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Global existence of solutions of the Dirichlet problem for the compressible Navier-Stokes equations

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Abstract. We prove existence of global in time regular solutions of the compressible Navier-Stokes equations in a domain $\Omega \subset \mathbf{R}^3$ with the vanishing Dirichlet boundary conditions. The solutions are close to a nontrivial static solutions. A key element of the proof is a special L_p-estimate for the linearized problem to show that the velocity belongs to $W^{2,1}_{r(loc)}(\Omega \times [0,\infty))$ with $r > 3$ which is the sharp result in this approach.

1 Introduction

The aim of this note is to analyze a stability of an equilibrium state of a viscous barotropic compressible fluid in a bounded three dimensional domain Ω given as a solution of

$$\nabla p(\bar{\varrho}) = \bar{\varrho} f \quad \text{in } \Omega, \tag{1.1}$$

with the following constitutive equation $p(\varrho) = a\varrho^\kappa$ with $\kappa > 1$ and $f(x) = \nabla\varphi(x)$. Moreover we assume that the solution of (1.1) is strictly positive, regular and is equal to

$$\bar{\varrho} = \left(\frac{\kappa - 1}{a\kappa}\varphi\right)^{\frac{1}{\kappa-1}}. \tag{1.2}$$

A perturbation of (1.1) is described by the evolutionary Navier-Stokes equations for compressible fluids. In the Eulerian coordinates they read

$$\begin{array}{lll} \varrho(v_t + (v\nabla)v) - \mu\Delta v - \nu\nabla div\, v + \gamma\nabla\sigma = \sigma f - \sigma\nabla\gamma & \text{in} & \Omega \times [0,T], \\ \sigma_t + div(\varrho v) = 0 & \text{in} & \Omega \times [0,T], \\ \sigma|_{t=0} = \varrho_0 - \bar{\varrho}, \quad v|_{t=0} = v_0 & \text{on} & \Omega, \\ v = 0 & \text{on} & \partial\Omega \times [0,T], \end{array} \tag{1.3}$$

where $v = (v^1, v^2, v^3)$ is the velocity of the fluid, ϱ is the density, $\sigma = \varrho - \bar{\varrho}$ is the perturbation of the density, ν, μ are constant positive viscosity coefficients and function γ is given by the relation

$$p(\varrho) - p(\bar{\varrho}) = \sigma \int_0^1 p'(\varrho + s(\bar{\varrho} - \varrho))ds = \sigma\gamma.$$

For solution of (1.3) we have the following conservation law which controls the L_2-norm (the proof one can find in the Appendix).

Energy Estimate. *Assume that $||\sigma||_{L_\infty}$ is sufficiently small. Then for sufficiently regular solutions of (1.3) we have*

$$||v(\cdot,t)||_{L_2(\Omega)} + ||\sigma(\cdot,t)||_{L_2(\Omega)} \leq B\left(||v_0||_{L_2(\Omega)} + ||\sigma_0||_{L_2(\Omega)}\right), \tag{1.4}$$

where B is independent of t.

Since we are interested in regular solutions of (1.3) we formulate the problem in the Lagrangian coordinates which are defined as initial data to the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=t_0} = \xi. \tag{1.5}$$

From (1.5) we obtain the relation between the Eulerian x and the Lagrangian ξ coordinates

$$x = \xi + \int_{t_0}^{t} u(\xi,t')dt' \equiv \mathcal{T}_{t_0}(\xi,t), \tag{1.6}$$

where $u(\xi,t) = v(x(\xi,t),t)$.

In the Lagrangian coordinates with initial time t_0 problem (1.3) takes the form

$$\begin{array}{lll} \eta u_t - \mu\Delta_u div_u u - \nu\nabla_u div_u u + \gamma\nabla_u\chi = \chi f - \chi\nabla\gamma & \text{in} & \Omega\times[t_0,T], \\ \chi_t + \eta div_u u = -\bar{\eta}_t & \text{in} & \Omega\times[t_0,T], \\ u|_{t=t_0} = v(\xi,t_0), \quad \chi|_{t=t_0} = \varrho(\xi,t_0) - \bar{\varrho}(\xi) & \text{on} & \Omega, \\ u = 0 & \text{on} & \partial\Omega\times[t_0,T], \end{array} \tag{1.7}$$

where $\eta(\xi,t) = \varrho(x(\xi,t),t)$, $\chi(\xi,t) = \eta(\xi,t) - \bar{\eta}(\xi,t)$, $\bar{\eta}(\xi,t) = \bar{\varrho}(\mathcal{T}_{t_0}(\xi,t))$, $\nabla_u = \frac{\partial\xi_i}{\partial x}\partial_{\xi_i}$, $div_u = \nabla_u\cdot$.

