On cylindrical symmetric flows through pipe-like domains

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Abstract. We prove existence of cylindrical symmetric solutions to the steady Navier-Stokes equations in bounded pipe-like domains in $\mathbf{R}^3$ with the slip boundary conditions. The result is shown for any large flows across the boundary assuming only a geometrical constraint on the shape of the domain which is independent of data. The simply connectedness of the domain is not required. The technique is based on a reformulation of the original problem and delivers us a new type of estimates in the Hölder spaces for this class of the solutions.

MS Classification: 35Q30, 76D03, 76D05

Key words: Navier-Stokes equations, large data and flux, inhomogeneous boundary data, geometrical constraint, subharmonic analysis.

1 Introduction

Our aim is to investigate the issue of existence for the model of incompressible flows of viscous Newtonian fluids in bounded pipe-like domains in $\mathbf{R}^3$ with large boundary data. The motion is governed by the steady Navier-Stokes equation with the slip boundary conditions involving friction between the boundary and the fluid. The system reads

$$\begin{array}{llll} v \cdot \nabla v - \nu \Delta v + \nabla p = F & & \text{in} & \Omega, \\ \operatorname{div} v = 0 & & \text{in} & \Omega, \\ n \cdot \mathbf{T}(v,p) \cdot \tau_k + f v \cdot \tau_k = 0 & k = 1,2 & \text{on} & \partial\Omega, \\ n \cdot v = d & & \text{on} & \partial\Omega, \end{array} \qquad (1.1)$$

where $v = (v_1, v_2, v_3)$ is the velocity of the fluid, p is the pressure, ν - the constant viscous coefficient, F - an external force, n and τ_k - the normal and tangent vectors to boundary $\partial\Omega$, f - the nonnegative friction coefficient which in general may be non-constant, $\mathbf{T}$ is the stress tensor of the Newtonian fluid

$$\mathbf{T}(v,p) = \nu \mathbf{D}(v) - pId = \{\nu(v_{i,j} + v_{j,i}) - p\delta_{ij}\}_{i,j=1,2,3}; \qquad (1.2)$$

data d describes the flow across the boundary and may be consider as follows

$$d = \begin{cases} 0 & \text{for} \quad x \in \Gamma_0 \\ v_{in/out} & \text{for} \quad x \in \Gamma_{in/out} \end{cases} \tag{1.3}$$

where Γ_0 and $\Gamma_{in/out}$ is a natural decomposition of boundary $\partial\Omega$ such that $\Gamma_0 = \{x \in \partial\Omega : d(x) = 0\}$ and $\Gamma_{in/out} = \partial\Omega \setminus \Gamma_0$.

By $(1.1)_2$ datum d satisfies the following compatibility condition

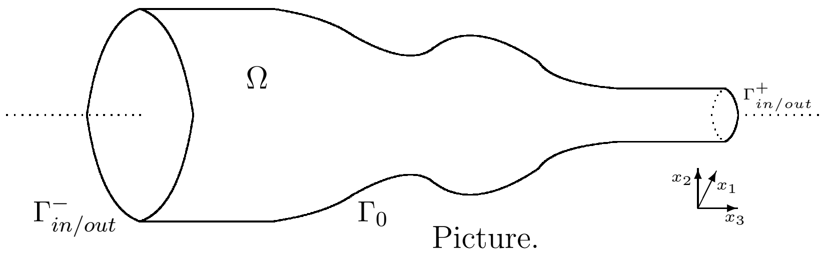
$$\int_{\partial\Omega} d(x)\, d\sigma = 0 \tag{1.4}$$

which controls the total flux through the pipe.

We want to investigate our system under the cylindrical symmetry. We require to domain Ω be cylindrical symmetric. As a model case we consider

$$\Omega = \{x \in \mathbf{R}^3 : \sqrt{x_1^2 + x_2^2} < R(x_3) \quad \text{and} \quad -L < x_3 < L\} \tag{1.5}$$

for a C^2-smooth positive function given on interval $[-L, L]$ and to avoid difficulties with regularity of solutions we assume that $R'(-L) = R'(L) = 0$, as well as $R''(-L) = R''(L) = 0$.



Picture.

In general, the boundary of the domain can be multiply connected, and for simplicity we assume that the first group of homotopy of Ω is trivial. There is no restriction on boundary data. Fluxes through each component of $\partial\Omega$ are not restricted, either.

Moreover, the velocity in the Cartesian coordinates reads

$$v = (x_1 v_r(w, x_3), x_2 v_r(w, x_3), v_3(w, x_3)), \tag{1.6}$$

where

$$w = \frac{x_1^2 + x_2^2}{2}. \tag{1.7}$$

The pressure, data d and the force are prescribed under the same symmetry

$$p = p(w, x_3), \quad d = d(w, x_2), \quad F = (x_1 F_r(w, x_3), x_2 F_r(w, x_3), F_3(w, x_3)). \tag{1.8}$$

In the standard cylindrical coordinate every considered quantities are independent of angle φ and φ-components of vectors are equal to zero. This type of symmetry is called "axially", too.

Definition 1.1. *Problem (1.1) with cylindrical constraints: (1.5), (1.6) and (1.8) we call the symmetric system.*

The goal of the paper is to prove existence of solutions to the symmetric system for large data (including large flux) under a geometrical constraint which restricts the shape of the domain. The bound is completely independent of largeness of data. The result of the paper is the following.

Theorem. *Let* $0 < a < 1$, $f \in L_\infty(\partial\Omega)$, F *and* d *be symmetric, and*

$$g = F_{3,w} - F_{r,w} \in L_2(\Omega), \quad d \in C^a(\Omega) \quad \text{and} \quad \nabla d / \sqrt{x_1^2 + x_2^2} \in L_\infty(\partial\Omega). \tag{1.9}$$

If the shape of domain Ω *fulfills*

$$A_0(\Omega) ||2\chi_1 - f/\nu||_{L_\infty(\partial\Omega)} < 1, \tag{1.10}$$

where

- $A_0(\Omega)$ *is a constant given by Definition 2.1,*
- $\chi_1 = -R''(x_3)/(1 + (R'(x_3))^2)^2$ *for* $x \in \Gamma_0$,
- $\chi_1 = 0$ *for* $x \in \Gamma_{in/out}$.

