Advances in Differential Equations Volume 10, Number 8 (2005), 861–900

ON THE STEFAN PROBLEM WITH SURFACE TENSION IN THE L_p FRAMEWORK

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Abstract. We prove the existence of unique regular local in time solutions to the quasi-stationary one-phase Stefan problem with the Gibbs-Thomson correction. The result is optimal with respect to L_p regularity and the obtained phase surface is a submanifold of the $W_p^{3,1}$-class. The proof is based on a Schauder-type estimate for a linearization of the original system.

1. Introduction

The Stefan problem with the Gibbs-Thomson correction is a classical model of the phase transition theory [3, 7, 19, 27, 31]. Describing the free boundary between two states of an examined material, the system takes into account the shape and geometrical features of the phase transition surface omitting other influences, which in an essential way improves the classical Stefan problem. Models of this type appear in mechanics, biology, or crystallography [3, 19, 14]. The most classical example describes the melting process of ice. However the idea of this approach can be found in models of tumor growth in medicine [14].

In this paper we want to concentrate on the one-phase quasi-stationary $(n+1)$-dimensional model. The studied system is called sometimes in the literature the Hele-Shaw or Mullins-Sekerka model.

Mathematical aspects of the model lead to a lot of difficulties. Basic questions concerning the well posedness were not solved until the 1990s. First approaches [4, 6, 13, 21] showed only some partial results. The knowledge about mathematical features of the system was not sufficient. The parabolic character has been first noted in [8], however only in [5, 9] has this property been applied effectively. The mentioned results proved existence of classical unique solutions in the Hölder class such that the free boundary $\partial\Omega_t \in$

Accepted for publication: February 2005.
AMS Subject Classifications: 35R35, 35K55, 35K99, 80A22, 74N20.

$C^{3+3a,1+a}$ with $a > 0$. Such regularity is a consequence of the fact that the examined system is equivalent to an abstract nonlocal nonlinear parabolic equation of the third order which governs the evolution of the free surface. The theory of abstract parabolic systems [1] and autonomous character of the structure of the equations imply C^∞ smoothness of solutions for any $t > 0$.

However the issue of regularity of the initial surface seems to be interesting and important for future study. In [9] it is required that the initial surface be in the C^{2+a} Hölder class. The first step in the L_p approach has been done in [11] for the evolutionary system; the authors showed existence of solutions in the Besov class B^s_{pp} with $p > n+4$ with an initial surface belonging to $B^{4-3/p}_{pp}$, where the evolving domain is a small perturbation of the half space.

In our paper we investigate the issue of well posedness in the L_p framework with a sharp regularity. However regularity aspects of the initial surface are not our only goal. We want to create an alternative approach to methods from [5, 9, 11]. The main tool of our technique is an estimate of Schauder type for a linearization of the studied system. This method can be more effective in future investigations if we will want to study models with a unisotropic correction of the Gibbs-Thomson type, which is more suitable to model some phenomena [16, 17, 18]. In such cases the L_p framework could be more helpful.

The parabolic character of the system enables one to prove some stability results of the equilibrium state globally in time [6, 10, 15].

In the present paper we concentrate on existence of regular local in time solutions guaranteeing the well posedness of the system for arbitrary initial domains.

We will investigate the quasi-stationary system governed by the following set of equations

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \Omega_t, \\ p = a\kappa & \text{on} & \partial\Omega_t, \\ \frac{\partial p}{\partial n} = -V_n & \text{on} & \partial\Omega_t, \\ \Omega_t|_{t=0} = \Omega_0, & & \end{array} \tag{1.1}$$

where p is the so-called temperature function, $a > 0$, n is the normal outward vector to $\partial\Omega_t$, the quantity κ defined on $\partial\Omega_t$ denotes the mean curvature of the surface; the chosen convention implies that for convex domains Ω the mean curvature κ is positive. In the local coordinate system it reads

$$\kappa = \Delta_{\partial\Omega_t}\psi(z'), \tag{1.2}$$

where $x(t) = (z', \psi(z'))$ describes locally the surface as a graph of the function ψ and $\Delta_{\partial\Omega_t}$ is the Laplace-Beltrami operator defined on $\partial\Omega_t$ given by the following explicit formula

$$\Delta_{\partial\Omega_t} = g^{-1/2}\frac{\partial}{\partial z^i}(g^{1/2}g^{ij}\frac{\partial}{\partial z_j}), \tag{1.3}$$

where $\{g_{ij}\}$ is the metric induced by the parameterization $x(t)$ and $g = det\{g_{ij}\}$. V_n denotes the normal outward velocity of the moving boundary and Ω is the initial domain which in general is a bounded subset of $\mathbf{R}^{n+1}$.

To underline the structure and features of the system we assume

$$\dim \partial\Omega_t = n \quad \text{and} \quad \dim \Omega_t = n+1. \tag{1.4}$$

This setting will be natural, since the most important examinations will be done on the free surface and analysis in the domain Ω_t has only an auxiliary character.

The phase transition surface can be described as follows

$$\begin{aligned}\partial\Omega_t = \Big\{ & x \in \mathbf{R}^{n+1} : \\ x(y,t) = y + \int_0^t & V_n(x(y,t'),t')(n(x(y,t'),t'))dt' \ \text{ for } \ y \in \partial\Omega_0\Big\}.\end{aligned} \tag{1.5}$$

The above implicit description is not effective in our method, so we find a simpler one. The surface $\partial\Omega_t$ can be described also by a scalar function

$$\varphi : \partial\Omega_0 \times (0,T) \to \mathbf{R} \tag{1.6}$$

as follows

$$\partial\Omega_t = \Big\{x \in \mathbf{R}^{n+1} : x(y,t) = y + \varphi(y,t)n_0(y) \ \text{ for } \ y \in \partial\Omega_0\Big\}, \tag{1.7}$$

where $n(\cdot)$ is the normal vector to $\partial\Omega_0$. Definition (1.7) of the free surface is more convenient to linearize the system, however it can be well defined locally in time only. We will apply such an approach, because it enables us to transform our system on the initial surface. However our analysis will require a modification of definition (1.7), since the regularity of our initial surface is too low.

From the geometrical point of view solutions to problem (1.1) define a map such that

$$\Phi_t(\Omega_0) = \Omega_t. \tag{1.8}$$

This definition is not unique and from some technical reasons we will require the following property of the transformation from (1.8)

$$\frac{\partial}{\partial n^0} f(\Phi_t^{-1}(y))\Big|_{y\in\partial\Omega_0} = \frac{\partial}{\partial n^t} f(\xi)\Big|_{\xi=\Phi_t(y)}. \tag{1.9}$$

The last relation says that the diffeomorphism preserves the normal vector to the domains. A choice of map Φ fulfilling (1.9) is trivial from the geometrical point of view.

The main result of our paper is the following theorem.

Theorem 1.1. *Let*

$$p > p_* = \max\{n, (n+5)/2\} \tag{1.10}$$

and the initial domain Ω *satisfy the following regularity assumption*

$$\partial\Omega_0 \in W_p^{3-3/p} \quad \textit{as a submanifold in } \mathbf{R}^{n+1}; \tag{1.11}$$

then there exists $T > 0$ *and a unique solution to problem* (1.1) *on the time interval* $(0,T)$ *such that*

$$\Big(\bigcup_{0<t<T} \partial\Omega_t \times \{t\}\Big) \in W_p^{3,1} \quad \textit{as a submanifold in } \mathbf{R}^{n+2}, \tag{1.12}$$

$$p \in L_p(0,T; W_p^{1+1/p}(\Omega_t)), \quad \textit{and} \quad \frac{\partial p}{\partial n} \in L_p\Big(\bigcup_{0<t<T} \partial\Omega_t \times \{t\}\Big). \tag{1.13}$$

Theorem 1.1 shows the well posedness of system (1.1), but the obtained solutions are not regular in the classical sense. The temperature function p is treated as a weak solution to equation $(1.1)_1$. We concentrate our attention on the evolution of the boundary. Applying the Green's function $\mathcal{G}_t$ for problem $(1.1)_{1,2}$ we obtain the following nonlocal initial problem

$$\begin{gathered} V_n + \frac{\partial}{\partial n}\left[\int_{\partial\Omega_t} a\kappa \frac{\partial}{\partial n_y}\mathcal{G}_t d\sigma_y\right] = 0, \\ \partial\Omega_t|_{t=0} = \partial\Omega_0, \end{gathered} \tag{1.14}$$

where p does not appear explicitly and system (1.14) describes only the evolution of the surface.

The results of Theorem 1.1 give us regular solutions to system (1.14). To see the type of system (1.14) let us consider the following equation

$$\phi_t + (-\Delta)^{3/2}\phi = 0, \tag{1.15}$$

which is a simplification of system (1.14). Hence systems (1.15) and (1.14) are parabolic of the third order. It follows that the spaces $W_p^{3,1}$ are optimal to search for the regular solutions in this class.

Our investigation will be done in the Sobolev-Slobodeckij function space class which is not equivalent to the Besov class B^s_{pp}; see Section 3; in particular Slobodeckij spaces are not an interpolation family; i.e., there is no relation of type (3.17) and $L_p \neq B^0_{pp}$. Comparing our result to [11], where similar techniques were applied, we have relaxed the required regularity of the initial surface, since we need $\partial\Omega_0 \in W_p^{3-3/p}$ with $p > p_*$ and in [11] they need $\partial\Omega_0 \in W_p^{4-3/p}$ with $p > n+4 > p_*$.

This difference follows from the alternative approach to system (1.1). Instead of an application of the theory of semigroups applied directly to the nonlinear system our approach is based on the Schauder analysis of a linearization of system (1.1) which allows us to consider $(1.1)_1$ in a weak sense.

The obtained result guarantees the classical character of the free surface. Theorem 1.1 implies that the mean curvature satisfies the following inclusion

$$\kappa \in W_p^{1,1/3} \subset L_\infty(0,T;L_p) \cap L_p(0,T;C^a), \tag{1.16}$$

however the curvature may not be bounded; the normal vector fulfills the following relation

$$n \in W_p^{2,2/3} \subset C^{a,a/3} \cap L_p(0,T;C^{1+b}) \tag{1.17}$$

for some $a, b > 0$.

To begin our proof we introduce a suitable linearization connected with the transformation of the whole problem into a rigid domain. In the next section of the paper we make precise our approach to the subject.

2. Linearization

Analysis of systems in domains with free boundaries leads to many difficulties; usually we transform them into a problem in a rigid domain as in [26] or [28]. In our case we use the map Φ_t to transform the system onto a domain Ω. We require that the boundary $\partial\Omega$ be sufficiently regular (at least C^3) and $\partial\Omega$ be close to the initial boundary $\partial\Omega_0$ in the $W_p^{3-3/p}$ norm. In the new coordinate system equations (1.1) take the following form

$$\begin{array}{lll} \Delta q = (\Delta - \Delta^t)q & \text{in} & \Omega, \\ q = a\tilde{\kappa} & \text{on} & \partial\Omega, \\ \dfrac{\partial q}{\partial n} = -v & \text{on} & \partial\Omega, \end{array} \tag{2.1}$$

where $y = \Phi_t^{-1}(x)$, $(\nabla^t)^{(i)} = \frac{\partial y^j}{\partial x_i}\partial_{y^j}$, $\Delta^t = \operatorname{div}^t \nabla^t$,

$$q(y,t) = p(\Phi_t(y),t) \tag{2.2}$$

and $\tilde{\kappa} = \kappa(\Phi_t(y),t)$. For the transformation Φ_t we require an analog of condition (1.9); i.e.,

$$\Phi_t^*(\frac{\partial q}{\partial n}) = \frac{\partial p}{\partial n}, \tag{2.3}$$

where $*$ is the standard pull-back operator; as well as

$$v(y,t) = V_n(\Phi_t(y),t).$$

The evolution of the system with respect to time is hidden in relations between v and $\tilde{\kappa}$.

The existence of solutions to problem (1.1) will follow from an auxiliary result. Then the proof of Theorem 1.1 will be a consequence of the standard application of the Banach fixed-point theorem. To realize this procedure we find a suitable linearization of system (2.1). The structure of (2.1) requires us to improve the standard methods from [20].

To apply more sophisticated analysis to our system it is necessary to introduce a special covering of the domain Ω. We take two collections of open sets: $\{\omega^k\}$ and $\{\Omega^k\}$ such that

$$\overline{\omega^k} \subset \Omega^k \subset \Omega, \qquad \bigcup_k \omega^k = \bigcup_k \Omega^k = \Omega$$

with $k \in \mathcal{M} \cup \mathcal{N}$. The index k belongs to one of two sets:

$$k \in \mathcal{M} \ \text{ if } \ \overline{\Omega^k} \cap \partial\Omega = \emptyset \ \text{ and } \ k \in \mathcal{N} \ \text{ if } \ \overline{\omega^k} \cap \partial\Omega \neq \emptyset.$$

Moreover,

$$\sup_k \operatorname{diam} \Omega^k \leq 2\lambda \tag{2.4}$$

for a small number λ which will be specified later. The magnitude of λ will be prescribed by the initial surface $\partial\Omega_0$. The cover number for collections $\{\omega^k\}$ and $\{\Omega^k\}$ is independent of the smallness of λ and this feature will be important in the next steps of our consideration.

These coverings define a partition of unity for the domain Ω. Let $\zeta^k : \Omega \to [0,1]$ be a smooth function such that

$$\zeta^k(x) = \begin{cases} 1 & \text{for} \quad x \in \omega^k \\ \in [0,1] & \text{for} \quad x \in \Omega^k \setminus \omega^k \\ 0 & \text{for} \quad x \in \Omega \setminus \Omega^k \end{cases} \tag{2.5}$$

and $|\nabla^\alpha \zeta^k| \leq c/\lambda^{|\alpha|}$, $1 \leq \sum_k (\zeta^k)^2 \leq N_0$. With the help of the functions ζ^k we define

$$\pi^k = \frac{\zeta^k}{\sum_l (\zeta^l)^2}. \tag{2.6}$$

By the definition of ζ^k, the functions π^k vanish for $x \in \Omega \setminus \Omega^k$; in addition, $|\nabla^\alpha \pi^k| \leq c/\lambda^{|\alpha|}$.