The transformation from (1.3) to (1.7) is necessary, because our result is done in the L_p-framework and in this approach there is no possibility to treat equation $(1.3)_2$ - so we consider $(1.7)_2$.

Let us define the global existence of solutions of problem (1.3). We say that problem (1.3) has a global in time regular solution if there exists $T > 0$ and following problems

$$\begin{array}{l} \eta^k u^k_t - \mu\Delta_{u^k} u^k - \nu\nabla_{u^k} u^k + \gamma^k\nabla_{u^k}\chi^k = \chi^k f - \chi^k\nabla_{u^k}\gamma^k, \\ \chi^k_t + \eta^k div_{u^k} u^k = -\bar{\eta}^k_t, \\ u^k = 0, \\ u^k|_{t=(k-1)T} = v(\mathcal{T}_{(k-2)T}(\xi,(k-1)T),(k-1)T), \\ \chi^k|_{t=(k-1)T} = \sigma(\mathcal{T}_{(k-2)T}(\xi,(k-1)T),(k-1)T) \end{array} \tag{1.8}$$

have solutions in $\Omega\times[(k-1)T,(k+1)T]$ for any $k > 0$.

The result of this paper is the following

Theorem A. *Let $r > 3$, $\varrho_0 - \bar{\varrho} \in W^1_r(\Omega)$, $v_0 \in W^{2-2/r}_r(\Omega)$ and*

$$||v_0||_{W^{2-2/r}_r(\Omega)} + ||\varrho_0 - \bar{\varrho}||_{W^1_r(\Omega)} \le M_0.$$

If M_0 is sufficiently small then for every $k > 0$ problem (1.8) has a unique solution such that

$$u^k \in W^{2,1}_r(\Omega\times[(k-1)T,(k+1)T]), \quad \eta^k - \bar{\eta} \in V_r(\Omega\times[(k-1)T,(k+1)T])$$

and the following estimate holds

$$||u^k||_{W^{2,1}_r(\Omega\times[(k-1)T,(k+1)T])} + ||\eta^k - \bar{\eta}||_{V_r(\Omega\times[(k-1)T,(k+1)T])} \le \delta(M_0),$$

where $||g||_{V_r(\Omega\times[0,T])} = ||g||_{W^{1,0}_r(\Omega\times[0,T])} + ||g_t||_{V_r(\Omega\times[0,T])}$.

To prove Theorem A we need two results.

The first one is the almost global in time existence of solutions to (1.7).

Theorem B (see [9]). *Let $r > 3$, $f = \nabla\varphi \in W^1_\infty(\Omega)$, $\varrho_0 - \bar{\varrho} \in W^1_r(\Omega)$, $v_0 \in W^{2-2/r}_r(\Omega)$, $\varrho_0 \ge \frac{1}{2}\inf\bar{\varrho}$ and $T > 0$ be given. There exist $\bar{M}_1(T)$ and $M_2(T,M_1)$, where $M_2(T,M_1) \to 0$ with $T \to \infty$ and $M_1 \to 0$, such that for $M_1 \le \bar{M}_1(T)$ and*

$$||v_0||_{W^{2-2/r}_r(\Omega)} + ||\varrho_0 - \bar{\varrho}||_{W^1_r(\Omega)} \le M_2(T,M_1),$$

then there exist solutions of (1.7) such that $u \in W^{2,1}_r(\Omega\times[0,T])$, $\chi \in V_r(\Omega\times[0,T])$ and the following estimate holds

$$||u||_{W^{2,1}_r(\Omega\times[0,T])} + ||\eta||_{V_r(\Omega\times[0,T])} \le M_1.$$

Moreover, $\bar{M}_1(T)$ ensures the boundedness $\eta(\xi,t) \ge \frac{1}{4}\inf\bar{\varrho}$ for each $(\xi,t) \in \Omega\times[0,T]$.

The above result delivers us the basic information about the existence in the function spaces as well as guarantees the lifespan of solutions for any time for suitable small initial data. The result is very technical, but in general is based on the standard methods - see [9].

