On a problem for the Navier-Stokes equations with the infinite Dirichlet integral

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Abstract. We study existence of global in time solutions to the Navier-Stokes equations in a two dimensional domain with an unbounded boundary. The problem is considered with slip boundary conditions involving nonzero friction. The main result shows a new L_∞-bound on the vorticity. A key element of the proof is the maximum principle for a reformulation of the problem. Under some restrictions on the curvature of the boundary and the friction the result for large data (including flux) with the infinite Dirichlet integral is obtained.

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1 Introduction

In the paper a model of motion of a viscous incompressible Newtonian fluid in a two space dimensional domain with an unbounded boundary is studied. To describe the phenomenon we apply the following evolution problem

$$\begin{array}{lll} v_t + v\cdot\nabla v - \nu\Delta v + \nabla p = 0 & \text{in} & \Omega\times[0,T], \\ \text{div } v = 0 & \text{in} & \Omega\times[0,T], \\ n\cdot v = 0, \quad n\cdot\mathbf{T}(v,p)\cdot\tau + f v\cdot\tau = 0 & \text{on} & \partial\Omega\times[0,T], \\ v \to v_\infty^{(i)}(y_2^{(i)}) \quad \text{as} \quad y_1^{(i)} \to +\infty & \text{in} & \Omega^{(i)}\times[0,T], \\ v|_{t=0} = v_{in.} & \text{on} & \Omega, \end{array} \tag{1.1}$$

where $v = (v^1, v^2)$ is the velocity of the fluid, p is the pressure, ν - the constant positive viscous coefficient, n and τ - the normal and tangent vectors to boundary $\partial\Omega$, $f \geq 0$ - the friction coefficient on the boundary and the stress tensor

$$\mathbf{T}(v,p) = \nu\mathbf{D}(v) - pId = \{\nu(v^i_j + v^j_i) - p\delta_{ij}\}_{i,j=1,2}.$$

Condition $(1.1)_3$ models a case when the tangential part of the velocity on the boundary can be nonzero and describes an influence of the boundary by involving the friction f which in general can be nonconstant. Such relations are called slip boundary conditions or Navier ones.

About domain Ω - see picture 1 - we require it to be simply connected, moreover

$$\Omega = \bigcup_{i=0}^{2} \Omega^{(i)}, \tag{1.2}$$

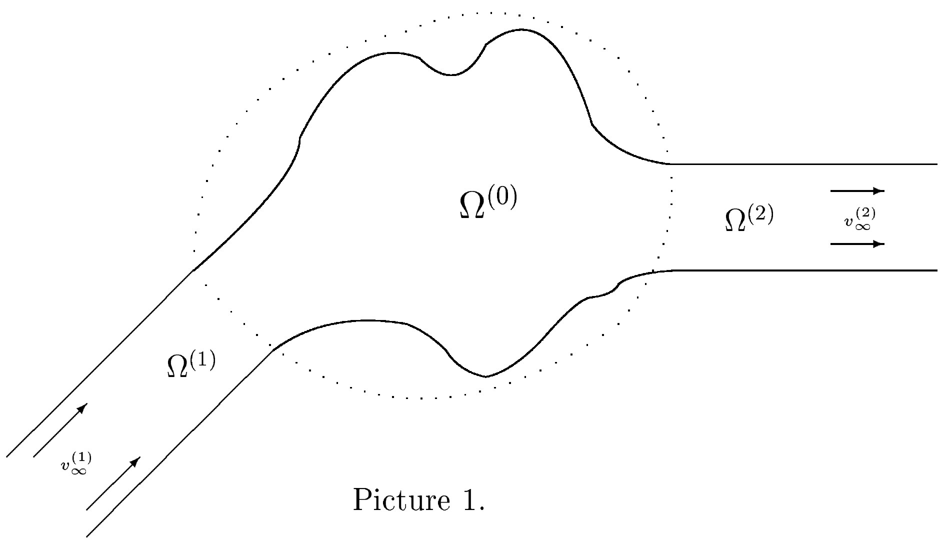
where $\Omega^{(0)}$ is bounded and the rest are infinite pipes which in their local coordinates $y^{(i)}$ can be described as follows

$$\Omega^{(i)} = \{(y^{(i)}_1, y^{(i)}_2) \in \mathbf{R}^2 : y^{(i)}_1 \in \mathbf{R}_+, y^{(i)}_2 \in (0, H^{(i)})\}, \tag{1.3}$$

where $H^{(i)}$ is the height of the i-th pipe. We also denote components of $\partial\Omega$

$$\bigcup_{i=1,2} \Gamma^{(i)} = \partial\Omega, \tag{1.4}$$

where $\Gamma^{(1)}$ is a upper connected part of boundary $\partial\Omega$ and $\Gamma^{(2)}$ is a lower one. For simplicity we assume that friction coefficient f on boundaries of $\Omega^{(1)}$ and $\Omega^{(2)}$ is constant.



Picture 1.

