On the inviscid limit of the Navier-Stokes equations for flows with large flux

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Abstract. We investigate the inviscid limit of solutions of the evolutionary and stationary Navier-Stokes equations in two dimensional bounded domains. The system is considered with the slip boundary conditions admitting flows across the boundary. Under a geometrical restriction on the shape of the domain, we prove the L_∞-bound on the vorticity of the velocity for any large boundary data. This estimate enables us to prove existence of solutions to the Navier-Stokes equations. Moreover the bound is independent of the viscosity and guarantees a strong convergence to the Eulerian limit. The result for the nonsteady case is global in time. In particular, the turbulence boundary layer does not appear in our approach in both cases.

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1 Introduction

The study of the zero viscosity limit for the evolutionary and stationary Navier-Stokes equations in two dimensional cases is the subject of this paper.

Results for problems in domains without boundaries as $\mathbf{R}^2$ or $\mathbf{T}^2$ [2,6,14] apply the maximum principle which holds for the vorticity equation. This way we obtain the L_∞-bound on the vorticity of the velocity independently of the viscous coefficient. This information leads us to a strong convergence of the solutions to an Eulerian flow.

A difficulty appears when we want to consider our flow in a bounded domain. A solution, to adapt the method for domains without boundaries, is to put boundary conditions $n\cdot v|_{\partial\Omega} = 0$ and $\operatorname{rot} v|_{\partial\Omega} = 0$ as in [5,6]. Then the structure of the vorticity problem is preserved and the maximum principle works, since the data control the vorticity on the boundary. This approach however does not fit well the Navier-Stokes equations in rigid domains and this type of boundary conditions are called "free boundary".

Searching for another approach, in this paper we examine the slip boundary condition which from the physical point of view neglects friction between the fluid and the surface of the boundary. To obtain sufficient information independent of the viscosity there is a need to find a reformulation of the problem. Thus, instead of the original Navier-Stokes equations we will consider the coupled system of the vorticity equation and the rot-div problem on the velocity. It can be done, since the slip boundary conditions enable us to compute the value of the vorticity at the boundary as a function of the curvature of the boundary and the tangent velocity. This boundary relation has been probably known by Navier [8], since he introduced the slip condition.

This new approach to the problem delivers not only the desired L_∞-bound on the vorticity, but also allows to consider nonhomogeneous boundary data, i.e. our model admits flows across the boundary. Assuming a geometrical constraint on the shape of the domain we are able to prove a novel estimate for solutions for any large boundary data independently of time and the viscosity (Theorem 1.1). The geometrical restriction depends only on features of the domain and is independent of the velocity data. This new result for the Navier-Stokes equations guarantees a strong limit to the Euler system, the same as for domains without boundaries (Theorem 1.2).

The important advantage of the presented technique is the independence of time. This feature allows us to obtain the same result for the stationary version of the problem (Theorems 1.3 and 1.4), of course, under the same geometrical constraint as for the evolutionary case, but also without any restriction on the nonhomogeneous boundary data.

The inviscid limit for the evolutionary Navier-Stokes equations with the slip boundary conditions has been studied earlier in [1]. The method from [1] was also based on the properties of the slip conditions and on the vorticity problem, but in an essential way it applied the energy estimate for the original equations. That is the reason the result is obtained only for homogeneous boundary data and finite time intervals. However there is no restriction on the shape of the domain.

Comparing to problems with Dirichlet boundary data, which are the most popular in the considerations for the Navier-Stokes equations the slip boundary conditions seem to be a good approximation of the Euler system. For the Dirichlet data, examination of the inviscid limit causes a creation of the turbulence boundary layer [2, 12, 13]. This phenomenon can be modeled by the Prandtl equations, but for this system there are only results for short time intervals [10, 11] and only for analytic data.

The presented result arose from the study on the problem of the optimal shape of obstacles for viscous flows with large flux [7].