The next result concerns the estimate for a linearization of (1.7) with constant coefficients.

$$\begin{array}{lll} cu_t - \mu\Delta u - \nu\nabla div\, u + a\nabla\eta = f & \text{in} & \Omega\times[0,T], \\ \eta_t + b\, div\, u = g & \text{in} & \Omega\times[0,T], \\ u=0 & \text{on} & \partial\Omega\times[0,T], \\ u|_{t=0}=0, \quad \eta|_{t=0}=0 & \text{on} & \Omega, \end{array} \tag{1.9}$$

where a, b and c are positive constants.

Theorem C (see [10]). *Solutions of (1.9) satisfy the following estimate*

$$\begin{array}{c} ||u||_{W^{2,1}_r(\Omega\times[0,T])} + ||\eta||_{V_r(\Omega\times[0,T])} \le A_0 \left(||f||_{L_r(\Omega\times[0,T])} + ||f||_{L_2(\Omega\times[0,T])} + \right. \\ \left. ||g||_{W^{1,0}_r(\Omega\times[0,T])} + ||g||_{W^{1,0}_2(\Omega\times[0,T])} + ||u||_{L_2(\Omega\times[0,T])} + ||\eta||_{L_2(\Omega\times[0,T])} \right), \end{array} \tag{1.10}$$

where $r \ge 2$ and A_0 is independent of T.

Estimate (1.10) is the heart of the method. To prove this bound we have to apply new properties of parabolic-hyperbolic system (1.9) which allows to treat it as a quasi-elliptic problem. The proof is complex, there is a need to apply nonstandard Schauder-type estimate to be concentrated to show independence of time for the constant A_0. That is the reason it is presented in other paper [10].

The first global in time existence result for equations of compressible viscous and heat conducting fluids was proved by Matsumura and Nishida for the Cauchy problem in [4]. A similar result for bounded domains, half spaces and exterior domains with vanishing Dirichlet conditions was proved also by Matsumura and Nishida in [5]. They obtained solutions in such classes that $v \in H^4$. Valli in [12] proved global in time existence for equations of barotropic viscous compressible fluids in bounded domains with no slip conditions such that $v \in H^3$. The above results were generalized by Valli and Zając̨zkowski in [13] to a general fluid with Dirichlet boundary conditions and for such motions that $v, \theta \in H^3$. Finally Kobayashi and Zając̨zkowski [3] proved the existence in anisotropic fractional spaces such that $v \in H^{2+\alpha,1+\alpha/2}$, where $\alpha \in (1/2, 1)$. An improvement, but in the whole space, has been done by Dachin [2], he shows the existence with the density in the Besov space $B^{3/2}_{2,1}(\mathbf{R}^3)$. Here we should note that all above results concern cases with the density close to a constant (or f is small). We have to underline that the above results are proved applying the energy method, so existence is obtained in the Sobolev-Hilbert type spaces. In the L_p-approach global existence was obtained in [11] using the theory of semigroups, where (under assumption of smallness of f) solutions are such that the velocity belongs to $C(0,\infty; W^3_r(\Omega))$.

Stability for nontrivial equilibrium states (generated by large external forces) has been studied by Matsumura and Padula in [6]. Assuming a special form of the constitutive equation, they obtain a priori estimate on higher norms of the solutions, using the standard energy method. Matsumura and Yamagata in [7] considered a barotropic case, but only for small quantity $\kappa - 1$ ($p(\varrho) = a\varrho^\kappa$). Their approach was also based on the energy method.

The technique applied in the paper is based on a different approach. To get global in time existence we need a conservation law, in our case it is the Energy Estimate, which controls smallness of one norm of the perturbation independently in time. Next, to estimate the whole norm of the solutions we apply Theorem C to localizations of the problem. Since the hyperbolic character of system (1.9) does not allow to apply interpolation theorems to estimate nonlinear terms, we need largeness of T and independence of constant A_0 to have smallness of quantity A_0/T - see estimation (3.6). This way we prove stability with no restriction on the constitutive equations (κ may be any) and largeness of the equilibrium state $\bar\varrho$. Moreover, the result is obtained with sharp regularity in the L_p-framework because the velocity belongs to $W^{2,1}_{r(loc)}(\Omega\times[0,\infty))$ with $r > 3$. The regularity is necessary for applying the Lagrangian coordinates because then the estimate

$$\int_0^t ||\nabla u(\cdot,t')||_{L_\infty(\Omega)} dt' \le c||u||_{W^{2,1}_r(\Omega\times[0,T])} \tag{1.11}$$

must hold.

