_{H^1_0}\alpha||_{H^1_0(\Omega)} + ||\Pi_{L_\infty}\alpha||_{L_\infty(\Omega)} \le A_0||2\chi - f/\nu||_{L_\infty(\partial\Omega)}||\Pi_{L_\infty}\alpha||_{L_\infty(\Omega)} +$$

$$2A^1/\nu||2\chi - f/\nu||_{L_\infty(\partial\Omega)}||F||_{L_2(\Omega)} + 2/\nu||F||_{L_2(\Omega)} + A^2||d||_{W^1_\infty(\partial\Omega)}. \tag{28}$$

Hence inequality (28) implies the following a priori bound (if condition (6) is satisfied)

$$||\alpha||_{L_\infty(\Omega)+H^1_0(\Omega)} \le (1 - A_0||2\chi - f/\nu||_{L_\infty(\partial\Omega)})^{-1}\cdot$$

$$\cdot\Big(2/\nu(1 + A^1||2\chi - f/\nu||_{L_\infty(\partial\Omega)})||F||_{L_2(\Omega)} + A^2||d||_{W^1_\infty(\partial\Omega)}\Big), \tag{29}$$

since $||\alpha||_{L_\infty(\Omega)+H^1_0(\Omega)} = \inf\{||a_1||_{L_\infty(\Omega)} + ||a_2||_{H^1_0(\Omega)} : a_1 \in L_\infty(\Omega),\ a_2 \in H^1_0(\Omega)$ and $a_1 + a_2 = \alpha\}$ and we have $\alpha = \Pi_{L_\infty}\alpha + \Pi_{H^1_0}\alpha$.

Bounds (24), (25) and (12) imply that $v \in C^a(\overline{\Omega})$ for any $0 < a < 1$, what finishes the proof of estimate (7).

Existence. A proof for the problem in a simply connected domain can be found in [13, Theorem 3.1], hence we show only main ideas of the proof of existence of solutions to (1), because it is almost the same as in [13]. The easiest approach to this issue is an application of the Leray-Schauder fixed point theorem. We construct a map $\Phi : C^a(\overline{\Omega}) \to C^a(\overline{\Omega})$ such that $\Phi(\tilde{v}) = v$, where v is the solution of (11) with α fulfilling (10) with $\tilde{v}$ instead of v. Existence for these linear problems follows form the classical techniques as the Galerkin method and the Green function for the Laplace operator. Map Φ is compact, since $a > 0$ and Ω is bounded. Boundedness of set $\Phi(v) = \lambda v$ for $\lambda \in [0, 1]$ is obtained in the same way as a priori estimate (29). Thus, we conclude existence of at least one fixed point of map Φ which define the weak solution to system (1). Theorem has been proved.

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