Then there exists at least one symmetric solution of the symmetric problem such that $v \in C^a(\Omega)$, $rot\, v \in L_\infty(\Omega) + H_0^1(\Omega)$ *and the following estimate is valid*

$$\begin{aligned} &||v||_{C^a(\Omega)} + ||v_r||_{L_\infty(\partial\Omega)} \leq c \left(||d, \tfrac{\nabla d}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)} + ||g||_{L_2(\Omega)} \right), \\ &||\tfrac{rot\, v}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\Omega)+H_0^1(\Omega)} \leq c \left(||d, \tfrac{\nabla d}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)} + ||g||_{L_2(\Omega)} \right). \end{aligned} \tag{1.11}$$

The obtained result describes a class of domains which admits existence of symmetric solutions to the steady Navier-Stokes equations with large flux across the boundary. The class is given by geometrical constraint (1.10). To see the geometrical character of (1.10) it is enough to note that quantities f

and ν are independent of features of domain Ω. Because of non-negativeness of f/ν we obtain the following condition

$$2A_0(\Omega)\min_{x\in\partial\Omega}\{\chi_1, 0\} > -1$$

which must be fulfilled by the shape of the domain.

Quantity χ_1 is one of main curvatures of surface Γ_0 and constant $A_0(\Omega)$ depends on the constant from the Poincare inequality for domain Ω. Thus, this relation restricts essentially the shape of the domain. Moreover, the magnitude of effective friction f/ν is bounded, too. Nevertheless, if condition (1.10) is fulfilled, we are able to obtain estimate (1.11) with no restriction on smallness of data. This bound has a linear form as the energy estimate for a case with homogeneous boundary data, but delivers us much more information.

To find an example of a non simply connected domain fulfilling the geometrical constraint, it is enough to consider

$$\Omega = \{x \in \mathbf{R}^3 : R_1 < |x| < R_2\},$$

then $R_2 - R_1$ describes the constant from the Poincare inequality and $1/R_1$, $1/R_2$ curvatures. Sufficiently smallness of quantity $(R_2 - R_1)/R_1$ guarantees possibility to satisfy condition (1.10). A perturbation of this set fulfills the condition, too. However, we concentrate the attention on example given by (1.5).

Since estimate (1.11) do not guarantees sufficient smoothness, Theorem delivers us only a weak solution to problem (1.1). The formulation of the weak sense is given at the end of section 3 - Definition 3.4. Increasing the regularity of data, adding suitable compatibility conditions we can obtain regular solutions applying the well known theory [4].

The slip boundary condition may be treated as an alternative to the Dirichlet condition for problems in rigid domains [5]. If friction f goes to infinity then $(1.1)_{3,4}$ become Dirichlet data. Also if effective friction f/ν is small, as in our case, slip boundary conditions seem to be a good supplement to the Navier-Stokes equations to study symmetric system (1.1) as an approximation of a flow of a perfect fluid [1, 8]. In the most of results involving $(1.1)_{3,4}$ are similar to ones for the Dirichlet conditions, since the approach is almost the same and is concentrated on examination of original problem $(1.1)_{1,2}$ [2, 10].

For the Navier-Stokes equations with the Dirichlet boundary conditions current methods prove existence of solutions only for small fluxes across the boundary, even for two dimensional cases [3, 4, 9], the technique works only for simply connected domains. The difficulty comes from the fact that to apply the energy method, we need to subtract from the solution an extension of boundary data to obtain a homogeneous boundary condition. This modification adds new terms to the equations and they can be estimated only under smallness of the flow across the boundary.

Comparing to the technique for the problem presented in this paper, we avoid any subtraction. We do not examine the original system, but study a reformulation of it. The method is based on the maximum principle for the vorticity and it follows that we work on the equation which preserves the physical structure, although the boundary data may be arbitrary large. One thing, which is required, is constraint (1.10).

The presented approach is an adaptation of the technique for the two dimensional case of system (1.1) studied in [7, 8]. The general ideas are similar and main new difficulties are hidden in a choice of the coordinates system (to avoid troubles with singularities in the equations) and appearance of a need to control some weighted norms of the solutions. This technique can not be adapt (in a direct way) to the full 3D system, since for non-symmetric cases we could not find the maximum principle for the equation on the vorticity. But some ideas of the method may be found in [12], where for the evolutionary system 3D stability of nontrivial cylindrical symmetric solutions is shown.

As a historical remark, let us note that first result concerns existence of evolutionary cylindrical symmetric solutions are due to Ladyzhenskaya, Ukhovskij and Yudovich [6,11] and their proofs are based on the analysis of the equation on the vorticity, too.

As we mentioned above, a key element of the proof of Theorem is a reformulation of the system. We do not investigate (1.1), but we examine the system on the vorticity. The equations read

$$v \cdot \nabla\alpha - \nu\Delta\alpha = \alpha \cdot \nabla v + \text{rot } F \quad \text{in } \Omega, \tag{1.12}$$

where $\alpha = \text{rot } v = \nabla \times v$ is the vorticity of the vector field. The structure of equation (1.12) does not allow to apply methods as the maximum principle. That is the main reason that we have to restrict our considerations to the narrow class of the cylindrical symmetric solutions.

By the required symmetry, function α takes the following form

$$\alpha = (x_2\theta(w, x_3), -x_1\theta(w, x_2), 0), \tag{1.13}$$

where θ is a scalar function given as follows

$$\theta = v_{3,w}(w, x_3) - v_{r,3}(w, x_3). \tag{1.14}$$

Inserting form (1.13) into (1.12) we get one scalar equation

$$v \cdot \nabla\theta - \nu\Delta\theta - 2\nu\theta_{,w} = g \qquad \text{in } \Omega, \tag{1.15}$$

where $g = F_{3,w} - F_{r,w}$.

