On the structure of flows through pipe-like domains fulfilling a geometrical constraint[1]

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Abstract. We study solutions of the steady Navier-Stokes equations in a bounded 2D domain with the slip boundary conditions admitting flow across the boundary. We show conditions guaranteeing uniqueness of the problem. Next, we examine the structure of such a solution considering an approximation given by a natural linearization. Suitable errors estimates are obtained, too.

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1 Introduction

In this note we would like to analyze a model of two dimensional flows of viscous incompressible fluids motion through bounded domains with nontrivial flows across the boundary. The motion is governed by the steady Navier-Stokes equations

$$\begin{array}{lll} v \cdot \nabla v - \nu \Delta v + \nabla p = F & \text{in} & \Omega, \\ \text{div } v = 0 & \text{in} & \Omega, \end{array} \tag{1.1}$$

where $v = (v^1, v^2)$ is the velocity of the fluid, p is the pressure, ν - the constant positive viscous coefficient, F - the external force and Ω is a simply connected domain, where our motion is studied.

As a supplement to (1.1) we take the slip boundary conditions involving friction

$$\begin{array}{lll} n \cdot \mathbf{T}(v,p) \cdot \tau + f v \cdot \tau = 0 & \text{on} & \partial\Omega, \\ n \cdot v = d & \text{on} & \partial\Omega, \end{array} \tag{1.2}$$

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where n and τ are the normal and tangent vectors to boundary $\partial\Omega$, f describes the friction between the fluid and the surface of the boundary, it is nonnegative, in general nonconstant and $\mathbf{T}(\cdot,\cdot)$ is the stress tensor of the Newtonian fluids, i.e.

$$\mathbf{T}(v,p) = \nu\mathbf{D}(v) - pId = \{\nu(v^i_{,j} + v^j_{,i}) - p\delta ij\}_{i,j=1,2}. \tag{1.3}$$

Datum d describes the inflow and outflow across the boundary.

This type of boundary relations may be treated as an alternative to the standard Dirichlet boundary data

$$u = D \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

The current theory for problem (1.1), (1.4) in general domains for nonzero data D delivers us existence of solutions only for small magnitudes of fluxes given by boundary data. For arbitrary large data we are able to find solutions only for problems in simply connected domains [3, 4-Chap. VIII, 6]. A difficulty is connected with the applications of the energy method which seems to be one natural way for the Dirichlet problem. In this approach there is a need to construct an extension of the boundary data, say $\mathcal{D}$, such that

$$\text{div } \mathcal{D} = 0, \quad \mathcal{D}|_{\partial\Omega} = D. \tag{1.5}$$

Moreover, the biggest obstacle to apply the energy method, comes from the obligation that field $\mathcal{D}$ must satisfy the following estimate

$$\left|\int_\Omega u \cdot \nabla\mathcal{D} u dx\right| \leq \delta ||u||_{H^1_0(\Omega)} \tag{1.6}$$

for any small δ and for all $u \in H^1_0(\Omega)$. Smallness of δ is determined by the magnitude of the viscosity. For arbitrary D we can find field $\mathcal{D}$ only for a simply connected domain.

To see the crucial feature of inequality (1.6) we are looking on the modification of equation (1.1) after a decomposition of the velocity as follows

$$v = \mathcal{D} + u, \tag{1.7}$$

then from (1.1) we get

$$\begin{array}{ll} v \cdot \nabla u - \nu\Delta u + \nabla p = F - u \cdot \nabla\mathcal{D} - \mathcal{D} \cdot \nabla\mathcal{D} + \nu\Delta\mathcal{D}, & \text{in} \quad \Omega, \\ \text{div } u = 0 & \text{in} \quad \Omega. \end{array} \tag{1.8}$$

By (1.7) the new searched function has good properties at the boundary, i.e.

$$u|_{\partial\Omega} = 0. \tag{1.9}$$

To obtain the energy estimate for (1.8) we need to bound the second term of the r.h.s. of $(1.8)_1$. Thus we apply (1.6) remembering that Ω is simply connected and field $\mathcal{D}$ can be constructed.

The uniqueness of solutions for the Dirichlet problem is prescribed only by smallness conditions on data D. It is a consequence of the energy approach which does not involve geometrical properties of domain Ω.

The inadequacy of the method for the Dirichlet problem (1.1),(1.4) follows form the modification of the original system. Because of general properties of the energy technique we need to have homogeneous boundary data, thus we introduce field $\mathcal{D}$ and consider function u instead of v. But this change leads to the appearance of bad term $-u \cdot \nabla \mathcal{D}$ in equation $(1.8)_1$ which is a consequence of the subtraction. The system loses its physical structure.

That is the reason to create a new approach to this issue. The technique should be based on the original equations and we have to avoid any subtractions for nonlinear systems. In this paper we present such a technique, but only for the slip boundary conditions (1.2). Moreover we will have to assume an extra features of the examined domain and smallness of effective friction f/ν. But then we obtain a priori estimate for the solutions to problem (1.1), (1.2) without any bound on largeness of boundary data d and external force F. The restriction is completely independent of largeness of the flow. And the bound has a linear form as the standard energy estimate for the homogeneous boundary data.