The functions $\pi^k\zeta^k$ define the partition of unity by the following formula

$$\sum_{k\in\mathcal{M}\cup\mathcal{N}} \pi^k\zeta^k = 1. \tag{2.7}$$

For each $k \in \mathcal{N}$ we pick one point $\xi^k \in \omega^k \cap \partial\Omega$ which will be the origin of the local coordinate system.

Next we introduce local coordinate systems connected with each ω^k for $k \in \mathcal{N}$. We find maps $Z_k : \mathbf{R}^{n+1}_+ \to \mathbf{R}^{n+1}$ such that

$$Z_k^{-1}(\Omega^k) \subset \mathbf{R}^{n+1}_+, \quad Z_k^{-1}(\Omega^k \cap \partial\Omega) \subset \mathbf{R}^n, \quad Z(0) = \xi^k. \tag{2.8}$$

Moreover, the diffeomorphism Z_k preserves the normal derivative at the boundary as Φ_t - see (2.3). Regularity of the diffeomorphism Z_k is controlled by the smoothness of ϕ. By considering truncations $Z_k|_{\mathbf{R}^n}$ we obtain an atlas of maps for $\partial\Omega$.

Now we are prepared to introduce the linearization of problem (2.1). Our main analysis will be concentrated on the following linear system

$$\begin{array}{lll} \Delta q = F & \text{in} & \Omega \times (0,T), \\ q = a\Lambda\psi + G & \text{on} & \partial\Omega \times (0,T), \\ \dfrac{\partial q}{\partial n} = -\partial_t\psi + H & \text{on} & \partial\Omega \times (0,T), \\ \psi|_{t=0} = \psi_0 & \text{on} & \partial\Omega, \end{array} \tag{2.9}$$

where

$$\Lambda\psi = \sum_k Z_k^*[\Delta' Z_k^{-1*}(\pi^k\zeta^k\psi)] \tag{2.10}$$

and

$$\Delta' = \frac{\partial^2}{\partial z_1^2} + ... + \frac{\partial^2}{\partial z_n^2}. \tag{2.11}$$

By (2.10), the definition of the operator Λ, it is possible to treat the boundary $\partial\Omega$ as a manifold instead of a submanifold. This point of view is important, since we look for optimal regularity of solutions to (2.9) and regularity of coefficients would cause some very unpleasant technical difficulties which by this setting are omitted.

The function $\psi : \partial\Omega \to \mathbf{R}$ describes the surface $\partial\Omega_t$ as follows

$$\partial\Omega_t \ni x(y,t) = y + n(y)\psi(y,t) \quad \text{with} \quad y \in \partial\Omega, \tag{2.12}$$

where $n(\cdot)$ is the normal vector to $\partial\Omega$. Hence the boundary can be described as a graph of the function ψ with respect to the surface $\partial\Omega$. It is worthwhile to underline that regularity of $\partial\Omega$ (as we assumed to be at least C^3) is necessary to obtain the desired regularity of the surface $\partial\Omega_t$. The same idea has been applied in [5] and [9]. The choice of Ω implies that the initial surface is given by

$$x(y,0) = n(y)\psi_0(y) \qquad y \in \partial\Omega \tag{2.13}$$

which can be realized as $\partial\Omega$ is sufficiently close to $\partial\Omega_0$ in the $W_p^{3-3/p}$ norm (to omit possible geometrical singularities). In case $\partial\Omega_0 \in C^3$ we may put $\partial\Omega = \partial\Omega_0$ and apply (1.7).

The above approach will guarantee that, if $\psi \in W_p^{3,1}$, then

$$\Big(\bigcup_{0\le t\le T} \partial\Omega_t \times \{t\}\Big) \in W_p^{3,1} \quad \text{as a submanifold in } \mathbf{R}^{n+2}. \tag{2.14}$$

To find the relation to (2.1) it is enough to insert the following quantities

$$F = (\Delta - \Delta^t)q, \quad G = (\Delta_{\partial\Omega_t} - \Lambda)\psi, \quad H = n^0(n^t - n^0)\partial_t\psi \tag{2.15}$$

into (2.9) to obtain system (2.1).

Because of the low regularity of the considered solutions we want to preserve some properties of equations $(1.1)_1$. For this reason we assume a form of the function F from system (2.9). By (2.15) we deduce that the right-hand side of $(2.9)_1$ should be in the following form

$$F = \sum_{k=1}^{m_0}\sum_{l=1}^{n+1} m_{kl}\partial_{x_l} q^k \tag{2.16}$$

for suitable functions m_{kl} and q^l. Also we require

$$\sum_{k=1}^{m_0}\sum_{l=1}^{n+1} n_l m_{kl} q^k = 0 \quad \text{on} \quad \partial\Omega, \tag{2.17}$$

where n_l is the l-th coordinate of the normal vector at the boundary. These conditions guarantees that for any $\pi \in C^\infty(\Omega)$

$$\int_\Omega \pi(\Delta - \Delta^t)q dx = -\int_\Omega (\nabla q\nabla\pi - \nabla^t q\nabla\pi)dx, \tag{2.18}$$

since the boundary term vanishes:

$$\int_{\partial\Omega} \pi\Big[\sum_{k=1}^{n+1} n_k \frac{\partial q}{\partial y_k} - \sum_{i,k,l=1}^{n+1} n_k \frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_i}\frac{\partial q}{\partial y_l}\Big]d\sigma = 0, \tag{2.19}$$

which we will see immediately. Let us look at the integrand in the last integral. Without loss of generality we assume that in the considered point the normal vector is $n = (0, ..., 0, 1)$, hence the integrand takes the form

$$\frac{\partial q}{\partial y_{n+1}} - \sum_{i,l=1}^{n+1} \frac{\partial y_{n+1}}{\partial x_i} \frac{\partial y_l}{\partial x_i} \frac{\partial q}{\partial y_l}. \tag{2.20}$$

However, condition (2.3) says that the normal derivative at the boundary is preserved, so

$$\sum_{i=1}^{n+1} \frac{\partial y_{n+1}}{\partial x_i} \frac{\partial y_l}{\partial x_i} = \delta_{n+1,l}. \tag{2.21}$$

From (2.21) quantity (2.20) vanishes, hence (2.19) is true. Thus condition (2.17) is a natural consequence of (2.3).

Now we state the main result for system (2.9) which will be a main tool in the proof of Theorem 1.1. In the following sections of the paper we prove the following result.

Theorem 2.1. *Let $p > p_* = \max\{3, n, (n+5)/2\}$ and $\partial\Omega \in W_p^{3-3/p}$. Let F be in the form* (2.16) *and fulfill condition* (2.17)*. If*

$$m_{kl} \in L_\infty(0, T; W_p^{2-3/p}(\Omega)), \quad q^l \in L_p(0, T; W_p^{1/p}(\Omega)),$$

$$G \in W_p^{1,0}(\partial\Omega \times (0, T)), \quad H \in L_p(\partial\Omega \times (0, T)), \ \ \psi_0 \in W_p^{3-3/p}(\partial\Omega),$$

then there exists a unique solution to (2.9) *such that*

$$\psi \in W_p^{3,1}(\partial\Omega \times (0, T)), \ \ p \in L_p(0, T; W_p^{1+1/p}(\Omega))$$

$$\text{and} \quad \frac{\partial p}{\partial n}\Big|_{\partial\Omega} \in L_p(\partial\Omega \times (0, T));$$

moreover,

$$\begin{aligned} &||\psi||_{W_p^{3,1}(\partial\Omega\times(0,T))} + ||p||_{L_p(0,T;W_p^{1+1/p}(\Omega))} + ||\frac{\partial p}{\partial n}|_{\partial\Omega}||_{L_p(\partial\Omega\times(0,T))} \\ &\leq C(T)\Big(\sum_{k=1}^{m_0}\Big(\sum_{l=1}^{n+1}||m_{kl}||_{L_\infty(0,T;W_p^{2-3/p}(\Omega))}||q^l||_{L_p(0,T;W_p^{1/p}(\Omega))}\Big) \\ &+||G||_{L_p(0,T;W_p^1(\partial\Omega))} + ||H||_{L_p(\partial\Omega\times(0,T))} + ||\psi_0||_{W_p^{3-3/p}(\partial\Omega)}\Big). \end{aligned} \tag{2.22}$$

Theorem 2.1 does not give regular solutions to system (2.9). We concentrate our attention at the boundary and treat the function p as an auxiliary quantity. To explain what we mean by a solution to $(2.9)_1$ we define the following weak formulation of this equation: the function q satisfies the following identity

$$\int_\Omega \nabla q \nabla \varphi dx = \int_{\partial\Omega} [-\partial_t \psi + H] \varphi d\sigma - \int_\Omega \sum_{k=1}^{m_0} \sum_{l=1}^{n+1} q^k \partial_{x_l}(m_{kl}\varphi) dx \qquad (2.23)$$

for $\varphi \in C^\infty(\Omega)$. We can replace $C^\infty(\Omega)$ by $W_q^{1-1/p}(\Omega)$ but it would make unnecessary complications. The regularity of the function p guarantees that the boundary equations are satisfied in the L_p sense.

From the regularity point of view the biggest obstacle is the existence of the trace of the normal derivative of the function p at the boundary. To obtain this information we need to add to our conditions in Theorem 2.1 the constraints (2.16) and (2.17).

This approach to equation $(2.9)_1$ will imply that we can look at system (2.9) as a local version of the following nonlocal equation

$$\begin{aligned} &\psi_t + \frac{\partial}{\partial n}\Big[\int_{\partial\Omega} \frac{\partial}{\partial n_y} \mathcal{G}[a\Lambda\psi + G] d\sigma_y + \int_\Omega \mathcal{G} F dy\Big] = H && \text{in} \quad \partial\Omega \times (0,T), \\ &\psi|_{t=0} = \psi_0 && \text{on} \quad \partial\Omega, \end{aligned} \qquad (2.24)$$

where $\mathcal{G}$ is the Green's function for the domain Ω. The system (2.24) is of parabolic type of the third order and the model case is given by (1.15). From this point of view Theorem 2.1 shows the existence of unique regular solutions to (2.24) with sharp regularity in the L_p framework.

The properties of the applied technique as well the linear character of the system enables us to improve the regularity of solutions such that $\psi \in W_p^{3m,m}$ for any $m \in \mathbf{N}$, assuming higher smoothness of the initial surface, and get classical solutions. We omit this issue and refer to paper [9].

The paper is organized as follows. In Section 3 we introduce the notation and basic results for the Besov-Slobodeckij spaces. Subsequently we investigate the Dirichlet problem for the Laplace operator with the sharp regularity. In Section 5 we solve a parabolic equation of the third order in the half space. Next we show existence of solutions to a system that is a simplification of (2.9). The proof of Theorem 2.1 is presented in Section 7 and Theorem 1.1 is proved in Section 8.

3. Notation and function spaces

Our investigation will be realized in the Besov space of Sobolev-Slobodeckij type. In this section we introduce some basic notation, definitions, and some auxiliary results which will be applied in our technique. For more details we refer to [2, 30].

First we introduce isotropic function spaces. Let $m \in \mathbf{R}_+$ and $p \geq 1$ and $d =$dim Q; then

$$||u||_{W_p^m(Q)} = \begin{cases} ||u||_{H_p^m(Q)} & \text{if} \quad m \in \mathbf{N} \\ ||u||_{B_{pp}^m} & \text{if} \quad m \in \mathbf{R}_+ \setminus \mathbf{N}, \end{cases} \tag{3.1}$$

where for $m \in \mathbf{N}$

$$||u||^p_{H_p^m(Q)} = \int_Q |u|^p dx + \sum_{|\alpha|=m} \int_Q |\partial^\alpha u|^p dx, \tag{3.2}$$

and for $m \in \mathbf{R}_+$

$$||u||^p_{B_{pp}^m(Q)} = ||u||^p_{H_p^{[m]_-}(Q)} + \sum_{|\alpha|=[m]_-} \int_Q dx \int_Q dx' \frac{|\partial^\alpha u(x) - \partial^\alpha u(x')|^p}{|x-x'|^{d+p(m-[m])}}, \tag{3.3}$$

where $[t]_-$ denotes the biggest integer number less than t.

If Q is the whole space we can define these spaces by the Fourier transform. Let

$$\hat{u}(\xi) = \mathcal{F}_x[u] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} u(x) dx \tag{3.4}$$

be the Fourier transform of the function u.

For any $m \in \mathbf{R}$ we define a generalization of the norm (3.2)

$$||u||_{H_p^m(\mathbf{R}^n)} = ||\mathcal{F}_x^{-1}[(1+|\xi|)^m \mathcal{F}[u]]||_{L_p(\mathbf{R}^n)}. \tag{3.5}$$

In order to define the Besov spaces B^s_{pp} we need to introduce the Paley-Littlewood decomposition. Let us assume we are given

$$\phi_k \in C_0^\infty(\mathbf{R}^n), \quad \phi_k \geq 0, \quad \sum_{k=0}^\infty \phi_k = 1; \tag{3.6}$$

moreover,

$$\text{supp } \phi_k \subset \{2^{k-n} \leq |\xi| \leq 2^{k+n}\} \tag{3.7}$$

and for any function u we define the following sequence

$$u_k = \mathcal{F}_x^{-1}[\phi_k \hat{u}]. \tag{3.8}$$

Then we introduce the following norm

$$||u||_{B^s_{pp}(\mathbf{R}^n)} = \Big(\sum_{k=0}^\infty 2^{ksp} ||u_k||^p_{L_p(\mathbf{R}^n)} \Big)^{1/p}, \tag{3.9}$$

which is equivalent to (3.3).

For $m \in \mathbf{N}$ and $p \geq 2$ we have the following imbedding $W_p^m(Q) \subset B_{pp}^m(Q)$ with the bound

$$||u||_{B_{pp}^m} \leq c||u||_{W_p^m}. \tag{3.10}$$

For the evolutionary problem we apply an isotropic space to underline the special character of the time direction [2]. Let

$$||u||_{W_p^{m,n}(Q\times(0,T))} = ||u||_{L_p(Q\times(0,T))}+$$

$$\Big(\int_0^T < u(\cdot,t) >_{W_p^m(\Omega)}^p dt\Big)^{1/p} + \Big(\int_Q < u(x,\cdot) >_{W_p^n(0,T)}^p\Big)^{1/p}, \tag{3.11}$$

where $< \cdot >_B$ denotes the main seminorm of the function space B.