The relation between (1.12) and (1.15) shows how crucial is the restriction to the cylindrical symmetry. This form not only transforms three equations on a full 3-D vector into one scalar equation on an 1-D unknown, but the first of all, it causes that term $\alpha \cdot \nabla v$ from the r.h.s. of (1.12) disappears. Owing to this feature, equation (1.15) possesses the maximum principle. This property is the main element of our technique.

Investigate boundary conditions. An essential advantage of slip conditions $(1.1)_{3,4}$ is possibility to compute tangent components of the vorticity vector on the boundary as functions of the tangent velocity. In general we have

$$\begin{aligned} \alpha \cdot \tau_2 &= (2\chi_1 - f/\nu) v \cdot \tau_1 + 2d_{,\tau_1}, \\ \alpha \cdot \tau_1 &= (f/\nu - 2\chi_2) v \cdot \tau_2 - 2d_{,\tau_2}, \end{aligned} \tag{1.16}$$

where frame (n, τ_1, τ_2) preserves the orientation of the boundary and χ_k denote curvatures of curves generated by tangent directions τ_k.

To obtain a suitable form of (1.16) compatible to the symmetry, we describe the boundary of Ω given by (1.5). In the studied case we put

$$\begin{aligned} \Gamma_0 &= \{x \in \partial\Omega : \sqrt{x_1^2 + x_2^2} = R(x_3) \text{ and } -L \le x_3 \le L\}, \\ \Gamma^-_{in/out} &= \{x \in \partial\Omega : \sqrt{x_1^2 + x_2^2} < R(-L) \text{ and } x_3 = -L\}, \\ \Gamma^+_{in/out} &= \{x \in \partial\Omega : \sqrt{x_1^2 + x_2^2} < R(L) \text{ and } x_3 = L\}. \end{aligned} \tag{1.17}$$

An interpretation of this assumption describes the considered model as a flow through a pipe with the inflow and outflow across bottoms.

Define condition (1.16) on Γ_0. On this part of the boundary the frame has the following form

$$\begin{aligned} n &= \frac{1}{\sqrt{1+(R'(x_3))^2}} \left(\frac{x_1}{R(x_3)}, \frac{x_2}{R(x_3)}, -R'(x_3) \right), \\ \tau_1 &= \frac{1}{\sqrt{1+(R'(x_3))^2}} \left(\frac{x_1 R'(x_3)}{R(x_3)}, \frac{x_2 R'(x_3)}{R(x_3)}, 1 \right), \quad \tau_2 = \frac{1}{R(x_3)}(x_2, -x_1, 0). \end{aligned} \tag{1.18}$$

Vector τ_2 is tangential to curve $l_{x_1,x_2}(x_3) = (x_1, x_2, R(x_3)) \subset \Gamma_0$.

Inserting forms (1.18) into (1.16) we obtain the following boundary condition

$$\theta = \frac{2\chi_1 - f/\nu}{R(x_3)} v \cdot \tau_1 \qquad \text{on } \Gamma_0, \tag{1.19}$$

where χ_1 is the curvature of curve $l_{x_1,x_2}(x_3)$ (by the symmetry it depends only on x_3 and it is given as in the statement of Theorem).

On $\Gamma^-_{in/out}$ and $\Gamma^+_{in/out}$ we have

$$n = (0,0,-1), \quad \tau_1 = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1, x_2, 0), \quad \tau_2 = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_2, -x_1, 0) \tag{1.20}$$

and

$$n = (0,0,1), \quad \tau_1 = \frac{1}{\sqrt{x_1^2+x_2^2}}(-x_1, -x_2, 0), \quad \tau_2 = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_2, -x_1, 0), \tag{1.21}$$

respectively. Inserting (1.20) and (1.21) into (1.16) we get

$$\theta = -\frac{f/\nu}{\sqrt{x_1^2+x_2^2}} v \cdot \tau_1 + \frac{2d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}} \qquad \text{on } \Gamma_{in/out}. \tag{1.22}$$

To close the system: (1.15), (1.19) and (1.22) we need information about the velocity. But this quantity satisfies the following elliptic problem

$$\begin{aligned} \text{rot } v = \alpha \quad & \text{in} \quad \Omega, \\ \text{div } v = 0 \quad & \text{in} \quad \Omega, \\ n \cdot v = d \quad & \text{on} \quad \Omega. \end{aligned} \tag{1.23}$$

Estimates for solutions to system (1.23) are crucial to close the estimation in the analysis of the vorticity system. In particular, there is a need of special bound in a weighted space for v on the very boundary. Such an exotic result we prove in Appendix - section 4, applying a nonstandard technique based on a feature of the subharmonic-type functions. Also we may omit this difficulty putting $f \equiv 0$ on $\Gamma_{in/out}$, then (1.22) turns into a given boundary datum. The uniqueness for (1.23) follows from the assumption that the first group of homotopy of Ω is trivial.

Thus, instead of the original symmetric system (1.1) we investigate the reformulation: (1.15), (1.19), (1.22) and (1.23).

We would like to avoid complications appearing in an approach in the cylindrical coordinates (as weighted spaces and singularities in the equations), that is the reason we study our system in the Cartesian coordinates.

Note that a cylindrical symmetric flow with a constant rotation around axle Ox_3 is not preserved by the nonlinear Navier-Stokes system. The third equation of (1.12) makes that either this rotation must be zero or the flow must be two dimensional, thus in our case trivial.

2 Notation

Throughout the paper all quantities are cylindrical symmetric, i.e.:
- scalars: $q = q(w, x_3)$,
- vectors: $h = (x_1 h_r(w, x_3), x_2 h_r(w, x_3), h_3(w, x_3))$,

where $w = \frac{x_1^2+x_2^2}{2}$.

To underline the symmetry we denote any functions space B by SB, i.e. $SB = \{b \in B : b$ is cylindrical symmetric$\}$ and norm $||\cdot||_{SB} = ||\cdot||_B$.