A similar method has been applied for the Cauchy problem in [8].

2 Notation

In our considerations we will need the anisotropic Sobolev spaces $W^{m,n}_r(Q_T)$ where $m,n \in R_+ \cup \{0\}$, $r \ge 1$ and $Q_T = Q\times(0,T)$ with the norm

$$\begin{array}{c} ||u||^r_{W^{m,n}_r(Q_T)} = \int_0^T \int_Q |u(x,t)|^r dxdt + \sum_{|m'|=[|m|]} \int_0^T dt \int_Q \int_Q \frac{|D^{m'}_x u(x,t) - D^{m'}_x u(x',t)|^r}{|x-x'|^{s+r(|m|-[|m|])}} dxdx' \\ + \sum_{0\le|m'|\le[|m|]} \int_0^T \int_Q |D^{m'}_x u(x,t)|^r dxdt + \sum_{0\le|n'|\le[|n|]} \int_0^T \int_Q |D^{n'}_t u(x,t)|^r dxdt \\ + \int_Q dx \int_0^T \int_0^T \frac{|D^{[n]}_t u(x,t) - D^{[n]}_t (x,t')|^r}{|t-t'|^{1+r(n-[n])}} dtdt', \end{array} \tag{2.1}$$

where $s = dim\, Q$, $[\alpha]$ - the integral part of α, $D^l_x = \partial^{l_1}_{x_1}...\partial^{l_2}_{x_s}$, where $l = (l_1, ..., l_s)$ a multiindex.

We also define space $V_r(Q_T)$ as the closure

$$V_r(Q_T) = \overline{C^\infty(Q_T)}^{||\cdot||_{V_r(Q_T)}},$$

where

$$||u||_{V_r(Q_T)} = ||u||_{W^{1,0}_r(Q_T)} + ||u_t||_{W^{1,0}_r(Q_T)}. \tag{2.2}$$

In the proof we use the following result for Sobolev spaces [1]. Let $u \in W^{m,n}_r(\Omega_T)$, $m, n \in \mathbf{R}_+$ then if $\sum_{i=1}^3(\alpha_i + \frac{1}{r} - \frac{1}{q})\frac{1}{m} + (\beta + \frac{1}{r} - \frac{1}{q})\frac{1}{n} < 1$ the following estimate holds

$$||D^\beta_t D^\alpha_x u||_{L_q(\Omega_T)} \le \varepsilon ||u||_{W^{m,n}_r(\Omega_T)} + c(\varepsilon)||u||_{L_2(\Omega_T)}, \tag{2.3}$$

where $q \ge r \ge 2$, $\varepsilon \in (0, 1)$ and $c(\varepsilon) \to \infty$ with $\varepsilon \to 0$.

During our considerations will use well known results like the imbedding theorems for Sobolev spaces. All constants are denoted by c.

3 Proof of Theorem A

Theorem A will be proved by the induction.

First step for $k = 1$ is an easily consequence of Theorem B.

We assume that solutions of (1.7) with initial time $t_0 = (k-2)T$ exist on $\Omega \times [(k-2)T, kT]$ and they satisfy

$$||u^{k-1}||_{W^{2,1}_r(\Omega\times[(k-1)T,kT])} + ||\eta^{k-1}||_{V_r(\Omega\times[(k-1)T,kT]} \le \delta. \tag{3.1}$$

Moreover by (3.1) and the trace theorem

$$||u^{k-1}||_{W^{2-2/r}_r(\Omega\times\{kT\})} + ||\eta^{k-1}||_{W^1_r(\Omega\times\{kT\})}$$

is so small that by Theorem B these solutions can be prolonged in time on $[kT, (k+1)T]$ and

$$||u^{k-1}||_{W^{2,1}_r(\Omega\times[kT,(k+1)T])} + ||\eta^{k-1}||_{V_r(\Omega\times[kT,(k+1)T])} \le \bar{\delta}. \tag{3.2}$$

Parameter T will be chosen later (see (3.11)). Quantities δ and $\bar{\delta}$ are sufficiently small.

Now we show estimates (3.1) and (3.2) for u^k and η^k.