Our approach takes into account geometrical features of the domain, hence the uniqueness conditions will depend on properties of Ω, not only on the magnitude of data.

A key idea is a reformulation of problem (1.1), (1.2). A crucial role will play the vorticity of the velocity

$$\alpha = \text{rot}\ v = v^2_{,1} - v^1_{,2}. \tag{1.10}$$

Note that it is possible only in two space dimensional case, since here the vorticity is a scalar function and in three dimensional case it is the whole vector.

Taking the rotation of $(1.1)_1$ we obtain a scalar equation

$$v \cdot \nabla\alpha - \nu\Delta\alpha = \text{rot } F \qquad \text{in} \quad \Omega. \tag{1.11}$$

Also the form of (1.11) follows from the fact that

$$\text{rot } (v \cdot \nabla v) = v \cdot \nabla\alpha, \tag{1.12}$$

but only in our case, since in general 3-D we have

$$\text{rot } (v \cdot \nabla v) = v \cdot \nabla \text{rot } v - \text{rot } v \cdot \nabla v. \tag{1.13}$$

We need to add a boundary relation to (1.11). We apply here an interesting feature of the slip boundary conditions. From (1.2) we compute the Dirichlet datum for the vorticity as follows

$$\alpha = (2\chi - f/\nu)v \cdot \tau + 2d_{,s} \quad \text{on} \quad \partial\Omega, \tag{1.14}$$

where χ is the curvature of Ω and s is the unit length parameter of $\partial\Omega$.

To close system (1.11), (1.14) we add the following elliptic problem on the velocity

$$\begin{array}{lll} \text{rot } v = \alpha & \text{in} & \Omega, \\ \text{div } v = 0 & \text{in} & \Omega, \\ n \cdot v = d & \text{on} & \partial\Omega. \end{array} \tag{1.15}$$

It is obvious that the coupled system (1.11), (1.14) and (1.15) as an equivalent to the original one (1.1), (1.2). Thus, we will investigate the reformulation, since this form is more suitable to reach our analysis.

The main advantage of this approach is the a priori estimate. The form of problem (1.11), (1.14) allows to apply the type of a maximum principle.

Take (1.11), multiply by

$$(\alpha - k^*)_+ = \max\{\alpha - k^*, 0\}, \tag{1.16}$$

where

$$k^* = \sup_{x \in \partial\Omega} (2\chi - f/\nu)v \cdot \tau + 2d_{,s}. \tag{1.17}$$

By definition (1.16) function $(\alpha - k^*)_+$ on the boundary is zero, hence we get

$$\int_\Omega \frac{1}{2} v \cdot \nabla(\alpha - k^*)_+ dx + \nu \int_\Omega |\nabla(\alpha - k^*)_+|^2 dx = -\int_\Omega F \cdot \nabla^\perp(\alpha - k^*)_+ \tag{1.18}$$

which implies

$$||\nabla(\alpha - k^*)_+||_{L_2(\Omega)} \leq \frac{1}{\nu}||F||_{L_2(\Omega)}. \tag{1.19}$$

The same bound we get for

$$(\alpha - k_*)_- = \min\{\alpha - k_*, 0\} \tag{1.20}$$

with

$$k_* = \inf_{x\in\partial\Omega}(2\chi - f/\nu)v\cdot\tau + 2d_{,s}, \tag{1.21}$$

i.e.

$$||\nabla(\alpha - k_*)_-||_{L_2(\Omega)} \leq \frac{1}{\nu}||F||_{L_2(\Omega)}. \tag{1.22}$$

This approach defines us the following decomposition of the vorticity

$$\alpha = \alpha_{L_\infty} + \alpha_{H_0^1}, \tag{1.23}$$

such that

$$\alpha_{L_\infty} \in L_\infty(\Omega) \quad \text{ and } \quad ||\alpha_{L_\infty}||_{L_\infty(\Omega)} \leq \max\{k^*, k_*\} \tag{1.24}$$

and

$$\alpha_{H_0^1} \in H_0^1(\Omega) \quad \text{ and } \quad ||\nabla\alpha_{H_0^1}||_{L_2(\Omega)} \leq \frac{1}{\nu}||F||_{L_2(\Omega)}. \tag{1.25}$$

This way we obtain an estimate for α in the $L_\infty(\Omega) + H_0^1(\Omega)$-space which reads

$$||\alpha||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq ||2\chi - f/\nu||_{L_\infty(\partial\Omega)}||v||_{C(\partial\Omega)} + c||d||_{W_\infty^1(\partial\Omega)} + \frac{1}{\nu}||F||_{L_2(\Omega)}. \tag{1.26}$$

Bound (1.26) follows from (1.24) and (1.25).