In particular,

$$SC(\Omega) = \{a \in C(\Omega) : a \text{ is cylindrical symmetric}\}, \tag{2.1}$$

$$SL_2(\Omega) = \{a \in L_2(\Omega) : a \text{ is cylindrical symmetric}\}, \tag{2.2}$$

where $C(\Omega)$ denotes the standard Banach space of continuous functions with norm $||a||_{C(\Omega)} = \sup_{x\in\Omega} |a(x)|$, and $L_2(\Omega)$ denotes the Hilbert space with norm $||a||_{L_2(\Omega)} = (\int_0^1 a^2 dx)^{1/2}$.

To simplify the notation, we use the following convention to denote the same norm of several quantities

$$||a_1, ..., a_n||_B = ||a_1||_B + ... + ||a_n||_B.$$

The essential constant in the formulation of Theorem is $A_0(\Omega)$ which depends on the constant from the Poincare Lemma. The precise description is given by the following definition.

Definition 2.1. *Constant $A_0(\Omega)$ is given as follows*

$$A_0(\Omega) = ||V_r||_{SC(\Omega)}, \tag{2.3}$$

where V is the solution to the following problem

$$\begin{aligned} rot\, V = (x_2, x_1, 0) \quad & in \quad \Omega, \\ div\, V = 0 \quad & in \quad \Omega, \\ n \cdot V = 0 \quad & on \quad \partial\Omega. \end{aligned} \tag{2.4}$$

Although we are trying to avoid weighed spaces, in the proof of Theorem we need the following results from [13, 14]. The first lemma concerns also non-symmetric case.

Lemma 2.1. *Let* $div\,\alpha = 0$ *and*

$$\alpha/\sqrt{x_1^2+x_2^2},\ \ \nabla\alpha/\sqrt{x_1^2+x_2^2}\ \ \in L_2(\Omega), \tag{2.5}$$

then the solution of the following problem

$$\begin{array}{rll} rot\,v = \alpha & in & \Omega, \\ div\,v = 0 & in & \Omega, \\ n\cdot v = 0 & on & \partial\Omega \end{array} \tag{2.6}$$

fulfills the following bound

$$||\tfrac{v}{\sqrt{x_1^2+x_2^2}}, \tfrac{\nabla v}{\sqrt{x_1^2+x_2^2}}, \tfrac{\nabla^2 v}{\sqrt{x_1^2+x_2^2}}||_{L_2(\Omega)} \le c||\tfrac{\alpha}{\sqrt{x_1^2+x_2^2}}, \tfrac{\nabla\alpha}{\sqrt{x_1^2+x_2^2}}||_{L_2(\Omega)}. \tag{2.7}$$

In particular, by the imbedding theorem

$$||\tfrac{v}{\sqrt{x_1^2+x_2^2}}||_{C(\Omega)} \le c||\tfrac{\alpha}{\sqrt{x_1^2+x_2^2}}, \tfrac{\nabla\alpha}{\sqrt{x_1^2+x_2^2}}||_{L_2(\Omega)}. \tag{2.8}$$

The next result concerns the symmetric case.

Lemma 2.2. *Let* $d, d_{,\tau_1}/\sqrt{x_1^2+x_2^2} \in SL_\infty(\partial\Omega)$, *then the solution of the following problem*

$$\begin{array}{rll} rot\,\mathcal{D} = 0 & in & \Omega, \\ div\,\mathcal{D} = 0 & in & \Omega, \\ n\cdot \mathcal{D} = d & on & \partial\Omega \end{array} \tag{2.9}$$

fulfills the following bound

$$||\mathcal{D}||_{C^a(\Omega)} + ||\tfrac{\mathcal{D}\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)} \le c||d, \tfrac{d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)}. \tag{2.10}$$

Note that estimate (2.10) is not a Schauder-type, but it is sufficient for our method.

Throughout the paper by letter c we denote generic constants, by A_k and B_k constants which are fixed in the proof.

3 The proof

We show existence for the coupled system: for the vorticity (1.15) with boundary data (1.19) and (1.22), and for the velocity (1.23).

Boundary condition $(1.23)_3$ defines us an affine space of velocity vectors where we are looking for solutions of our problem.

Definition 3.1. *The following affine space*

$$\Xi = \{v \in SC(\Omega) : n \cdot v|_{\partial\Omega} = d\} \tag{3.1}$$

we call the velocity vector space.

A tool to prove existence of solutions to the symmetric problem (1.1) is the Schauder fixed point theorem. Since the examined system is nonlinear and data are large, we do not expect uniqueness of solutions.

Define a map

$$\Phi : \Xi \to \Xi, \quad \Phi(v) = \tilde{v}, \tag{3.2}$$

where function $\tilde{v}$ is a solution to the following elliptic system

$$\begin{array}{rcl} \text{rot } \tilde{v} = \alpha & \text{in} & \Omega, \\ \text{div } \tilde{v} = 0 & \text{in} & \Omega, \\ n \cdot \tilde{v} = d & \text{on} & \partial\Omega. \end{array} \tag{3.3}$$

And a vector α is equal to

$$\alpha = (x_2\theta(w, x_3), -x_1\theta(w, x_3), 0), \tag{3.4}$$

where a scalar function θ satisfies the following problem

$$\begin{array}{lcr} v \cdot \nabla\theta - \nu\Delta\theta - 2\nu\theta_{,w} = g & \text{in} & \Omega, \\ \theta = \frac{2\chi_1 - f/\nu}{R(x_3)} v \cdot \tau_1 & \text{on} & \Gamma_0, \\ \theta = -\frac{f/\nu}{\sqrt{x_1^2+x_2^2}} v \cdot \tau_1 + \frac{2d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}} & \text{on} & \Gamma_{in/out}, \end{array} \tag{3.5}$$

where $g = F_{3,w} - F_{r,3}$.

The required regularity makes us define such a sense of solutions only for problem (3.5), for (3.3) we are able to find a regular solution.

Examine problem (3.5). First we introduce a function which cuts off the inhomogeneity from the boundary to find an existence for this system.