Take system (1.7) with initial time $t_0 = (k-1)T$. By (3.2) solutions exist on time interval $[(k-1)T, (k+1)T]$. We prove estimate (3.1) for u^k and η^k.

Cut off initial data using a smooth function $\zeta_k : \mathbf{R}_+ \to [0, 1]$ such that

$$\zeta_k(t) = \begin{cases} 1 & \text{for} \quad t \ge Tk \\ 0 & \text{for} \quad t \le (k-1)T \end{cases}$$

and $|\zeta'_k| \le \frac{2}{T}$.

Introduce new variables $U^k = \zeta_k u^k$ and $X^k = \zeta_k \chi^k$ which satisfy the following problem

$$\begin{array}{lll} \eta U^k_t - \mu\Delta U^k - \nu\nabla div\, U^k + \gamma\nabla X^k = F - \eta\zeta'_k U^{k-1} & \text{in} & O_k, \\ X^k_t + \eta\, div\, U^k = G - \zeta'_k X^{k-1} & \text{in} & O_k, \\ U^k = 0 & \text{on} & \partial O_k, \\ U^k|_{t=(k-1)T} = 0, \quad X^k|_{t=(k+1)T} = 0 & \text{on} & \Omega, \end{array} \tag{3.3}$$

where

$$\begin{array}{l} O_k = \Omega \times [(k-1)T, (k+1)T], \quad \partial O_k = \partial\Omega \times [(k-1)T, (k+1)T], \\ F = \mu(\Delta_u - \Delta)U^k + \nu(\nabla_u div_u - \nabla div)U^k + X^k\nabla_u\gamma + \gamma(\nabla_u - \nabla)X^k, \\ G = \eta(div - div_u)U^k - \bar{\eta}_t \end{array}$$

and $U^{k-1}(\xi, t) = v(\mathcal{T}_{(k-1)T}(\xi, t), t)$, $X^{k-1}(\xi, t) = \sigma(\mathcal{T}_{(k-1)T}(\xi, t), t)$.

Since η and γ are not close to any constants we need to localize our problem. Introduce a cover of domain Ω.

Since $\partial\Omega$ is smooth we can find the following two collections of sets with smooth boundaries: $\{\omega^{(l)}\}_{l\in L}$ and $\{\Omega^{(l)}\}_{l\in L}$ (with finite L) such that $\omega^{(l)} \subset \Omega^{(l)}$ for all $l \in L$ and

$$\bigcup_{l\in L}\omega^{(l)} = \bigcup_{l\in L}\Omega^{(l)} = \Omega, \quad \sup_{l\in L} \operatorname{diam} \omega^{(l)} \le \lambda, \quad \sup_{l\in L} \operatorname{diam} \Omega^{(l)} \le 2\lambda.$$

With each $l \in L$ is connected a smooth function $\phi_l : \Omega \to [0,1]$ such that

$$\phi_l = \begin{cases} 1 & \text{for} \quad x \in \omega^{(l)} \\ 0 & \text{for} \quad x \in \Omega \setminus \Omega^{(l)} \end{cases}$$

and $|\nabla^k \phi_l| \le \frac{c}{\lambda^k}$. By ξ_l we denote the "center" of $\omega^{(l)}$ (a point inside).

Using these functions we obtain a localization of problem (3.3)

$$\begin{array}{l} \bar\varrho(x_l)(\phi_l U^k)_t - \mu\Delta(\phi_l U^k) - \nu\nabla div(\phi_l U^k) + \gamma(x_l)\nabla(\phi_l X^k) = F' + \phi_l F - \phi_l \eta \zeta_k' U^{k-1}, \\ (\phi_l X^k)_t + \bar\varrho(x_l) div(\phi_l U^k) = G' + \phi_l G - \phi_l \zeta_k' X^{k-1}, \\ \phi_l U^k|_{\partial\Omega^{(l)}} = 0, \quad \phi_l U^k|_{t=(k-1)T} = 0, \quad \phi_l X^k|_{t=(k-1)T} = 0, \end{array} \tag{3.4}$$

where

$$\begin{array}{l} F' = (\bar\eta(\xi_l,(k-1)T) - \eta)\,(\phi_l U^k)_t + \mu\left(\phi_l \Delta U^k - \Delta(\phi_l U^k)\right) + \\ \nu\left(\phi_l \nabla div U^k - \nabla div(\phi_l U^k)\right) + \gamma(\phi_l \nabla X^k - \nabla(\phi_l X^k)) + (\gamma(\bar\eta(\xi_l,(k-1)T)) - \gamma)\nabla(\phi_l X^k), \\ G' = \phi_l\,(\bar\eta(\xi_l,(k-1)T) - \eta)\, div U^k + \bar\eta(\xi_l,(k-1)T)\left(div(\phi_l U^k) - \phi_l div U^k\right) \end{array}$$

and $x_l = x(\xi_l,(k-1)T)$, $\bar\varrho(x_l) = \bar\eta(\xi_l,(k-1)T)$.