First, we introduce the evolutionary Navier-Stokes equation in a two dimensional bounded domain Ω with the slip boundary conditions

$$\begin{array}{lll} v_t + v\cdot\nabla v - \nu\Delta v + \nabla p = 0 & \text{in} & \Omega\times(0,T), \\ \operatorname{div} v = 0 & \text{in} & \Omega\times(0,T), \\ n\cdot\mathbf{T}(v,p)\cdot\tau = 0 & \text{on} & \partial\Omega\times(0,T), \\ n\cdot v = d & \text{on} & \partial\Omega\times(0,T), \\ v|_{t=0} = v_0 & \text{on} & \Omega, \end{array} \tag{1.1}$$

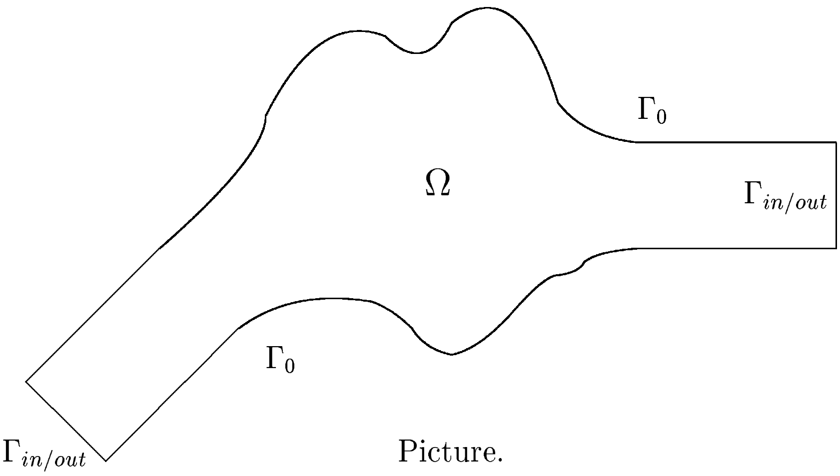
where $v = (v^1, v^2)$ is the velocity of the fluid, p is the pressure, ν is the constant positive viscous coefficient, n and τ are the normal and tangent vectors to boundary $\partial\Omega$, $\mathbf{T}$ is the stress tensor for the Newtonian fluid

$$\mathbf{T}(v,p) = \nu\mathbf{D}(v) - pId = \left\{\nu(v^i_{,j} + v^j_{,i}) - p\delta_{ij}\right\}_{i,j=1,2}, \tag{1.2}$$

where $\mathbf{D}$ is the deformation tensor; v_0 is a divergence free initial datum and d describes the flow across the boundary

$$d = \begin{cases} 0 & \text{on} \quad \Gamma_0\times(0,T) \\ v_{in/out} & \text{on} \quad \Gamma_{in/out}\times(0,T) \end{cases} \tag{1.3}$$

From the physical interpretation reasons we divide the boundary into two elements: one where there is no flow across $\partial\Omega$ and the second one where such a flow is possible, i.e. we introduce Γ_0 and $\Gamma_{in/out}$ such that $\overline{\Gamma}_0\cup\overline{\Gamma}_{in/out} = \partial\Omega$ and $\Gamma_0 = \{x\in\partial\Omega : d(x,t) = 0 \text{ for } t\in(0,T)\}$, $\Gamma_{in/out} = \partial\Omega\setminus\Gamma_0$. Domain Ω is simply connected and the boundary is piecewise smooth such that interior angles between smooth parts are $\pi/2$ to avoid difficulties with regularity of solution near corners - see the picture.



Picture.

To keep well posedness of the boundary data we require to exist a sufficiently smooth divergence free vector field $\mathcal{D}$ such that

$$\begin{array}{lll} \operatorname{div} \mathcal{D} = 0 & \text{in} & \Omega \times (0,T), \\ \mathcal{D} \cdot n = d & \text{on} & \partial\Omega \times (0,T). \end{array} \tag{1.4}$$

To have an agreement with $(1.1)_2$, we put the following compatibility conditions on boundary data d

$$\int_{\partial\Omega} d(x,t)d\sigma = 0 \quad \text{for} \quad t \in (0,T) \tag{1.5}$$

which controls the total flux through the domain and on the initial datum v_0

$$\operatorname{div} v_0 = 0 \quad \text{and} \quad v_0|_{\partial\Omega} = d. \tag{1.6}$$

The first result of our paper concerns the existence of global in time solutions with estimates independent of viscous coefficient ν.