$v_\infty^{(i)}$ are velocities on the inlet and outlet of domain Ω and they are proportional to a flow w_f and in our considerations they are time independent. Velocity w_f is a solution to equations $(1.1)_{1,2,3}$ in a straight pipe and in a model case (the height of the pipe is equal to 1, i.e. $\Omega = \mathbf{R} \times (0,1)$) one can easily check that

$$w_f(y_2) = \left(F \left[-\frac{6f}{f+6} y_2^2 + \frac{6f}{f+6} y_2 + \frac{6}{f+6} \right], 0 \right), \tag{1.5}$$

where F is the value of the flux of the flow. We see that if $f = 0$ then we obtain the constant flow $(F, 0)$ and if $f \to \infty$ we get the Poiseuille flow. In the second case our condition $(1.1)_3$ changes to the zero Dirichlet boundary datum. So $v_\infty^{(i)}$ and $\alpha_\infty^{(i)} = \operatorname{rot} v_\infty^{(i)}$ satisfy $(1.1)_{1,2,3,4}$ for $\Omega = \mathbf{R} \times (0, H^{(i)})$, in particular we see that

$$v_\infty^{(i)} \cdot \nabla \alpha_\infty^{(i)} = 0, \quad \Delta \alpha_\infty^{(i)} = 0 \quad \text{in} \quad \mathbf{R} \times (0, H^{(i)}) \tag{1.6}$$

and

$$\alpha_\infty^{(i)} = v_\infty^{(i)} \cdot \tau(-f) \quad \text{on} \quad \mathbf{R} \times \{0, H^{(i)}\}. \tag{1.7}$$

To complete the statement of the problem we need to add a compatibility condition on the total flux which must hold by $(1.1)_2$, i.e. we require

$$\sum_{i=1}^{2} \int_0^{H^{(i)}} {v_\infty^{(i)}}^1 (y_2^{(i)}) dy_2^{(i)} = 0. \tag{1.8}$$

The aim of this paper is to analyze the existence of solutions to problem (1.1). We prove two results. The first one is the following.

Theorem A. *Let div* $v_{in.} = 0$ *and*

$$v_{in.}|_{\Omega^{(0)}} \in L_2(\Omega^{(i)}), \quad v_{in.}|_{\Omega^{(i)}} - v_\infty^{(i)} \in L_2(\Omega^{(i)}) \tag{1.9}$$

for $i = 1, 2$, *moreover* $f - 2\chi > -B(\Omega)$, *where* χ *is the curvature of boundary* $\partial\Omega$ *and* $B(\Omega)$ *is a constant depending on* Ω. *Then there exists unique weak global in time solution of problem (1.1) such that*

$$v \in H^{1,0}_{loc}(\Omega \times (0, \infty)) \cap L_{\infty(loc)}(0, \infty; L_{2(loc)}(\Omega)), \tag{1.10}$$

satisfying the following estimate

$$\sum_{i=1}^{2} \left(||v - v_\infty^{(i)}||_{L_\infty(0,T;L_2(\Omega^{(i)}))} + ||\nabla v - \nabla v_\infty^{(i)}||_{L_2(\Omega^{(i)} \times (0,T))} \right)$$

$$\leq Ce^{Tc(v_\infty)}(||v_{in.}||_{L_2(\Omega^{(0)})} + \sum_{i=1}^{2} ||v_{in.} - v_\infty^{(i)}||_{L_2(\Omega^{(i)})} + c(v_\infty)), \tag{1.11}$$

where $c(v_\infty)$ *is a constant depending on the data at infinity.*

Theorem A is proved by the standard Galerkin method applied to a reformulation of the original problem. A proof is given in section 3 and the sense of the weak solution follows from (3.4) and (3.15). It gives us well posedness of the problem for large data for all times. This result is as an auxiliary theorem for the next one which will concern some global features of the solutions. Also the following corollary can be easily concluded.

Corollary of Theorem A. *Let assumptions of Theorem A be fulfilled, moreover* $\partial\Omega \in C^\infty$. *If initial data are smooth* ($v_{in.}|_{\Omega^{(0)}} \in H^m(\Omega^{(i)})$, $v_{in.}|_{\Omega^{(i)}} - v_\infty^{(i)} \in H^m(\Omega^{(i)})$ *with* $m > 1$*), then solutions of (1.1) are also smooth* ($\nabla^m v \in V_{loc}^{1,0}(\Omega \times (0,T))$ *with estimates analogous to (1.11)).*

The proof of Corollary will be omitted. Since Ω is a two dimensional domain and (1.11) holds, by the standard technique [7, 12] one can conclude thesis of the Corollary assuming sufficient regularity of the boundary.

Finally we have.

Theorem B. *Let assumptions of Theorem A be fulfilled and additionally* $rot\ v_{in.} \in L_\infty(\Omega)$. *If*

$$A(\Omega)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)} < 1, \tag{1.12}$$

where χ *is the curvature of boundary* $\partial\Omega$ *and* $A(\Omega)$ *is a constant depending only on the constant form the Poincare inequality for domain* Ω *- see (2.7), then*

$$rot\ v \in L_\infty(\Omega \times (0,\infty)) \quad and \quad v \in C^{\alpha,0}(\Omega \times (0,\infty))$$

for $0 < \alpha < 1$ *and the following estimate holds*

$$||rot\ v||_{L_\infty(\Omega\times(0,\infty))} + ||v||_{C^{\alpha,0}(\Omega\times(0,\infty))} \leq c(1 - A(\Omega)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)})^{-1}\cdot$$

$$\cdot(A(\Omega)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)}c(v_\infty) + ||rot\, v_{in.}||_{L_\infty(\Omega)}). \tag{1.13}$$

Theorem B delivers us a new estimate for solutions to the Navier-Stokes equations in pipe-like domains. Estimate (1.13) is a generalization of the L_∞-bound for the vorticity which is well known for domains without boundaries. This technique works if we assume the geometrical constraint (1.12).