Next, we recall basic results for Besov spaces.

Theorem 3.1 (the Marcinkiewicz-Mikhlin theorem, see [23]). *Suppose that the function $\Phi : \mathbf{R}^m \setminus \{0\} \to \mathbf{C}$ is smooth enough and there exists $M > 0$ such that for every point $x \in \mathbf{R}^m$ we have*

$$|x_{j_1}x_{j_2}...x_{j_k}|\Big|\frac{\partial^k\Phi}{\partial x_{j_1}\partial x_{j_2}...\partial x_{j_k}}\Big| \leq M,\ 0 \leq k \leq m,\ 1 \leq j_1 < j_2 < ... < j_k \leq m.$$

Then the operator

$$Pg(x) = (2\pi)^{-m}\int_{R^m} dy e^{ixy}\Phi(y)\int_{R^m} e^{-iyz}g(z)dz$$

is bounded in $W_p^s(\mathbf{R}^m)$ and

$$||Pg||_{W_p^s(R^m)} \leq A_{p,m}M||g||_{W_p^s(R^m)}. \tag{3.12}$$

Proposition 3.2 (imbedding theorem, see [2]). *Let $u \in W_r^{m,n}(\Omega_T)$, $m, n \in \mathbf{R}_+$, then if $\kappa = \sum_{i=1}^3(\alpha_i + \frac{1}{r} - \frac{1}{q})\frac{1}{m} + (\beta + \frac{1}{r} - \frac{1}{q})\frac{1}{n} < 1$, the following estimate holds*

$$||D_t^\beta D_x^\alpha u||_{L_q(\Omega_T)} \leq \varepsilon||u||_{W_r^{m,n}(\Omega_T)} + c(\varepsilon)||u||_{L_p(\Omega_T)}, \tag{3.13}$$

where $q \geq r \geq p$, $\varepsilon \in (0,1)$ and $c(\varepsilon) \sim \varepsilon^{-a(\alpha,\beta,m,n,p,q,r)}$.

Proposition 3.3 (trace theorem, see [2]).

(i) Let $u \in W_r^m(Q)$ and $m > 1/r$, then

$$u|_{\partial Q} \in W_r^{m-1/r}(\partial Q) \quad \text{and} \quad ||u|_{\partial Q}||_{W_r^{m-1/r}(\partial Q)} \leq c||u||_{W_r^m(Q)}; \tag{3.14}$$

(ii) if $u \in W_r^{3m,m}(Q \times (0,T))$ and $3m > 3/r$, then

$$u|_{t=0} \in W_r^{3m-3/r}(Q) \quad \text{and} \quad ||u|_{t=0}||_{W_r^{3m-3/r}(Q)} \leq c||u||_{W_r^{3m,m}(Q\times(0,T))}, \tag{3.15}$$

moreover,

$$\sup_{0\le t\le T} ||u(\cdot,t)||_{W_r^{3m-3/r}(Q)} \le c\left(< u >_{W_r^{3m,m}(Q\times(0,T))} + ||u(\cdot,0)||_{W_r^{3m-3/r}(Q)}\right), \tag{3.16}$$

where c *in* (3.16) *is independent of* T.

Our method will require a result based on the interpolation theory of function spaces of Besov type. From [30] we introduce the following interpolation relations for the Besov spaces

$$(B^{s_1}_{pp}, B^{s_2}_{pp})_{\theta,p} = B^s_{pp}, \tag{3.17}$$

where $s = \theta s_1 + (1-\theta)s_2$ for $0 < \theta < 1$. Moreover, the following result holds.

Proposition 3.4. (Marcinkiewicz interpolation theorem, see [30]). *Let* $1 < p < \infty$, $0 < s_1 < s_2$, $0 < q_1 < q_2$, *and let* T *be a linear operator defined on* $B^{s_1}_{pp} + B^{s_2}_{pp}$ *into* $B^{q_1}_{pp} + B^{q_2}_{pp}$. *Suppose* T *is bounded on* $B^{s_1}_{pp}$ *and* $B^{s_2}_{pp}$; *i.e.,*

$$\begin{aligned} ||Tf||_{B^{q_1}_{pp}} &\le ||T||_{L(B^{s_1}_{pp};B^{q_1}_{pp})} ||f||_{B^{s_1}_{pp}}, \\ ||Tf||_{B^{q_2}_{pp}} &\le ||T||_{L(B^{s_2}_{pp};B^{q_2}_{pp})} ||f||_{B^{s_2}_{pp}}. \end{aligned} \tag{3.18}$$

Then $T : B^s_{pp} \to B^q_{pp}$ *is bounded, where* $s = \theta s_1 + (1-\theta)s_2$ *and* $q = \theta q_1 + (1-\theta)q_2$ *and*

$$||Tf||_{B^q_{pp}} \le ||T||^{\theta}_{L(B^{s_1}_{pp};B^{q_1}_{pp})} ||T||^{1-\theta}_{L(B^{s_2}_{pp};B^{q_2}_{pp})} ||f||_{B^s_{pp}}. \tag{3.19}$$

4. The Laplace operator

Important tools in our technique will be results for the Dirichlet problem for the Laplace operator. We will need estimates of solutions to problems in bounded domains as well as for model cases in the half and whole space.

We consider

$$\begin{aligned} \Delta p = f \quad &\text{in} \quad \Omega, \\ p = g \quad &\text{on} \quad \partial\Omega. \end{aligned} \tag{4.1}$$

Additionally, we assume the following form of the right-hand side of $(4.1)_1$

$$f = \sum_{k=1}^{m_0} \sum_{l=1}^{n+1} m_{kl} \partial_{x_l} q^k, \tag{4.2}$$

and

$$\sum_{k=1}^{m_0} \sum_{l=1}^{n+1} n_l m_{kl} q^k = 0 \quad \text{on} \quad \partial\Omega \tag{4.3}$$

as the trace distribution, where n_l is the l-th coordinate of the normal vector to $\partial\Omega$ ($n = (n_1, ..., n_{n+1})$) and $m_0 \in \mathbf{N} \setminus \{0\}$. Conditions (4.2) and (4.3) are a consequence of our assumptions of Theorem 2.1, (2.15)-(2.16).

Theorem 4.1. *Let $p > p_*$, $\partial\Omega \in W_p^{3-3/p}$, $g \in W_p^1(\partial\Omega)$, $q^l \in W_p^{1/p}(\Omega)$, and $m_{kl} \in W_p^{2-3/p}(\Omega)$. Then there exists a unique solution to problem (4.1) such that*

$$p \in W_p^{1+1/p}(\Omega) \quad \textit{and} \quad \frac{\partial p}{\partial n}\Big|_{\partial\Omega} \in L_p(\partial\Omega);$$

moreover

$$||p||_{W_p^{1+1/p}(\Omega)} + ||\frac{\partial p}{\partial n}||_{L_p(\partial\Omega)}$$
$$\leq c\Big(\sum_{k=1}^{m_0}(\sum_{l=1}^{n+1}||m_{kl}||_{W_p^{2-3/p}(\Omega)}||q^l||_{W_p^{1/p}(\Omega)}) + ||g||_{W_p^1(\partial\Omega)}\Big). \tag{4.4}$$

The above result plays a crucial role in the investigation of the linear system (2.9). It concerns a limiting case for the boundary-value problem from the regularity point of view. In general if $g \in W_p^1(\partial\Omega)$ from (4.1) we can not control the trace of the normal derivative of the solution at the boundary, because for a general function from $W_p^{1+1/p}(\Omega)$ the function $\frac{\partial p}{\partial n}|_{\partial\Omega}$ does not exist. However, adding conditions (4.2) and (4.3) onto the form of f, we are able to obtain this information.

Applying other analytical techniques [24, 29] we are able to obtain similar results for the Laplace operator. Theorem 4.1 is an important tool in our method and some parts of its proof will be applied directly in the next sections, hence we present here some details of the proof.

The first step of the proof of Theorem 4.1 is to solve the problem in the whole space

$$\begin{array}{c} \Delta p = f \quad \text{in} \quad \mathbf{R}^{n+1}, \\ p \to 0 \quad \text{as} \quad |x| \to +\infty, \end{array} \tag{4.5}$$

where $f = \sum_{k=1}^{m_0} \sum_{l=1}^{n+1} m_{kl} \partial_{x_l} q^k$.

Lemma 4.1. *Let $q^l \in W_p^{1/p}(\mathbf{R}^{n+1})$ and $m_{kl} \in W_p^{2-3/p}(\mathbf{R}^{n+1})$; additionally we assume that each function m_{kl} has compact supports; i.e.,*

$$supp\, m_{kl} \subset B(0, R), \tag{4.6}$$

where $R > 0$ is a fixed number. Then

$$p \in W_p^{1+1/p}(\mathbf{R}^{n+1}) \quad \textit{and} \quad p|_{x_{n+1}=0} \in W_p^1(\mathbf{R}^n). \tag{4.7}$$

Proof. Here we prove Theorem 4.1 only for $n + 1 \geq 3$, omitting the case $n + 1 = 2$. It is connected with the behavior of the fundamental solution for the Laplace operator in the whole space. In these cases the compact support of the right-hand side of (4.5) enables us to show boundedness of the $L_p(\mathbf{R}^{n+1})$ the norm of solutions. For the case $n+1=2$, there is a need to modify slightly the method and consider norms of solutions over a fixed ball in $\mathbf{R}^{n+1}$. But it would not change the statement of Theorem 4.1.

First we split function f as follows

$$f = f_1 + f_2, \tag{4.8}$$

where

$$f_1 = \sum_{k=1}^{m_0}\sum_{l=1}^{n+1} \partial_{x_l}(m_{kl}q^l), \qquad f_2 = -\sum_{k=1}^{m_0}\sum_{l=1}^{n+1} q^l \partial_{x_l} m_{kl}. \tag{4.9}$$

We will investigate two systems connected with this decomposition

$$\begin{gathered} \Delta p_1 = f_1, \qquad \Delta p_2 = f_2 \quad \text{in} \quad \mathbf{R}^{n+1}, \\ p_1,\, p_2 \to 0 \quad \text{as} \quad |x| \to \infty. \end{gathered} \tag{4.10}$$

Solutions to (4.10) define a solution to system (4.5) in the following way

$$p = p_1 + p_2. \tag{4.11}$$

Let us solve the first system from (4.10). To show regularity of p_1 we need to prove that

$$m_{kl}q^l \in W_p^{1/p}(\mathbf{R}^{n+1}). \tag{4.12}$$

This imbedding follows from two facts (see Proposition 3.2). The first one is

$$W_p^{2-3/p}(\mathbf{R}^{n+1}) \subset L_\infty(\mathbf{R}^{n+1}), \tag{4.13}$$

if $p > (n+4)/2$. And the second one is

$$W_p^{2-3/p}(\mathbf{R}^{n+1}) \subset W_\infty^{1/p}(\mathbf{R}^{n+1}), \tag{4.14}$$

if $p > (n+5)/2$. Recalling the definition of the Slobodeckij space (3.2), and that $p > p_*$, we obtain (4.12).

These considerations lead us to the estimate for p_1. Applying the Fourier transform we obtain the explicit form of the solution

$$p_1 = \mathcal{F}^{-1}\Big[\sum_{k,l=1}^{n+1} \frac{i\xi_k}{-|\xi|^2}\mathcal{F}[m_{kl}q^l]\Big]. \tag{4.15}$$

The Marcinkiewicz theorem, Proposition 3.1, and the definition of $W_p^{1/p}$ given by the Palay-Littlewood decomposition give the desired bound

$$p_1 \in W_p^{1+1/p}(\mathbf{R}^{n+1}). \tag{4.16}$$

By the trace theorem Proposition 3.3 (i) we obtain from (4.16) also that

$$p_1|_{x_{n+1}} \in W^1_p(\mathbf{R}^n). \tag{4.17}$$

Next, we study p_2. Since f_2 is given by (4.9) we obtain

$$(\partial_{x_k} m_{kl})q^l \in L_{p/2}(\mathbf{R}^{n+1}) \tag{4.18}$$

which follows from trivial imbeddings

$$\partial_{x_k} m_{kl} \in W^{1-3/p}_p(\mathbf{R}^{n+1}) \subset L_p(\mathbf{R}^{n+1}) \quad \text{and} \quad q^l \in W^{1/p}_p(\mathbf{R})^{n+1} \subset L_p(\mathbf{R})^{n+1}. \tag{4.19}$$

The same procedure as one used for p_1 will yield

$$p_2 = \mathcal{F}^{-1}\Big[\frac{1}{-|\xi|^2}\mathcal{F}[f_2]\Big]. \tag{4.20}$$

Thus, (4.18) guarantees that

$$p_2 \in W^2_{p/2}(\mathbf{R}^{n+1}). \tag{4.21}$$

Again, applying the imbedding theorem we conclude that

$$W^2_{p/2}(\mathbf{R})^{n+1} \subset W^{1+1/p}_p(\mathbf{R}^{n+1}), \tag{4.22}$$

if $p > n$. Analogously to p_1 we obtain $p_2|_{x_{n+1}=0} \in W^1_p(\mathbf{R}^n)$. By (4.16), (4.17), (4.21), and (4.22) the proof of Lemma 4.1 is finished. □

Next, we study the following problem in the half space

$$\begin{array}{lll} \Delta p = f & \text{in} & \mathbf{R}^{n+1}_+, \\ p_{,x_{n+1}} = 0 & \text{on} & \mathbf{R}^n, \\ p \to 0 & \text{as} & |x| \to \infty, \end{array} \tag{4.23}$$

where $f = \sum_{m=1}^{m_0} \sum_{l=1}^{n+1} m_{kl}\partial_{x_l} q^k$ and supp $m_{kl} \in B(0,R) \cap \mathbf{R}^{n+1}_+$; moreover condition (4.3) takes the following form

$$\sum_{k=1}^{m_0} m_{k,n+1} q^k\Big|_{x_{n+1}=0} = 0 \quad \text{on} \ \mathbf{R}^n. \tag{4.24}$$

The compatibility condition to (4.23) necessary for solvability ($\int_{\mathbf{R}^{n+1}_+} f dx = 0$) is fulfilled because of (4.24).