By Theorem C we get the estimate for $\phi_l U^k$ and $\phi_l X^k$

$$\begin{array}{c} ||\phi_l U^k||_{W_r^{2,1}(O_k^{(l)})} + ||\phi_l U^k||_{V_r(O_k^{(l)})} \le A_0 \Big(||F' + \phi_l F||_{L_r(O_k^{(l)})} + ||G' + \phi_l G||_{W_r^{1,0}(O_k^{(l)})} + \\ ||\phi_l \eta \zeta_k' U^{k-1}||_{L_r(O_k^{(l)})} + ||\phi_l \zeta_k' X^{k-1}||_{W_r^{1,0}(O_k^{(l)})} + ||\phi_l u||_{L_2(O_k^{(l)})} + ||\phi_l \chi||_{L_2(O_k^{(l)})} \Big), \end{array} \tag{3.5}$$

where $O^{(l)} = \Omega^{(l)} \times [(k-1)T,(k+1)T]$.

Two last terms of the r.h.s. of (3.5) we estimate by the Energy Estimate. The rest (without fourth one) we easily estimate using the interpolation theorems (see Appendix). Only one term makes some difficulties.

$$\begin{array}{c} ||\phi_l \zeta_k' X^{k-1}||_{W_r^{1,0}(\Omega^{(l)}\times[(k-1)T,kT])} \le \\ \frac{2}{T}||\nabla \phi_l X^{k-1}||_{L_r(\Omega^{(l)}\times[(k-1)T,kT])} + \frac{2}{T}||\nabla X^{k-1}||_{L_r(\Omega^{(l)}\times[(k-1)T,kT])}, \end{array} \tag{3.6}$$

where we used $|\zeta_k'| \le \frac{2}{T}$. As we see to estimate the first term of the r.h.s. of (3.6) we can apply an interpolation theorem, but for the second one we can't such a chance. To treat this we have to take sufficiently large T (A_0 is independent of T) to find suitable boundedness. Here we should underline that the choice of the magnitude of T depends only on A_0.

Hence one gets

$$\begin{array}{c} ||\phi_l U^k||_{W_r^{2,1}(O_k)} + ||\phi_l X^k||_{V_r(O_k)} \le \\ A_0 \Big(c\Big(\lambda + ||\chi||_{V_r(O_k)} + T^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_k)}\Big) ||\phi_l U^k||_{W_r^{2,1}(O_k)} + \\ c\Big(\varepsilon + ||\chi||_{V_r(O_k)} + T^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_k)}\Big)(||U^k||_{W_r^{2,1}(O_k)} + ||X^k||_{V_r(O_k)}) \\ + c(\varepsilon,\lambda)(||u||_{L_2(O_k)} + ||\chi||_{L_2(O_k)}) + \frac{4}{T}||\nabla X^{k-1}||_{L_r(\Omega^{(l)}\times[(k-1)T,kT])} \\ + \varepsilon(||U^{k-1}||_{W_r^{2,1}(\Omega^{(l)}\times[(k-1)T,kT])} + ||X^{k-1}||_{V_r(\Omega^{(l)}\times[(k-1)T,kT])})\Big). \end{array} \tag{3.7}$$