It is worth to underline that to prove it, no energy method has been applied. Estimate (1.13) does not give any information about convergence to data at infinity, but if Theorem A holds bound (1.11) control condition $(1.1)_4$ for all times.

In our considerations we will not examine system (1.1). Since in the studied case the Dirichlet integral $\int_\Omega \nabla v : \nabla v dx = \infty$, there is a need to find a different approach. Applying a property of boundary conditions $(1.1)_3$ we describe completely the Dirichlet problem on the vorticity of the velocity, which in two dimensions is a scalar function

$$\alpha = \text{rot } v = v_1^2 - v_2^1 \tag{1.14}$$

satisfying the following problem

$$\begin{array}{lll} \alpha_t + v \cdot \nabla\alpha - \nu\Delta\alpha = 0 & \text{in} & \Omega \times [0,T], \\ \alpha = v \cdot \tau(2\chi - f/\nu) & \text{on} & \partial\Omega \times [0,T], \\ \alpha \to \alpha_\infty^{(i)} \quad \text{as} \quad y_1^{(i)} \to \infty & \text{in} & \Omega^{(i)} \times [0,T], \\ \alpha|_{t=0} = \alpha_{in.} = \text{rot } v_{in.} & \text{on} & \Omega, \end{array} \tag{1.15}$$

where χ is the curvature of the boundary $\partial\Omega$ and $\alpha_\infty^{(i)} = \text{rot } v_\infty^{(i)}$ - see (1.5). To obtain $(1.15)_2$ it is enough to differentiate the first condition of $(1.1)_3$ with respect to the length parameter and use the second one (see [2]).

Problem (1.15) has better properties then (1.1). First of all we have here the maximum principle which is the crucial tool in our proof. To complete problem (1.5) we need to describe the velocity by the vorticity and this information is given by the following problem

$$\begin{array}{lll} \text{rot } v = \alpha & \text{in} & \Omega, \\ \text{div } v = 0 & \text{in} & \Omega, \\ n \cdot v = 0 & \text{on} & \partial\Omega, \\ v^1(y^{(i)}) \to {v_\infty^{(i)}}^1(y^{(i)}) \ \text{ as } \ y_1^{(i)} \to \infty & \text{in} & \Omega^{(i)}. \end{array} \tag{1.16}$$

Since we require to domain Ω be simply connected, by the Poincare Lemma and $(1.7)_2$, vector v can be described by a scalar function φ in the following way

$$v = \nabla^\perp \varphi = (-\partial_{x_2}\varphi, \partial_{x_1}\varphi). \tag{1.17}$$

In the mathematical fluid dynamics slip boundary conditions are not so popular as the problems with Dirichlet or Neumann type [3, 6, 11]. In general this relation can be treated as an alternative approach to describe phenomena in rigid domains which for some cases it seems to be more natural. This type of boundary conditions, in particular cases of small effective friction f/ν, has good properties to investigate system (1.1) as an approximation of motion of a perfect fluid [2, 9].

In this paper we present interesting properties of these flows. For small friction and small curvatures of the boundary (by (1.12) we require only smallness of the negative curvatures, because for positive ones we can choose suitable f to (1.12) be satisfied) we are able to find the maximum principle for the equation on the vorticity system. From the mathematical point of view this property is quite unusual for problems dealing with Navier-Stokes equations in domains with boundaries. But our result holds only in two dimensions, in three dimensional case there appears an extra term $v\nabla\alpha$ in $(1.15)_1$ which destroys the structure of the problem, although slip conditions still define boundary data of the vorticity [13].

System (1.1) is a modification of Leray's problem concerning the Navier-Stokes equations $(1.1)_{1,2}$ with nonslip boundary conditions and the Poiseuille flow at infinity [5 chap-XI]. Questions of the existence in the steady case or long time behavior for nonsteady equations for large data are still open. We have results just for small fluxes [1, 4, 5 chap-XI, 10]. The difficulty is hidden in the total energy. For such problems the Dirichlet integral $\int_\Omega \nabla v : \nabla v dx = \infty$. An approach to such problem was presented in [8], but without conditions in infinity.

If $f > 0$ then also by (1.5) rot $v_\infty^{(i)} \neq 0$, hence the Dirichlet integral is also infinite. To avoid difficulties similar to those appearing in Leray's problem we study the reformulation of (1.1) given by the coupled system (1.15) and (1.16). Then if condition (1.12) is fulfilled we obtain an a priori estimate (1.13) which is uniform in time and is not connected with the energy of the system.