We want to extend problem (4.23) onto the whole space $\mathbf{R}^{n+1}$ by a symmetry which preserves the boundary condition $(4.23)_2$.

For this purpose we define $\tilde{f} : \mathbf{R}^{n+2} \to \mathbf{R}$ such that

$$\tilde{f} = \tilde{f}_1 + \tilde{f}_2, \tag{4.25}$$

where $\tilde{f}_1 = \sum_{k=1}^{m_0} \sum_{l}^{n+1} \partial_{x_l}(\tilde{m}_{kl}\tilde{q}^k)$ and $\tilde{f}_2 = -\sum_{k=1}^{m_0} \sum_{l=1}^{n+1} \tilde{q}^k \partial_{x_l}\tilde{m}_{kl}$ with

$$q^l(x,t) = \begin{cases} q^l(x', x_{n+1}, t) & \text{for} \quad x_{n+1} \geq 0 \\ q^l(x', -x_{n+1}, t) & \text{for} \quad x_{n+1} < 0 \end{cases} \tag{4.26}$$

for $l = 1, ..., n+1$; and

$$m_{kl}(x,t) = \begin{cases} m_{kl}(x', x_{n+1}, t) & \text{for} \quad x_{n+1} \geq 0 \\ m_{kl}(x', -x_{n+1}, t) & \text{for} \quad x_{n+1} < 0 \end{cases} \tag{4.27}$$

for $k = 1, ..., m_0$, $l = 1, ..., n$; and

$$m_{kn+1}(x,t) = \begin{cases} m_{kn+1}(x', x_{n+1}, t) & \text{for} \quad x_{n+1} \geq 0 \\ -m_{kn+1}(x', -x_{n+1}, t) & \text{for} \quad x_{n+1} < 0 \end{cases} \tag{4.28}$$

for $k = 1, ..., m_0$.

Lemma 4.2. *Let $\tilde{f}$ be given by* (4.25), *where $m_{kl} \in W_p^{2-3/p}(\mathbf{R}_+^{n+1})$ and $q^l \in W_p^{1/p}(\mathbf{R}_+^{n+1})$; then $\tilde{f}$ fulfills the conditions of Lemma* 4.1.

Proof. One point which is unclear is the regularity with respect to x_{n+1}. However, condition (4.24) guarantees that the symmetry of (4.28) preserves $W_p^{2-3/p}$ regularity for $m_{n+1,l}$. Then the considerations from Lemma 4.1 end the proof of Lemma 4.2.

Lemma 4.3. *Let us suppose that the assumptions of Lemma* 4.2 *are fulfilled; then there exists a weak solution to* (4.23) *such that*

$$p \in W_p^{1+1/p}(\mathbf{R}_+^{n+1}) \quad \textit{and} \quad p|_{x_{n+1}=0} \in W_p^1(\mathbf{R}^n). \tag{4.29}$$

Proof. To show this result we use the method of symmetry. We consider a modification of problem (4.5); namely

$$\begin{array}{c} \Delta\tilde{p} = \tilde{f} \quad \text{in} \quad \mathbf{R}^{n+1}, \\ \tilde{p} \to 0 \quad \text{as} \quad |x| \to \infty, \end{array} \tag{4.30}$$

where $\tilde{f}$ is given by (4.25). The symmetry implies that $\tilde{p}$ fulfills

$$\tilde{p}(x', x_{n+1}) = \tilde{p}(x', -x_{n+1}). \tag{4.31}$$

It follows that

$$\tilde{p}|_{\{x_{n+1}>0\}} = p \qquad \text{and} \qquad \frac{\partial\tilde{p}}{\partial x_{n+1}} = 0 \quad \text{on} \quad x_{n+1} = 0. \tag{4.32}$$

Applying the Marcinkiewicz theorem (Proposition 3.1) and Lemma 4.1 we obtain (recalling $1/p + 1/q = 1$)

$$||\nabla\tilde{p}||_{W_p^{1+1/p}} \leq c||f||_{(W_q^{1-1/p})^*}. \tag{4.33}$$

The compactness of the support of function f enables us to find bounds on lower norms of $\tilde{p}$. In particular, by the trace theorem (Proposition 3.3) we get

$$\tilde{p}|_{x_{n+1}} \in W^1_p(\mathbf{R}^n). \tag{4.34}$$

The proof of Lemma 4.3 is finished. □

Lemmas 4.1 and 4.3 deliver the following result in the half space

$$\begin{array}{lll} \Delta p = f & \text{in} & \mathbf{R}^{n+1}, \\ p = g & \text{on} & \mathbf{R}^n, \\ p \to 0 & \text{as} & |x| \to \infty. \end{array} \tag{4.35}$$

Lemma 4.4. *Let f be given as in* (4.23) *and fulfill* (4.24). *If*

$$m_{kl} \in W^{2-3/p}_p(\mathbf{R}^{n+1}_+), \quad q^l \in W^{1/p}_p(\mathbf{R}^{n+1}_+),$$

then there exists a unique solution to (4.35) *such that*

$$p \in W^{1+1/p}_p(\mathbf{R}^{n+1}_+) \quad \textit{and} \quad \left.\frac{\partial p}{\partial x_{n+1}}\right|_{x_{n+1}=0} \in L_p(\mathbf{R}^n),$$

and the following estimate is valid

$$\begin{aligned} &< p >_{W^{1+1/p}_p(\mathbf{R}^{n+1}_+)} + || \frac{\partial p}{\partial x_{n+1}}|_{x_{n+1}=0}||_{L_p(\mathbf{R}^n)} \\ &\le c \sum_{k=1}^{m_0} \sum_{l=1}^{n+1} ||m_{kl}\partial_{x_l} q^k||_{(W^{1-1/p}_q(\mathbf{R}^{n+1}_+))^*}. \end{aligned} \tag{4.36}$$

Proof. The conditions on f give us a solution for problem (4.23), then we are able to reduce system (4.35) into the following one

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \mathbf{R}^{n+1}, \\ p = g & \text{on} & \mathbf{R}^n, \\ p \to 0 & \text{as} & |x| \to \infty, \end{array} \tag{4.37}$$

where

$$p_{new} = p_{old} - p_L, \quad g_{new} = g_{old} - p_L|_{x_{n+1}=0}, \tag{4.38}$$

where p_{old} is the solution to (4.35) and p_L to (4.23) given by Lemma 4.3. The estimates hold by (4.29).

Applying the Fourier transform to system (4.37) with respect to tangential directions ($\hat{p} = \mathcal{F}^{-1}_x[p]$) we obtain

$$\begin{array}{lll} (\partial^2_{x_{n+1}} - |\xi|^2)\hat{p} = 0 & \text{in} & \mathbf{R}^n \times (0,\infty), \\ \hat{p} = \hat{g} & \text{on} & \mathbf{R}^n, \\ \hat{p} \to 0 & \text{as} & |x| \to \infty. \end{array} \tag{4.39}$$

Solving $(4.39)_1$ and applying $(4.39)_3$ and $(4.39)_2$ we obtain the explicit formula

$$\hat{p}(\xi, x_{n+1}) = \hat{g}(\xi) e^{-|\xi| x_{n+1}}. \tag{4.40}$$

Let us show $p \in W_p^{1+1/p}(\mathbf{R}_+^{n+1})$. We apply the well-known classical result [30]; see also [25].

Proposition 4.1. *Let $g \in W_p^{k-1/p}(\mathbf{R}^n)$ for $k = 1, 2$ and $p > 1$; then the solution to* (4.37) *exists and fulfills the following bound*

$$< p >_{W_p^k(\mathbf{R}_+^{n+1})} \leq c||g||_{W_p^{k-1/p}(\mathbf{R}^n)} \tag{4.41}$$

for $k = 1, 2$.

The application of Proposition 4.1 is based on the Marcinkiewicz interpolation theorem for $k = 1$ and $k = 2$ - see Proposition 3.4. From (4.41) we obtain the following bound

$$< p >_{(W_p^1(\mathbf{R}_+^{n+1}), W_p^2(\mathbf{R}_+^{n+1}))_{\theta,p}} \leq c||g||_{(W_p^{1-1/p}(\mathbf{R}^n), W_p^{2-1/p}(\mathbf{R}^n))_{\theta,p}}. \tag{4.42}$$

Taking $\theta = 1 - 1/p$ and recalling Proposition 3.4 and (3.17) with bound (3.10) we obtain the following estimate on the solution to (4.37)

$$< p >_{W_p^{1+1/p}(\mathbf{R}_+^{n+1})} \leq c||g||_{B_{p,p}^1(\mathbf{R}^n)} \leq c||g||_{W_p^1(\mathbf{R}^n)}. \tag{4.43}$$

To end the proof we find the bound on the normal derivative at the boundary. By (4.40) we get

$$\left.\frac{\partial p}{\partial x_{n+1}}\right|_{x_{n+1}=0} = \mathcal{F}^{-1}[-|\xi|\hat{g}]. \tag{4.44}$$

By properties of the space $W_p^1(\mathbf{R}^n) = H_p^1(\mathbf{R}^n)$ we conclude that

$$\left.\frac{\partial p}{\partial x_{n+1}}\right|_{x_{n+1}=0} \in L_p(\mathbf{R}^n). \tag{4.45}$$

To conclude the second part of (4.37) we recall that $\frac{\partial p_L}{\partial x_{n+1}}|_{x_{n+1}=0} = 0$. Lemma 4.4 is proved.

Proof of Theorem 4.1. First, we show existence of weak solutions to problem (4.1). Since the function $g \in W_p^1(\partial\Omega)$, in particular we have a trivial imbedding into $W_2^{1/2}(\partial\Omega)$. It follows that g is a trace of a function belonging to $W_2^1(\Omega)$. A similar property holds for the function f which can be treated as an element of $(W_2^1(\Omega))^*$. The elementary theory of the calculus of variations [12] gives the following result.

Proposition 4.2 (Existence of weak solutions to (1)). *There exists a function $p \in W_2^1(\Omega)$ such that $p|_{\partial\Omega} = g$ and*

$$\int_\Omega \nabla p \nabla \phi dx = -\int_\Omega f \phi dx \tag{4.46}$$

for any function $\phi \in C_0^\infty(\Omega)$. Moreover,

$$||p||_{W_2^1(\Omega)} \leq c(||f||_{W_2^{-1}(\Omega)} + ||g||_{W_2^{1/2}(\partial\Omega)}). \tag{4.47}$$

Next we improve the regularity of solutions to (4.46). Recalling the coverings from Section 2 we study two possibilities.

If $k \in \mathcal{M}$, then we consider the following problem

$$\Delta(\zeta^k p) = \zeta^k f + \nabla\zeta^k \nabla p + \operatorname{div}(p\nabla\zeta^k) \quad \text{in} \quad \mathbf{R}^{n+1}. \tag{4.48}$$

Then applying Lemma 4.1 and the interpolation theorem we get the following bound

$$\begin{aligned} ||\zeta^k p||_{W_p^{1+1/p}(\mathbf{R}^{n+1})} &\leq \epsilon ||p||_{W_p^{1+1/p}(supp\,\zeta^k)} + c(\epsilon)||p||_{W_2^1(supp\,\zeta^k)} \\ &+ ||\zeta^k f||_{(W_q^{1-1/p}(\mathbf{R}^{n+1}))^*}. \end{aligned} \tag{4.49}$$

If $k \in \mathcal{N}$, then we obtain the following problem in the half space

$$\begin{array}{ll} \Delta Z_k^*(\zeta^k p) = (\Delta - \Delta_x) Z_k^*(\zeta^k p) + Z_k^*[\zeta^k f + \nabla\zeta^k \nabla p + \operatorname{div}(p\nabla\zeta^k)] & \text{in } \mathbf{R}_+^{n+1}, \\ Z_k^*[\zeta^k p] = Z_k^*[\zeta^k g] & \text{on } \mathbf{R}^n. \end{array} \tag{4.50}$$

Lemma 4.4 gives the following bound for solutions to (4.50)

$$\begin{aligned} ||Z_k^*(\zeta^k p)||_{W_p^{1+1/p}(\mathbf{R}_+^{n+1})} &+ \left\| \frac{\partial}{\partial y_{n+1}} Z_k^*(\zeta^k p)|_{y_{n+1}=0} \right\|_{L_p(\mathbf{R}^n)} \\ &\leq c \Bigg(||(\Delta - \Delta_x) Z_k^*(\zeta^k p)||_{(W_q^{1-1/p}(\mathbf{R}_+^{n+1}))^*} + \\ ||Z_k^*[\zeta^k f + \nabla\zeta^k \nabla p + \operatorname{div}(p\nabla\zeta^k)]||&_{(W_q^{1-1/p}(\mathbf{R}_+^{n+1}))^*} + ||\zeta^k g||_{W_p^1(\partial\Omega)} \Bigg). \end{aligned} \tag{4.51}$$

The above estimate is valid if we are able to control the structure of the right-hand side of $(4.50)_1$. Note that

$$\begin{aligned} (\Delta - \Delta_x) Z_k^*(\zeta^k p) &= \sum_{i=1}^{n+1} \left(\partial_{y_i}^2 - (\sum_{j=1}^{n+1} \frac{\partial y_j}{\partial x_i} \partial_{y_i})^2 \right) Z_k^*(\zeta^k p) \\ &= \sum_{i,j=1}^{n+1} \left(\delta_{ij} - \frac{\partial y_j}{\partial x_i} \right) \partial_{y_j} \Big[\sum_{m=1}^{n+1} \frac{\partial y_m}{\partial x_i} \partial_{y_m} \Big] Z_k^*(\zeta^k p) \end{aligned}$$

$$+\sum_{i=1}^{n+1}\partial_{y_i}\Big(\sum_{m=1}^{n+1}(\delta_{im}-\frac{\partial y_m}{\partial x_i}\Big)\partial_{y_m})Z_k^*(\zeta^k p)$$

has a form

$$=\sum_{m=1}^{m_0}\sum_{l=1}^{n+1}m_{kl}\partial_{y_l}q^k, \tag{4.52}$$

where $\frac{\partial y_i}{\partial x_j}$ denote elements of the Jacobi matrix of the transformation into the local coordinates system.

By the form of (4.52) we deduce that the right-hand side of $(4.50)_1$ satisfies condition (4.2). To get condition (4.3) it is enough to recall condition (2.17) from Section 2. Thus the application of Lemma 4.4 was correct.