Next, the standard theory of the elliptic operators deliver us the following bound for the solutions of (1.15)

$$||v||_{C(\Omega)} \leq A(\Omega)||\alpha||_{L_\infty(\Omega)+H_0^1(\Omega)} + c||d||_{W_\infty^1(\Omega)}. \tag{1.27}$$

Estimate (1.26) and (1.27) lead to the main a priori bound

$$\begin{gathered} ||v||_{C(\Omega)} + ||\alpha||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq \\ B(1 - A(\Omega)||2\chi - f/\nu||_{L_\infty(\partial\Omega)})^{-1}(||d||_{W_\infty^1(\Omega)} + \tfrac{1}{\nu}||F||_{L_2(\Omega)}), \end{gathered} \tag{1.28}$$

if the following condition is valid

$$GC = A(\Omega)||2\chi - f/\nu||_{L_\infty(\partial\Omega)} < 1. \tag{1.29}$$

The above constraint delivers us a geometrical restriction on the shape of the domain. Since quantity f/ν is nonnegative and may be chosen independently from the domain, to fulfill condition (1.29) we need only the negative part of the curvature. Hence (1.29) implies that the domain must fulfill the following relation

$$A(\Omega) \inf_{x\in\partial\Omega} \min\{\chi, 0\} > -1. \tag{1.30}$$

Then we are able to find suitable f and ν to condition (1.29) be satisfied.

Constant $A(\Omega)$ is optimal and depends only on geometrical features of domain Ω. In general $A(\Omega) \sim H$, where parameter H is defined in the following way

$$H = \min_{\phi\in[0,\pi)} h_\phi, \tag{1.31}$$

where

$$h_\phi = \max_{y_1\in\mathbf{R}} \lambda_1\{(y_1, y_2) : y_2 \in \mathbf{R}) \cap \Omega)\}, \tag{1.32}$$

where λ_1 is the one dimensional Lebesgue measure and y is the coordinate system obtained from the original x-coordinates by the rotation on angel ϕ.

The obtained bound guarantees the following existence result.

Theorem A. *Let* $0 < a < 1$, $F \in L_2(\Omega)$ *and* $d \in W^1_\infty(\partial\Omega)$.
If

$$A(\Omega)||2\chi - f/\nu||_{L_\infty(\partial\Omega)} < 1, \tag{1.33}$$

then there exists at least one solution to problem (1.1), (1.2) such that

$$v \in C^a(\Omega) \quad \textit{and} \quad rot\, v \in L_\infty(\Omega) + H^1_0(\Omega), \tag{1.34}$$

and the following estimate is valid

$$\begin{gathered} ||v||_{C^a(\Omega)} + ||(rot\, v)_{L_\infty}||_{L_\infty(\Omega)} + ||(rot\, v)_{H^1_0}||_{H^1_0(\Omega)} \leq \\ B(||d||_{W^1_\infty(\partial\Omega)} + \tfrac{1}{\nu}||F||_{L_2(\Omega)}) = S_0, \end{gathered} \tag{1.35}$$

where $rot\, v = (rot\, v)_{L_\infty} + (rot\, v)_{H^1_0}$, $(rot\, v)_{L_\infty} \in L_\infty(\Omega)$, $(rot\, v)_{H^1_0} \in H^1_0(\Omega)$ *and* B *is independent of* ν.

We skip a proof of Theorem A, since it follows from the a priori bound (1.28) proved at the beginning. The obtained regularity of the solutions guarantees the existence by the standard application of the Leray-Schauder fixed point theorem. A proof of similar result can be found in [5-Theorem 3.1].

As an alternative to this assumption for bounded datum d, we may assume existence of such a vector field $\mathcal{D}$ that

$$n \cdot \mathcal{D}|_{\partial\Omega} = d, \quad \text{div}\, \mathcal{D} = 0 \quad \text{and} \quad \text{rot}\, \mathcal{D} = 0 \tag{1.36}$$

and

$$\mathcal{D} \in W^1_\infty(\Omega) \cap W^1_\infty(\partial\Omega). \tag{1.37}$$

A construction of such field we present in section 4.

One of the aims of the presented note is the description of conditions which imply uniqueness of the solutions.

In section 3 we prove the following theorem.

Theorem B. *Let assumptions of Theorem A be fulfilled and*

$$A(\Omega)\left(\frac{S_0}{\nu}(|\Omega|^{1/2} + H) + ||2\chi - f/\nu||_{L_\infty(\partial\Omega)}\right) < 1 \tag{1.38}$$

or

$$\frac{B_2^2(1+H)^2}{\nu}\left(B_1^2||d||_{L_\infty(\partial\Omega)} + S_0\right) < 1. \tag{1.39}$$

Then the system (1.1)-(1.2) admits only one solution.

A comparison of these two conditions shows that the standard energy approach neglects the influence of the features of the domain. (1.39) implies that S_0 and $||d||_{L_\infty}$ have to be small. Condition (1.38) takes into account quantities h and $|\Omega|$ as well as condition (1.33) and admits large data (provided smallness of H and $|\Omega|$).

A natural consequence of the uniqueness result is an analysis of the structure of such solutions. Since the very norms of them are relatively small we want to find an approximation of them as series. In section 4 we prove the following result.

