The Navier-Stokes equations and the maximum principle

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Abstract. We consider the steady Navier-Stokes equations in a two dimensional infinite channel with slip boundary conditions. Provided a geometrical constraint on the domain we are able to show existence of solutions for arbitrary large velocity data, including the flux of flow. The crucial point of the proof is the maximum principle for a reformulation of the problem. This technique gives a new type of estimates in the Hölder space for the solutions with the infinite Dirichlet integral.

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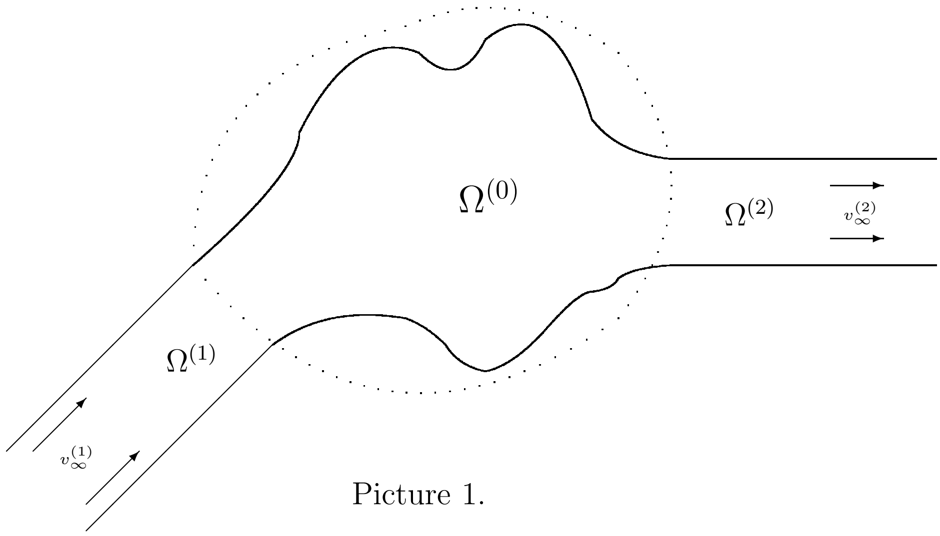
1 Introduction

We consider a flow of an incompressible viscous fluid in a two dimensional infinite channel. This model is described by the steady Navier-Stokes equations with a boundary conditions involving the friction. The problem reads

$$\begin{array}{lll} v\cdot\nabla v - \mathrm{div}\ \mathbf{T}(v,p) = F & \text{in} & \Omega, \\ \mathrm{div}\ v = 0 & \text{in} & \Omega, \\ n\cdot v = 0, \ \ n\cdot\mathbf{T}(v,p)\cdot\tau + fv\cdot\tau = 0 & \text{on} & \partial\Omega, \\ v \to v_\infty^{(i)}(y_2^{(i)}) \ \text{ as } \ y_1^{(i)} \to +\infty & \text{in} & \Omega^{(i)}, \end{array} \tag{1.1}$$

where $v = (v^1, v^2)$ is the velocity of the fluid, p is the pressure, F is an external force, n and τ - the normal (outward) and tangent vectors to boundary $\partial\Omega$, $\mathbf{T}$ is the stress tensor and $\mathbf{T}(v,p) = \nu\mathbf{D}(v) - pId$, where $\mathbf{D}(v) = \{v^i_{,j} + v^j_{,i}\}_{i,j=1,2}$, ν is the constant positive viscous coefficient and $f \geq 0$ is the friction coefficient at the boundary which may be, in general, non constant.



Picture 1.

We consider a domain Ω (see picture 1) which is simply connected and can be described as follows

$$\Omega = \Omega^{(0)} \cup \Omega^{(1)} \cup \Omega^{(2)}, \tag{1.2}$$

where $\Omega^{(0)}$ is bounded and the rest are infinite pipes which in their local coordinates $y^{(i)}$ can be described in the following way

$$\Omega^{(i)} = \{(y_1^{(i)}, y_2^{(i)}) \in \mathbf{R}^2 : y_1^{(i)} \in \mathbf{R}_+, y_2^{(i)} \in (0, H^{(i)})\}, \tag{1.3}$$

where $H^{(i)}$ is the height of the i-th pipe. Quantities $v_\infty^{(i)}$ are velocities on the inlet and outlet of domain Ω and they are proportional to a flow w_f. Velocity w_f is a solution to equations $(1.1)_{1,2,3}$ with constant f in a perfect straight pipe and in a model case (the height of the pipe is equal to 1 - $\Omega = \mathbf{R} \times (0,1)$) one can easily check that

$$w_f(y_2) = \left(TF\left[-\frac{6f}{f+6}y_2^2 + \frac{6f}{f+6}y_2 + \frac{6}{f+6}\right], 0\right), \tag{1.4}$$

where TF is the flux of the flow. This scalar quantity characterizes the flow described by system (1.1) and by $(1.1)_{2,3}$ is well defined. We see that if $f = 0$ then we obtain the constant flow $(TF, 0)$ and if $f \to \infty$ we get the Poiseuille flow. In the last case our condition $(1.1)_3$ changes to the zero Dirichlet datum.

We put on boundaries of pipes $\Omega^{(1)}$ and $\Omega^{(2)}$ constant friction f (the coefficient is the same on each of component of $\partial\Omega^{(i)}$). On the part for $\partial\Omega^{(0)}$ it may be non constant.

To keep the agreement with $(1.1)_2$ we add the compatibility condition on the conservation of the total flux of flow. The following constraint should be assumed

$$\int_0^{H^{(1)}} v_\infty^{(1)\,1}(y_2^{(1)})dy_2^{(1)} + \int_0^{H^{(2)}} v_\infty^{(2)\,1}(y_2^{(2)})dy_2^{(2)} = 0. \tag{1.5}$$

The aim of this paper is to prove existence of solutions. The main result is the following.

Theorem 1.1 *Let $\partial\Omega \in C^2$, $F \in L_2(\Omega)$ and $0 < a < 1$. If*

$$A(\Omega)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)} < 1, \tag{1.6}$$

where $A(\Omega)$ depends only on properties of domain Ω - see Lemma 2.2 and χ is the curvature of boundary $\partial\Omega$, then there exists at least one weak-$*$ solution of problem (1.1) such that $v \in C^a(\Omega)$, $rot\, v \in L_\infty(\Omega) + H_0^1(\Omega)$ and the following estimate is valid

$$||v||_{C^a(\Omega)} + ||rot\, v||_{L_\infty(\Omega)+H_0^1(\Omega)} \le c(D_\infty + ||F||_{L_2(\Omega)}), \tag{1.7}$$

where $D_\infty = ||v_\infty^{(1)}||_{C(0,H^{(1)})} + ||v_\infty^{(2)}||_{C(0,H^{(2)})}$.