Theorem 1.1. *Let $\nu > 0$, $0 \leq a < 1$, $\mathcal{D} \in C^{1,a/2}(\Omega \times (0,\infty))$ and $\operatorname{rot} v_0 \in L_\infty(\Omega)$. If the boundary of the domain satisfies the following condition*

$$2A(\Omega)||\chi||_{L_\infty(\partial\Omega)} < 1, \tag{1.7}$$

where χ is the curvature of $\partial\Omega$ and $A(\Omega)$ is a constant depending only on properties of Ω - see Definition 2.2. Then there exists a unique global in time *solution of problem (1.1) such that* $v \in C^{a,a/2}(\Omega \times (0,\infty))$ *and the following bound holds*

$$||v||_{C^{a,a/2}(\Omega\times(0,\infty))} + \sup_{t\in(0,\infty)} ||rot\, v(\cdot,t)||_{L_\infty(\Omega)} \leq S, \qquad (1.8)$$

where S *is independent of* ν.

Condition (1.7) restricts essentially the class of considered domains. The curvature of the boundary need to be suitably small comparing it to the constant from the Poincare inequality which determines $A(\Omega)$ - see Lemma 2.2. It causes that shapes of domains are close to the example presented on the picture. But on the other hand there is no bound on largeness of initial data v_0 and the inflow and outflow given by data d. Hence condition (1.7) delivers a geometrical constraint on the shape of domains which admits viscous flows with any large flux, but of course, only for the slip boundary conditions $(1.1)_{3,4}$. This novel result is an exception to results for problems with the Dirichlet boundary data, where we have restrictions on the magnitude of fluxes [3,9]. Condition (1.7) and a new approach enable us to find a priori estimate (1.8). A proof is based on the maximum principle for the system for the vorticity and a property of the the slip boundary condition. This way bound (1.8) has a linear form which neglects the nonlinear term. The same phenomenon holds for the standard energy estimate. Nevertheless, in our case we obtain strong information independently of viscous coefficient ν and for nonhomogeneous boundary data, but under the restriction of geometrical constraint (1.7).

Note that estimate (1.8) is not a Schauder-type, since we required to have $\mathcal{D} \in C^{1,a/2}$. This is a consequence of features of the reformulation as well as the ill posedness of the elliptic problems in the L_∞-space. However, the obtained bound is sufficient to increase regularity to get smooth solutions, if data will be regular enough.

Theorem 1.1 follows from Theorem 3.1 from section 3, in this part we define precisely the sense of the solutions. Having such a type of estimates we obtain the next result of our paper.

Theorem 1.2. *Let assumptions of Theorem 1.1 be fulfilled, then for* $t \in (0,\infty)$ *and a subsequence* $\nu_k \to 0$

$$v^{\nu_k}(\cdot,t) \to v_E(\cdot,t) \quad \textit{strongly in } C^a(\Omega), \qquad (1.9)$$

where v^{ν_k} *is the solution of problem (1.1) given by Theorem 1.1 with viscous* *coefficient* ν_k *and function* v_E *satisfies the Euler system*

$$\begin{array}{lll} v_{E,t} + v_E \cdot \nabla v_E + \nabla p_E = 0 & in & \Omega \times (0,T), \\ div\ v_E = 0 & in & \Omega \times (0,T), \\ n \cdot v_E = d & on & \partial\Omega \times (0,T), \\ v_E|_{t=0} = v_0 & on & \Omega, \end{array} \tag{1.10}$$

where d *and* v_0 *are the same as for the viscous flow. Moreover, the following estimate holds*

$$||v_E(\cdot,t)||_{C^{a,a/2}(\Omega\times(0,\infty))} + \sup_{t\in(0,\infty)} ||rot\ v_E(\cdot,t)||_{L_\infty(\Omega)} \le S. \tag{1.11}$$

We would like to underline that estimate (1.11) is the same strong as for the the case when domain Ω is the whole $\mathbf{R}^2$. Although in our system, since the shape of the domain is nontrivial, the flow of the fluid is nontrivial, too, even for large time.