The essential part of the estimations are bounds for $\frac{\partial y_i}{\partial x_j}$. By properties of the map we know that

$$|\frac{\partial y_i}{\partial x_j}-\delta_{ij}|\leq c\lambda \quad \text{on} \quad \text{supp } \zeta^k, \tag{4.53}$$

where λ is prescribed by the covering Ω^k; see Section 2.

Since imbeddings (4.13), (4.14), and (4.22) are compact we are able to apply the interpolation estimates - see Proposition 3.2 - and obtain for representation (4.52) the following local bound

$$||m_{kl}\partial_{y_l}q^k||_{W_q^{1-1/p}(supp\,\zeta^k))^*}\leq(\epsilon+c(\epsilon)\lambda)||m_{kl}||_{W_p^{2-3/p}(supp\,\zeta^k)}||q^l||_{W_p^{1/p}(supp\,\zeta^k)} \tag{4.54}$$

for suitable small $\epsilon>0$. Since $\partial\Omega\in W_p^{3-3/p}$ implies that $\frac{\partial y_i}{\partial x_j}\in W_p^{2-3/p}$, bound (4.51) takes the following form

$$||Z_k^*(\zeta^k p)||_{W_p^{1+1/p}(supp\,\zeta^k)}\leq c\Big((\epsilon+c(\epsilon)\lambda)||p||_{W_p^{1+1/p}(supp\,\zeta^k)}$$

$$+||f||_{(W_q^{1-1/p}(supp\,\zeta^k))^*}+||g||_{W_p^1(\partial\Omega\cap supp\,\zeta^k)}+c(\epsilon,\lambda)||p||_{W_2^1(supp\,\zeta^k)}\Big). \tag{4.55}$$

The cover number for the collection $\{\Omega^k\}$ is finite, so choosing sufficiently small ϵ and λ, recalling (4.47), we obtain that solutions given by Proposition 4.2 fulfill

$$||p||_{W_p^{1+1/p}(\Omega)}+||\frac{\partial p}{\partial n}||_{L_p(\partial\Omega)}\leq c(||f||_{(W_q^{1-1/p}(\Omega))^*}+||g||_{W_p^1(\partial\Omega)}). \tag{4.56}$$

By (4.56) and (4.54) we obtain (4.4). The constraint in (4.53) depends on covers for domains Ω. Theorem 4.1 is proved.

5. Half space

The goal of this part is to analyze the following linear system in the half space

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \mathbf{R}^{n+1}_+ \times (0,\infty), \\ p = a\Delta'\phi + g & \text{on} & \mathbf{R}^n \times (0,\infty), \\ p_{,x_{n+1}} = -\partial_t \phi + h & \text{on} & \mathbf{R}^n \times (0,\infty), \\ \phi|_{t=0} = \phi_0 & \text{on} & \mathbf{R}^n, \\ p \to 0 \quad \text{as} \quad |x| \to \infty, & & \end{array} \tag{5.1}$$

where Δ' is given by (2.11).

System (5.1) is a certain form of linearization of (2.9). In this section we prove the following result.

Theorem 5.1. *Let $h, \nabla' g \in L_p(\mathbf{R}^n \times (0,\infty))$; then there exists a unique solution to system* (5.1) *such that*

$$\phi \in W^{3,1}_{p(loc)}(\mathbf{R}^n \times (0,\infty)) \quad \textit{and} \quad p \in L_{p(loc)}(0,\infty; W^{1+1/p}_{p(loc)}(\mathbf{R}^{n+1}_+))$$

and

$$\begin{aligned} &< \phi >_{W^{3,1}_p(\mathbf{R}^n \times (0,T))} + < p >_{L_p(0,\infty; W^{1+1/p}_p(\mathbf{R}^{n+1}_+))} \\ &\le c \left(||f, \nabla' g||_{L_p(\mathbf{R}^n \times (0,\infty))} + < \phi_0 >_{W^{3-3/p}_p(\mathbf{R}^n)} \right). \end{aligned} \tag{5.2}$$

The main information about the system will be obtained from the following parabolic problem

$$\begin{array}{lll} \partial_t \phi + a\mathcal{F}^{-1}_x[|\xi|^3 \hat{\phi}] = m & \text{in} & \mathbf{R}^n \times (0,+\infty), \\ \phi|_{t=0} = \phi_0 & \text{on} & \mathbf{R}^n. \end{array} \tag{5.3}$$

The analysis of system (5.1) will imply that it is a local version of nonlocal system (5.3), hence Theorem 5.1 will be a consequence of the following lemma.

Lemma 5.1. *Let $m \in L_p(\mathbf{R}^n \times (0,\infty))$ and $\phi_0 \in W^{3-3/p}_p(\mathbf{R}^n)$; then there exists a unique solution to* (5.3) *such that $\phi \in W^{3,1}_{p(loc)}(\mathbf{R}^n \times (0,\infty))$ and*

$$||u_t, \nabla^3 u||_{L_p(\mathbf{R}^n \times (0,\infty))} \le c(||m||_{L_p(\mathbf{R}^n \times (0,\infty))} + ||\phi_0||_{W^{3-3/p}_p(\mathbf{R}^n)}). \tag{5.4}$$

Proof. To solve the system we consider the following auxiliary problem

$$\begin{array}{lll} \partial_t \phi + a\mathcal{F}^{-1}_x[|\xi|^3 \hat{\phi}] = m & \text{in} & \mathbf{R}^n \times (0,+\infty), \\ \phi|_{t=0} = 0 & \text{on} & \mathbf{R}^n. \end{array} \tag{5.5}$$

Regularity of m and the uniqueness of the problem imply that we are able to extend (5.5) onto $t<0$ by zero. Then we obtain the following system in the whole space

$$\partial_t \tilde{\phi} + a\mathcal{F}_x^{-1}[|\xi|^3 \mathcal{F}_x[\tilde{\phi}]] = \tilde{m} \qquad \text{in} \quad \mathbf{R}^{n+1}. \tag{5.6}$$

After the Fourier transform with respect to time ($t \leftrightarrow \tau$) equation (5.6) reads

$$(i\tau + a|\xi|^3)\mathcal{F}_{x,t}[\tilde{\phi}] = \mathcal{F}_{x,t}[\tilde{m}] \qquad \text{in} \quad \mathbf{R}^{n+1}. \tag{5.7}$$

Hence from (5.7) we get

$$\mathcal{F}_{x,t}[\tilde{\phi}] = \frac{\mathcal{F}_{x,t}[\tilde{m}]}{i\tau + a|\xi|^3}. \tag{5.8}$$

The Marcinkiewicz theorem, Proposition 3.1, implies that $\tilde{\phi} \in W^{3,1}_{p(loc)}$ and

$$||\tilde{\phi}_t, \nabla^3 \tilde{\phi}||_{L_p(\mathbf{R}^{n+1})} \leq c||\tilde{m}||_{L_p(\mathbf{R}^{n+1})}. \tag{5.9}$$

The second step of the proof of Lemma 5.1 is to consider system (5.3) with $m \equiv 0$

$$\begin{array}{lll} \partial_t \phi + a\mathcal{F}_x^{-1}[|\xi|^3 \hat{\phi}] = 0 & \text{in} & \mathbf{R}^n \times (0,+\infty), \\ \phi|_{t=0} = \phi_0 & \text{on} & \mathbf{R}^n. \end{array} \tag{5.10}$$

Solving (5.10) we get

$$\hat{\phi} = \hat{\phi}_0 e^{-a|\xi|^3 t}. \tag{5.11}$$

Let us show that $\nabla^3 \phi \in L_p(\mathbf{R}^n \times (0,\infty))$. By formula (5.11) the solution reads

$$\phi(x,t) = \mathcal{F}_x^{-1}[e^{-a|\xi|^3 t}] *_x \phi_0(x). \tag{5.12}$$

The third derivative of the function ϕ has the form

$$\partial_{x^i}\partial_{x^j}\partial_{x^k}\phi(x,t) = \mathcal{F}_x^{-1}[i\xi_k e^{-a|\xi|^3 t}] * (\partial_{x^i}\partial_{x^j}\phi_0) \tag{5.13}$$

and by the assumptions $\partial_{x^i}\partial_{x^j}\phi_0 \in W_p^{1-3/p}(\mathbf{R}^n)$ with $p>3$. Let

$$\Gamma(y,t) = \mathcal{F}_x^{-1}[i\xi_k e^{-a|\xi|^3 t}]. \tag{5.14}$$

By the elementary properties of the Fourier transform we have

$$\int_{\mathbf{R}^n} \Gamma(y,t)dy = 0 \quad \text{for} \quad t \in \mathbf{R}_+. \tag{5.15}$$

Moreover, note that for each $t \in \mathbf{R}_+$ the function $i\xi_k e^{-a|\xi|^3 t}$ will determine the features of $\Gamma(\cdot,t)$, although it is not in the Schwarz class.

Examine the norm of (5.13). Let

$$I(t) = ||\partial_{x^i}\partial_{x^j}\partial_{x^k}\phi(x,t)||_{L_p(\mathbf{R}^n)} \leq || \int_{\mathbf{R}^n} \Gamma(y,t)(\psi(x-y,t)-\psi(x))dy||_{L_p(\mathbf{R}^n)};$$

where $\psi = \partial_{x^i}\partial_{x^j}\phi_0(x)$ and we used relation (5.15);

$$\leq \int_{\mathbf{R}^n} |\Gamma(y,t)|N(y)dy, \tag{5.16}$$

where

$$N(y) = ||\psi(\cdot - y) - \psi(\cdot)||_{L_p(\mathbf{R}^n)}. \tag{5.17}$$

We split the integral (5.16) into two parts

$$I_1(t) = \int_{|y|\leq t^{1/3}} |\Gamma(y,t)|N(y)dy, \tag{5.18}$$

$$I_2(t) = \int_{|y|\geq t^{1/3}} |\Gamma(y,t)|N(y)dy. \tag{5.19}$$

Let us consider I_1. Applying the Hölder inequality we get

$$|I_1(t)| \leq (\int_{|y|\leq t^{1/3}} |\Gamma(y,t)|N^p(y)dy)^{1/p}(\int_{\mathbf{R}^n} |\Gamma(y,t)|dy)^{1-1/p}. \tag{5.20}$$

To estimate I_1 we need a good representation of the kernel. Note

$$\Gamma(x,t) = i\int_{\mathbf{R}^n} e^{ix\xi}\xi_k e^{-a|\xi|^3 t}d\xi;$$

changing the coordinates: $\zeta = t^{1/3}\xi$ we obtain that this is equal to

$$it^{-(1/3+n/3)}\int_{\mathbf{R}^n} e^{ixt^{-1/3}\zeta}\zeta_k e^{-a|\zeta|^3}d\zeta = t^{-(n+1)/3}f(x/t^{1/3}), \tag{5.21}$$

where

$$f(w) = i\int_{\mathbf{R}^n} e^{iw\xi}\xi_k e^{-a|\xi|^3}d\xi. \tag{5.22}$$

The elementary features of (5.22) give the following bound

$$\sup_{w\in\mathbf{R}^n}(1+|w|^{n+2})|f(w)| < +\infty, \tag{5.23}$$

since $|\xi|^3 \in C^2$. Hence, in particular, from (5.23) we conclude

$$\int_{\mathbf{R}^n} |\Gamma(x,t)|dx = t^{-(n+1)/3}\int_{\mathbf{R}^n} |f(x/t^{1/3})|dx = t^{-1/3}\int_{\mathbf{R}^n} |f(w)|dw = ct^{-1/3}. \tag{5.24}$$

Thus, (5.20) and (5.24) imply

$$|I_1(t)| \leq ct^{-1/3(1-1/p)}\Big(\int_{|y|\leq t^{1/3}} |\Gamma(y,t)|N^p(y)dy\Big)^{1/p}. \tag{5.25}$$

Take the norm of $I_1(\cdot)$

$$\Big(\int_0^\infty |I_1(t)|^p dt\Big)^{1/p} = c\Big(\int_0^\infty dt \int_{|y|\le t^{1/3}} dy\; t^{-1/3(p-1)}|\Gamma(y,t)|N^p(y)\Big)^{1/p}; \tag{5.26}$$

by the Fubini theorem, this equals

$$c\Big(\int_{\mathbf{R}^n} dy \int_{|y|^3}^{+\infty} t^{-1/3(p-1)}|\Gamma(y,t)|N^p(y)\Big)^{1/p}. \tag{5.27}$$

Examine the integral

$$\int_{|y|^3}^\infty t^{-1/3(p-1)}|\Gamma(y,t)|dt = \int_{|y|^3}^\infty t^{-1/3(p+n)}|f(y/t^{1/3})|dt;$$

denoting $y=\omega|y|$ for $\omega\in S^{n-1}$ and taking $s=|y|/t^{1/3}$ $(t=|y|^3/s^3)$ we get;

$$|y|^{3-n-p}\int_0^1 s^{p+n}s^{-4}|f(\omega s)|ds. \tag{5.28}$$

The last integral is finite, because $p>1$, $n\ge 1$, and $f(0)=0$, hence by (5.27)

$$\Big(\int_0^\infty |I_1(t)|^p dt\Big)^{1/p} \le c\Big(\int_{\mathbf{R}^n} \frac{N^p(y)}{|y|^{n+p(1-3/p)}}dy\Big)^{1/p}. \tag{5.29}$$

To examine I_2 we proceed similarly as for I_1

$$|I_2(t)| \le \Big(\int_{|y|\ge t^{1/3}} |y|^{1-p}\,|\Gamma(y,t)|N^p(y)dy\Big)^{1/p}\Big(\int_{\mathbf{R}^n} |y|\,|\Gamma(y,t)|dy\Big)^{1-1/p}. \tag{5.30}$$

But by (5.21) and (5.23) we obtain

$$\int_{\mathbf{R}^n} |x|\,|\Gamma(x,t)|dx = t^{-(n+1)/3}\int_{\mathbf{R}^n} |x|\,|f(x/t^{1/3})|dx = \int_{\mathbf{R}^n} |w|\,|f(w)|dw = c. \tag{5.31}$$