Theorem 1.1 gives a solvability to problem (1.1) for large velocity data, there is no bound on the L_2-norm of force F and the magnitude of the flux described by data $v_\infty^{(i)}$. We require only condition (1.6) to be fulfilled. It restricts a class of domains, where Theorem 1.1 is able to apply - see Lemma 2.2. However geometrical constraint (1.6) is completely independent of quantities D_∞ and $||F||_{L_2(\Omega)}$. The result gives a new type of estimate for solutions to the Navier-Stokes equations in domains with boundaries, which is not connected with the energy of the system. This type of bound has been known only for the problem in the whole $\mathbf{R}^2$ - see [6,8]. In this special case we find the bound which neglects the nonlinear term. That is the reason, (1.7) has a linear character the same as the standard energy estimate for problems with homogeneous boundary data.

The estimate (1.7), which is the main information known about the solutions, implies no properties of behavior at infinity, hence the result proves only existence of weak-$*$ solutions, because we are not able to control convergence to data as $|x| \to \infty$. And it seems to be not so obvious if (1.7) together with properties of the system can give such a information. The proof is based on an approximation of the problem in bounded domains. In this case we are able to show regular solutions - see Theorem 3.1. Since for the each step of the approximation the domain is bounded, estimate (1.7) guarantees also strong compactness. Next, taking a suitable subsequence of approximations we obtain a solution to problem (1.1). The definition of the sense of the obtained solution is given in section 4 - Definition 4.2.

Let us note that weakness of solutions is only at infinity, thus one can prove the following result which arises naturally from Theorem 1.1.

Theorem 1.2. *Let $F \in C^\infty(\overline{\Omega}) \cap L_2(\Omega)$ and $\partial\Omega \in C^\infty$, then the solutions given by Theorem 1.1 are smooth, i.e. $v \in C^\infty(\overline{\Omega})$.*

We omit the proof of Theorem 1.2, because it is just a technical consequence which follows from the well-known theory [9,14].

The problem considered in this paper can be treated as a kind of Leray's problem [5 chap XI], i.e. $(1.1)_{1,2,4}$ with $v = 0$ on the boundary (or $f \to \infty$). The existence for the classical Leray problem is still open for large data and there are only results for small fluxes [1,4,5 chap XI]. Difficulties are hidden in the energy of the system, the so-called Dirichlet integral, which in this model is infinite. That is the reason that the energy methods works only for small data [5 chap XI]. An approach to this problem, but without conditions on infinity, with large fluxes has been done in [10].

In our paper we examine slip boundary condition $(1.1)_3$ which is an alternative to the no slip constraint [3,7,13]. And if $f > 0$ then rot $v_\infty \neq 0$ which implies that the Dirichlet integral $\int_\Omega \nabla v : \nabla v dx = \infty$ as in Leray's problem. But in our case if condition (1.6) is fulfilled, which is connected with a strong condition on the shape of the domain and smallness of the friction coefficient f, we apply estimates of the solutions in $L_\infty(\Omega)$ and $C^a(\Omega)$-spaces omitting difficulties with the infinite energy of the system.

The slip boundary conditions are a natural supplement to the Navier-Stokes equations when we want to approximate the perfect flow by a viscous one. These properties one can find in [2,11], they are based on an estimate of type (1.7).

The presented technique arises from the feature of the slip boundary conditions, which enables to define the vorticity on the very boundary. This property has been applied in 3D for a flow in a cylinder in [15].

A key element of the proofs is the reformulation of the problem. The form of equation (1.1) are not suitable to consider them in function spaces of type

$C(\Omega)$ or $L_\infty(\Omega)$. We apply here the equations on the vorticity of the velocity. Using the feature of slip boundary condition $(1.1)_3$ we describe completely the Dirichlet datum on the vorticity on the boundary of the domain - to get $(1.8)_2$ it is enough to differentiate the first condition of $(1.1)_3$ with respect to the length parameter of $\partial\Omega$ and combine it with the second one. Hence we get the following problem

$$\begin{array}{lll} v\cdot\nabla\alpha-\nu\Delta\alpha=g & \text{in} & \Omega, \\ \alpha=v\cdot\tau(2\chi-f/\nu) & \text{on} & \partial\Omega, \\ \alpha\to\alpha_\infty^{(i)} \ \text{ as } \ y_1^{(i)}\to\infty & \text{in} & \Omega^{(i)}, \end{array} \tag{1.8}$$

where $g=\text{rot}\ F$ and

$$\alpha=\text{rot}\ v=v^2_{,1}-v^1_{,2} \tag{1.9}$$

which is a scalar, χ is the curvature of $\partial\Omega$ and $\alpha_\infty^{(i)}(y_2^{(i)})=-\partial_{y_2^{(i)}}({v_\infty^{(i)}}^1(y_2^{(i)}))$.

In the two dimensional case equation (1.8) has a special form. In the three dimensions there is an extra term $\alpha\cdot\nabla v$ in equation $(1.8)_1$. Here the vorticity is a scalar, not a vector as in 3D and this term vanishes. It causes that problem (1.8) possesses the maximum principle. This property appears not so often in problems dealing with the Navier-Stokes equations.

Next, to complete problem (1.8) there is a need to describe the velocity having the vorticity and it is given by the following problem

$$\begin{array}{lll} \text{rot}\ v=\alpha & \text{in} & \Omega, \\ \text{div}\ v=0 & \text{in} & \Omega, \\ n\cdot v=0 & \text{on} & \partial\Omega, \\ v^1(y^{(i)})\to{v_\infty^{(i)}}^1(y^{(i)}) \ \text{ as } \ y_1^{(i)}\to\infty & \text{in} & \Omega^{(i)}. \end{array} \tag{1.10}$$

Since we require domain Ω to be simply connected, by the Poincare Lemma and $(1.10)_2$, vector v can be described by a scalar function φ in the following way

$$v=\nabla^\perp\varphi=(-\partial_{x_2}\varphi,\partial_{x_1}\varphi). \tag{1.11}$$

This form simplifies equations $(1.10)_{1,2}$ to the Laplace operator, which makes our considerations easier.

Since any solution to (1.8)-(1.10) defines a solution to the original system, we investigate the coupled system (1.8) and (1.10), instead of analyze problem (1.1).

The paper is organized as follows. In section 2 we define some notations and quantities which will be necessary in the proofs. Next we consider the approximation of the problem and show existence for it. In section 4 we prove Theorem 1.1.