Note that in Theorem 1.2 we choose a subsequence $\nu_k \to 0$ which define as limit v_E. Since the boundary data for (1.10) are nonhomogeneous we lose the uniqueness of solutions to the Euler system. This behavior follows from the hyperbolic character of the Euler system. To hold the uniqueness there is a need to control the vorticity at a part of the boundary, but we lose this information, since the convergence to the limit from the Navier-Stokes equations is done for a weak formulation of the system - see Theorem 3.2.

A key element of the proof of Theorem 1.2 is the type of the boundary conditions. Relation $(1.3)_3$ says that there is no friction between the boundary and the fluid. This feature should hold for the perfect fluid to keep the balance of the energy. That is the reason the neglect of the friction seems to be physical reasonable to treat system (1.1) as an approximation of the Euler model. However, the technique works if condition (1.7) is fulfilled, there appears a question: what is the physical interpretation of this restriction? Another question concerns the uniqueness of the limit and it does not seem obvious.

An advantage of this approach to the Navier-Stokes is that we obtain a similar result for the steady flow which in our case is nontrivial since data of the inflow and outflow may be any.

The steady version of problem (1.1) reads

$$\begin{array}{lll} v\cdot\nabla v-\nu\Delta v+\nabla p=0 & \text{in} & \Omega,\\ \text{div }v=0 & \text{in} & \Omega,\\ n\cdot\mathbf{T}(v,p)\cdot\tau=0 & \text{on} & \partial\Omega,\\ n\cdot v=d & \text{on} & \partial\Omega. \end{array} \tag{1.12}$$

The same as in the evolutionary case, there is a need to introduce an extension of boundary data to keep regularity of the solutions. Thus, we assume that there exists a vector field $\mathcal{D}$ fulfilling the stationary version of condition (1.4) and (1.5).

For the issue of existence to system (1.12) we have the following result.

Theorem 1.3. *Let $\nu>0$, $0\le a<1$ and $\mathcal{D}\in C^1(\Omega)$. If*

$$2A(\Omega)||\chi||_{L_\infty(\partial\Omega)}<1,$$

then there exists at least one solution to problem (1.12) such that $v\in C^a(\Omega)$ and rot $v\in L_\infty(\Omega)$ with the following estimate

$$||v||_{C^a(\Omega)}+||rot\ v||_{L_\infty(\Omega)}\le S, \tag{1.13}$$

where S is independent of ν.

A proof of Theorem 1.3 can be found in [7, Theorem 3.1 with $F\equiv 0$ and $f\equiv 0$]. In general, it is based on the same methods as for the evolutionary case presented in section 3.

The consequence of Theorem 1.3 is the next result.

Theorem 1.4. *Let assumptions of Theorem 1.3 be fulfilled, then for a subsequence $\nu_k\to 0$ with $k\to\infty$*

$$\begin{array}{rll} v^{\nu_k}\to v_E & \textit{strongly in} & C^a(\Omega),\\ rot\ v^{\nu_k}\rightharpoonup rot\ v_E & \textit{weakly-}*\ \textit{in} & L_\infty(\Omega), \end{array} \tag{1.14}$$

where v^ν is a solution of problem (1.12) given by Theorem 1.3 with viscous coefficient ν and v_E satisfies the steady Euler system

$$\begin{array}{lll} v_E\cdot\nabla v_E+\nabla p_E=0 & in & \Omega,\\ div\ v_E=0 & in & \Omega,\\ n\cdot v_E=d & on & \partial\Omega, \end{array} \tag{1.15}$$

with the following estimate

$$||v_E||_{C^a(\Omega)}+||rot\ v_E||_{L_\infty(\Omega)}\le S. \tag{1.16}$$

It is unknown, if v_E given by Theorem 1.4 is a potential flow or not (i.e. if $v_E = \nabla\psi : \Delta\psi = 0$ and $\nabla\psi \cdot n|_{\partial\Omega} = d$). The convergence to the limit is done in a weak formulation for system (1.12), hence we do not control condition $(1.12)_3$ - the gradient of the solution at the boundary. Nevertheless, Theorem 1.4 seems to be an interesting new result, since in the standard approach for the evolutionary case we apply the energy estimate and for steady flows this method is not valid. The technique to obtain (1.12) is based on the maximum principle the same as to get (1.8).