Hence from (5.30) and (5.31) we get

$$\Big(\int_0^\infty |I_2(t)|^p dt\Big)^{1/p} = c\Big(\int_0^\infty dt \int_{|y|\ge t^{1/3}} dy\; |y|^{1-p}\,|\Gamma(y,t)|N^p(y)\Big)^{1/p};$$

and by the Fubini theorem, this equals

$$c\Big(\int_{\mathbf{R}^n} dy \int_0^{|y|^3} dt|y|^{1-p}\,|\Gamma(y,t)|N^p(y)\Big)^{1/p}. \tag{5.32}$$

And again we consider

$$\int_0^{|y|^3} |y|^{1-p}\,|\Gamma(y,t)|dt = \int_0^{|y|^3} t^{-1/3(n+1)}|y|^{1-p}\,|f(y/t^{1/3})|dt;$$

denoting $y=\omega|y|$ for $\omega\in S^{n-1}$ and taking $s=|y|/t^{1/3}$ ($t=|y|^3/s^3$) we get

$$|y|^{3-n-p}\int_1^\infty s^{n+1}s^{-4}|f(\omega s)|ds. \tag{5.33}$$

by (5.23) we get the convergence of the integral from (5.33). Thus from (5.32) we have

$$\Big(\int_0^\infty |I_2(t)|^p dt\Big)^{1/p}\le c\Big(\int_{\mathbf{R}^n}\frac{N^p(y)}{|y|^{n+p(1-3/p)}}dy\Big)^{1/p}. \tag{5.34}$$

Summing (5.29) and (5.34) we obtain

$$\Big(\int_0^\infty |I(t)|^p dt\Big)^{1/p}\le$$

$$c\Big(\int_{\mathbf{R}^n\times\mathbf{R}^n}\frac{|\phi_0(x-y)-\phi_0(x)|^p}{|y|^{n+p(1-3/p)}}dxdy\Big)^{1/p}=c<\phi_0>_{W_p^{3-3/p}(\mathbf{R}^n)}. \tag{5.35}$$

Thus, for the solution to (5.10) we have proved the following estimate

$$||\phi_{,t},\nabla^3\phi||_{L_p(\mathbf{R}^n\times(0,\infty))}\le c<\phi_0>_{W_p^{3-3/p}(\mathbf{R}^n)}, \tag{5.36}$$

where the bound for ϕ_t is obtained from equation $(5.10)_1$.

To conclude inequality (5.4), it is enough to note that the solutions to problem (5.2) are the sum of solutions of (5.5) and (5.10), and apply bounds (5.9) and (5.36). Lemma 5.1 is proved. □

Proof of Theorem 5.1. After the Fourier transform with respect to x' system (5.1) reads as

$$\begin{array}{lcr}(-|\xi|^2+\partial^2_{x_{n+1}})\hat p=0 & \text{in} & \mathbf{R}^{n+1}_+\times(0,\infty),\\ \hat p=-a|\xi|^2\hat p+\hat g & \text{on} & \mathbf{R}^n\times(0,\infty),\\ \hat p_{,x_{n+1}}=-\partial_t\hat\phi+\hat h & \text{on} & \mathbf{R}^n\times(0,\infty),\\ \hat\phi|_{t=0}=\hat\phi_0 & \text{on} & \mathbf{R}^n,\\ \hat p\to 0 & \text{as} & x_{n+1}\to\infty,\end{array} \tag{5.37}$$

where

$$\hat{\cdot}=\int_{\mathbf{R}^n}e^{-ix'\xi}\cdot(x')dx'.$$

Solving $(5.37)_{1,5}$ we obtain

$$\hat p(\xi,x_n,t)=\hat P(\xi,t)e^{-|\xi|x_{n+1}} \tag{5.38}$$

for a function $P(x',t)$. From (5.38) it follows that the boundary conditions $(5.37)_{2,3}$ take the following form

$$\begin{array}{lcl}\hat P=-a|\xi|^2\hat\phi+\hat g & \text{in} & \mathbf{R}^n,\\ -|\xi|\hat P=-\partial_t\hat\phi+\hat h & \text{in} & \mathbf{R}^n.\end{array} \tag{5.39}$$

Reducing P in (5.39) we obtain the following parabolic equation

$$\partial_t\hat{\phi} + a|\xi|^3\hat{\phi} = \hat{h} + |\xi|\hat{g} \tag{5.40}$$

with initial datum $(5.37)_4$.

Lemma 5.1 guarantees existence of solutions to (5.40) and the following bound holds

$$||\phi_t, \nabla^3\phi||_{L_p(\mathbf{R}^n\times(0,\infty))} \le c(||\nabla' g, h||_{L_p(\mathbf{R}^n\times(0,\infty))} + ||\phi_0||_{W_p^{3-3/p}(\mathbf{R}^n)}). \tag{5.41}$$

Next, equation $(5.39)_1$ implies that

$$\nabla' P \in L_p(\mathbf{R}^n \times (0,\infty)). \tag{5.42}$$

By Lemma 4.4 we deduce the following bound

$$< p >_{L_p(0,\infty;W_p^{1+1/p}(\mathbf{R}^{n+1}))} \le c(||\nabla' g, h||_{L_p(\mathbf{R}^n\times(0,\infty))} + ||\phi_0||_{W_p^{3-3/p}(\mathbf{R}^n)}). \tag{5.43}$$

The information about the normal derivative of the function p at the boundary is taken from $(5.39)_2$. The proof of Theorem 5.1 is complete. □

6. The reduced problem

The linear problem in the general form of (2.9) can be reduced by the procedure described in the next section into the following system

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \Omega \times (0,T), \\ p = a\Lambda\phi & \text{on} & \partial\Omega \times (0,T), \\ \dfrac{\partial p}{\partial n} + \partial_t\phi = h & \text{on} & \partial\Omega \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & \partial\Omega. \end{array} \tag{6.1}$$

We want to show existence of solutions for this system. By considerations from Section 5 we conclude that (6.1) is equivalent to the following nonlocal parabolic equation

$$\begin{array}{lll} L\phi = h & \text{in} & \partial\Omega \times (0,T), \\ \phi|_{t=0} = \phi_0 & \text{on} & \partial\Omega. \end{array} \tag{6.2}$$

A nonlocal character of system (6.2) causes difficulties which require technical tricks related to a suitable localization. We use here the notation from Section 2 connected with covering sets $\{\omega^k\}$ and $\{\Omega^k\}$.

We want to show existence of a regular solution to problem (6.2) by the technique of the regularizer for parabolic systems taken from [20]. We prove the following result.

Theorem 6.1. *Let $p > p_*$, $\partial\Omega \in W_p^{3-3/p}$. Moreover, assume*

$$h \in L_p(\partial\Omega \times (0,T)) \quad \textit{and} \quad \phi_0 \in W_p^{3-3/p}(\partial\Omega);$$

then there exists a unique solution to problem (6.2) *such that* $\phi \in W_p^{3,1}(\partial\Omega \times (0,T))$ *and*

$$||\phi||_{W_p^{3,1}(\partial\Omega\times(0,T))} \leq C(T)(||h||_{L_p(\partial\Omega\times(0,T))} + ||\phi||_{W_p^{3-3/p}(\partial\Omega)}). \tag{6.3}$$

In order to prove Theorem 6.1 we construct the regularizer for system (6.2). Let

$$\phi^k = R^k(\zeta^k h, \zeta^k \phi_0), \tag{6.4}$$

where $\phi^k = Z_k^{-1^*}[\bar{\phi}^k]$ and $\bar{\phi}^k$ is a solution to the following problem

$$\begin{array}{lll} \Delta\bar{p}^k = 0 & \text{in} & \mathbf{R}_+^{n+1} \times (0,T), \\ \bar{p}^k = a\Delta'\bar{\phi}^k & \text{on} & \mathbf{R}^n \times (0,T), \\ \bar{p}^k_{,x_n} + \partial_t\bar{\phi}^k = Z_k^*[\zeta^k h] & \text{on} & \mathbf{R}^n \times (0,T), \\ \bar{\phi}^k|_{t=0} = Z_k^*[\zeta^k\phi_0] & \text{on} & \mathbf{R}^n. \end{array} \tag{6.5}$$

Definition 6.1. An operator

$$R : L_p(\partial\Omega \times (0,T)) \times W_p^{3-3/p}(\partial\Omega) \to W_p^{3,1}(\Omega \times (0,T))$$

such that

$$R(h,\phi_0) = \sum_k \pi^k \phi^k \tag{6.6}$$

we call a *regularizer*.

To find good properties of the operator R there is a need to introduce the following quantity

$$\beta = c(\epsilon + c(\epsilon)\lambda + T^a/\lambda^3), \tag{6.7}$$

where $a > 0$, $c(\epsilon)$ is given in Proposition 3.2 and T is the length of the time interval $(0,T)$.

We will require that the quantity β be sufficiently small. This is possible since for arbitrary ϵ we can find λ sufficiently small to control the smallness of term $c(\epsilon)\lambda$, and by choosing T we control the expression for β.

The operator R enables us to construct an inverse operator to L. First we need the following result.

Lemma 6.1. *We have*

$$LRh = h + \mathcal{T}h \tag{6.8}$$

and the norm of the operator $\mathcal{T}$ is small and controlled by β if the time interval $(0,T)$ is sufficiently short.

Proof. We restate the right-hand side of (6.8) as follows

$$LRh = \sum_k (L(\pi^k \phi^k) - \pi^k L\phi^k) + \sum_k \pi^k L\phi^k. \tag{6.9}$$

By the definition of the operator L we see that

$$L\phi^k = \partial_t \phi^k + \frac{\partial p}{\partial n}, \tag{6.10}$$

where p fulfills the following problem

$$\begin{array}{lll} \Delta p = 0 & \text{in} & \Omega, \\ p = a\Lambda\phi^k & \text{on} & \partial\Omega. \end{array} \tag{6.11}$$

After the localization and transformation into the half space we obtain from (6.10) and (6.11) the following system

$$\begin{array}{lll} \Delta(\bar\pi^k \bar p) = F_1 & \text{in} & \mathbf{R}^{n+1}_+ \times (0,T), \\ \bar\pi^k \bar p = a\Delta'(\bar\pi^k \bar\phi^k) + G_1, & \text{on} & \mathbf{R}^n \times (0,T), \\ (\bar\pi^k \bar p)_{,x_{n+1}} + \partial_t(\bar\pi^k \bar\phi^k) = \bar\pi^k Z^{-1*}[L\phi^k] & \text{on} & \mathbf{R}^n \times (0,T), \\ \bar\pi^k \bar\phi^k|_{t=0} = \bar\pi^k \bar\phi_0 & \text{on} & \mathbf{R}^n, \end{array} \tag{6.12}$$

where

$$F_1 = (\Delta - \Delta_x)(\bar\pi^k \bar p^k) + Z_k^{-1*}[2\nabla\bar\pi^k \nabla p + (\Delta\bar\pi^k)p],$$
$$G_1 = -2a\nabla'\bar\pi^k \nabla'\bar\phi^k - a(\Delta'\bar\pi^k)\bar\phi^k.$$

There appears a natural decomposition of the solution to (6.12)

$$\bar\pi^k \bar p = p_0 + p_1, \tag{6.13}$$

where p_1 is a solution to

$$\Delta p_1 = F_1 \ \text{ in } \ \mathbf{R}^{n+1}_+, \quad p_1 = G_1 \quad \text{on } \ \mathbf{R}^n \tag{6.14}$$

and

$$\Delta p_0 = 0 \ \text{ in } \ \mathbf{R}^{n+1}_+, \quad p_0 = a\Delta'(\bar\pi^k \bar\phi) \quad \text{on } \ \mathbf{R}^n. \tag{6.15}$$

To see properties of this decomposition let us look at a modification of problem (6.5). After multiplying by $\bar\pi^k$ we get

$$\begin{array}{lll} \Delta(\bar\pi^k \bar q) = F_2 & \text{in} & \mathbf{R}^{n+1}_+ \times (0,T), \\ \bar\pi^k \bar q = a\Delta'(\bar\pi^k \bar\phi^k) + G_1, & \text{on} & \mathbf{R}^n \times (0,T), \\ (\bar\pi^k \bar q)_{,x_{n+1}} + \partial_t(\bar\pi^k \bar\phi^k) = \bar\pi^k Z^{-1*}[L\phi^k] & \text{on} & \mathbf{R}^n \times (0,T), \\ \bar\pi^k \bar\phi^k|_{t=0} = \bar\pi^k \bar\phi^k{}_0 & \text{on} & \mathbf{R}^n, \end{array} \tag{6.16}$$

where $F_2 = 2\nabla\bar\pi^k \nabla\bar q + (\Delta\bar\pi^k)\bar q$, and $G_2 = G_1$ as in (6.12). And again we split the solution to (6.16)

$$\pi^k \bar q = q_0 + q_1, \tag{6.17}$$

where q_1 is a solution to

$$\Delta q_1 = F_2 \text{ in } \mathbf{R}^{n+1}_+, \quad q_1 = G_2 \quad \text{on } \mathbf{R}^n \tag{6.18}$$

and

$$\Delta q_0 = 0 \text{ in } \mathbf{R}^{n+1}_+, \quad q_0 = a\Delta'(\bar\pi^k \bar\phi^k) \quad \text{on } \mathbf{R}^n. \tag{6.19}$$

System (6.16) gives us the following relation

$$(q_0 + q_1)_{,x_{n+1}} + \partial_t(\bar\pi^k \bar\phi^k) = Z_k^{-1*}[\pi^k \zeta^k h] \quad \text{on } x_{n+1} = 0. \tag{6.20}$$

Note that by (6.15) and (6.19) we have $p_0 = q_0$, hence comparing (6.20) and $(6.12)_3$ we obtain that

$$Z_k^{-1*}[\pi^k L\phi^k] = Z_k^{-1*}[\pi^k \zeta^k h] + (p_1 - q_1)_{,x_{n+1}}. \tag{6.21}$$

Combining information from (6.10) and (6.21) we conclude that

$$\sum_k \pi^k L\phi^k = h + \sum_k Z_k^{-1*}[(q_1 - p_1)_{,x_{n+1}}]. \tag{6.22}$$

To find a suitable estimate of the second term of the right-hand side of (6.22) we need some estimate on p and q. Since p is given by (6.11) and

$$\Lambda\phi^k \in W^{1,1/3}_p(\partial\Omega \times (0,T)) \subset L_\infty(0,T; L_p(\partial\Omega)) \subset L_\infty(0,T; L_2(\partial\Omega)) \tag{6.23}$$

and the same property holds for q, we conclude that

$$p, q \in W^{1+1/p,1/3}_p(\partial\Omega \times (0,T)) \subset L_\infty(0,T; L_2(\Omega)). \tag{6.24}$$

Estimates following from (6.24) imply the smallness of lower norms of p and q with respect to a positive power of T. This information can be obtained, since (6.24) follows from the trace theorem, Proposition 3.3 (ii), where a crucial fact is that the constant in estimate (3.16) is independent of T.