2 Notation

In the paper we try to use the standard notations [7, 12]. We recall

$$||f||^2_{H^m(\Omega)} = \sum_{0\le|\alpha|\le m} \int_\Omega |\partial_x^\alpha f|^2 dx \tag{2.1}$$

for $m \in \mathbf{N}$, where $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$, $|\alpha| = \alpha_1 + \alpha_2$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$;

$$||f||^2_{H^{1,0}(\Omega)} = ||f||^2_{L_2(\Omega\times(0,T))} + ||\nabla f||^2_{L_2(\Omega\times(0,T))}, \tag{2.2}$$

where $\nabla = (\partial_{x_1}, \partial_{x_2})$;

$$||f||^2_{V^{1,0}(\Omega\times(0,T))} = \sup_{0\le t\le T} ||f(\cdot,t)||^2_{L_2(\Omega)} + ||\nabla f||^2_{L_2(\Omega\times(0,T))}, \tag{2.3}$$

$$||f||_{C^{\alpha,0}(\Omega\times(0,T))} = ||f||_{C(\Omega\times(0,T))} + \sup_{t\in(0,T)} \sup_{x,y\in\Omega;x\neq y} \frac{|f(x,t)-f(y,t)|}{|x-y|^\alpha}. \tag{2.4}$$

In the statement of Theorems A and B there are two constants $A(\Omega)$ and $B(\Omega)$, which depend only on properties of domain Ω and they are defined by the following problem

$$\begin{array}{lcl} \Delta\varphi = f & \text{in} & \Omega, \\ \varphi = 0 & \text{on} & \partial\Omega, \\ \varphi \to 0 \quad \text{as} \quad |x|\to\infty & & \end{array} \tag{2.5}$$

And constants come from the below estimates for solutions to problem (2.5)

$$||\frac{\partial\varphi}{\partial n}||^2_{L_2(\partial\Omega)} \le B(\Omega)||f||^2_{L_2(\Omega)}, \tag{2.6}$$

$$||\nabla\varphi||_{C(\Omega)} \le A(\Omega)||f||_{L_\infty(\Omega)}. \tag{2.7}$$

The constants depend on heights of pipes and diam $\Omega^{(0)}$. Estimate (2.6) follows from the energy method, the Schauder estimates and the trace theorem. Inequality (2.7) is proved section 4 - estimate (4.13) for solutions of problem (4.5).

In proofs by letter c we denote a generic constant. By A, B, ... we denote constants fixed in each proof of lemmas and by

$$c(v_\infty) \le c\sum_{i=1}^{2} ||v_\infty^{(i)}||_{W^1_\infty(0,H^{(i)})}. \tag{2.8}$$

3 Proof of Theorem A

To prove the existence, we need the following lemma.

Lemma 3.1. *There exists a smooth vector field* $\bar v : \Omega \to \mathbf{R}^2$ *satisfying* $(1.16)_{2,3,4}$ *such that*

$$\bar v = v_\infty^{(i)} \quad in \quad \Omega^{(i)}, \tag{3.1}$$

and

$$||\bar v||_{C^2(\Omega)} \le c(v_\infty). \tag{3.2}$$

Proof. Let us construct $\bar{v} = \nabla^{\perp}\bar{\phi}$ which satisfies $(1.16)_{2,3,4}$. In each pipe $\Omega^{(i)}$ for $i = 1, ..., K$ we define

$$\bar{v}|_{\Omega^{(i)}} = \nabla^{\perp}\bar{\phi}|_{\Omega^{(i)}} = v_{\infty}^{(i)} = \nabla^{\perp}\phi_{\infty}^{(i)}, \tag{3.3}$$

where $\phi_{\infty}^{(i)}$ - potential of $v_{\infty}^{(i)}$ - is defined up to a constant. By condition $(1.16)_3$ we see that $\bar{\phi}$ is constant along each $\Gamma^{(i)}$, so $\bar{\phi}$ in one pipe uniquely defines potential in the next one. Let $\bar{\phi} = 0$ on $\Gamma^{(1)}$. By (3.3) we have $\bar{\phi}$ on $\Omega^{(1)}$. This causes that constant of $\phi_{\infty}^{(2)}$ is defined uniquely. So by (3.3) function $\bar{\phi}$ is defined also in $\Omega^{(2)}$. In particular, $\bar{\phi}$ has been defined on $\Gamma^{(2)}$. Note that this procedure is well defined since the total flux - see assumption (1.8) - is conserved ($\bar{\phi}$ on $\Gamma^{(2)}$ defined by $\bar{\phi}$ on $\Gamma^{(1)}$ gives zero as we assumed). To complete the construction we need to describe $\bar{\phi}$ in $\Omega^{(0)}$, since this part is bounded the extension can be any, but smooth and conserving norms. Thus we get $\bar{v} = \nabla^{\perp}\bar{\phi}$ satisfying $(1.16)_{2,3,4}$ and $||\bar{v}||_{C^2(\Omega)} \le c(v_{\infty})$. Lemma 3.1 is proved.

To obtain an estimate guaranteeing existence for all times we need to modify solutions to be integrable in L_2-norms. To achieve this we postulate

$$\alpha = \bar{\alpha} + \beta \quad \text{and} \quad v = \bar{v} + u, \tag{3.4}$$

where $\bar{\alpha} = \text{rot}\ \bar{v}$ and they are defined by Lemma 3.1. Having such forms, using (1.15), (1.16) and (1.17) we state the following problem for β and u

$$\begin{array}{lll}
\beta_t + v \cdot \nabla\beta - \nu\Delta\beta = -u \cdot \nabla\bar{\alpha} - \bar{v} \cdot \nabla\bar{\alpha} + \nu\Delta\bar{\alpha} & \text{in} & \Omega \times [0, T], \\
\beta = u \cdot \tau(2\chi - f) + [\bar{v} \cdot \tau(2\chi - f) - \bar{\alpha}] & \text{on} & \partial\Omega \times [0, T], \\
\Delta\varphi = \beta & \text{in} & \Omega \times [0, T], \\
\varphi = 0 & \text{on} & \partial\Omega \times [0, T], \\
\varphi,\ \beta \to 0 \quad \text{as} \quad |x| \to \infty & & \\
\beta|_{t=0} = \alpha_{in.} - \bar{\alpha} = \beta_{in.} & \text{on} & \Omega,
\end{array} \tag{3.5}$$

where $u = \nabla^{\perp}\varphi$.