Lemma 3.1. *There exists an interior smooth vector function $\mathcal{B}$ such that $\mathcal{B}$ is a solution to the following problem*

$$\begin{array}{lll} -\nu\Delta\mathcal{B} - 2\nu\mathcal{B}_{,w} = 0 & in & \Omega, \\ \mathcal{B} = \frac{2\chi_1 - f/\nu}{R(x_3)} v\cdot\tau_1 & on & \Gamma_0, \\ \mathcal{B} = \frac{-f/\nu}{\sqrt{x_1^2+x_2^2}} v\cdot\tau_1 + \frac{2d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}} & on & \Gamma_{in/out} \end{array} \tag{3.6}$$

and the following bound is valid

$$\begin{array}{c} ||\mathcal{B}||_{L_\infty(\Omega)} \leq ||\frac{2d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\Gamma_{in/out})} + \\ \max\left\{||\frac{2\chi_1 - f/\nu}{R(x_3)} v\cdot\tau_1||_{L_\infty(\Gamma_0)}, ||\frac{f/\nu}{\sqrt{x_1^2+x_2^2}} v\cdot\tau_1||_{L_\infty(\Gamma_{in/out})}\right\}. \end{array} \tag{3.7}$$

Proof. The interior smoothness of field $\mathcal{B}$ is obvious. To get bound (3.7) it is enough to apply the maximum principle. Note that because of the symmetry $dx = 2\pi dx_3 dw$.

Having Lemma 3.1 we consider function θ as follows

$$\theta = \eta + \mathcal{B}, \tag{3.8}$$

where η is a solution of the following problem

$$\begin{array}{lll} v\cdot\nabla\eta - \nu\Delta\eta - 2\nu\eta_{,w} = g - v\cdot\nabla\mathcal{B} & \text{in} & \Omega, \\ \eta = 0 & \text{in} & \partial\Omega. \end{array} \tag{3.9}$$

For problem (3.9) we define a weak solution.

Definition 3.2. *We say that $\eta \in SH_0^1(\Omega)$ is a weak solution to problem (3.9) iff the following identity*

$$\begin{array}{c} (v\cdot\nabla\eta, \varphi)_{L_2(\Omega)} + \nu(\nabla\eta, \nabla\varphi)_{L_2(\Omega)} - 2\nu(\eta_{,w}, \varphi)_{L_2(\Omega)} = \\ -(v\cdot\nabla\varphi, \mathcal{B})_{L_2(\Omega)} + (g, \varphi)_{L_2(\Omega)} \end{array} \tag{3.10}$$

holds for any $\varphi \in SH_0^1(\Omega)$.

Note that function η as well as $\mathcal{B}$ may not vanish on axle Ox_3. Hence we should underline that third term of l.h.s. of (3.10) should develop into boundary terms, but it is well posed, since by the trace theorem they are well defined. Of course, function θ vanishes on axle Ox_3, but it follows from the definition and the symmetry.

Lemma 3.2. *There exists a weak solution of problem (3.9) in the sense of Definition 3.2 such that η fulfills the following bound*

$$||\nabla\eta||_{H_0^1(\Omega)} + \left(\int_{-L}^{L} \eta^2(0,0,x_3)dx_3\right)^{1/2} \leq c(||g||_{L_2(\Omega)} + || \, |v| \, |\mathcal{B}| \, ||_{L_2(\Omega)}). \quad (3.11)$$

Proof. A proof of Lemma 3.2 follows from the Galerkin method and the energy estimate (3.11). To get this bound it is enough to insert $\varphi = \eta$ into formulation (3.10).

The above considerations lead us to an existence of solutions to problem (3.5) in the following sense.

Definition 3.3. *We say that a function θ is a weak solution of system (3.5), if $\theta \in SL_\infty(\Omega) + SH_0^1(\Omega)$ and θ fulfills (3.5) in the sense of Definition 3.2 and form (3.8).*

Obtained estimates are not sufficient for our investigation, there is a need of more precise ones which would have a linear form. That is the reason, we decompose function θ.

Introduce

$$B\Theta = \begin{cases} \theta - B^* & \text{for} \quad \theta(x) > B^* \\ 0 & \text{for} \quad B_* \leq \theta(x) \leq B^* \\ \theta - B_* & \text{for} \quad \theta(x) < B_* \end{cases} \quad (3.12)$$

where

$$\begin{aligned} B^* &= \max\left\{\sup_{y\in\Gamma_0} \tfrac{2\chi_1 - f/\nu}{R(x_3)} v\cdot\tau_1, \sup_{y\in\Gamma_{in/out}} -\tfrac{f/\nu}{\sqrt{x_1^2+x_2^2}} v\cdot\tau_1 + \tfrac{2d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}}\right\}, \\ B_* &= \min\left\{\inf_{y\in\Gamma_0} \tfrac{2\chi_1 - f/\nu}{R(x_3)} v\cdot\tau_1, \inf_{y\in\Gamma_{in/out}} -\tfrac{f/\nu}{\sqrt{x_1^2+x_2^2}} v\cdot\tau_1 + \tfrac{2d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}}\right\}. \end{aligned} \quad (3.13)$$

Quantity $B\Theta$ enables us to formulate the next result which gives the desired bounds.