2 Notation

We try to use the standard notations [9,14].

$$||f||^2_{H^1(\Omega)} = \int_\Omega |\nabla f|^2 dx + \int_\Omega |f|^2 dx, \tag{2.1}$$

where $\nabla = (\partial_{x_1}, \partial_{x_2})$;

$$||f||_{C^a(\Omega)} = \sup_{x\in\Omega} |f(x)| + \sup_{x,y\in\Omega; x\neq y} \frac{|f(x) - f(y)|}{|x-y|^a}. \tag{2.2}$$

For domain Ω - Picture 1 and (1.2), (1.3) we define for $1 \leq p < \infty$

$$||f||_{L_{p(loc)}(\Omega)} = \sup_{k\in\mathbf{N}} ||f||_{L_p(O_k)} = \sup_{k\in\mathbf{N}} \left(\int_{O_k} |f|^p dx \right)^{1/p}, \tag{2.3}$$

where $\{O_k\}_{k=1}^\infty$ is a cover of Ω such that

$$2H \leq \text{diam } O_k \leq 3H, \tag{2.4}$$

for every $k \in \mathbf{N}$, where

$$H = \max\{H^{(1)}, H^{(2)}, \text{diam } \Omega^{(0)}\}. \tag{2.5}$$

The same we define $W^l_{p(loc)}$

$$||f||_{W^l_{p(loc)}(\Omega)} = \sup_{k\in\mathbf{N}} \left(\sum_{0\leq|\alpha|\leq l} ||\partial^\alpha_x f||_{L_p(O_k)} \right), \tag{2.6}$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N} \times \mathbf{N}$, $|\alpha| = \alpha_1 + \alpha_2$, $\partial^\alpha_x = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2}$.

The crucial quantity in the statement of condition (1.6) is the constant from the Poincare inequality.

Lemma 2.1. *Let domain Ω fulfills conditions from section 1. If $u \in W^1_p(\Omega)$ and $u|_{\partial\Omega} = 0$, then the following estimate holds*

$$||u||_{L_p(\Omega)} \leq c(\Omega) ||\nabla u||_{L_p(\Omega)} \tag{2.7}$$

for $1 \leq p \leq \infty$*, where* $c(\Omega) \sim H$*.*

Next, we introduce constant $A(\Omega)$ from (1.6) by the following lemma. A proof can be found in [12 - section 4].

Lemma 2.2. *Let domain* Ω *fulfills conditions from section 1 and* $h \in L_\infty(\Omega)$*. Then the weak solution of the following problem*

$$\begin{array}{ll} \Delta\varphi = h & in \quad \Omega, \\ \varphi = 0 & on \quad \partial\Omega, \\ \varphi \to 0 \quad as \quad |x| \to \infty & \end{array} \tag{2.8}$$

satisfies the bound

$$||\nabla\varphi||_{C(\Omega)} \leq A(\Omega)||h||_{L_\infty(\Omega)}, \tag{2.9}$$

where $A(\Omega) \sim H$*.*

To see relation $A(\Omega) \sim H$, it is enough to take (2.8) with $f \equiv 1$ and Ω - the straight pipe, then $A(\Omega) = 1/2H$. This relation describes the restriction of condition (1.6). If we considered a case with non simply connected domain, the curvature would be $\chi \sim 1/H$. So $A(\Omega)\chi \sim$ constant, which excludes such a case. Of course, this consideration shows only that in general there is no large class of non simply connected domain fulfilling the geometrical condition.

3 The approximation

In order to prove existence for problem (1.8)-(1.10) there is a need to construct an approximation in a bounded domain to avoid difficulties with a lack of compactness in the unbounded case. First we define an approximation of domain Ω by introducing the following set of domains

$$\Omega_N = \Omega^{(0)} \cup \{\bigcup_{i=1}^{2} \Omega_N^{(i)}\}, \tag{3.1}$$

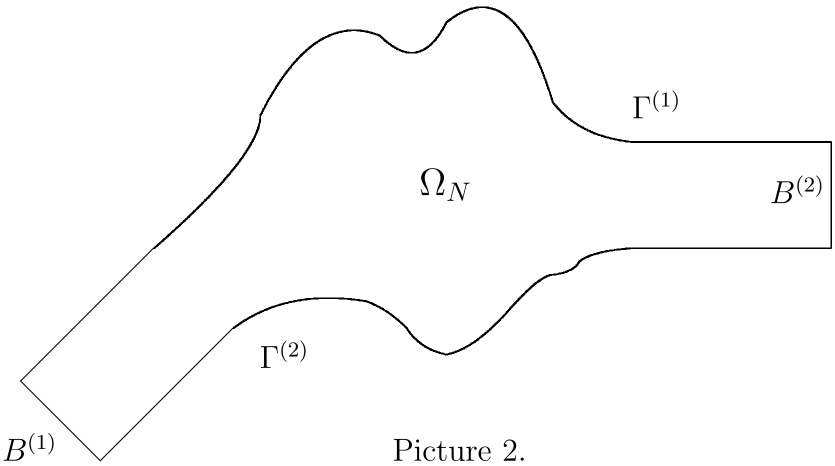
where

$$\Omega_N^{(i)} = \{(y_1^{(i)}, y_2^{(i)}) \in \mathbf{R}_+^2 : 0 < y_1 < N,\ y_2 \in (0, H^{(i)})\}, \tag{3.2}$$

where $y^{(i)}$ are the local coordinates of the i-th pipe and

$$\partial\Omega_N = \{\bigcup_{i=1}^{2} \Gamma_N^{(i)}\} \cup \{\bigcup_{i=1}^{2} B^{(i)}\}, \tag{3.3}$$

where $\Gamma_N^{(1)}$ is the upper part of $\partial\Omega_N$ and $\Gamma^{(2)}$ is the lower one and $B_N^{(i)}$ is a "bottom" of $\Omega_N^{(i)}$ - see Picture 2.



Picture 2.

For each domain Ω_N we define an approximation of problem (1.8)

$$\begin{array}{lll} v_N \cdot \nabla\alpha_N - \nu\Delta\alpha_N = g_N & \text{in} & \Omega_N, \\ \alpha_N = b_N & \text{on} & \partial\Omega_N, \end{array} \tag{3.4}$$

where $g_N = g|_{\Omega_N} = \text{rot } F|_{\Omega_N} = \text{rot } F_N$,

$$b_N = \begin{cases} v_N \cdot \tau(2\chi - f/\nu) & \text{on} \quad \Gamma_N^{(i)} \;\; i = 1,2 \\ \alpha_\infty^{(i)} & \text{on} \quad B_N^{(i)} \;\; i = 1,2 \end{cases} \tag{3.5}$$

and analogically for problem (1.10)

$$\begin{array}{lll} \text{rot } v_N = \alpha_N & \text{on} & \Omega_N, \\ \text{div } v_N = 0 & \text{on} & \Omega_N, \\ v_N \cdot n = d_N & \text{on} & \partial\Omega_N, \end{array} \tag{3.6}$$

where ($n^{(i)}$ - the normal vector to $B^{(i)}$)

$$d_N = \begin{cases} 0 & \text{on} \quad \Gamma_N^{(i)} \;\; i = 1,2 \\ {v_\infty^{(i)}}^1 \cdot n^{(i)} & \text{on} \quad B_N^{(i)} \;\; i = 1,2 \end{cases} \tag{3.7}$$

We should underline that conditions on $B_N^{(i)}$ have to be defined by flow w_f - see (1.4), because on $\Gamma_N^{(i)} \cap B_N^{(i)}$ and $\Gamma_N^{(i)} \cap B_N^{(i-1)}$ condition $(3.4)_2$ must be well defined, and it holds, since $\alpha_\infty^{(i)} = v_\infty^{(i)} \cdot \tau(-f)$ which follows from definition of w_f, (3.7), flatness of edges near the corners and that the interior angle is $\pi/2$. Hence datum b_N is bounded and satisfies the estimate

$$||b_N||_{L_\infty(\partial\Omega_N)} \leq ||f/\nu - 2\chi||_{L_\infty(\partial\Omega)}||v_N||_{C(\partial\Omega_N)} + cD_\infty. \tag{3.8}$$

Index N will be omitted in below considerations, because it is fixed and it should cause no misunderstanding.