Lemma 3.2. *If solutions to problem (3.5) are sufficiently regular then they satisfy the following inequality*

$$\sup_{0 \le t \le T} ||\nabla\varphi(\cdot, t)||^2_{L_2(\Omega)} + \int_0^T \int_{\Omega} \beta^2 dx dt \le C e^{Tc(v_{\infty})} \Big(||\nabla\varphi_{in.}||_{L_2(\Omega)} + c(v_{\infty}) \Big) \tag{3.6}$$

for all $T > 0$ and C independent of T.

Proof. We multiply $(3.5)_1$ by φ and integrate over Ω, getting

$$\begin{aligned} -\textstyle\int_\Omega \nabla\varphi_t\cdot\nabla\varphi dx - \int_\Omega v\cdot\nabla\varphi\beta dx + \nu\int_\Omega \nabla\beta\cdot\nabla\varphi dx = \\ -\textstyle\int_\Omega u\cdot\nabla\bar\alpha\varphi dx - \int_\Omega \bar v\cdot\nabla\bar\alpha\varphi dx - \nu\int_\Omega \Delta\bar\alpha\varphi dx. \end{aligned} \tag{3.7}$$

The second term of the l.h.s. of (3.7) is equal to

$$-\int_\Omega (u+\bar v)\cdot\nabla\varphi\beta dx = -\int_\Omega \bar v\cdot\nabla\varphi\beta dx - \int_\Omega \nabla^\perp\varphi\cdot\nabla\varphi\beta dx. \tag{3.8}$$

The last term of (3.8) vanishes, so it can be bound as follows

$$\left|\int_\Omega v\cdot\nabla\varphi\beta dx\right| \le \frac{\nu}{4}\int\beta^2 dx + c(v_\infty)\int_\Omega |\nabla\varphi|^2 dx. \tag{3.9}$$

The third one takes the form

$$\nu\int_\Omega \nabla\beta\cdot\nabla\varphi dx = -\nu\int_\Omega \beta^2 dx + \nu\int_{\partial\Omega}\beta\nabla\varphi\cdot n d\sigma;$$

but by $(3.5)_2$ we get;

$$-\nu\int_\Omega \beta^2 dx - \nu\int_{\partial\Omega}(f-2\chi)(u\cdot\tau)^2 d\sigma - \nu\int_{\partial\Omega}(\bar v\cdot\tau(-f)-\bar\alpha)u\cdot\tau d\sigma. \tag{3.10}$$

The first term of the r.h.s. of (3.7) vanishes, because

$$-\int_\Omega \nabla^\perp\varphi\cdot\nabla\bar\alpha\varphi dx = \int_\Omega \nabla^\perp\varphi\cdot\nabla\varphi\bar\alpha dx = 0.$$

To bound the last two terms of the r.h.s. of (3.7), note that $\bar v\cdot\nabla\bar\alpha$, $\Delta\bar\alpha \in L_2(\Omega)$ and $\bar v\cdot\tau(-f)-\bar\alpha \in L_2(\partial\Omega)$. It follows from properties of v_∞, α_∞ - see (1.6), (1.7) and the construction of $\bar v$ - Lemma 3.1, which guarantees supp $\bar v\cdot\nabla\bar\alpha$, $\Delta\bar\alpha \subset \Omega^{(0)}$ and supp $\bar v\cdot\tau(-f)-\bar\alpha \subset \partial\Omega\cap\partial\Omega^{(0)}$. Hence, since $||\varphi||_{L_2(\Omega)} \le c||\beta||_{L_2(\Omega)}$ we have

$$\left|\int_\Omega \bar v\cdot\nabla\bar\alpha\varphi dx\right| + \nu\left|\int_\Omega \Delta\bar\alpha\varphi dx\right| \le \frac{\nu}{4}\int_\Omega \beta^2 dx + c(v_\infty). \tag{3.11}$$

This way equality (3.7) gives the following inequality

$$\frac{d}{dt}\int_\Omega |\nabla\varphi|^2 dx + \nu\int_\Omega \beta^2 dx + \nu\int_{\partial\Omega}(f-2\chi)(u\cdot\tau)^2 d\sigma \le$$

$$|\int_{\partial\Omega}(\bar{v}\cdot\tau(-f)-\bar{\alpha})u\cdot\tau d\sigma|+c(v_\infty)\int_\Omega|\nabla\varphi|^2dx+c(v_\infty), \qquad (3.12)$$

but $||u||^2_{L_2(\partial\Omega)}\leq B(\Omega)||\beta||^2_{L_2(\Omega)}$ (here we need to assume that $f-2\chi>-B(\Omega)$ - see (3.12)). So by the Gronwall inequality from (3.12) we conclude

$$\int_\Omega|\nabla\varphi(\cdot,t)|^2dx\leq\exp\{tc(v_\infty)\}\int_\Omega|\nabla\varphi_{in.}|^2dx+c(v_\infty)\exp\{tc(v_\infty)\}. \quad (3.13)$$

Using this information again to (3.12) we get

$$\int_0^T\int_\Omega\beta^2dxdt\leq Ce^{Tc(v_\infty)}(||\nabla\varphi_{in.}||^2_{L_2(\Omega)}+c(v_\infty)). \qquad (3.14)$$

By $(3.5)_3$ $||\nabla\varphi_{in.}||_{L_2(\Omega)}\leq c||\beta||_{H^{-1}(\Omega)}$. From (3.13) and (3.14) we obtain Lemma 3.2.