Lemma 3.3. *The weak solution of problem (3.5) satisfies the following estimates*

$$||B\Theta||_{H_0^1(\Omega)} \leq c||g||_{L_2(\Omega)}, \quad (3.14)$$

$$\begin{aligned} ||\theta - B\Theta||_{L_\infty(\Omega)} \leq || \tfrac{2d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}} ||_{L_\infty(\Gamma_{in/out})} + \\ \max\left\{|| \tfrac{2\chi_1 - f/\nu}{R(x_3)} v\cdot\tau_1 ||_{L_\infty(\Gamma_0)}, || \tfrac{f/\nu}{\sqrt{x_1^2+x_2^2}} v\cdot\tau_1 ||_{L_\infty(\Gamma_{in/out})}\right\}. \end{aligned} \quad (3.15)$$

Proof. Bound (3.15) follows from the definition of $B\Theta$ - (3.12). To obtain (3.14), we write (3.10) in the following form

$$(v\cdot\nabla\theta,\varphi)_{L_2(\Omega)}+\nu(\nabla\theta,\nabla\varphi)_{L_2(\Omega)}-2\nu(\theta_{,w},\varphi)_{L_2(\Omega)}=(g,\varphi)_{L_2(\Omega)}. \tag{3.16}$$

Inserting $\varphi=B\Theta$ into (3.16), we get

$$\begin{array}{c}(v\cdot\nabla B\Theta,B\Theta)_{L_2(\Omega)}+\nu(\nabla B\Theta,\nabla B\Theta)_{L_2(\Omega)}\\ -2\nu(B\Theta_{,w},B\Theta)_{L_2(\Omega)}=(g,B\Theta)_{L_2(\Omega)}.\end{array} \tag{3.17}$$

This equality gives (3.14), because the first term vanishes, since by definition (3.12), trace $B\Theta$ on the boundary is zero and the third term is nonnegative, since

$$\begin{array}{c}-2\nu(B\Theta_{,w},B\Theta)_{L_2(\Omega)}=-\nu\int_\Omega(B\Theta)^2_{,w}dw\,2\pi dx_3\\ =2\pi\nu\int_{-L}^L(B\Theta)^2(0,0,x_3)dx_3\geq 0.\end{array}$$

The next consideration concerns the velocity problem (3.3). We want to find a precise bound on the velocity on the boundary to close estimates (3.14) and (3.15). By the feature of regularity of the vorticity we consider the velocity as the following sum

$$\tilde v=y+z+\mathcal{D}, \tag{3.18}$$

where y satisfies the following problem

$$\begin{array}{rcl}\text{rot } y=\beta & \text{in} & \Omega,\\ \text{div } y=0 & \text{in} & \Omega,\\ n\cdot y=0 & \text{on} & \partial\Omega\end{array} \tag{3.19}$$

with β equal to

$$\beta=(x_2\,B\Theta,-x_1\,B\Theta,0); \tag{3.20}$$

and z satisfies the following one

$$\begin{array}{rcl}\text{rot } z=\gamma & \text{in} & \Omega,\\ \text{div } z=0 & \text{in} & \Omega,\\ n\cdot z=0 & \text{on} & \partial\Omega\end{array} \tag{3.21}$$

with γ equal to

$$\gamma=(x_2(\theta-B\Theta),-x_1(\theta-B\Theta),0); \tag{3.22}$$

and $\mathcal{D}$ is the extension of data d given by Lemma 2.2.

First we examine system (3.19).

Lemma 3.4. *Let* $0 < a < 1$ *and* β *fulfills the following inclusions*

$$(\sqrt{x_1^2 + x_2^2})^{-1}\beta \in SL_2(\Omega), \qquad \left(\sqrt{x_1^2 + x_2^2}\right)^{-1} \nabla\beta \in SL_2(\Omega), \tag{3.23}$$

then the solution to problem (3.19) satisfies the following bounds

$$||y||_{C^a(\Omega)} \le c||\beta||_{H^1(\Omega)}, \quad ||\tfrac{y}{\sqrt{x_1^2+x_2^2}}||_{C(\Omega)} \le c||\tfrac{\beta}{\sqrt{x_1^2+x_2^2}}, \tfrac{\nabla\beta}{\sqrt{x_1^2+x_2^2}}||_{L_2(\Omega)}. \tag{3.24}$$

A proof of Lemma 3.4 follows from Lemma 2.1.

Lemmas 3.3 and 3.4 imply the next result.

Lemma 3.5. *Let* β *be given by (3.20) then*

$$||\tfrac{y\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{C(\Omega)} \le c||g||_{L_2(\Omega)}. \tag{3.25}$$

Examine (3.21). By the definition of γ - (3.22), the r.h.s. of $(3.21)_1$ belongs to $SL_\infty(\Omega)$.

Lemma 3.6. *Let* $0 < a < 1$ *and* γ *be given by (3.22), then the solution of (3.21) satisfies the following bounds*

$$||z||_{C^a(\Omega)} \le c||\theta - B\Theta||_{L_\infty(\Omega)}, \quad ||\tfrac{z\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{C(\partial\Omega)} \le c||\theta - B\Theta||_{L_\infty(\Omega)}. \tag{3.26}$$

Proof. Since $\gamma \in L_\infty(\Omega)$ and this type of elliptic systems has not optimal regularity in L_∞, we imbed $L_\infty(\Omega)$ in $L_p(\Omega)$ for sufficiently large $p < \infty$. Then by the standard theory we obtain that the solution of problem (3.27) satisfies the following estimate

$$||z||_{W_p^1(\Omega)} \le c||\gamma||_{L_p(\Omega)}. \tag{3.27}$$

This implies that $v \in C^a(\Omega)$, if $a + 3/p < 1$. Hence we get the first bound of (3.26). Moreover, since $\mathrm{dist}(\Gamma_0, Ox_3) \ge \min_{y\in[-L,L]} R(x_3) > 0$ we conclude from (3.27) that

$$||\tfrac{z\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{C(\Gamma_0)} \le c||\gamma||_{L_\infty(\Omega)}. \tag{3.28}$$

However to end the proof of Lemma 3.6, we need the same information on $\Gamma_{in/out}$, which seems to be not obvious for $x_1^2+x_2^2 \to 0$. In Appendix - Lemma 4.1 we prove a result - estimate (4.2) - which gives the desired bound

$$||\tfrac{z\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{C(\Gamma_{in/out})} \le c||\theta - B\Theta||_{L_\infty(\Omega)}. \tag{3.29}$$

A proof of (3.29) is based on a type of subharmonic analysis and gives only the bound on the boundary.

By Lemmas 3.5 and 3.6 and from (3.18) and Lemma 2.2 we obtain the next result.