To show existence for problem (3.4)-(3.7) we apply the Schauder fixed point theorem. First we define a set of functions, where the solutions will be searched for. Define

$$\begin{array}{c}\Xi = \Big\{v \in C(\Omega_N) : ||v_N||_{C(\Omega_N)} \leq M,\ v \cdot n|_{\partial\Omega_N} = d_N, \\ \text{div } v = 0 \text{ in the distributional sense in } \Omega_N\Big\},\end{array} \tag{3.9}$$

where M will be obtained from considerations in the proof. It is well seen that Ξ is a convex subset of $C(\Omega_N)$. Next, follows the standard procedure, we construct a map

$$\Phi : \Xi \to \Xi, \qquad \Phi(v) = \tilde{v}. \tag{3.10}$$

Function $\tilde{v}$ is given as a solution to the following problem

$$\begin{array}{lll}\text{rot } \tilde{v} = \alpha & \text{in} & \Omega_N, \\ \text{div } \tilde{v} = 0 & \text{in} & \Omega_N, \\ \tilde{v} \cdot n = d & \text{on} & \partial\Omega_N,\end{array} \tag{3.11}$$

where α is a solution of the next one

$$\begin{array}{lll}v \cdot \nabla\alpha - \nu\Delta\alpha = g & \text{in} & \Omega_N, \\ \alpha = b & \text{on} & \partial\Omega_N,\end{array} \tag{3.12}$$

where $b = b(v)$ is defined by (3.5) with v. A fixed point of map Φ will be a solution to problem (3.4)-(3.7).

The main result of this section is the following.

Theorem 3.1. *Let $F_N \in L_2(\Omega_N)$, $0 \leq a < 1$ and assumption (1.6) be fulfilled, then problem (3.4)-(3.7) has a solution v as a fixed point of map Φ and satisfies the bound*

$$||v_N||_{C^a(\Omega_N)} \leq c(D_\infty + ||F||_{L_2(\Omega)}), \tag{3.13}$$

where the r.h.s. of (3.13) is independent of N.

Theorem 3.1 deliver the solutions to problem (3.4)-(3.7) in the following sense.

Definition 3.1. *We say that* $(v_N, \alpha_N) \in C^a(\Omega_N) \times (L_\infty(\Omega_N) + H_0^1(\Omega_N))$ *is the weak-∗ solution to problem (3.4)-(3.7) iff the pair satisfies the following identity*

$$L_N(v_N, \alpha_N, \psi) = 0 \tag{3.14}$$

for any $\psi \in V_N$*, where*

$$\begin{array}{c} L_N(v_N, \alpha_N, \psi) = \\ \nu \int_{\Omega_N} \nabla\gamma_N \cdot \nabla\psi dx - \nu \int_{\Omega_N} \beta_N \Delta\psi dx - \nu \int_{\partial\Omega_N} (f/\nu - 2\chi) v_N \cdot \tau \frac{\partial\psi}{\partial n} d\sigma \\ - \int_{\Omega_N} v_N \cdot \nabla\psi(\beta_N + \gamma_N) dx + \int_{\Omega_N} F_N \cdot \nabla^\perp \psi dx, \end{array} \tag{3.15}$$

with $\operatorname{rot} v_N = \alpha_N = \beta_N + \gamma_N$ *such that* $\beta_N \in L_\infty(\Omega_N)$, $\gamma_N \in H_0^1(\Omega_N)$, $v_N \in \Xi$*; and*

$$V_N = \left\{ f \in W_1^2(\Omega_N) : f|_{\partial\Omega_N} = 0 \right\}.$$

It is worth to note that Theorem 3.1 guarantees the regularity for the solutions to problem (3.4)-(3.7), i.e. the following proposition is true.

Corollary from Theorem 3.1. *Let* $\partial\Omega \in C^\infty$ *and* $F_N \in C^\infty(\Omega_N)$*, then the solutions of problem (3.4)-(3.7) given by Theorem 3.1 are smooth, i.e.* $v_N \in C^\infty(\overline{\Omega}_N)$.

In order to prove Theorem 3.1 we need the following results. First, let us consider the second problem - (3.12).

Lemma 3.1. *Let* $g = \operatorname{rot} F$, $F \in L_2(\Omega_N)$, $v \in \Xi$ *and* $b \in L_\infty(\partial\Omega_N)$*, then there exists unique solution to problem (3.12) such that* $\alpha \in L_\infty(\Omega_N) + H_0^1(\Omega_N)$.

Proof. Since v and b are only bounded we need to describe a sense of solution α. To reformulate problem (3.12), take an extension of boundary datum b as a solutions of the Laplace problem

$$\begin{array}{lll} \Delta\bar{b} = 0 & \text{in} & \Omega_N, \\ \bar{b} = b & \text{on} & \partial\Omega_N. \end{array} \tag{3.16}$$

Existence of solutions $\bar{b}$ as a smooth function inside Ω_N is obvious. And by the maximum principle we conclude the estimate

$$||\bar{b}||_{L_\infty(\Omega_N)} \le ||b||_{L_\infty(\partial\Omega_N)}. \tag{3.17}$$

Bound (3.17) guarantees the uniqueness for solutions to (3.16) as well the continuous dependence from the data. Then we take vorticity α as follows

$$\alpha = \beta + \bar{b}, \tag{3.18}$$

where β is a solution to the following problem

$$\begin{array}{lll} v \cdot \nabla\beta - \nu\Delta\beta = g - v \cdot \nabla\bar{b} & \text{in} & \Omega_N, \\ \beta = 0 & \text{on} & \partial\Omega_N. \end{array} \tag{3.19}$$

For (3.19) we define weak solutions.

Definition 3.2. *We say that $\beta \in H^1_0(\Omega_N)$ is a weak solution of problem (3.17) iff the identity*

$$(v \cdot \nabla\beta, \psi)_{L_2(\Omega_N)} + \nu(\nabla\beta, \nabla\psi)_{L_2(\Omega_N)} = (v \cdot \nabla\psi, \bar{b})_{L_2(\Omega_N)} - (F, \nabla^\perp\psi)_{L_2(\Omega_N)} \tag{3.20}$$

holds for any $\psi \in H^1_0(\Omega_N)$.