Theorem C. *Let S_0 be sufficiently small, then the solutions of problem (1.1)-(1.2) have the following form*

$$v = u^0 + u^1 + ... + u^N + u \tag{1.40}$$

for any $N \in \mathbf{N}$*, moreover* u^l *is a solution of a linear system which depends only on* $u^{l-1}, ..., u^0$ *and data; additionally*

$$||u^l||_{C^a(\Omega)} \leq \Xi^l \tag{1.41}$$

for $l = 0, ..., N$*, and also the rest*

$$||u||_{C^a(\Omega)} \leq B\Xi^{N+1}, \tag{1.42}$$

where $\Xi = B_0 S_0$ *and* B_0 *depends on the constants of the problem.*

Corollary of Theorem C. *The solution given by Theorem C can be expressed by the following series*

$$v = \sum_{l=0}^{\infty} u^l. \tag{1.43}$$

The above statement follows from estimate of Theorem C the fact that $\Xi < 1$.

The key element of Theorem C is the bound (1.14). The method applied to studied system makes us possible obtain information in the Hölder class. Of course, it can be done, since we have already had Theorem A with bound (1.31). Thus, Theorem C is a consequence of Theorems A and B. Applying the standard energy approach we would obtain a weaker result with less information about the error of the approximation.

2 Notation

Throughout the paper we try to use the standard notation [7].

By $L_p(\Omega)$ we denote the Lebesgue space of integrable functions with the p-power with the following norm

$$||f||_{L_p(\Omega)} \leq \left(\int_\Omega |f(x)|^p dx \right)^{1/2} \tag{2.1}$$

for $1 \leq p < \infty$, and for $p = \infty$

$$||f||_{L_\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} |f(x)|. \tag{2.2}$$

By $C^a(\Omega)$ we denote the space of Hölder continuous functions with norm

$$||f||_{C^a(\Omega)} = \sup_{x\in\Omega} |f(x)| + \sup_{x,y\in\Omega; x\neq y} \frac{|f(x) - f(y)|}{|x-y|^a}. \tag{2.3}$$

Moreover $X + Y$, for X, Y Banach spaces, denotes the space

$$X + Y = \{z = x + y : x \in X \quad \text{and} \quad y \in Y\} \tag{2.4}$$

and

$$||z||_{X+Y} = \inf\{||x||_X + ||y||_Y : x + y = z\}. \tag{2.5}$$

Also, the well known theory [1,2,7] delivers us the following auxiliary results.

Lemma 2.1. *Let $\partial\Omega \in C^2$ and $u \in W^1_p(\Omega)$, then the trace*

$$\tilde{u} = u|_{\partial\Omega} \tag{2.6}$$

is well defined as a function belongs to $W^{1-1/p}_p(\partial\Omega)$ and

$$||\tilde{u}||_{W^{1-1/p}_p(\partial\Omega)} \leq c||u||_{W^1_p(\Omega)}. \tag{2.7}$$

Lemma 2.2. *(Korn inequality) Let $u \in H^1(\Omega)$, moreover $div\, u = 0$ and $n \cdot u|_{\partial\Omega} = 0$, then*

$$||u||_{H^1(\Omega)} \leq c||\mathbf{D}(u)||_{L_2(\Omega)}. \tag{2.8}$$

Lemma 2.3. *Let $b \in W^{2-1/p}_p(\partial\Omega)$, then the solution of the following problem*

$$\begin{array}{ll} \Delta\varphi = 0 & in \quad \Omega, \\ \varphi = b & on \quad \partial\Omega \end{array} \tag{2.9}$$

satisfies the following bound

$$||\varphi||_{W^2_p(\Omega)} \leq c||b||_{W^{2-1/p}_p(\partial\Omega)}. \tag{2.10}$$

3 Proof of Theorem B

In this part we prove Theorem B. Two conditions (1.38) and (1.39) follow from two different approaches to the issue of uniqueness. The first one is based on the analysis of the coupled system (1.11), (1.14) and (1.15).

Take two solutions of problem (1.1), say, v^1 and v^2. They fulfill the following systems

$$\begin{array}{lll} v^i \cdot \nabla \alpha^i - \nu \Delta \alpha^i = \mathrm{rot}\, F & \text{in} & \Omega, \\ \alpha^i = (2\chi - f/\nu) v^i \cdot \tau + 2d_{,s} & \text{on} & \partial\Omega, \\ \mathrm{rot}\, v^i = \alpha^i & \text{in} & \Omega, \\ \mathrm{div}\, v^i = 0 & \text{in} & \Omega, \\ n \cdot v^i = d & \text{on} & \partial\Omega \end{array} \tag{3.1}$$

for $i = 1, 2$.

Introduce

$$V = v_1 - v_2 \qquad \text{and} \qquad R = \alpha_1 - \alpha_2, \tag{3.2}$$

then after the subtraction of system (3.1) for $i = 1$ and $i = 2$ we get

$$\begin{array}{lll} v^1 \cdot \nabla R - \nu \Delta R = -V \cdot \nabla \alpha^2 & \text{in} & \Omega, \\ R = (2\chi - f/\nu) V \cdot \tau & \text{on} & \partial\Omega, \\ \mathrm{rot}\, V = R & \text{in} & \Omega, \\ \mathrm{div}\, V = 0 & \text{in} & \Omega, \\ n \cdot V = 0 & \text{on} & \partial\Omega. \end{array} \tag{3.3}$$

We repeat the estimation for the a priori bound (1.16)-(1.28) for the solutions to obtain a suitable condition to control the uniqueness.