Lemma 3.7. *The solutions to problem (3.3) satisfies*

$$\begin{aligned} ||\tilde{v}||_{C^a(\Omega)} &\leq c(||\theta||_{L_\infty(\Omega)+H_0^1(\Omega)} + ||d||_{C^a(\partial\Omega)}), \\ ||\tfrac{\tilde{v}\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{C(\partial\Omega)} &\leq A_0(\Omega)||\theta - B\Theta||_{L_\infty(\Omega)} + c||\tfrac{d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}}||_{C(\partial\Omega)}. \end{aligned} \tag{3.30}$$

Combining Lemmas 3.3 and 3.7, we obtain.

Lemma 3.8. *The following estimates hold*

$$\begin{aligned} ||\tfrac{\tilde{v}\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\Omega)} &\leq A_0(\Omega)||2\chi_1 - f/\nu||_{L_\infty(\partial\Omega)}||\tfrac{v\cdot\tau}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)} + B_1X, \\ ||\tilde{v}||_{C^a(\Omega)} &\leq A_1(a)A_0(\Omega)||2\chi_1 - f/\nu||_{L_\infty(\partial\Omega)}||\tfrac{v\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)} + B_2X, \end{aligned} \tag{3.31}$$

where

$$X = ||g||_{L_2(\Omega)} + ||\tfrac{d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\Gamma_{in/out})} + ||\tfrac{\mathcal{D}\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)}. \tag{3.32}$$

The last result defines us the geometrical constraint. Note that $A_0(\Omega)$ in $(3.31)_1$ is given by Definition 2.1. To obtain suitable properties of map Φ we need to find a subset of $C(\Omega)$ which is preserved under acting of map Φ.

By $(3.31)_1$ we see that

$$\text{if} \quad ||\tfrac{v\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)} \leq M_0, \quad \text{then} \quad ||\tfrac{\tilde{v}\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)} \leq M_0, \tag{3.33}$$

where

$$M_0 = (1 - A_0(\Omega)||2\chi_1 - f/\nu||_{L_\infty(\partial\Omega)})^{-1}B_1X. \tag{3.34}$$

Relation (3.33) is valid, if the domain fulfills the following condition

$$A_0(\Omega)||2\chi_1 - f/\nu||_{L_\infty(\partial\Omega)} < 1. \tag{3.35}$$

Relation (3.35) is our condition (1.10) from Theorem.

By Lemma 3.8, we conclude also

$$||\tilde{v}||_{C^a(\Omega)} \leq A_2(a)M_0. \tag{3.36}$$

Hence, we define a convex subset of set Ξ, which we call Ξ_0, in the following way

$$\Xi_0 = \left\{ v \in \Xi : ||v||_{C^a(\Omega)} \leq A_2(a)M_0 \text{ and } \left\| v \cdot \tau_1/\sqrt{x_1^2+x_2^2} \right\|_{C(\partial\Omega)} \leq M_0 \right\}. \tag{3.37}$$

Then

$$\Phi : \Xi_0 \to \Xi_0; \tag{3.38}$$

moreover we see that map Φ is continuous and compact (in the sense of $C^a(\Omega)$). Thus, by the Schauder fixed point theorem we conclude existence of at least one vector $v_* \in \Xi_0$ such that

$$\Phi(v_*) = v_*. \tag{3.39}$$

To end the proof it is enough to note that since $v_* \in \Xi_0$, by definition (3.37), (3.32) and (2.10) we get bounds $(1.11)_1$. $(1.11)_2$ follows form (3.4) and Lemma 3.3.

Finally, by Definitions 3.2 and 3.3 we formulate a sense of solutions to system (1.1) given as a fixed point of map Φ. Note that by definition (1.4), the symmetry and smoothness of the solutions function θ vanishes on axle Ox_3.

Definition 3.4. *We say that v is a weak solution to problem (1.1), iff*

$$v \in SC^a(\Omega),\, v \cdot \tau_1/\sqrt{x_1^2+x_2^2}\Big|_{\partial\Omega} \in SL_\infty(\partial\Omega) \quad \text{and} \quad \theta \in SL_\infty(\Omega) + SH_0^1(\Omega) \tag{3.40}$$

and the following identity

$$\begin{array}{c} (v\cdot\nabla\theta_{H^1},\psi)_{L_2(\Omega)} - (v\cdot\nabla\psi,\theta_{L_\infty})_{L_2(\Omega)} + 2\nu(\theta,\psi_{,w})_{L_2(\Omega)} + \\ \nu(\nabla\theta_{H^1},\nabla\psi)_{L_2(\Omega)} - \nu(\theta_{L_\infty},\Delta\psi)_{L_2(\Omega)} - \nu\int_{\Gamma_0}\frac{f/\nu-2\chi_1}{R(x_3)}v\cdot\tau_1\frac{\partial\psi}{\partial n}d\sigma \\ -\int_{\Gamma_{in/out}}\left(\frac{f}{\sqrt{x_1^2+x_2^2}}v\cdot\tau_1 - \frac{2d_{,\tau_1}}{\sqrt{x_1^2+x_2^2}}\right)\frac{\partial\psi}{\partial n}d\sigma - (g,\psi) = 0 \end{array} \tag{3.41}$$

holds for any $\psi \in SW_1^2(\Omega) \cap SH_0^1(\Omega) \cap \{\psi|_{x=(0,0,x_3)} = 0\}$, where $\theta = \theta_{L_\infty} + \theta_{H^1}$ such that $\theta_{L_\infty} \in SL_\infty(\Omega)$ and $\theta_{H^1} \in SH_0^1(\Omega)$.

4 Appendix

In this section we prove a result about an estimate for the solutions of problem (1.23) in a weighted spaces. The proof is based on the standard isotropic theory and elements of theory of subharmonic functions. Underline that the below result holds only for the cylindrical symmetric case.

Lemma 4.1. *Let $\sigma \in SL_\infty(\Omega)$, then the solution of the following problem*

$$\begin{array}{lll} rot\ v = \gamma & in & \Omega, \\ div\ v = 0 & in & \Omega, \\ n \cdot v = 0 & on & \partial\Omega, \end{array} \tag{4.1}$$

where $\gamma = (x_2\sigma, -x_1\sigma, 0)$, fulfills the following bound

$$||\tfrac{v\cdot\tau_1}{\sqrt{x_1^2+x_2^2}}||_{L_\infty(\partial\Omega)} \le c||\sigma||_{L_\infty(\Omega)}, \tag{4.2}$$

where τ_1 is defined as in (1.18), (1.20) and (1.21).