By the Lax-Milgram theorem we deduce the existence from the energy estimate which follows from formulation (3.20)

$$||\beta||_{H^1_0(\Omega_N)} \leq c\left(||F||_{L_2(\Omega_N)} + \left(\int_{\Omega_N} |v|^2|\bar{b}|^2 dx\right)^{1/2}\right). \tag{3.21}$$

Since v and $\bar{b}$ are bounded in $L_\infty(\Omega_N)$-norms and Ω_N is bounded the last integral in the r.h.s. of (3.21) is finite. This bound, of course, strongly depends on index N and it is useful just in this step to clarify the existence for (3.12). The uniqueness of problem (3.12) follows from the linearity of the system and boundary condition $(3.19)_2$. Lemma 3.1 is proved.

In this way we defined and proved existence for problem (3.12) as a sum of solutions of (3.16) and (3.19), hence $\alpha \in L_\infty(\Omega_N) + H^1_0(\Omega_N)$, but we want to improve this information, because estimate (3.21) depends on N and we look for a bound which are independent of N.

Lemma 3.2. *Let $g = rot\, F$, $F \in L_2(\Omega_N)$ and $b \in L_\infty(\partial\Omega_N)$. Then there exists unique solution of problem (3.12) in the sense of (3.18) and (3.20) such that $\alpha \in L_\infty(\Omega_N) + H^1_0(\Omega_N)$ and the following bounds*

$$\begin{array}{c} ||\alpha[b]||_{H^1_0(\Omega_N)} \leq c||F||_{L_2(\Omega_N)}, \\ ||\alpha - \alpha[b]||_{L_\infty(\Omega_N)} \leq ||f/\nu - 2\chi||_{L_\infty(\partial\Omega)}||v||_{C(\partial\Omega_N)} + cD_\infty \end{array} \tag{3.22}$$

hold, where constants in (3.22) are independent of N.

Proof. By (3.18) and (3.20) we write

$$(v \cdot \nabla\alpha, \psi)_{L_2(\Omega_N)} + \nu(\nabla\alpha, \nabla\psi)_{L_2(\Omega_N)} = -(F, \nabla^{\perp}\psi)_{L_2(\Omega_N)}. \tag{3.23}$$

Put

$$\alpha[b] = \begin{cases} \alpha(x) - B^* & \text{if} \quad \alpha(x) > B^* \\ 0 & \text{if} \quad B_* \le \alpha(x) \le B^* \\ \alpha(x) - B_* & \text{if} \quad \alpha(x) \le B_* \end{cases} \tag{3.24}$$

where

$$B^* = \sup_{y \in \partial\Omega_N} b(y) \quad \text{and} \quad B_* = \inf_{y \in \partial\Omega_N} b(y). \tag{3.25}$$

By definition (3.24) we see that quantity $\alpha - \alpha[b] \in L_\infty(\Omega_N)$ and

$$||\alpha - \alpha[b]||_{L_\infty(\Omega_N)} \le ||b||_{L_\infty(\partial\Omega_N)}. \tag{3.26}$$

Also by the definition, function $\alpha[b]$ vanishes on the boundary, hence it can be put in (3.23) instead of ψ, by properties of the supports of function $\alpha[b]$, we have

$$(v \cdot \nabla\alpha[b], \alpha[b])_{L_2(\Omega_N)} + \nu(\nabla\alpha[b], \nabla\alpha[b])_{L_2(\Omega_N)} = -(F, \nabla\alpha[b])_{L_2(\Omega_N)}. \tag{3.27}$$

Equality (3.27) gives the following bound

$$||\nabla\alpha[b]||_{H_0^1(\Omega_N)} \le c||F||_{L_2(\Omega_N)}. \tag{3.28}$$

Thus the vorticity can be split into two term

$$\alpha = (\alpha - \alpha[b]) + \alpha[b], \quad \text{where} \quad \alpha - \alpha[b] \in L_\infty, \quad \alpha[b] \in H_0^1(\Omega_N). \tag{3.29}$$

Lemma 3.2 is proved.

Now, we examine problem (3.11).

Lemma 3.3. *Let α be given by Lemma 3.2 and d by (3.7) with $v \in \Xi$, then there exists unique solution to (3.11) such that $\tilde{v} \in C^a(\Omega_N)$.*

Proof. By the form of the vorticity, we see that it is easier to consider solution $\tilde{v}$ as the sum

$$\tilde{v} = v_1 + v_2, \tag{3.30}$$

where v_1 satisfies the problem

$$\begin{array}{lll} \text{rot } v_1 = \alpha[b] & \text{in} & \Omega_N, \\ \text{div } v_1 = 0 & \text{in} & \Omega_N, \\ n \cdot v_1 = 0 & \text{on} & \partial\Omega_N \end{array} \tag{3.31}$$

and vector v_2 fulfills the following one

$$\begin{array}{lll} \text{rot } v_2 = \alpha - \alpha[b] & \text{in} & \Omega_N, \\ \text{div } v_2 = 0 & \text{in} & \Omega_N, \\ n \cdot v_2 = d_N & \text{on} & \partial\Omega_N. \end{array} \tag{3.32}$$

We are interested in finding estimates for v_i in $C(\Omega_N)$, but also we look for a compactness which in this case will be connected with the embedding in $C^a(\Omega_N)$. Let us consider first problem (3.31). By estimate $(3.22)_1$ the r.h.s. of $(3.31)_1$ belongs to $H_0^1(\Omega_N)$ and the norm depends only on given data - the external force and the domain. Since Ω_N is simply connected v_1 may be represented by the potential

$$v_1 = \nabla^\perp \varphi_1 = (-\partial_{x_2}\varphi_1, \partial_{x_1}\varphi_1) \tag{3.33}$$

up to a constant. Next, look on condition $(3.31)_3$. Applying form (3.33) we get on $\partial\Omega_N$ the following relation

$$v_1 \cdot n = n \cdot \nabla^\perp \varphi_1 = \frac{d}{ds}\varphi_1 = 0, \tag{3.34}$$

where s is the unit length parameter of curve $\partial\Omega_N$. By (3.33) and (3.34) we restate system (3.31) as the following scalar problem

$$\begin{array}{lll} \Delta\varphi_1 = \alpha[b] & \text{in} & \Omega_N, \\ \varphi_1 = 0 & \text{on} & \partial\Omega_N. \end{array} \tag{3.35}$$

Since the angles in irregular points of $\partial\Omega_N$ are equal to $\pi/2$, then by the symmetry argument and the classical theory we obtain the following bound

$$||\varphi_1||_{H^3(\Omega_N)} \leq c(\Omega)||\alpha[b]||_{H_0^1(\Omega_N)}, \tag{3.36}$$

where constant $c(\Omega)$ depends only on the constant from the Poincare inequality - Lemma 2.1 and by $(3.22)_1$ and the embedding theorem for $0 \leq a < 1$ we get

$$||v_1||_{C^a(\Omega_N)} \leq c||F||_{L_2(\Omega_N)}. \tag{3.37}$$

Next, we take the second problem - (3.32). We want to find a field satisfying $(3.32)_{2,3}$. There is a need of the following result.