Let

$$K^* = \sup_{x \in \partial\Omega} (2\chi - f/\nu) V \cdot \tau. \tag{3.4}$$

Take $(3.3)_1$, multiply by

$$(R - K^*)_+ = \max\{R - K^*, 0\}, \tag{3.5}$$

then integrating over Ω, we get

$$\nu \int_\Omega |\nabla (R - K^*)_+|^2 dx \le \left| \int_\Omega V \cdot \nabla \alpha^2 (R - K^*)_+ dx \right|. \tag{3.6}$$

The last integral we divide using the decomposition of vorticity α^2. By Theorem A we know that

$$\alpha^2 = \alpha^2_{L_\infty} + \alpha^2_{H^1_0}, \tag{3.7}$$

where $\alpha^2_{L_\infty} \in L_\infty(\Omega)$ and $\alpha^2_{H^1_0} \in H^1_0(\Omega)$. Then

$$\begin{aligned} \textstyle\int_\Omega V \cdot \nabla\alpha^2 (R - K^*)_+ dx = \\ - \textstyle\int_\Omega V \cdot \nabla(R - K^*)_+ \alpha^2_{L_\infty} dx + \int_\Omega V \cdot \nabla\alpha^2_{H^1_0}(R - K^*)_+ dx. \end{aligned} \tag{3.8}$$

The first term is bounded as follows

$$||V\alpha^2_{L_\infty}||_{L_2(\Omega)}||\nabla(R - K^*)_+||_{L_2(\Omega)} \le S_0|\Omega|^{1/2}||V||_{L_\infty(\Omega)}||\nabla(R - K^*)_+||_{L_2(\Omega)}. \tag{3.9}$$

The second one we treat in the following way

$$||V \cdot \nabla\alpha^2_{H^1_0}||_{L_2(\Omega)}||(R - K^*)_+||_{L_2(\Omega)} \le S_0 H||V||_{L_\infty(\Omega)}||\nabla(R - K^*)_+||_{L_2(\Omega)}. \tag{3.10}$$

Thus, we get

$$||\nabla(R - K^*)_+||_{L_2(\Omega)} \le \frac{S_0}{\nu}(|\Omega|^{1/2} + H)||V||_{L_\infty(\Omega)}. \tag{3.11}$$

The same bound we obtain for

$$(R - K_*)_- = \min\{R - K_*, 0\} \tag{3.12}$$

with

$$K_* = \inf_{x\in\partial\Omega}(2\chi - f/\nu)V \cdot \tau. \tag{3.13}$$

Thus

$$||\nabla(R - K_*)_-||_{L_2(\Omega)} \le \frac{S_0}{\nu}(|\Omega|^{1/2} + H)||V||_{L_\infty(\Omega)}. \tag{3.14}$$

Next, we find an estimate for the solution of $(3.3)_{3,4,5}$, analogue to (1.27)

$$\begin{aligned} ||V||_{C(\Omega)} \le A(\Omega)\cdot \\ \cdot\left(||\nabla(R - K_*)_-||_{L_2(\Omega)} + ||\nabla(R - K^*)_+||_{L_2(\Omega)} + \max\{|K^*|, |K_*|\}\right)||V||_{C(\Omega)}. \end{aligned} \tag{3.15}$$

Applying (3.4), (3.11), (3.13) and (3.14), we get

$$||V||_{C(\Omega)} \leq A(\Omega)\left(\frac{S_0}{\nu}(|\Omega|^{1/2}+H)+||2\chi-f/\nu||_{L_\infty(\partial\Omega)}\right)||V||_{C(\Omega)}. \quad (3.16)$$

Thus V is equal zero if

$$A(\Omega)\left(\frac{S_0}{\nu}(|\Omega|^{1/2}+H)+||2\chi-f/\nu||_{L_\infty(\partial\Omega)}\right)<1 \quad (3.17)$$

which shows (1.38).

Note that the above condition depends on $|\Omega|$. It is possible to remove this dependence by the localization of problem (3.3). The next result delivers us a different information, independently from $|\Omega|$.

We would like to compare above result with a condition which follows from the standard energy method applied to the uniqueness issue.

Using the same notation as for (3.3), from (1.1) and (1.2) we get

$$\begin{array}{lll} v^1\cdot\nabla V-\nu\Delta V+\nabla P=-V\cdot\nabla v^2 & \text{in} & \Omega, \\ \text{div } V=0 & \text{in} & \Omega, \\ n\cdot\mathbf{T}(V,P)\cdot\tau+fV\cdot\tau=0 & \text{on} & \partial\Omega, \\ n\cdot V=0 & \text{on} & \partial\Omega \end{array} \quad (3.18)$$

Multiplying $(3.18)_1$ by V, remembering that

$$-\nu\Delta V+\nabla P=-\text{div } \mathbf{T}(V,P), \quad (3.19)$$

we get

$$\int_\Omega v^1\nabla VV dx+\nu\int_\Omega|\mathbf{D}(V)|^2dx+\int_{\partial\Omega}f(V\cdot\tau)^2d\sigma=-\int_\Omega V\cdot\nabla v^2V dx, \quad (3.20)$$

hence

$$\nu\int_\Omega|\mathbf{D}(V)|^2dx+\int_{\partial\Omega}f(V\cdot\tau)^2d\sigma\leq\int_{\partial\Omega}|d||V|^2d\sigma+\int_\Omega|v^2||V||\nabla V|dx. \quad (3.21)$$

The first term of the r.h.s. of (3.21) can be estimated from the trace theorem as follows

$$\int_{\partial\Omega}|d|\,|V|^2d\sigma\leq||d||_{L_\infty(\partial\Omega)}||V||^2_{L_2(\partial\Omega)}\leq B_1^2||d||_{L_\infty(\partial\Omega)}||V||^2_{H^1(\Omega)}, \quad (3.22)$$

where constant B_1 comes from the trace theorem - see Lemma 2.1.