By (6.14) and (6.18) we have

$$\begin{array}{lll} \Delta(p_1 - q_1) = F_1 - F_2 & \text{in} & \mathbf{R}^{n+1}_+, \\ p_1 - q_1 = 0 & \text{on} & \mathbf{R}^n. \end{array} \tag{6.25}$$

We are interested only in finding information about

$$(p_1 - q_1)_{,x_{n+1}}|_{x_{n+1}=0}.$$

The main result from Section 4 delivers the following bound on quantity (6.26) below

$$\begin{aligned} ||(p_1 - q_1)_{,x_{n+1}}|_{x_{n+1}=0}||_{L_p(\mathbf{R}^n)} \le c(&||(\Delta - \Delta_x)(\bar\pi^k \bar p^k)||_{(W^{1-1/p}_q(\mathbf{R}^{n+1}_+))^*} \\ &+ ||Z_k^{-1*}[2\nabla\bar\pi^k \nabla\bar p + (\Delta\bar\pi^k)\bar p]||_{(W^{1-1/p}_q(\mathbf{R}^{n+1}_+))^*} \\ &+ ||2\nabla\bar\pi^k \nabla\bar q + (\Delta\bar\pi^k)\bar q||_{(W^{1-1/p}_q(\mathbf{R}^{n+1}_+))^*}). \end{aligned} \tag{6.26}$$

By estimate (4.54) from Section 4 we obtain that

$$||(\Delta - \Delta_x)(\bar{\pi}^k \bar{p}^k)||_{(W_q^{1-1/p}(supp\,\bar{\pi}^k))^*}$$

$$\leq (\epsilon + c(\epsilon)\lambda)||\frac{\partial y}{\partial x}||_{W_p^{2-3/p}}||\bar{\pi}^k \bar{p}^k||_{W_p^{1/p}(\mathbf{R}_+^{n+1})}, \tag{6.27}$$

where $\frac{\partial y}{\partial x}$ denotes the Jacobi matrix for the local diffeomorphism Z_k.

From (6.24) and Proposition 3.2 we are able to find a bound on the second and third term of the right-hand side of (6.26)

$$||[2\nabla\bar{\pi}^k\nabla\bar{p} + (\Delta\bar{\pi}^k)\bar{p}]||_{L_p(0,t;(W_q^{1-1/p}(\mathbf{R}_+^{n+1}))^*)}$$

$$+||2\nabla\bar{\pi}^k\nabla\bar{q} + (\Delta\bar{\pi}^k)\bar{q}||_{L_p(0,T;(W_q^{1-1/p}(\mathbf{R}_+^{n+1}))^*)}$$

$$\leq \frac{T^a}{\lambda^2}(||h||_{L_p(supp\,\pi^k\times(0,T))} + ||\phi_0||_{W_p^{3-3/p}(supp\,\pi^k)}), \tag{6.28}$$

where the constant a is defined by Proposition 3.2 and $0 < a < 1/p$.

Summing (6.27) and (6.28), we get

$$||(p_1 - q_1)_{,x_{n+1}}|_{x_{n+1}=0}||_{L_p(\mathbf{R}^n\times(0,T))} \leq$$

$$c\Big(\epsilon + c(\epsilon)\lambda + \frac{T^a}{\lambda^2}\Big)(||h||_{L_p(supp\,\pi^k\times(0,T))} + ||\phi_0||_{W_p^{3-3/p}(supp\,\pi^k)}). \tag{6.29}$$

To end the proof we study the first term of the right-hand side of (6.9); using the interpolation theorem, Proposition 3.2, we get

$$||\sum_k(\pi^k\phi^k) - \pi^k L\phi^k||_{L_p(\Omega\times(0,T))}$$

$$\leq c\sum_k||\,|\nabla\pi^k||\nabla\phi^k| + |\nabla^2\pi^k||\phi^k|\,||_{W_p^{1,0}(\Omega\times(0,T))}$$

$$\leq c(1/\lambda||\nabla^2\phi||_{L_p(\Omega\times(0,T))} + 1/\lambda^2||\nabla\phi||_{L_p(\Omega\times(0,T))} + 1/\lambda^3||\phi||_{L_p(\Omega\times(0,T))})$$

$$\leq c/\lambda^3\Big(T^{1/p^*}||\phi_0||_{W_p^{3-3/p}(\Omega)} + T^{1/p}||\phi||_{W_p^{3,1}(\Omega\times(0,T))}\Big)$$

$$\leq c\beta(||h||_{L_p(\Omega\times(0,T))} + ||\phi_0||_{W_p^{3-3/p}(\Omega)}). \tag{6.30}$$

Taking into account (6.29) and (6.30) we conclude that

$$||\mathcal{T}|| \leq c(\epsilon + c(\epsilon)\lambda + \frac{T^a}{\lambda^3}) = c\beta. \tag{6.31}$$

The smallness of T provides the smallness of the norm $||\mathcal{T}||$. Lemma 6.1 is proved.

Lemma 6.2. *We have*

$$RL\phi = \phi + \mathcal{W}\phi \tag{6.32}$$

and the norm of the operator $\mathcal{W}$ is small and controlled by β if the time interval $(0,T)$ is sufficiently small.

Proof. We have

$$RL\phi = \sum_k \pi^k Z_k^{-1*} R^k[\zeta^k {Z_k}^*[\phi]] - \pi^k Z_k^{-1*} R^k[{Z_k}^*[\zeta^k L\phi]] \\ + \sum_k \pi^k Z_k^{-1*} R^k[{Z_k}^*[L(\zeta^k \phi)]]. \tag{6.33}$$

To find a suitable form of the last term we consider $R^k[{Z_k}^*[L(\zeta^k\phi)]]$. After a localization we get

$$\begin{array}{lll} \Delta q = (\Delta - \Delta_x) q & \text{in} & \mathbf{R}_+^{n+1} \times (0,T), \\ q = a\Delta'(\bar{\zeta}^k \bar{\phi}) & \text{in} & \mathbf{R}^n \times (0,T), \\ q_{,x_{n+1}} + \partial_t(\bar{\zeta}^k\bar{\phi}) = Z_k^*[L(\zeta^k\phi)] & \text{in} & \mathbf{R}^n \times (0,T), \\ \bar{\zeta}^k \bar{\phi}|_{t=0} = \bar{\zeta}^k \bar{\phi}_0 & \text{on} & \mathbf{R}^n. \end{array} \tag{6.34}$$

By analogy to Lemma 6.1 we split the solution of system (6.34) into

$$q = q_0 + q_1 \tag{6.35}$$

where q_1 is a solution to

$$\Delta q_1 = (\Delta - \Delta_x) q \quad \text{in } \mathbf{R}_+^{n+1}, \quad q_1 = 0 \quad \text{on } \mathbf{R}^n \tag{6.36}$$

and

$$\Delta q_0 = 0 \quad \text{in } \mathbf{R}_+^{n+1}, \quad q_0 = a\Delta'(\zeta\bar{\phi}) \quad \text{on } \mathbf{R}^n. \tag{6.37}$$

But on the other hand we know that

$$\begin{array}{lll} \Delta \bar{p} = 0 & \text{in} & \mathbf{R}_+^{n+1} \times (0,T), \\ \bar{p} = a\Delta'(\zeta\bar{\phi}) & \text{in} & \mathbf{R}^n \times (0,T), \\ \bar{p}_{,x_{n+1}} + \partial_t \phi = L[\zeta^k \pi\phi] & \text{in} & \mathbf{R}^n \times (0,T), \\ \bar{\zeta}^k \bar{\phi}|_{t=0} = \bar{\zeta}^k \phi_0 & \text{on} & \mathbf{R}^n. \end{array} \tag{6.38}$$

By $(6.38)_{1,2}$ and (6.37) we see that $q_0 = \zeta^k\phi$. Hence,

$$\sum_k \pi^k Z_k^{-1*} R^k[Z_k^*[L(\zeta\phi)]] = \phi + \sum_k \pi^k Z_k^{-1*} R^k[\bar{p}_{1,x_{n+1}}]. \tag{6.39}$$

To obtain the estimate on $\mathcal{W}$ it is enough to repeat the estimates from the proof of Lemma 6.1 and obtain

$$||\mathcal{W}|| \le c\beta. \tag{6.40}$$

Proof of Theorem 6.1. Lemmas 6.1 and 6.2 give us the following relations

$$\begin{array}{c} LRh = h + \mathcal{T}h, \\ RL\phi = \phi + \mathcal{W}\phi. \end{array} \tag{6.41}$$

For suitably small time T we have estimates (6.31) and (6.40) guaranteeing that the operators $(1+\mathcal{T})$ and $(1+\mathcal{W})$ are invertible and

$$(1+\mathcal{T})^{-1} \quad \text{and} \quad (1+\mathcal{W})^{-1} \tag{6.42}$$

are bounded. Hence we get

$$LR(1+\mathcal{T})^{-1}\tilde{h}=\tilde{h}, \qquad (1+\mathcal{W})^{-1}RL\tilde{\phi}=\tilde{\phi} \tag{6.43}$$

which implies that

$$R(1+\mathcal{T})^{-1}=(1+\mathcal{W})^{-1}R=L^{-1} \tag{6.44}$$

Thus the solution to (6.2) is given by

$$\phi=L^{-1}h. \tag{6.45}$$

The estimate of the norm of the operator L^{-1} follows from the boundedness of the operator R and (6.31), (6.40). Theorem 6.1 is proved only for a time interval $[0,T_*]$. To obtain the general result it is enough to apply the trace theorem Proposition 3.3 (ii) and continue the solution onto the intervals $[T_*,2T_*],[2T_*,3T_*],...,$ since the constants in (6.31) and (6.40) are independent of data (the boundary is fixed). This procedure causes the constant from (2.21) to depend on T exponentially. Theorem 6.1 is proved.

7. Proof of Theorem 2.1

In this section we summarize the results from Sections 4, 5, and 6, and we prove Theorem 2.1.

Let us consider the following system

$$\begin{array}{lll} \Delta p=F & \text{in} & \Omega\times(0,T),\\ p=a\Lambda\psi+G & \text{on} & \partial\Omega\times(0,T),\\ \dfrac{\partial p}{\partial n}=-\partial_t\psi+H & \text{on} & \partial\Omega\times(0,T),\\ \psi|_{t=0}=\psi_0 & \text{on} & \partial\Omega. \end{array} \tag{7.1}$$

Due to the assumptions of Theorem 2.1 the functions F and G fulfill the conditions of Theorem 4.1 from Section 4. Hence, the solution to the problem

$$\begin{array}{lll} \Delta p_1=F & \text{in} & \Omega,\\ p_1=G & \text{on} & \partial\Omega \end{array} \tag{7.2}$$

satisfies the following bound

$$\begin{aligned} &||p_1||_{L_p(0,T;W_p^{1/p}(\Omega))}+||\frac{\partial p_1}{\partial n}|_{\partial\Omega}||_{L_p(\partial\Omega\times(0,T))}\\ &\le c(||F||_{L_p(0,T;(W_q^{1-1/p}(\Omega))^*)}+||G||_{L_p(0,T;W_p^1(\partial\Omega))}), \end{aligned} \tag{7.3}$$

where $1/p + 1/q = 1$. Applying the solution to problem (7.2) we reduce system (7.1) to the following

$$\begin{array}{lcr} \Delta p_2 = 0 & \text{in} & \Omega \times (0,T), \\ p_2 = a\Lambda\psi & \text{on} & \partial\Omega \times (0,T), \\ \dfrac{\partial p_2}{\partial n} = -\partial_t \psi + H_2 & \text{on} & \partial\Omega \times (0,T), \\ \psi|_{t=0} = \psi_0 & \text{on} & \partial\Omega, \end{array} \tag{7.4}$$

where

$$p_2 = p - p_1 \tag{7.5}$$

and

$$H_2 = H - \frac{\partial p_1}{\partial n}|_{\partial\Omega}. \tag{7.6}$$

By (7.3) the datum H_2 belongs to $L_p(\partial\Omega \times (0,T))$, hence we are able to apply Theorem 6.1 to find a solution of system (7.4). By the results from Sections 4 and 6 we conclude the existence of a solution to (7.4) such that

$$\begin{gathered} \psi \in W_p^{3,1}(\partial\Omega \times (0,T)), \quad p_2 \in L_p(0,T; W_p^{1+1/p}(\Omega)), \\ \frac{\partial p_2}{\partial n} \in L_p(\partial\Omega \times (0,T)) \end{gathered} \tag{7.7}$$

which fulfills the following bound

$$\begin{aligned} ||\psi||_{W_p^{3,1}(\partial\Omega\times(0,T))} + ||p_2||_{L_p(0,T;W_p^{1/p}(\Omega))} + ||\frac{\partial p_2}{\partial n}||_{L_p(\partial\Omega\times(0,T))} \\ \le c(T)(||H_2||_{L_p(\partial\Omega\times(0,T))} + ||\psi_0||_{W_p^{3-3/p}(\partial\Omega)}). \end{aligned} \tag{7.8}$$

Combining estimates (7.3) and (7.8), remembering about (7.5) and (7.6), we get the following bound

$$\begin{aligned} ||\psi||_{W_p^{3,1}(\partial\Omega\times(0,T))} + ||p||_{L_p(0,T;W_p^{1/p}(\Omega))} + ||\frac{\partial p}{\partial n}||_{L_p(\partial\Omega\times(0,T))} \\ \le c(T)\Big(||F||_{L_p(0,T;(W_q^{1-1/p}(\Omega))^*)} + ||G||_{L_p(0,T;W_p^1(\partial\Omega))} \\ + ||H||_{L_p(\partial\Omega\times(0,T))} + ||\psi_0||_{W_p^{3-3/p}(\partial\Omega)}\Big). \end{aligned} \tag{7.9}$$

Recalling considerations from Section 4, Lemma 4.1, we find the estimate on the norm of F and obtain (2.22). Theorem 2.1 is proved.