Considerations for the steady case are presented in section 4.

To sketch proofs of the results we concentrate on the main ideas of a priori estimate on the vorticity of the velocity in the evolutionary case. The bound for the steady case is almost the same. These considerations are a motivation of our paper.

To underline dependence of ν we rewrite (1.1) as follows

$$\begin{array}{lll} v^\nu_t + v^\nu \cdot \nabla v^\nu - \nu\Delta v^\nu + \nabla p^\nu = 0 & \text{in} & \Omega \times (0,\infty), \\ \text{div } v^\nu = 0 & \text{in} & \Omega \times (0,\infty), \\ n \cdot \mathbf{D}(v^\nu) \cdot \tau = 0 & \text{on} & \partial\Omega \times (0,\infty), \\ n \cdot v^\nu = d & \text{on} & \partial\Omega \times (0,\infty), \\ v^\nu|_{t=0} = v_0 & \text{on} & \Omega. \end{array} \tag{1.17}$$

Condition $(1.17)_3$ follows from $(1.1)_3$ and definition (1.2).

Since the flux through the boundary is nonzero we need to find an approach to avoid a subtraction and work on the original solutions to preserve the physical structure of the system. To get this we apply a reformulation of problem (1.17). Let

$$\alpha^\nu = \text{rot } v^\nu = v^{\nu 2}_{,1} - v^{\nu 1}_{,2} \tag{1.18}$$

be the vorticity of the velocity. Since the considered flow is two dimensional, the vorticity is a scalar function and the equation on α^ν reads

$$\alpha^\nu_{,t} + v^\nu \cdot \nabla\alpha^\nu - \nu\Delta\alpha^\nu = 0 \qquad \text{in} \qquad \Omega \times (0,\infty). \tag{1.19}$$

Next, applying the property of the slip boundary condition $(1.1)_{3,4}$ we compute the value of the vorticity at the boundary

$$\alpha^\nu|_{\partial\Omega\times(0,\infty)} = b(v^\nu) = \left\{ \begin{array}{lll} 2\chi v^\nu \cdot \tau & \text{on} & \Gamma_p \times (0,\infty) \\ 2\chi v^\nu \cdot \tau + \alpha_{in/out} & \text{on} & \Gamma_{in/out} \times (0,\infty) \end{array} \right. \tag{1.20}$$

where $\alpha_{in/out} = 2\frac{\partial}{\partial s} d$ and s is the unit length parameter for curve $\partial\Omega$; note that the assumption of smoothness of field $\mathcal{D}$ guarantees the continuity of data $b(v)$ on the whole $\partial\Omega$ including corners (to get (1.20) it is enough to differentiate $(1.1)_4$ with respect to parameter s and use $(1.1)_3$).

To complete (1.19) and (1.20) we add the initial datum

$$\alpha^\nu|_{t=0} = \text{rot } v_0 \qquad \text{on} \qquad \Omega. \tag{1.21}$$

For problem (1.19), (1.20) and (1.21) the maximum principle gives the following bound

$$\begin{aligned} \sup_{t\in(0,\infty)} ||\alpha^\nu(\cdot,t)||_{L_\infty(\Omega)} \leq ||2\chi||_{L_\infty(\partial\Omega)} \sup_{t\in(0,\infty)} ||v(\cdot,t)||_{C(\partial\Omega)} + \\ \sup_{t\in(0,\infty)} ||\alpha_{in/out}(\cdot,t)||_{C(\partial\Omega)} + ||\text{rot } v_0||_{L_\infty(\Omega)}. \end{aligned} \tag{1.22}$$