Lemma 3.4. *There exists a smooth vector field* $\bar{v} : \Omega_N \to \mathbf{R}^2$ *satisfying* $(3.29)_{2,3}$ *such that*

$$||\bar{v}||_{C^1(\Omega_N)} \leq cD_\infty. \tag{3.38}$$

Proof. Construct a field $\bar{v} = \nabla^{\perp}\bar{\phi}$. In pipes $\Omega^{(1)}$ and $\Omega^{(2)}$ we define $\bar{\phi}|_{\Omega^{(i)}} = \varphi^{(i)}_{\infty}$, where $\varphi^{(i)}_{\infty}$ are potentials of $v^{(i)}_{\infty}$. Since $\bar{\phi}$ must be constant on $\Gamma^{(1)}$ and $\Gamma^{(2)}$ which follows from (3.34), the potential described in one pipe determines $\bar{\phi}$ in the other uniquely. By (1.5) the construction is well defined. Hence we got $\bar{v}$ on $\Omega^{(1)} \cup \Omega^{(2)} \cup \Gamma^{(1)} \cup \Gamma^{(2)}$ and we need to define it in $\Omega^{(0)}$, but since this domain is bounded we take any extension of $\bar{\phi}$ - smooth, conserving the norm. This way we obtain Lemma 3.4 with estimate (3.38). To show Corollary from Theorem 3.1 we need Lemma 3.4 for $C^{\infty}(\Omega_N)$. Since the extension in $\Omega^{(0)}$ can preserve also C^{∞}-smoothness an analog of (3.38) for $C^{\infty}(\Omega_N)$ can be easily found. Lemma 3.4 is proved.

Having Lemma 3.4 we examine v_2 - solution of (3.32) as the following sum

$$v_2 = \bar{v} + u, \tag{3.39}$$

where u satisfies the following problem

$$\begin{array}{lll} \mathrm{rot}\ u = \alpha - \alpha[b] - \mathrm{rot}\ \bar{v} & \text{in} & \Omega_N, \\ \mathrm{div}\ u = 0 & \text{in} & \Omega_N, \\ n \cdot u = 0 & \text{on} & \partial\Omega_N. \end{array} \tag{3.40}$$

To solve (3.40) we need the following lemma.

Lemma 3.5. *Let $G \in L_{\infty}(\Omega_N)$ and $2 < p < \infty$. Then the following elliptic system*

$$\begin{array}{lll} rot\ u = G & in & \Omega_N, \\ div\ u = 0 & in & \Omega_N, \\ n \cdot u = 0 & on & \partial\Omega_N \end{array} \tag{3.41}$$

has unique solution which belongs to $W^1_p(\Omega_N)$ such that

$$||u||_{C^a(\Omega_N)} \leq c||G||_{L_{\infty}(\Omega_N)} \tag{3.42}$$

for $0 \leq a < 1 - 2/p$, where c depends on Ω, p and is independent of N.

Proof. The same as for (3.31) we introduce a potential for field $v = \nabla^{\perp}\varphi_2 = (-\partial_{x_2}\varphi_2, \partial_{x_1}\varphi_2)$, then (3.41) takes the following form

$$\begin{array}{lll} \Delta\varphi_2 = G & \text{in} & \Omega_N, \\ \varphi_2 = 0 & \text{on} & \Omega_N. \end{array} \tag{3.43}$$

Since the theory of the Laplace operator is ill posed in the L_{∞}-approach, we consider function G as a element of $L_{p(loc)}(\Omega_N)$ - see definition (2.3). And by a localization along the channel (to omit dependence from index N), using standard techniques we get (see section 4 in [12])

$$||\varphi_2||_{W^2_{p(loc)}(\Omega)} \leq c||G||_{L_{p(loc)}(\Omega_N)} \tag{3.44}$$

- see definition (2.6); and by the imbedding theorem we conclude (3.42). Lemma 3.5 is proved.

From (3.37), (3.38) and (3.42) we conclude that $\tilde{v} \in C^a(\Omega_N)$. The uniqueness of the solutions follows from simply connectedness of domain Ω_N. Lemma 3.3 is proved.

By Lemma 3.3 we conclude the following result for problem (3.11).

Lemma 3.6. *The solutions to problem (3.11) satisfy the following estimate*

$$||\tilde{v}||_{C^a(\Omega_N)} \leq A_a(\Omega_N)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)}||\alpha - \alpha[b]||_{L_\infty(\Omega_N)} + c(D_\infty + ||F||_{L_2(\Omega)}). \tag{3.45}$$

Proof of Theorem 3.1. By the construction we have $\tilde{v} \in \Xi$. Applying Lemma 3.2 to 3.6 we obtain the following estimate

$$||\tilde{v}||_{C^a(\Omega_N)} \leq A_a(\Omega_N)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)}||v||_{C(\partial\Omega_N)} + c(D_\infty + ||F||_{L_2(\Omega_N)}). \tag{3.46}$$

The main condition in Theorem 3.1 - assumption (1.6) arises naturally from estimate (3.46). To obtain the bound it is required (recall Lemma 2.2)

$$A(\Omega)||f/\nu - 2\chi||_{L_\infty(\partial\Omega)} < 1.$$

Inequality (3.46) describes us also quantity M - see (3.10). Put

$$M = (1 - A(\Omega)||f/\nu - 2\chi||_{L_\infty(\partial\Omega_N)})^{-1}c(D_\infty + ||F||_{L_2(\Omega_N)}), \tag{3.47}$$

then by (3.43) with $a = 0$ we see that

$$\text{if} \qquad ||v||_{C(\Omega_N)} \leq M, \qquad \text{then} \qquad ||\tilde{v}||_{C(\Omega_N)} \leq M. \tag{3.45}$$

To show existence of a fixed point there is a need of compactness of map Φ. Inequality (3.46) gives estimates in $C^a(\Omega_N)$ for $a > 0$, so since Ω_N is bounded, imbedding $C^a(\Omega_N) \subset C(\Omega_N)$ is compact which implies compactness of map Φ. Continuity of Φ is obvious by the linearity of systems (3.11) and (3.12).

By the Schauder fixed point theorem we conclude the existence of v_* such that $v_* \in \Xi$ and $\Phi(v_*) = v_*$. Hence (3.48) by (3.46) gives also (3.13). Theorem 3.1 is proved.

By the above considerations we see that solutions given by Theorem 3.1 satisfies Definition 3.1. Is is enough to combine Definition 3.2 and the distributional form of problem (3.16). The information about the vorticity is obtained from Lemma 3.2 with bound (3.22).