Weak formulation for problem (3.5)

We say that φ satisfies (3.5) in the weak sense if and only if

$$\varphi\in H_0^{2,0}(\Omega\times(0,T)),\quad \nabla\varphi\in L_\infty(0,T;L_2(\Omega))$$

and the following identity

$$\begin{aligned}&\int_\Omega\nabla\varphi_t\cdot\nabla\psi dx+\nu\int_\Omega\Delta\varphi\Delta\psi dx+\int_{\partial\Omega}(f-2\chi)\frac{\partial\varphi}{\partial n}\frac{\partial\psi}{\partial n}d\sigma+\int_\Omega v\cdot\nabla\psi\Delta\varphi dx\\ &=\int_\Omega u\cdot\nabla\bar{\alpha}\psi dx+\int_\Omega\bar{v}\cdot\nabla\bar{\alpha}\psi dx-\int_\Omega\nu\Delta\bar{\alpha}\psi dx-\nu\int_{\partial\Omega}[\bar{v}\cdot\tau(f-2\chi)-\bar{\alpha}]\psi d\sigma\end{aligned} \qquad (3.15)$$

holds for any $\psi\in H_0^{2,1}(\Omega\times(0,T))$ in distributional sense on time integral $[0,T)$ (i.e. $\psi(\cdot,T)\equiv 0$), where $u=\nabla^\perp\varphi$ and $v=\bar{v}+u$.

Next we prove.

Lemma 3.3. *If $\nabla\varphi_{in.}\in L_2(\Omega)$ then there exists unique weak solution on time interval* $[0,1]$.

Proof. Having a priori estimate (3.6) it is enough to show existence for short time. To prove it we apply a standard technique - the Galerkin method. Since $H_0^2(\Omega)$ is Hilbertian and separable, we introduce a base $\{w_i\}_{i=1}^\infty$

$$H_0^2(\Omega)=\overline{\text{span}\{w_1,w_2,...,w_n,...\}}^{||\cdot||_{H^2(\Omega)}},$$

also by the Gramm-Schmidt procedure we require to

$$\int_\Omega\nabla w_i\cdot\nabla w_jdx=\delta_{ij}. \qquad (3.16)$$

And we define a finite dimensional subspace of $H_0^2(\Omega)$

$$V^N = \text{span}\{w_1, ..., w_N\}. \tag{3.17}$$

The next step is the construction of approximations of solutions. Introduce

$$\varphi^N(x,t) = \sum_{j=1}^{N} d_j^N(t) w_j(x). \tag{3.18}$$

To find coefficients $\{d_j^N\}_{i=1,\dots,N}$ we solve the following ordinary differential system

$$\begin{array}{c} \frac{d}{dt} d_k^N(t) + \int_\Omega \nu \Delta\varphi^N \Delta w_k dx + \nu \int_{\partial\Omega} (f-2\chi) \frac{\partial\varphi^N}{\partial n} \frac{\partial\psi}{\partial n} d\sigma \\ + \int_\Omega v^N \cdot \nabla w_k \Delta\varphi^N dx = \int_\Omega u^N \cdot \nabla\bar{\alpha} w_k dx + \int_\Omega \bar{v} \cdot \nabla\bar{\alpha} w_k dx \\ - \int_\Omega \nu \Delta\bar{\alpha} w_k dx - \nu \int_{\partial\Omega} [\bar{v} \cdot \tau(f-2\chi) - \bar{\alpha}] w_k d\sigma, \\ d_k^N(0) = \int_\Omega \nabla\varphi_{in.} \cdot \nabla w_k dx, \quad v^N = \bar{v} + u^N, \quad u^N = \nabla^\perp \varphi^N \end{array} \tag{3.19}$$

for $k = 1, ..., N$.

The above problem has unique solution on time interval $T_N > 0$. Next, following the standard procedure we need to find an estimate to control T_N independently of N. To bound φ_N we repeat all steps as for (3.13) and (3,14), but with φ^N - instead of φ. And we obtain the same bound

$$\sup_{0\le t\le T} ||\nabla\varphi^N||^2_{L_2(\Omega)} + \int_0^T \int_\Omega |\nabla^2\varphi^N|^2 dx \le Ce^{c(v_\infty)T}(||\nabla\varphi_{in.}||^2_{L_2(\Omega)} + c(v_\infty)). \tag{3.20}$$

Hence for fixed T, say $T = 1$, we have uniform bound for $\nabla\varphi^N$ in $V^{1,0}(\Omega \times (0,1))$. Thus there exists a subsequence $\{\varphi^{N_k}\}_{k=1}^\infty$ such that

$$\begin{array}{c} \varphi^{N_k} \rightharpoonup \varphi_* \quad \text{weakly in} \quad H^{2,0}(\Omega \times (0,1)), \\ \nabla\varphi^{N_k} \rightharpoonup \nabla\varphi_* \quad *\text{-weakly in} \quad L_\infty(0,1; L_2(\Omega)) \end{array}$$

as $k \to \infty$.