The first term of the r.h.s. of (1.22) is unknown, but the velocity satisfies

$$\begin{array}{lll} \text{rot } v^\nu = \alpha^\nu & \text{in} & \Omega\times(0,\infty), \\ \text{div } v^\nu = 0 & \text{in} & \Omega\times(0,\infty), \\ n\cdot v^\nu = d & \text{on} & \partial\Omega\times(0,\infty). \end{array} \tag{1.23}$$

For problem (1.23) one can show the following estimate

$$\begin{aligned} \sup_{t\in(0,\infty)} ||v^\nu(\cdot,t)||_{C(\Omega)} \leq \\ A(\Omega)\sup_{t\in(0,\infty)} ||\alpha^\nu(\cdot,t)||_{L_\infty(\Omega)} + c\sup_{t\in(0,\infty)} ||v_{in/out}(\cdot,t)||_{C^1(\partial\Omega)}, \end{aligned} \tag{1.24}$$

where $A(\Omega)$ is the same constant as in Theorem 1.1 and the definition is given in section 2 - Definition 2.2.

Inserting (1.24) to (1.22), we conclude the desired uniformly with respect to viscous coefficient ν bound

$$\begin{aligned} \sup_{t\in(0,\infty)} ||\alpha^\nu(\cdot,t)||_{L_\infty(\Omega)} + \sup_{t\in(0,\infty)} ||v^\nu(\cdot,t)||_{C(\Omega)} \leq \\ c(1-A(\Omega)||2\chi||_{L_\infty(\partial\Omega)})^{-1}\left(\sup_{t\in(0,\infty)} ||d(\cdot,t)||_{C^1(\partial\Omega)} + ||\text{rot } v_0||_{L_\infty(\Omega)}\right) \end{aligned} \tag{1.25}$$

which is valid only if

$$A(\Omega)||2\chi||_{L_\infty(\partial\Omega)} < 1.$$

Hence, we showed a sketch of the proof of bound (1.8). To get (1.13) the procedure is almost the same.

Now, we converge to the limit. For a subsequence, if $\nu\to 0$ then

$$\begin{array}{lll} \alpha^\nu \rightharpoonup \alpha^* & \text{weakly-}* \text{ in} & L_\infty(\Omega\times(0,\infty)), \\ v^\nu \rightharpoonup v^* & \text{weakly-}* \text{ in} & L_\infty(0,\infty;C(\Omega)). \end{array} \tag{1.26}$$

Thus, formally limit v^* satisfies the following problem

$$\begin{array}{lll} v^*_t + v^* \cdot \nabla v^* + \nabla p^* = 0 & \text{in} & \Omega \times (0,\infty), \\ \text{div } v^* = 0 & \text{in} & \Omega \times (0,\infty), \\ n \cdot v^* = d & \text{on} & \partial\Omega \times (0,\infty), \\ v^*|_{t=0} = v_0 & \text{on} & \Omega \end{array} \tag{1.27}$$

which follows from (1.17) by the formal insert $\nu = 0$.

This estimate shows that the investigation of the coupled systems for the vorticity: (1.19), (1.20) and (1.21); and for the velocity (1.23) is more effective, than the study of the original problem. That is the reason we examine this reformulation instead of (1.1).

2 Notation

In the paper we try to use standard notations. Nevertheless, we recall some definition.

$B(0,T)$ denotes the space of bounded functions on interval $(0,T)$

$$B(0,T) = \{f \in \mathcal{M}(0,T) : \sup_{t\in(0,T)} |f(t)| < \infty\} \tag{2.1}$$

with the norm

$$||f||_{B(0,T)} = \sup_{t\in(0,T)} |f(t)|. \tag{2.2}$$

$V^{1,0}(\Omega \times (0,T))$ denotes a space from the theory of parabolic system such that

$$V^{1,0}(\Omega \times (0,T)) = \{f \in \mathcal{M}(\Omega \times (0,T)) : \\ f \in B(0,T; L_2(\Omega)) \ \text{ and } \ \nabla f \in L_2(\Omega \times (0,T))\} \tag{2.3}$$

with the norm

$$||f||_{V^{1,0}(\Omega\times(0,T))} = ||f||_{B(0,T;L_2(\Omega))} + ||\nabla f||_{L_2(\Omega\times(0,T))}. \tag{2.4}$$