4 Proof of Theorem 1.1

In this part of the paper we prove Theorem 1.1. We want to adapt the result of the previous section to obtain a solution to problem (1.1). Theorem 3.1 constructed us the sequence of solutions which we want to treat as an approximation of a solution of the problem.

For $N \in \mathbf{N}$ we have

$$v_N : \Omega_N \to \mathbf{R}^2 \tag{4.1}$$

with the uniform in N bound

$$||v_N||_{C^a(\Omega_N)} \leq c(D_\infty + ||F||_{L_2(\Omega)}). \tag{4.2}$$

Moreover, the vorticity of the velocity can be split in the following way

$$\text{rot } v_N = \alpha_N = \beta_N + \gamma_N \ \in L_\infty(\Omega_N) + H^1_0(\Omega_N), \tag{4.3}$$

where $\beta_N \in L_\infty(\Omega_N)$, $\gamma_N \in H^1_0(\Omega_N)$ and the following independent of N estimates hold

$$||\beta_N||_{L_\infty(\Omega_N)} \leq c(D_\infty + ||F||_{L_2(\Omega)}), \quad ||\gamma_N||_{H^1_0(\Omega_N)} \leq c||F||_{L_2(\Omega)}. \tag{4.4}$$

The decomposition of the vorticity defines also the following form of the velocity

$$v_N = V(\beta_N) + V(\gamma_N), \tag{4.5}$$

where $V(\beta_n)$ and $V(\gamma_N)$ satisfy the following equations

$$\begin{array}{llll} \text{rot } V(\beta_N) = \beta_N, & \text{rot } V(\gamma_N) = \gamma_N & \text{in} & \Omega_N, \\ \text{div } V(\beta_N) = 0, & \text{div } V(\gamma_N) = 0 & \text{in} & \Omega_N, \\ n \cdot V(\beta_N) = d_N, & n \cdot V(\gamma_N) = 0 & \text{on} & \partial\Omega_N. \end{array} \tag{4.6}$$

First, we show existence of the distributional solution to problem (1.8)-(1.10).

Definition 4.1. *We say that* $(v,\alpha) \in C^a(\Omega) \times (L_\infty(\Omega) + H^1_0(\Omega))$ *is the distributional solution of problem (1.8)-(1.10) iff the pair satisfies the following identity*

$$L(v,\alpha,\psi) = 0 \tag{4.7}$$

for every $\psi \in C^\infty_0(\Omega)$ *(i.e. supp* ψ *is compact and* $\psi|_{\partial\Omega} = 0$*), where*

$$\begin{aligned} L(v,\alpha,\psi) = \nu \textstyle\int_\Omega \nabla\gamma \cdot \nabla\psi dx - \nu \int_\Omega \beta\Delta\psi dx - \nu \int_{\partial\Omega}(f-2\chi)v\cdot\tau\frac{\partial\psi}{\partial n}d\sigma \\ - \textstyle\int_\Omega v\cdot\nabla\psi\beta dx - \int_\Omega v\cdot\nabla\psi\gamma dx + \int_\Omega F\cdot\nabla^\perp\psi dx, \end{aligned} \tag{4.8}$$

where $rot\, v = \alpha = \beta + \gamma$ *with* $\beta \in L_\infty(\Omega)$*,* $\gamma \in H^1_0(\Omega)$ *and* $v \in \Xi_\infty$*, where*

$$\Xi_\infty = \{v \in C(\Omega) : v\cdot n|_{\partial\Omega} = 0, \textit{the flux of the flow is equal TF},$$

$$div\, v = 0 \;\; \textit{in the distributional sense}\}. \tag{4.9}$$

In the definition of set Ξ_∞ we require to prescribe the flux of the flow defined by field v. Since we are able to obtain only the weak-$*$ solutions to preserve features of conditions at infinity $(1.1)_4$ we assume (4.9). A similar setting has been done in [10].

Fixed $N \in \mathbf{N}$ and consider only test functions ψ_N such that supp $\psi_N \subset\subset \Omega_N$. Then for any $k > N$ results of Theorem 3.1 implies that by Definition 3.1 we have

$$L(v_k, \alpha_k, \psi_N) = 0. \tag{4.10}$$

The solution will be a limit of a subsequence of approximations. First we choose a subsequence from

$$\{\gamma_k\}^\infty_{k=N} \subset H^1_0(\Omega_N). \tag{4.11}$$

By the estimate (4.4) and weak compactness of bounded sets in $H^1_0(\Omega_N)$ one can find a subsequence $\{\gamma_{k_n}\}^\infty_{n=1}$ such that

$$\gamma^{(N)}_{k_n} \rightharpoonup \gamma^{(N)}_* \;\; \text{weakly} \;\; \text{in } H^1_0(\Omega_N) \tag{4.12}$$

with the estimate

$$||\gamma^{(N)}_*||_{H^1_0(\Omega_N)} \leq c||F||_{L_2(\Omega)}. \tag{4.13}$$

REMARK. Next we consider only indices k_n, so we omit n and we will write k instead of k_n. The same we will do after each choice of a subsequence in our considerations.

Take sequence $\{\beta_k\}_{k=N}^{\infty} \subset L_\infty(\Omega_N)$. By estimate (4.4) and weak-$*$ compactness of bounded sets in $L_\infty(\Omega_N)$ there exists a subsequence $\{\beta_{k_n}\}_{n=1}^{\infty}$ such that

$$\beta_{k_n} \rightharpoonup \beta_*^{(N)} \quad \text{weakly-}* \text{ in } \ L_\infty(\Omega_N) \tag{4.14}$$

and

$$||\beta_*^{(N)}||_{L_\infty(\Omega_N)} \le c(D_\infty + ||F||_{L_2(\Omega)}). \tag{4.15}$$

From (4.12) and (4.14) we describe a limit of the searched subsequence of the approximation of the vorticity as follows

$$\alpha_k = \beta_k + \gamma_k \rightharpoonup \beta_*^{(N)} + \gamma_*^{(N)} = \alpha_*^{(N)} \quad \text{weakly-}* \text{ in } \ L_\infty(\Omega_N) + H_0^1(\Omega_N). \tag{4.16}$$

For a suitable subsequence we obtain

$$v_k \to v_*^{(N)} \quad \text{strongly in} \quad C^a(\Omega_N) \tag{4.17}$$

with the independent of N estimate and

$$||v_*^{(N)}||_{C^a(\Omega_N)} \le c(D_\infty + ||F||_{L_2(\Omega)}). \tag{4.18}$$

By the choice of subsequences for α_N we deduce that in the distributional sense rot $v_* = \alpha_*$ and div $v_* = 0$. In particular pair (v_*, α_*) satisfies $(1.10)_{1,2,3}$. Hence to prove that v_* and α_* defined by (4.16) and (4.17) are a solution of problem (1.8)-(1.10) it is enough to show that this pair fulfills equation (1.8). The flux is preserved, too.