By Lemma 2.2 we get

$$\int_{\partial\Omega} |d|\,|V|^2 d\sigma \leq B_1^2 B_2^2 (1+H)^2 ||d||_{L_\infty(\partial\Omega)} ||\mathbf{D}(V)||^2_{L_2(\Omega)}. \tag{3.23}$$

The second term we treat

$$\int_{\Omega} |v^2|\,|V|\,|\nabla V| dx \leq B_2^2 (1+H)^2 S_0 ||\mathbf{D}(V)||^2_{L_2(\Omega)}. \tag{3.24}$$

Thus, we get

$$||\mathbf{D}(V)||^2_{L_2(\Omega)} \leq \frac{B_2^2(1+H)^2}{\nu} \Big(B_1^2 ||d||_{L_\infty(\partial\Omega)} + S_0 \Big) ||\mathbf{D}(V)||_{L_2(\Omega)}. \tag{3.25}$$

Hence relation (3.25) delivers us the uniqueness of the solutions if

$$\frac{B_2^2(1+H)^2}{\nu} \Big(B_1^2 ||d||_{L_\infty(\partial\Omega)} + S_0 \Big) < 1. \tag{3.26}$$

Conditions (3.17) and (3.26) fulfill the assertion of Theorem B.

4 Expansion of the solution

In this part we investigate the structure of the unique solutions. By Theorem B, it restricts our attention only to cases for small data.

The first component of the expansion of the velocity will be a vector $\mathcal{D}$. This function is defined as a solution to the following problem

$$\begin{array}{lll} \text{rot}\, \mathcal{D} = 0 & \text{in} & \Omega, \\ \text{div}\, \mathcal{D} = 0 & \text{in} & \Omega, \\ n \cdot \mathcal{D} = d & \text{on} & \partial\Omega. \end{array} \tag{4.1}$$

Lemma 4.1. *Let $d \in W^1_\infty(\partial\Omega)$, then there exists a unique solution of problem (4.1) such that $\mathcal{D} \in W^1_p(\Omega)$ for any $2 < p < \infty$ and the following estimate is valid*

$$||\mathcal{D}||_{W^1_p(\Omega)} \leq c(\Omega, p) ||d||_{W^1_\infty(\partial\Omega)}. \tag{4.2}$$

In particular, if $0 < a < 1 - 2/p$, we have

$$||\mathcal{D}||_{C^a(\Omega)} \leq c(\Omega, p) ||d||_{W^1_\infty(\partial\Omega)}. \tag{4.3}$$

Proof. Since our domain is simply connected, by $(4.1)_2$ and the Poincare Lemma there exists a scalar function (stream function) φ_0 such that

$$\mathcal{D} = (-\partial_{x_2}\varphi_0, \partial_{x_1}\varphi_0). \tag{4.4}$$

By boundary condition $(4.1)_3$ and form (4.4) we get

$$n \cdot \mathcal{D} = \frac{d}{ds}\varphi_0 = 0, \tag{4.5}$$

where s is the unit length parameter of curve $\partial\Omega$. Since function φ_0 is defined up a constant we obtain the following elliptic problem

$$\begin{array}{lll} \Delta\varphi_0 = 0 & \text{in} & \Omega, \\ \varphi_0|_{\partial\Omega} = b & \text{on} & \partial\Omega, \end{array} \tag{4.6}$$

where

$$b(s) = \int_{s_0}^{s} d(t)dt \tag{4.7}$$

for a fixed point $s_0 \in \partial\Omega$ and $b(s_0) = 0$.

By the assumptions

$$b \in W^2_\infty(\partial\Omega) \quad \text{and} \quad ||b||_{W^2_\infty(\partial\Omega)} \leq c||d||_{W^1_\infty(\partial\Omega)}. \tag{4.8}$$

The theory of the Schauder estimates of solutions for problem (4.6) is ill posed for the L_∞-space. That is the reason, we embed $W^2_\infty(\partial\Omega)$ into $W^2_p(\partial\Omega)$ for sufficiently large finite p ($2 < p < \infty$). Moreover to avoid unnecessary complications we embed space $W^2_p(\partial\Omega)$ into the trace space $W^{2-1/p}_p(\partial\Omega)$, i.e.

$$W^2_p(\partial\Omega) \subset W^{2-1/p}_p(\partial\Omega) \text{ and } ||b||_{W^{2-1/p}_p(\partial\Omega)} \leq c||b||_{W^2_p(\partial\Omega)}. \tag{4.9}$$

Then the standard theory [1,2] for the Laplace operator (Lemma 2.3) leads us to the following bound on function φ_0

$$||\varphi_0||_{W^2_p(\Omega)} \leq c(\Omega)||d||_{W^{2-1/p}_p(\partial\Omega)}. \tag{4.10}$$

Then by definition (4.4) and the imbedding theorems we get (4.2) and (4.3). Lemma 4.1 is proved.