$C^{a,b}(\Omega \times (0,T))$ denotes the following Hölder-type space

$$C^{a,b}(\Omega \times (0,T)) = \{f \in C(\Omega \times (0,T)) : f(\cdot,t) \in C^a(\Omega) \text{ for } t \in (0,T) \\ \text{and } \ f(x,\cdot) \in C^b(0,T) \text{ for } x \in \Omega\} \tag{2.5}$$

with the norm

$$||f||_{C^{a,b}(\Omega\times(0,T))} = ||f||_{C(\Omega\times(0,T))} + \sup_{t\in(0,T)} \sup_{x,y\in\Omega, x\neq y} \frac{|f(x,t)-f(y,t)|}{|x-y|^a}$$

$$+\sup_{x\in\Omega} \sup_{t,t'\in(0,T), t\neq t'} \frac{|f(x,t)-f(x,t')|}{|t-t'|^b}. \tag{2.6}$$

Moreover

$$||f||_{W_p^1(\Omega)} = ||f,\nabla f||_{L_p(\Omega)}, \quad ||f||_{W_p^{2,1}(\Omega\times(0,T))} = ||f,\nabla^2 f, f_t||_{L_p(\Omega\times(0,T)} \tag{2.7}$$

and if $f|_{\partial\Omega\times(0,T))} = 0$, then

$$||f||_{H_0^{1,1}(\Omega\times(0,T))} = ||\nabla f, f_t||_{L_2(\Omega\times(0,T))} \tag{2.8}$$

Definition 2.1. *Let $p > 2$. Let us define an affine space Ξ of velocity vectors:*

(i) for the nonsteady system

$$\Xi = \{v \in B(0,\infty; C(\Omega)\cap W_p^1(\Omega)) : div\ v = 0, \quad n\cdot v|_{\partial\Omega\times(0,\infty)} = d\}; \tag{2.9}$$

(ii) for the steady system

$$\Xi = \{v \in C(\Omega)\cap W_p^1(\Omega) : div\ v = 0, \quad n\cdot v|_{\partial\Omega} = d\}, \tag{2.10}$$

where n is the normal vector to $\partial\Omega$. Space Ξ is convex and complete.

Next, we define the constant appearing in geometric constraint (1.7).

Definition 2.2. *Constant $A(\Omega)$ is defined as the optimal constant satisfying the following inequality*

$$||\nabla\varphi||_{C(\Omega)} \le A(\Omega)||f||_{L_\infty(\Omega)}, \tag{2.11}$$

where function φ is a solution to the following problem

$$\begin{array}{lll} \Delta\varphi = f & in & \Omega, \\ \varphi = 0 & on & \partial\Omega \end{array} \tag{2.12}$$

for any $f \in L_\infty(\Omega)$.

By letter c we denote generic constants and by S - constants which depend on norms of data d and initial datum v_0, and they are independent of viscous coefficient ν and length of time interval $(0,T)$.

3 The evolutionary system

The aim of this part is to define a sense of solutions for the nonsteady problem and to prove the existence of them. Next, having suitable estimates we pass to the Euler limit. To obtain our results we consider a reformulation of problem (1.1). As in section 1, instead of (1.17), we investigates system: (1.19), (1.20), (1.21) for the vorticity and (1.23) for the velocity. This approach loses no information about original problem (1.1). We recall that the vorticity is given by (1.18).

Definition 3.1. *We say that $v \in \Xi$ is a global in time weak-$*$ solution of problem (1.1), iff $v \in B(0,\infty; C(\Omega))$, rot $v = \alpha \in B(0,\infty; L_\infty(\Omega))$ and the following identity*

$$\begin{array}{c} -\int_0^T \int_\Omega \alpha \varphi_t dx dt + \int_\Omega \alpha(x,T)\varphi(x,T)dx - \int_\Omega \alpha_0(x)\varphi(x,0)dx \\ +\nu \int_0^T \int_\Omega \alpha \Delta\varphi dx dt - 2\nu \int_0^T \int_{\partial\Omega} \chi b(v) \frac{\partial\varphi}{\partial n} - \int_0^T \int_\Omega v\alpha \cdot \nabla\varphi dx dt = 0 \end{array} \tag{3.1}$$

holds for every $\varphi \in W_1^{2,1}(\Omega \times (0,\infty)) \cap \{\varphi|_{\partial\Omega\times(0,\infty)} = 0\}$ and any $T > 0$, where space Ξ is defined by (2.9) and $b(\cdot)$ by (1.20).