8. Proof of Theorem 1.1

In this part of the paper we prove Theorem 1.1. For this purpose we will find a local in time solution to problem (2.9) by a standard application of the Banach fixed-point theorem. This well-known method requires a certain smallness of data. In our case, since we are interested in finding solutions with sharp regularity, smallness of the lifespan is not sufficient. There is a need to introduce an extension of initial data.

Let $\bar{q}$ and $\bar{\psi}$ be given as solutions to the following problem

$$\begin{array}{lll} \Delta\bar{q} = 0 & \text{in} & \Omega\times(0,T), \\ \bar{q} = a\Lambda\bar{\psi} & \text{on} & \partial\Omega\times(0,T), \\ \dfrac{\partial\bar{q}}{\partial n} = -\partial_t\bar{\psi} & \text{on} & \partial\Omega\times(0,T), \\ \psi|_{t=0} = \psi_0 & \text{on} & \partial\Omega. \end{array} \tag{8.1}$$

By Theorem 2.1 we obtain the solution to system (8.1) fulfilling the following inclusions

$$\bar{q}\in L_p(0,T;W_p^{1+1/p}(\Omega)), \quad \frac{\partial\bar{q}}{\partial n}\in L_p(\partial\Omega\times(0,T)), \quad \bar{\psi}\in W_p^{3,1}(\partial\Omega\times(0,T)). \tag{8.2}$$

Since we do not require keeping any restrictions on the magnitude of the initial datum ψ_0, the norms of quantities (8.2) are not small even for arbitrary small T. However we are able to find T_1 such that for fixed $\delta_1 > 0$

$$<\bar{q}>_{L_p(0,T_1;W_p^{1+1/p}(\Omega))} + \|\frac{\partial\bar{q}}{\partial n}\|_{L_p(\partial\Omega\times(0,T_1))} + <\bar{\psi}>_{W_p^{3,1}(\partial\Omega\times(0,T_1))} \le \delta_1, \tag{8.3}$$

where $\delta_1 \to 0$ as $T_1 \to 0$, which follows form elementary features of the Lebesgue integral.

We look for the solution to (2.9) in the following form

$$q = u + \bar{q}, \qquad \psi = f + \bar{\psi}. \tag{8.4}$$

Then functions u and f satisfy the following system

$$\begin{array}{lll} \Delta u = F_1 & \text{in} & \Omega\times(0,T), \\ u = a\Lambda f + G_1 & \text{in} & \partial\Omega\times(0,T), \\ \dfrac{\partial u}{\partial n} = -\partial_t f + H_1 & \text{on} & \partial\Omega\times(0,T), \\ f|_{t=0} = 0 & \text{on} & \partial\Omega, \end{array} \tag{8.5}$$

where analogously to (2.15) we have

$$F_1 = (\Delta - \Delta^t)u + (\Delta - \Delta^t)\bar{q}, \quad G_1 = (\Delta_{\partial\Omega_t} - \Lambda)f + (\Delta_{\partial\Omega_t} - \Lambda)\bar{\psi},$$
$$H_1 = n^0(n^t - n^0)\partial_t f + n^0(n^t - n^0)\partial_t\bar{\psi}.$$

Next, we look for a solution to problem (8.5) as a fixed point to the following map

$$\mathcal{M}(\bar{u}, \bar{f}) = (u, f), \tag{8.6}$$

where

$$(\bar{u}, \bar{f}) \in \Xi \tag{8.7}$$

and

$$\begin{array}{c} \Xi = \{(u,f) \in L_p(0,T; W_p^{1+1/p}(\Omega)) \times W_p^{3,1}(\partial\Omega \times (0,T)) : \\ f|_{t=0} = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} \in L_p(\partial\Omega \times (0,T)) \\ ||u||_{L_p(0,T;W_p^{1+1/p}(\Omega))} + ||\frac{\partial u}{\partial n}||_{L_p(\partial\Omega\times(0,T))} + ||f||_{W_p^{3,1}(\partial\Omega\times(0,T))} \leq \delta\}; \end{array} \tag{8.8}$$

the functions u and f from (8.6) are given as solutions to the following problem

$$\begin{array}{lll} \Delta u = \bar{F}_1 & \text{in} & \Omega \times (0,T), \\ u = a\Lambda f + \bar{G}_1 & \text{in} & \partial\Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = -\partial_t f + \bar{H}_1 & \text{on} & \partial\Omega \times (0,T), \\ f|_{t=0} = 0 & \text{on} & \partial\Omega, \end{array} \tag{8.9}$$

where

$$\bar{F}_1 = (\Delta - \bar{\Delta}^t)\bar{u} + (\Delta - \bar{\Delta}^t)\bar{q}, \quad \bar{G}_1 = (\Delta_{\overline{\partial\Omega_t}} - \Lambda)\bar{f} + (\Delta_{\overline{\partial\Omega_t}} - \Lambda)\bar{\psi},$$
$$\bar{H}_1 = n^0(\bar{n}^t - n^0)\partial_t \bar{f} + n^0(\bar{n}^t - n^0)\partial_t\bar{\psi}$$

and $\overline{\Delta}^t$ is the Laplace operator for the domain $\bar{\Omega}^t$ which is obtained by the function $\bar{f} + \bar{\psi}$, $\Delta_{\overline{\partial\Omega^t}}$ is the Laplace-Beltrami operator on the boundary of $\bar{\Omega}^t$, and $\bar{n}^t$ is the normal vector to $\partial\bar{\Omega}^t$.

To simplify the notation we introduce the following quantity

$$||(u,f)||_{\Xi} = ||u||_{L_p(0,T;W_p^{1+1/p}(\Omega))} + ||\frac{\partial u}{\partial n}||_{L_p(\partial\Omega\times(0,T))} + < f >_{W_p^{3,1}(\partial\Omega\times(0,T))} . \tag{8.10}$$

By Proposition 3.3 (ii), if $f|_{t=0} = 0$, then for bounded T the following quantities are equivalent

$$< f >_{W_p^{3,1}(\partial\Omega\times(0,T))} \simeq ||f||_{W_p^{3,1}(\partial\Omega\times(0,T))}. \tag{8.11}$$

By Theorem 2.1 we obtain the estimate for the solutions to problem (8.9) in the class of regularity described by (2.22). Let us find a suitable bound on

$$||\bar{F}_1||_{L_p(0,T;(W_q^{1-1/p}(\Omega))^*)}. \tag{8.12}$$

Since the features of the operator $\Delta - \bar{\Delta}^t$ are described by $\bar{f} + \bar{\psi} - \psi_0$ (see relations (1.7), (2.12), (2.13), and (8.4)), to examine the first term of the right-hand side of (8.12) we recall the form (4.52). Terms $\frac{\partial y_i}{\partial x_j} \in$

$L_\infty(0,T;W_p^{2-3/p}(\Omega))$ are prescribed by $\nabla(\bar f-\bar\psi-\psi_0)$. To find the desired bound we need to examine a modification of the following expression

$$(\delta_{ij}-\frac{\partial y_i}{\partial x_i})\partial_k(\partial_m u). \tag{8.13}$$

By considerations in Section 4 we deduce that

$$|\delta_{ij}-\frac{\partial y_i}{\partial x_i}| \simeq |\nabla(\bar f-\bar\psi)| \tag{8.14}$$

and we find

$$||(\delta_{ij}-\frac{\partial y_i}{\partial x_i})\partial_k(\partial_m u)||_{L_p(0,T;(W_q^{1-1/p}(\Omega))^*)} \le$$
$$c||\delta_{ij}-\frac{\partial y_i}{\partial x_i}||_{L_\infty(0,T;W_p^{2-3/p}(\partial\Omega))}||\nabla u||_{L_p(0,T;W_p^{1/p}(\Omega))}. \tag{8.15}$$

Hence one can show the following bound

$$||\bar F_1||_{L_p(0,T;(W_q^{1-1/p}(\Omega))^*)} \le c(||(\bar u,\bar f)||^2_\Xi+||(\bar q,\bar\psi)||^2_\Xi). \tag{8.16}$$

Take $\bar G_1$ from (8.9). To analyze the behavior of $\Delta_{\overline{\partial\Omega^t}}-\Lambda$ we see that it is completely determined by the function

$$\bar f+\bar\psi. \tag{8.17}$$

Moreover, by the definition of ψ_0 and choice of domain Ω, for each element of the covering Ω^k, we have

$$||(\bar f+\bar\psi)(\cdot,0)||_{W_p^{3-3/p}(\partial\Omega)} < \epsilon \tag{8.18}$$

for sufficiently small (fixed at the beginning) $\epsilon>0$. In the local coordinate system coefficients of the main symbol of $\Delta_{\overline{\partial\Omega^t}}-\Lambda$ belong to $W_p^{2,2/3}$, so by the imbedding properties

$$W_p^{2,2/3}(\partial\Omega\times(0,T))\subset C^{b,b/3}(\partial\Omega\times(0,T)) \tag{8.19}$$

for a certain $b>0$, if $p>p_*$. Then by (8.18), (8.19), Propositions 3.2, and 3.3 we have

$$||(\Delta_{\overline{\partial\Omega^t}}-\Lambda)\bar f||_{L_p(0,T;W_p^1(\partial\Omega))}$$
$$\le c(||\nabla(\bar f+\bar\psi)\nabla^3\bar f||_{L_p(\partial\Omega\times(0,T))}+||\nabla(\bar f+\bar\psi)^2\nabla^2\bar f||_{L_p(\partial\Omega\times(0,T))}$$
$$\le c(\epsilon+\lambda^b+T^{b/3})||\bar\psi||_{W_p^{3,1}(\partial\Omega\times(0,T))}||\bar f||_{W_p^{3,1}(\partial\Omega\times(0,T))}+c||\bar f||^2_{W_p^{3,1}(\partial\Omega\times(0,T))}. \tag{8.20}$$

In a similar way we show that

$$||n_t-n_0||_{L_\infty}\le c(||\psi-\psi_0||_{W_p^{3-3/p}(\partial\Omega)}+||\bar f||_{W_p^{3-3/p}(\partial\Omega)}). \tag{8.21}$$

Summing up we obtain the following estimate for solutions to system (8.9)

$$||(u,f)||_\Xi \le c(\epsilon+\lambda^b+T^{b/3}+||(\bar u,\bar f)||_\Xi)||(\bar u,\bar f)||_\Xi$$
$$+c||(\bar q,\bar\psi)||_\Xi^2+c(\lambda^b+T^{b/3})^2||\psi_0||^2_{W_p^{3-3/p}(\partial\Omega)}. \tag{8.22}$$

The above inequality guarantees that if λ and T are sufficiently small compared to the magnitude of norm of the initial datum, we are able to find sufficiently small δ such that

$$||(u,f)||_\Xi \le \delta. \tag{8.23}$$

The next step is to show the contraction property of the map $\mathcal{M}:\Xi\to\Xi$.

From (8.6) we obtain

$$\mathcal{M}(\bar u_1,\bar f_1)-\mathcal{M}(\bar u_2,\bar f_2)=(u_1-u_2,f_1-f_2), \tag{8.24}$$

where u_1-u_2 and f_1-f_2 satisfy the following problem

$$\begin{array}{lll} \Delta(u_1-u_2)=\bar F_2 & \text{in} & \Omega\times(0,T),\\ u_1-u_2=a\Lambda(f_1-f_2)+\bar G_2 & \text{on} & \partial\Omega\times(0,T),\\ \dfrac{\partial(u_1-u_2)}{\partial n}=-\partial_t(f_1-f_2)+\bar H_2 & \text{on} & \partial\Omega\times(0,T),\\ f_1-f_2|_{t=0}=0 & \text{on} & \partial\Omega, \end{array} \tag{8.25}$$

where

$$\bar F_2=(\Delta-\bar\Delta_1^t)\bar u_1-(\Delta-\bar\Delta_2^t)\bar u_2+(\bar\Delta_2^t-\Delta_1^t)\bar q,$$
$$\bar G_2=(\Delta_{\overline{\partial\Omega}_1^t}-\Lambda)\bar f_1-(\Delta_{\overline{\partial\Omega}_2^t}-\Lambda)\bar f_2+(\Delta_{\overline{\partial\Omega}_1^t}-\Delta_{\overline{\partial\Omega}_2^t})\bar\psi,$$
$$\bar H_2=n^0(\bar n_1^t-n^0)\partial\bar f_1-n^0(\bar n_2^t-n^0)\partial\bar f_2+n^0(\bar n_1^t-\bar n_2^t)\partial\bar\psi.$$

Repeating the estimations for the bound (8.23) we obtain

$$||(u_1-u_2,f_1-f_2)||_\Xi\le$$
$$c(||(\bar u_1,\bar f_1)||_\Xi+||(\bar u_2,\bar f_2)||_\Xi+||(\bar q,\bar\psi)||_\Xi+\epsilon+\lambda^b+T^{b/3})||(\bar u_1-\bar u_2,\bar f_1-\bar f_2)||_\Xi. \tag{8.26}$$

By the smallness of δ (this magnitude can be decreased) we are able to find sufficiently small T such that

$$||\mathcal{M}(\bar u_1,\bar f_1)-\mathcal{M}(\bar u_2,\bar f_2)||_\Xi\le\frac12||(\bar u_1-\bar u_2,\bar f_1-\bar f_2)||_\Xi. \tag{8.27}$$

Hence there exists a unique fixed point (u_*,f_*) of the map $\mathcal{M}$ satisfying the bound

$$||(u_*,f_*)||_\Xi\le\delta \tag{8.28}$$

with δ as in (8.23).

By the whole procedure we obtain the solution to (2.1)

$$q=u_*+\bar q,\qquad \psi=f_*+\bar\psi. \tag{8.29}$$

Relations (1.7), (2.2), and (2.13) describe the solution to the original problem (1.1). The meaning of equations $(2.1)_1$ is prescribed by the weak formulation (2.23) in the end of Section 2. Theorem 1.1 is proved.

Acknowledgment. The author would like to thank Piotr Rybka for useful discussions during the preparation of the paper. The work has been supported by Polish KBN grant No. 1 PO3A 037 28.

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