Next, we study the behavior of the vorticity. A linearization of the vorticity problem delivers us the following system

$$\begin{aligned} -\nu\Delta\alpha_0 &= \text{rot } F \quad \text{in} \quad \Omega, \\ \alpha_0|_{\partial\Omega} &= 2d_{,s} \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.11}$$

For the above system we have the following result

Lemma 4.2. *Let* $F \in L_2(\Omega)$, $d_{,s} \in L_\infty(\partial\Omega)$ *then there exists a unique solution of problem (4.11) such that*

$$\alpha_0 \in L_\infty(\Omega) + H_0^1(\Omega) \tag{4.12}$$

and the following estimate is valid

$$||\alpha_0||_{L_\infty(\Omega)+H_0^1(\Omega)} \le c\left(||d_{,s}||_{L_\infty(\partial\Omega)} + \frac{1}{\nu}||F||_{L_2(\Omega)}\right). \tag{4.13}$$

Proof. To obtain bound (4.13) we repeat the method as for (1.26). Multiplying $(4.11)_1$ by

$$(\alpha_0 - k^*)_+ = \max\{\alpha_0 - k^*, 0\} \tag{4.14}$$

with

$$k^* = \sup_{x\in\partial\Omega} 2d_{,s}, \tag{4.15}$$

integrate over Ω, we get

$$||\nabla(\alpha_0 - k^*)_+||_{L_2(\Omega)} \le \frac{1}{\nu}||F||_{L_2(\Omega)}. \tag{4.16}$$

The same we repeat for the negative part of α_0 with $k_* = \inf_{x\in\partial\Omega} 2d_{,s}$. This way we get (4.13). Lemma 4.2 is shown.

As we see term α_0 gives a new term to the velocity expansion of the same order as $\mathcal{D}$. Hence again we study the velocity problem. Examine the following system

$$\begin{aligned} \text{rot } u_0 &= \alpha_0 \quad \text{in} \quad \Omega, \\ \text{div } u_0 &= 0 \quad\ \ \text{in} \quad \Omega, \\ n \cdot u_0 &= 0 \quad\ \ \text{on} \quad \partial\Omega. \end{aligned} \tag{4.17}$$

Lemma 4.3. *Let α_0 be given by Lemma 4.2, then there exists a unique solution to problem (4.17) such that $u_0 \in C^a(\Omega)$ and*

$$||u_0||_{C^a(\Omega)} \leq c||\alpha_0||_{L_\infty(\Omega)+H_0^1(\Omega)}. \tag{4.18}$$

To prove the above result it is enough to repeat the proof of Lemma 4.1, remembering that $H_0^1(\Omega) \subset L_p(\Omega)$ for any $p < \infty$, if dim $\Omega = 2$.

Our investigation we start from the following form of the solutions. We decompose the velocity using functions $\mathcal{D}$ and u_0 as follows

$$v = \mathcal{D} + u_0 + u \tag{4.19}$$

and the vorticity in the following way

$$\alpha = \alpha_0 + \beta. \tag{4.20}$$

Lemma 4.4. *Let $\mathcal{D}$, u_0 and α_0 be given by Lemmas 4.1, 4.2 and 4.3, then*

$$\begin{array}{l} ||v - (\mathcal{D} + u_0)||_{C^a(\Omega)} \leq S_2||\alpha_0||_{L_\infty(\Omega)+H_0^1(\Omega)}, \\ ||\alpha - \alpha_0||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq S_2||\alpha_0||_{L_\infty(\Omega)+H_0^1(\Omega)}. \end{array} \tag{4.21}$$

Proof. A subtraction of (1.11),(1.14) and (4.11) gives

$$\begin{array}{lll} v \cdot \nabla\beta - \nu\Delta\beta = -v \cdot \nabla\alpha_0 & \text{in} & \Omega, \\ \beta = (2\chi - f/\nu)u \cdot \tau & \text{on} & \partial\Omega. \end{array} \tag{4.22}$$

The same as for the issue of uniqueness we treat problem (4.22) and obtain the following bound

$$||\beta||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq S_1||\alpha_0||_{L_\infty(\Omega)+H_0^1(\Omega)} + ||2\chi - f/\nu||_{L_\infty(\partial\Omega)}||u||_{C(\Omega)}. \tag{4.23}$$

To get information about the velocity, note that perturbation u satisfies the following system

$$\begin{array}{lll} \text{rot } u = \beta & \text{in} & \Omega, \\ \text{div } u = 0 & \text{in} & \Omega, \\ n \cdot u = 0 & \text{on} & \partial\Omega. \end{array} \tag{4.24}$$

Using the result of Lemma 4.3 we conclude

$$||u||_{C^a(\Omega)} \leq A(\Omega)||\beta||_{L_\infty(\Omega)+H_0^1(\Omega)}. \tag{4.25}$$

Thus we get

$$||\beta||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq (1 - A(\Omega)||2\chi - f/\nu||_{L_\infty(\partial\Omega)})^{-1} S_1 ||\alpha_0||_{L_\infty(\Omega)+H_0^1(\Omega)} \quad (4.26)$$

and

$$||u||_{C^a(\Omega)} \leq A(\Omega)(1 - A(\Omega)||2\chi - f/\nu||_{L_\infty(\partial\Omega)})^{-1} S_1 ||\alpha_0||_{L_\infty(\Omega)+H_0^1(\Omega)}. \quad (4.27)$$

Lemma 4.4 is proved.