Limit φ_* is a solution to (3.5) in the sense of formulation (3.15). By classical results for the 2D Navier-Stokes equations [12] we control the convergence for the nonlinear term in (3.15). Also this theory guarantees uniqueness of the solutions obtained in this way. Lemma 3.3 is proved.

To finish the proof of Theorem A it is enough to note that the estimate given by Lemma 3.2 guarantees boundedness of norms of weak solutions for any time $T > 0$ and by Lemma 3.3 the solution can be extended on time interval $[0, T+1]$, so we get existence for all $T > 0$.

Theorem A gives insufficient information about behavior of solutions for $t \to \infty$. This question will be partially addressed in the next section.

4 Proof of Theorem B

First, we consider the problem on the vorticity (1.14) to find the L_∞-bound. To obtain it we apply the maximum principle for system (1.15). By a straightforward application of this method we get

$$||\alpha||_{L_\infty(\Omega\times(0,T))} \leq ||\alpha_{in.}||_{L_\infty(\Omega)} + \sup_{i=1,2} ||\alpha_\infty^{(i)}||_{L_\infty(0,H^{(i)})}$$

$$+||f-2\chi||_{L_\infty(\partial\Omega\times(0,T))}||v||_{L_\infty(\partial\Omega\times(0,T))}. \tag{4.1}$$

To prove (4.1) it is enough to multiply $(1.15)_1$ by $(\alpha-\bar{k})_+ = \max\{\alpha-\bar{k},0\}$ and integrate over Ω with

$$\bar{k} = \max\left\{\sup_{x\in\Omega}\alpha_{in.}(x); \sup_{i=1,2}\left(\sup_{y\in(0,H^{(i)})}\alpha_\infty^{(i)}\right); \sup_{x\in\partial\Omega\times(0,T)} v\cdot\tau(2\chi-f)\right\}. \tag{4.2}$$

Since cross-sections of domain Ω are uniformly bounded we easily conclude that $||(\alpha-\bar{k})_+||_{L_2(\Omega)} = 0$. The same we have for $(\alpha-\underline{k})_-$ with a suitable $\underline{k}$. This way we obtain (4.1).

Next we study the elliptic problem (1.16). To solve it we note that for simplification nonhomogenity from $(1.16)_4$ should be removed. By $(1.16)_2$ it is enough to consider a potential (stream function) of the velocity. So we search for v as follows

$$v = \bar{v} + u, \tag{4.3}$$

where $\bar{v}$ is defined by Lemma 3.1 and u satisfies

$$\begin{array}{lll} \text{rot } u = \alpha - \text{rot } \bar{v} & \text{in} & \Omega, \\ \text{div } u = 0 & \text{in} & \Omega, \\ n\cdot u = 0 & \text{on} & \partial\Omega, \\ u^1(y^{(i)}) \to 0 \text{ as } y_1^{(i)} \to \infty & \text{in} & \Omega^{(i)}. \end{array} \tag{4.4}$$

Using the potential $(u = \nabla^\perp\varphi)$, problem (4.4) takes the form

$$\begin{array}{lll} \Delta\varphi = \alpha - \text{rot } \bar{v} & \text{in} & \Omega, \\ \varphi = 0 & \text{on} & \partial\Omega, \\ \varphi \to 0 \quad \text{as} \quad |x| \to \infty. & & \end{array} \tag{4.5}$$

It is important to underline that here we do not study the existence to problem (4.5) - this information is already done by Theorem A. We just look for a suitable estimate for them. To find a L_∞-bound on $\nabla\varphi$ we have to localize the problem, because the domain is unbounded.

Introduce a decomposition of domain Ω

$$\Omega = \Omega^{(0)} \cup \left(\bigcup_{(i,j)=\{1,2\}\times\mathbf{N}} \omega^{(i,j)} \right), \tag{4.6}$$

where $\omega^{(i,j)} = [jL, (j+1)L] \times [0, H^{(i)}]$ in local coordinates $\Omega^{(i)}$, where L is a parameter and will be specified later. Next we define a smooth functions $\eta^{(i,j)} : \Omega \to [0,1]$ such that for $j \geq 1$

$$\eta^{(i,j)} = 1 \quad \text{for} \quad x \in \omega^{(i,j)}, \qquad \text{supp}\ \eta^{(i,j)} \subset \omega^{(i,j-1)} \cup \omega^{(i,j)} \cup \omega^{(i,j+1)} \tag{4.7}$$

and $|\nabla^\alpha \eta^{(i,j)}| \leq c/L^{|\alpha|}$. And we define $\eta^{(0,0)}$ in the following way

$$\eta^{(0,0)} = 1 \quad \text{for} \quad x \in \Omega^{(0)} \subset \Omega^{(0)} \cup \left(\bigcup_{i=1,2} \omega(i,0) \right),$$

$$\text{supp}\ \eta^{(0,0)} \subset \Omega^{(0)} \cup \left(\bigcup_{i=1,2} \omega(i,0) \cup \omega(i,1) \right)$$

and $|\nabla^\alpha \eta^{(0,0)}| \leq c/L^{|\alpha|}$.