By (4.16) and (4.17)

$$\alpha_*^{(N)} = \beta_*^{(N)} + \gamma_*^{(N)} \ \text{ and } \ v_*^{(N)} \tag{4.19}$$

are defined on Ω_N. To obtain these quantities well defined on the whole Ω we apply the induction.

In the step for Ω_{N+1} we consider only these subsequences which have been chosen in the step for Ω_N. This method guarantees that

$$\alpha_*^{(N+1)}|_{\Omega_N} = \alpha_*^{(N)} \quad \text{and} \quad v_*^{(N+1)}|_{\Omega_N} = v_*^{(N)}. \tag{4.20}$$

Since

$$\bigcup_{N \in \mathbf{N}} \Omega_N = \Omega \tag{4.21}$$

and estimates (4.13), (4.15) and (4.18) are independent of N, by (4.20) we conclude existence of function α_* and v_* defined on Ω such that

$$\alpha_*|_{\Omega_N} = \alpha_*^{(N)} \quad \text{and} \quad v_*|_{\Omega_N} = v_*^{(N)} \quad \text{for } N \in \mathbf{N} \tag{4.22}$$

being the distributional solution to problem (1.8)-(1.10) in the sense of Definition 4.1.

Now we want to improve Definition 4.1, we define the weak-$*$ solution in $L_\infty(\Omega) + H_0^1(\Omega)$ for problem (1.8)-(1.10).

Definition 4.2. *We say that $(v, \alpha) \in C^a(\Omega) \times (L_\infty(\Omega) + H_0^1(\Omega))$ is the weak-$*$ solution of problem (1.8)-(1.10) iff the pair satisfies the following identity*

$$L(v, \alpha, \psi) = 0 \tag{4.23}$$

for every $\psi \in V$, where $\operatorname{rot} v = \alpha = \beta + \gamma$ with $\beta \in L_\infty(\Omega)$, $\gamma \in H_0^1(\Omega)$, $v \in \Xi_\infty$ and

$$\begin{array}{c} V = \{f \in \mathcal{M}(\Omega): \ \Delta f \in L_1(\Omega), \ \ \nabla f \in L_1(\Omega), \\ \nabla f \in L_2(\Omega), \ \ \frac{\partial f}{\partial n} \in L_1(\partial\Omega), \ \ f|_{\partial\Omega} = 0\}. \end{array} \tag{4.24}$$

The definition of space V must contain all inclusions, because the theory of the Laplace operator in the L_1-spaces is ill posed. But $C_0^\infty(\Omega)$ is dense in Banach space V, i.e. the following proposition holds.

Proposition 4.1. *For any $\varepsilon > 0$ and any $\psi \in V$ there exist functions $\psi_r \in C_0^\infty(\Omega)$ and $\psi_\varepsilon \in V$ such that $\psi = \psi_r + \psi_\varepsilon$ and*

$$||\psi_\varepsilon||_V = ||\Delta\psi_\varepsilon||_{L_1(\Omega)} + ||\nabla\psi_\varepsilon||_{L_1(\Omega)} + ||\nabla\psi_\varepsilon||_{L_2(\Omega)} + ||\frac{\partial\psi_\varepsilon}{\partial n}||_{L_1(\partial\Omega)} < \varepsilon. \tag{4.25}$$

To show Theorem 1.1 it is enough to show that $L(v_*, \alpha_*, \psi) = 0$ for every $\psi \in V$. By Proposition 4.1 we can split $\psi = \psi_r + \psi_\varepsilon$. Then by (4.13), (4.15), (4.17) and (4.8) we immediately conclude

$$|L(v_*, \alpha_*, \psi_\varepsilon)| < cM^2\varepsilon. \tag{4.26}$$

Thus we can restrict our attention to regular part of ψ. Since $\psi_r \in C_0^\infty(\Omega)$ then there exists $l \in \mathbf{N}$ such that supp $\psi_r \subset\subset \Omega_l$.

By (4.8) we have

$$L(v_*, \alpha_*, \psi) = L(v_*, \alpha_*, \psi_r) + L(v_*, \alpha_*, \psi_\varepsilon) = L(v_*, \alpha_*, \psi_\varepsilon), \tag{4.27}$$

since (v_*, α_*) fulfill Definition 4.1.

From arbitrary smallness of ε and (4.26) we conclude

$$L(v_*, \alpha_*, \psi) = 0 \tag{4.28}$$

for any $\psi \in V$. Hence the pair satisfies Definition 4.2 Theorem 1.1 has been proved.

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References

1. Amick, C.J.; Fraenkel, L.E.: Steady solutions of the Navier-Stokes equations representing plane flow in channels of various types, Acta Math. 144, (1980) 83-152

2. Clopeau T.; Mikelić A.; Robert R.: On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions, Nonlinearity 11 (1998), 1625-1636

3. Farwig, R.: Stationary solutions of compressible Navier-Stokes equations with slip boundary conditions, Comm. PDE 14, (1989) 1579-1606

4. Fontelos, M.A.; Friedman, A.: Stationary Non-Newtonian Fluid Flows in Channels-like and Pipe-like Domains, Arch. Rational Mech. Anal. 151 (2000) 1-43

5. Galdi, G.P.: An introduction to the mathematical theory of the Navier-Stokes equations, Springer-Verlag, 1994

6. Giga, Y.; Matsui, S.; Sawada, O.: Global Existence of Two-Dimensional Navier-Stokes Flow with Nondecaying Initial Velocity, J. Math. Fluid Mech. 3, (2002) 302-315

7. Itoh, S.; Tanaka N.; Tani A.: The initial value problem for the Navier-Stokes equations with general slip boundary condition, Adv. Math. Sci. Appl. 4, (1994) 51-69

8. Kato, T.; Fujita, H.: Maximum principle for vorticity in viscous fluid, J. Phys. Soc. Japan 14, (1959) 545

9. Ladyzhenskaya, O.A.: The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1966

10. Ladyzhenskaya, O.A.; Solonnikov, V.A.: Determination of solutions of boundary value problems for steady-state Stokes and Navier-Stokes equations in domains having an unbounded Dirichlet integral, Zap. Nauch. Sem. Len. Ot. Mat. Inst. Steklov (LOMI) 96, (1980) 117-160 (in Russian)

11. Mucha P.B.: On the inviscid limit of the Navier-Stokes equations for flows with large flux, Nonlinearity 16, 5, (2003) 1715-1732

12. Mucha P.B.: On a problem for the Navier-Stokes equations with the infinite Dirichlet integral, to appear in ZAMP.

13. Solonnikov, V.A.; Scadilov, V.E.: On a boundary value problem for a stationary system of Navier-Stokes equations, Trudy Mat. Inst. Steklov. 125 (1973) 186-199

14. Temam,R.: Navier-Stokes Equations. North Holland, Amsterdam (1977)

15. Zajaczkowski, W.M.: On global special solutions for Navier-Stokes equations with boundary slip condition in a cylindrical domain, to appear.