If we choose covers $\{\omega^{(l)}\}$ and $\{\Omega^{(l)}\}$ of domain Ω (and this choice is independent for each step of the induction) with sufficiently small λ, then by (3.2) we have

$$A_0 c\Big(\lambda^\alpha + ||\chi||_{V_r(\Omega\times[(k-1)T,(k+1)T])} + T^{\frac{r-1}{r}}||u||_{W_r^{2,1}(\Omega\times[(k-1)T,(k+1)T])}\Big) \le \frac{1}{2}$$

and get

$$\begin{array}{c} ||\phi_l U^k||_{W_r^{2,1}(O_k)} + ||\phi_l X^k||_{V_r(O_k)} \le \\ 2A_0 \Big(c\Big(\varepsilon + ||\chi||_{V_r(O_k)} + T^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_k)}\Big)(||U^k||_{W_r^{2,1}(O_k)} + ||X^k||_{V_r(O_k)}) \\ + c(\varepsilon,\lambda)(||u||_{L_2(O_k)} + ||\chi||_{L_2(O_k)}) + \frac{4}{T}||\nabla X^{k-1}||_{L_r(\Omega^{(l)}\times[(k-1)T,kT])} \\ + \varepsilon(||U^{k-1}||_{W_r^{2,1}(\Omega^{(l)}\times[(k-1)T,kT])} + ||X^{k-1}||_{V_r(\Omega^{(l)}\times[(k-1)T,kT])})\Big). \end{array} \tag{3.8}$$

Since the multiplicity of the cover is less then N_0, we can sum inequalities (3.8) over $l \in L$ and get

$$\begin{array}{c} ||U^k||_{W_r^{2,1}(O_k)} + ||X^k||_{V_r(O_k)} \le \\ 2A_0 N_0 \Big(c(\varepsilon + ||\chi||_{V_r(O_k)} + T^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_k)})(||U^k||_{W_r^{2,1}(O_k)} + ||X^k||_{V_r(O_k)}) \\ + c(\varepsilon,\lambda)(||u||_{L_2(O_k)} + ||\chi||_{L_2(O_k)}) + \frac{4}{T}||\nabla X^{k-1}||_{L_r(\Omega\times[(k-1)T,kT])} \\ + \varepsilon(||U^{k-1}||_{W_r^{2,1}(\Omega\times[(k-1)T,kT])} + ||X^{k-1}||_{V_r(\Omega\times[(k-1)T,kT])})\Big). \end{array} \tag{3.9}$$

Recalling (3.2) - $\bar{\delta}$ is sufficiently small - we have

$$2A_0N_0c(\varepsilon + ||\chi||_{V_r(\Omega\times[(k-1)T,(k+1)T])} + T^{\frac{r-1}{r}}||u||_{W^{2,1}_r(\Omega\times[(k-1)T,(k+1)T])}) \leq \frac{1}{2}.$$

So boundedness (3.9) gives

$$\begin{gathered} ||U^k||_{W^{2,1}_r(\Omega\times[kT,(k+1)T])} + ||X^k||_{V_r(\Omega\times[kT,(k+1)T])} \leq \\ 4A_0N_0\Big((\varepsilon + \tfrac{4}{T})(||U^{k-1}||_{W^{2,1}_r(\Omega\times[(k-1)T,kT])} + ||X^{k-1}||_{V_r(\Omega\times[(k-1)T,kT])}) + \\ + c(\varepsilon,\lambda,T)(||u||_{L_2(\Omega\times[(k-1)T,(k+1)T])} + ||\chi||_{L_2(\Omega\times[(k-1)T,(k+1)T])})\Big) \end{gathered} \tag{3.10}$$

From (3.1) we get that if δ sufficiently small then

$$||U^{k-1}||_{W^{2,1}_r(\Omega\times[(k-1)T,kT])} + ||X^{k-1}||_{V_r(\Omega\times[(k-1)T,kT])} \leq 2\delta.$$

Having ε so small and T so large that (by Theorem C constant A_0 is independent of T)

$$4A_0N_0(\varepsilon + \frac{4}{T}) \leq \frac{1}{4}, \tag{3.11}$$

by the Energy Estimate we get

$$||U^k||_{W^{2,1}_r(\Omega\times[kT,(k+1)T])} + ||X^k||_{V_r(\Omega\times[kT,(k+1)T])} \leq \tfrac{1}{2}\delta + c(\varepsilon,\lambda,T)(||v_0||_{L_2(\Omega)} + ||\sigma_0||_{L_2(\Omega)}). \tag{3.12}$$

Now if M_0 (see Theorem A) is so small that

$$c(\varepsilon,\lambda,T)(||v_0||_{L_2(\Omega)} + ||\sigma_0||_{L_2(\Omega)}) \leq \frac{1}{2}\delta$$

then

$$||U^k||_{W^{2,1}_r(\Omega\times[kT,(k+1)T])} + ||X^k||_{V_r(\Omega\times[kT,(k+1)T])} \leq \delta. \tag{3.13}$$