Proof. Since this type of operators does not allow Schauder estimates in L_∞-spaces and thesis of Lemma 4.1 involves such information, we apply here a theory of sub(super)harmonic-type functions to get the bound only on the very boundary.

Consider the following problem

$$\begin{array}{lll} \text{rot } v^* = \gamma^* & \text{in} & \Omega, \\ \text{div } v^* = 0 & \text{in} & \Omega, \\ n \cdot v^* = 0 & \text{on} & \partial\Omega, \end{array} \tag{4.3}$$

where $\gamma^* = (x_2\sigma^*, -x_1\sigma^*, 0)$ and σ^* is a constant defined in the following way

$$\sigma^* = \text{ess } \sup_{x\in\Omega} \sigma(x). \tag{4.4}$$

This follows that v^* is a smooth function, since the r.h.s. of $(4.3)_1$ smooth. Thus, $v^* \in SC^1(\Omega)$, but the symmetry implies that $v^* = (x_1v_r^*, x_2v_r^*, v_3^*)$. Hence the regularity of v^* gives the following bound

$$||v_r^*||_{C(\Omega)} \le c|\sigma^*|. \tag{4.5}$$

On the other hand, by the symmetry, $(4.1)_1$ may be written as a scalar equation

$$v_{3,w} - v_{r,3} = \sigma.$$

Hence, if we examine v as the following sum

$$v = u + v^*, \tag{4.6}$$

we get that vector field u satisfies

$$\begin{array}{rll} u_{3,w} - u_{r,w} = \sigma - \sigma^* \leq 0 & \text{in} & \Omega, \\ \text{div } u = 0 & \text{in} & \Omega, \\ n \cdot u = 0 & \text{on} & \partial\Omega. \end{array} \tag{4.7}$$

Equation $(4.7)_2$ implies

$$2(wu_r)_{,w} + u_{3,3} = 0. \tag{4.8}$$

By the Poincare Lemma there exists a scalar function φ such that

$$\varphi_{,3} = -2wu_r, \quad \varphi_{,w} = u_3. \tag{4.9}$$

Inserting form (4.9) into boundary condition $(4.7)_3$ we get

$$\partial_{\tau_1}\varphi = 0 \quad \text{on } \partial\Omega. \tag{4.10}$$

Since function φ is defined up a constant we may choose $\varphi = 0$ on the boundary (the first group of homotopy of Ω is trivial). Hence inserting form (4.9) into (4.8) we get the following problem

$$\begin{array}{rll} \varphi_{,ww} + \frac{\varphi_{,33}}{2w} \leq 0 & \text{in} & \Omega, \\ \varphi = 0 & \text{on} & \partial\Omega. \end{array} \tag{4.11}$$

System (4.11) describes us φ as a subharmonic-type function. This allows to obtain the sign of φ. Multiplying $(4.11)_1$ by $\min\{\varphi, 0\}$ and integrating over Ω we conclude that

$$\varphi \geq 0 \quad \text{in } \Omega. \tag{4.12}$$

Hence in particular from (4.12) we are given information about the normal derivative on the boundary. Because function φ has minimum on $\partial\Omega$ then

$$\frac{\partial\varphi}{\partial n} \leq 0 \quad \text{on } \partial\Omega. \tag{4.13}$$

Thus, on $\Gamma^-_{in/out}$ the normal vector is equal to $n = (0, 0, -1)$, hence by (4.9)

$$u_r = \frac{1}{2w}\frac{\partial\varphi}{\partial n} \leq 0 \quad \text{on } \Gamma^-_{in/out}. \tag{4.14}$$

Recalling (4.6) and (4.5) we obtain

$$v_r = u_r + v_r^* \leq c|\sigma^*| \quad \text{on } \Gamma^-_{in/out}. \tag{4.15}$$

The same consideration for $\Gamma^+_{in/out}$ with $n = (0, 0, 1)$ gives

$$u_r = -\frac{1}{2w}\frac{\partial\varphi}{\partial n} \geq 0 \quad \text{on} \;\; \Gamma^+_{in/out}. \tag{4.16}$$

which follows that

$$v_r = u_r + v^*_r \leq -c|\sigma^*| \qquad \text{on} \;\; \Gamma^+_{in/out}. \tag{4.17}$$

Now, we need the bounds from other sides. Repeating the consideration for (4.3) we investigate

$$\begin{array}{lll} \text{rot } v_* = \gamma_* & \text{in} & \Omega, \\ \text{div } v_* = 0 & \text{in} & \Omega, \\ n \cdot v_* = 0 & \text{on} & \partial\Omega. \end{array} \tag{4.18}$$

with $\gamma_* = (x_2\sigma_*, -x_1\sigma_*, 0)$, where

$$\sigma_* = \text{ess} \inf_{x \in \Omega} \sigma(x). \tag{4.19}$$

The same as for v^* we show the following bound

$$||v_{*r}||_{C(\Omega)} \leq c|\sigma_*|. \tag{4.20}$$

The consideration for $v - v_*$ leads to the following estimates

$$v_r \geq -c|\sigma_*| \quad \text{on} \;\; \Gamma^-_{in/out} \quad \text{and} \quad v_r \leq c|\sigma_*| \quad \text{on} \;\; \Gamma^+_{in/out}. \tag{4.21}$$

From (4.15), (4.17), (4.21) and (4.4), (4.19) we obtain

$$||v_r||_{L_\infty(\Gamma_{in/out})} \leq c||\sigma||_{L_\infty(\Omega)}. \tag{4.22}$$

Since (4.22) proves (4.2). Lemma 4.1 is shown.

Acknowledgment. The work has been supported by Polish KBN grant No 2 PO3A 002 23.

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