Now we localize problem (4.5). We examine $\eta^{(i,j)}\varphi$. By (4.7) we see that

$$\begin{array}{lll} \Delta(\eta^{(i,j)}\varphi) = F_{ij} & \text{in} & \text{supp}\ \eta^{(i,j)}, \\ \eta^{(i,j)}\varphi = 0 & \text{on} & \partial(\text{supp}\ \eta^{(i,j)}), \end{array} \tag{4.8}$$

where

$$F_{ij} = \eta^{(i,j)}(\alpha - \text{rot}\ \bar{v}) + 2\nabla\eta^{(i,j)}\nabla\varphi + (\Delta\eta^{(i,j)})\varphi.$$

By the theory of the Laplace operator problem (4.8) is ill posed in the L_∞-space, so we choose a large $p < \infty$ and we consider it in L_p-spaces. In this approach we get the following estimate

$$||\varphi||_{W^2_p(\omega^{(i,j)})} \leq c(L) \Big(||\eta^{(i,j)}(\alpha - \text{rot}\ \bar{v})||_{L_p(\Omega)} + $$

$$||2\nabla\eta^{(i,j)}\nabla\varphi||_{L_p(\Omega)} + ||(\Delta\eta^{(i,j)})\varphi||_{L_p(\Omega)} \Big). \tag{4.9}$$

By the standard energy method constant $c(L)$ for $p = 2$ is independent of L and depends on the constant from the Poincare inequality. We show that

$$c(L) \leq cL^{1/2-1/p}. \tag{4.10}$$

For general p the Schauder estimates give the following bound

$$||\nabla^2\eta^{(i,j)}\varphi||_{L_p(supp\,\eta^{(i,j)})} \leq c(||\Delta(\eta^{(i,j)}\varphi)||_{L_p(supp\,\eta^{(i,j)})} + ||\eta^{(i,j)}\varphi||_{L_p(supp\,\eta^{(i,j)})}) \tag{4.11}$$

where the constant is independent of diameters of the domain and depends only on the shape of it.

To find the L_p-bound we apply the energy estimate to solutions to (4.8). We have

$$||\eta^{(i,j)}\varphi||_{H^1_0(supp\,\eta^{(i,j)})} \leq c||F_{ij}||_{L_2(supp\,\eta^{(i,j)})}, \tag{4.12}$$

but

$$c||F_{ij}||_{L_2(supp\,\eta^{(i,j)})} \leq cL^{1/2-1/p}||F_{ij}||_{L_p(supp\,\eta^{(i,j)})}. \tag{4.13}$$

The imbedding theorem says that

$$||\eta^{(i,j)}\varphi||_{L_p(supp\,\eta^{(i,j)})} \leq c||\eta^{(i,j)}\varphi||_{H^1_0(supp\,\eta^{(i,j)})}. \tag{4.14}$$

Hence

$$||\eta^{(i,j)}\varphi||_{L_p(supp\,\eta^{(i,j)})} \leq cL^{1/2-1/p}||F_{ij}||_{L_p(supp\,\eta^{(i,j)})} \tag{4.15}$$

which with (4.11) shows (4.10). Let us underline that all above constants depend on the constant from the Poincare inequality.

Introduce the following quantity

$$M = \sup_{(i,j)\in I} ||\varphi||_{W^2_p(\omega^{(i,j)})}, \tag{4.16}$$

where $I = \{(0,0), \{1,2\} \times \mathbf{N}\}\}$.

By properties of functions $\eta^{(i,j)}$ we easily deduce that second and third terms of the r.h.s. of (4.9) can be estimated as follows

$$||2\nabla\eta^{(i,j)}\nabla\varphi||_{L_p(\Omega)} + ||(\Delta\eta^{(i,j)})\varphi||_{L_p(\Omega)} \leq c(L)L^{-1}M \tag{4.17}$$

for sufficiently large L. Hence (4.17) with (4.9) and (4.10) give

$$M \leq A_1L^{1/2}||\alpha - \text{rot }\bar{v}||_{L_\infty(\Omega)} + BL^{1/2-1/p}L^{-1}M \tag{4.18}$$

and if we choose L so large that $BL^{-1/2-1/p} < 1$, then by the imbedding theorem if $p > 2$, (4.18) implies

$$||\nabla\varphi||_{C(\Omega)} \leq A_2||\alpha - \text{rot }\bar{v}||_{L_\infty(\Omega)}. \tag{4.19}$$

Recalling (4.3) and Lemma 3.1 we obtain

$$||v||_{C(\Omega)} \leq A(\Omega)||\alpha||_{L_\infty(\Omega)} + c(v_\infty). \tag{4.20}$$

Inserting (4.20) to (4.1) we get

$$||\alpha||_{L_\infty(\Omega\times(0,T))} \leq ||f/\nu - 2\chi||_{L_\infty(\partial\Omega)} A(\Omega)(||\alpha||_{L_\infty(\Omega\times(0,T)} + c(v_\infty))$$

$$+c(v_\infty) + ||\alpha_{in.}||_{L_\infty(\Omega)}. \tag{4.21}$$

Hence if $||f/\nu - 2\chi||_{L_\infty(\partial\Omega)}A(\Omega) < 1$ then we obtain a priori estimate

$$||\alpha||_{L_\infty(\Omega\times(0,T))} \leq (1 - A(\Omega)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)})^{-1}\cdot$$

$$\cdot(A(\Omega)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)}c(v_\infty) + ||\alpha_{in.}||_{L_\infty(\Omega)} + c(v_\infty)). \tag{4.22}$$

Theorem B has been proved.

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