Next, we want to construct an approximation of the solutions of higher order. Introduce the following notations describing us the expansion up to the k-th order

$$\begin{aligned} v &= v^k + u = u^0 + u^1 + ... + u^k + u, \\ \alpha &= \alpha^k + \beta = \beta^0 + \beta^1 + ... + \beta^k + \beta, \end{aligned} \quad (4.28)$$

where $u_0 = \mathcal{D} + u_0$ and β^0 is defined by Lemma 4.4 and u^k and β for $k \geq 1$ are given as solutions to the following problem

$$\begin{array}{lll} v^{k-1} \cdot \nabla\beta^k - \nu\Delta\beta^k = -u^{k-1} \cdot \nabla\alpha^{k-1} & \text{in} & \Omega, \\ \beta^k = (2\chi - f/\nu) u^k \cdot \tau & \text{on} & \partial\Omega, \\ \text{rot } u^k = \beta^k & \text{in} & \Omega, \\ \text{div } u^k = 0 & \text{in} & \Omega, \\ n \cdot u^k = 0 & \text{on} & \partial\Omega. \end{array} \quad (4.29)$$

Lemma 4.5. *The solutions of (4.29) satisfy the following bounds*

$$\begin{aligned} ||u^k||_{C^a(\Omega)} + ||\beta^k||_{L_\infty(\Omega)+H_0^1(\Omega)} &\leq \Xi^k, \\ ||v - v^k||_{C^a(\Omega)} + ||\alpha - \alpha^k||_{L_\infty(\Omega)+H_0^1(\Omega)} &\leq B\Xi^{k+1}. \end{aligned} \quad (4.30)$$

Proof. The same as for (1.26) we find the estimates for β^k and v^k

$$||\beta^k||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq B||u^{k-1}\alpha^{k-1}||_{L_2(\Omega)} \leq \Xi^{k-1} B S_0. \quad (4.31)$$

Hence

$$||u^k||_{C^a(\Omega)} \leq A\Xi^{k-1} B S_0. \quad (4.32)$$

From (4.31) and (4.32) we conclude $(4.30)_1$.

To show bounds on the errors note that summing systems (4.29) over k for $k = 0, 1, ..., m$ we get

$$\begin{array}{lll} v^{m-1} \cdot \nabla \alpha^m - \nu \Delta \alpha^m = 0 & \text{in} & \Omega, \\ \alpha^m = (2\chi - f/\nu) v^m \cdot \tau + 2d_{,s} & \text{on} & \partial\Omega, \\ \text{rot } v^m = \alpha^m & \text{in} & \Omega, \\ \text{div } v^m = 0 & \text{in} & \Omega, \\ n \cdot v^m = d & \text{on} & \partial\Omega. \end{array} \tag{4.33}$$

Then subtracting (1.11), (1.14) and (1.15) from (4.33) we obtain

$$\begin{array}{lll} v^{m-1} \cdot \nabla \beta - \nu \Delta \beta = -u \cdot \nabla \alpha - u^m \cdot \nabla \alpha & \text{in} & \Omega, \\ \beta = (2\chi - f/\nu) u \cdot \tau & \text{on} & \partial\Omega, \\ \text{rot u} = \beta & \text{in} & \Omega, \\ \text{div u} = 0 & \text{in} & \Omega, \\ n \cdot u = 0 & \text{on} & \partial\Omega. \end{array} \tag{4.34}$$

The estimate of solutions for (4.34) gives

$$||\beta||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq ||2\chi - f/\nu||_{L_\infty(\partial\Omega)} ||u||_{C(\Omega)} + B||u^m \cdot \nabla\alpha||_{L_2(\Omega)}, \tag{4.35}$$

but again

$$||u||_{C(\Omega)} \leq A(\Omega)||\beta||_{L_\infty(\Omega)+H_0^1(\Omega)} \tag{4.36}$$

which follows (since $A(\Omega)||2\chi - f/\nu||_{L_\infty(\partial\Omega)} < 1$)

$$||\beta||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq B||u^m \cdot \nabla\alpha||_{L_2(\Omega)}. \tag{4.37}$$

Thus

$$||\beta||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq B\Xi^m||\nabla\alpha||_{L_2(\Omega)}. \tag{4.38}$$

Solving $(4.34)_{3,4,5}$ we get

$$||u||_{C^a(\Omega)} \leq B\Xi^{m+1}. \tag{4.39}$$

Theorem C is proved.

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