Since $U^k = u^k$ and $X^k = \chi^k$ for $t \in [kT,(k+1)T]$ estimate (3.13) proves (3.1) from this induction step. By smallness of δ we can continue solutions on $[(k+1)T,(k+2)T]$ with boundedness

$$||u^k||_{W^{2,1}_r(\Omega\times[(k+1)T,(k+2)T])} + ||\chi^k||_{V_r(\Omega\times[(k+1)T,(k+2)T])} \leq \bar{\delta}. \tag{3.14}$$

Theorem A is proved.

4 Appendix

First we prove the Energy Estimate.

Proof of Energy Estimate. Multiplying $(1.3)_1$ by v, integrating over Ω, using $(1.3)_2$ we easily get

$$\frac{d}{dt}\int_\Omega \left[\frac{1}{2}\varrho v^2 + \frac{a}{\kappa-1}\varrho^\kappa - \varrho\varphi\right]dx + \mu\int_\Omega |\nabla v|^2 dx + \nu\int_\Omega |div\, v|^2 dx = 0. \tag{4.1}$$

Next we examine a functional

$$I(\sigma) = \int_\Omega \left[\frac{a}{\kappa-1}(\bar{\varrho}+\sigma)^\kappa - (\bar{\varrho}+\sigma)\varphi\right]dx - \int_\Omega \left[\frac{a}{\kappa-1}\bar{\varrho}^\kappa - \bar{\varrho}\varphi\right]dx.$$

Since $||\sigma||_{L_\infty}$ is small and $\kappa > 1$, from (1.2), using the elementary calculus of variation we get

$$a_1||\sigma||^2_{L_2(\Omega)} \leq I(\sigma) \leq a_2||\sigma||^2_{L_2(\Omega)}. \tag{4.2}$$

So form (4.1) we easily obtain

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho v^2 + I(\sigma)\right)dx \leq 0 \tag{4.3}$$

which together with (4.2) gives (1.4).

Next we find estimates for terms of the r.h.s of (3.5).

Estimates on F, F', G and G'. Here we estimate few terms of F, F', G and G', the rest one can estimate similarly.

$$||\mu(\Delta-\Delta_u)U^k||_{L_r}\le cT^{\frac{r-1}{r}}||u||_{W^{2,1}_r}\left(||U^k||_{W^{2,1}_r}+||\nabla U^k||_{L_r(0,T;L_\infty(\Omega))}\right)$$
$$\le cT^{\frac{r-1}{r}}||u||_{W^{2,1}_r}\left(||U^k||_{W^{2,1}_r}+||u||_{L_r(0,T;L_2(\Omega))}\right);$$

$$||X^k\nabla_u\gamma||_{L_r}\le c||X^k||_{L_\infty}\le\varepsilon||X^k||_{V_r}+c(\varepsilon)||\chi||_{L_\infty(0,T;L_2(\Omega))};$$

$$||\bar\eta_t||_{W^{1,0}_r}=||\nabla\bar\eta\cdot u||_{W^{1,0}_r}\le\varepsilon\left(||U^{k-1}||_{W^{2,1}_r}+||U^k||_{W^{2,1}_r}\right)+c(\varepsilon)||u||_{L_\infty(0,T;L_2(\Omega))}.$$

To estimate the first term of F' we note that

$$\bar\eta(\xi_l,(k-1)T)-\eta(\xi,t)=\bar\eta(\xi_l,(k-1)T)-\bar\eta(\xi,t)-\chi(\xi,t)$$

and since $\bar\varrho(\cdot)$ is sufficiently regular

$$||\bar\eta(\xi_l,(k-1)T)-\bar\eta(\xi,t)||_{L_\infty}\le c||\xi_l-\xi+\textstyle\int_{(k-1)T}^t u(\cdot,t')dt'||_{L_\infty}\le c\lambda+cT^{\frac{r-1}{r}}||u||_{W^{2,1}_r}.$$

So we obtain

$$||(\bar\eta(\xi_l,(k-1)T)-\eta)(\phi_lU^k)_t||_{L_r}\le c(\lambda+T^{\frac{r-1}{r}}||u||_{W^{2,1}_r}+||\chi||_{V_r})||\phi_lU^k||_{W^{2,1}_r}.$$

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