Remark. In Definition 3.1 we use test functions from $W_1^{2,1}$, but to get more precise formulation we should use space $V = \{\phi : \phi_t, \Delta\phi \in L_1(\Omega \times (0,\infty)), \phi(\cdot,t) \in L_1(\Omega), \frac{\partial\phi}{\partial n} \in L_1(\partial\Omega \times (0,\infty))\}$, because the theory for L_1 is ill posed and $V \neq W_1^{2,1}$. But this simplification does not change the final result.

Theorem 3.1 *Let $\nu > 0$, $0 \le a < 1$ and $2A(\Omega)||\chi||_{L_\infty(\partial\Omega)} < 1$, then there exists a unique global in time weak-$*$ solution to problem (1.1) such that*

$$||v||_{B(0,\infty;C^a(\Omega))} + ||\alpha||_{B(0,\infty;L_\infty(\Omega))} \le S, \tag{3.2}$$

where bound (3.2) is independent of ν.

Proof. Construct a map $\Phi : \Xi \to \Xi$ such that

$$\Phi(v) = \tilde{v}, \tag{3.3}$$

where $\tilde{v}$ is a solution of the following elliptic problem

$$\begin{array}{lll} \text{rot } \tilde{v} = \alpha & \text{in} & \Omega \times (0,\infty), \\ \text{div } \tilde{v} = 0 & \text{in} & \Omega \times (0,\infty), \\ n \cdot \tilde{v} = d & \text{on} & \partial\Omega \times (0,\infty) \end{array} \tag{3.4}$$

and α is a solution of the following parabolic system

$$\begin{array}{lll} \alpha_t + v\cdot\nabla\alpha - \nu\Delta\alpha = 0 & \text{in} & \Omega\times(0,\infty), \\ \alpha = b(v) & \text{on} & \partial\Omega\times(0,\infty), \\ \alpha|_{t=0} = \alpha_0 & \text{on} & \Omega, \end{array} \tag{3.5}$$

where $b(\cdot)$ is given by (1.20).

First, we consider problem (3.5) on the vorticity. To make the considerations easier we introduce a function $\bar{b}$ as a solution to the following parabolic problem

$$\begin{array}{lll} \bar{b}_{,t} - \nu\Delta\bar{b} = 0 & \text{in} & \Omega\times(0,\infty), \\ \bar{b} = b(v) & \text{on} & \partial\Omega\times(0,\infty), \\ \bar{b}|_{t=0} = \alpha_0 & \text{on} & \Omega. \end{array} \tag{3.6}$$

By the elementary theory function $\bar{b}$ is smooth inside $\Omega\times(0,\infty)$ and by the maximum principle we have

$$||\bar{b}||_{B(0,\infty;L_\infty(\Omega))} \le \max\left\{||b||_{B(0,\infty;L_\infty(\partial\Omega))}, ||\alpha_0||_{L_\infty(\Omega))}\right\}. \tag{3.7}$$

This way we consider vorticity α in the form

$$\alpha = \beta + \bar{b}, \tag{3.8}$$

where β satisfies the following system

$$\begin{array}{lll} \beta_t + v\cdot\nabla\beta - \nu\Delta\beta = -v\cdot\nabla\bar{b} & \text{in} & \Omega\times(0,\infty), \\ \beta = 0 & \text{on} & \partial\Omega\times(0,\infty), \\ \beta|_{t=0} = 0 & \text{on} & \Omega. \end{array} \tag{3.9}$$

Now we define a standard weak solution for system (3.9) in $V^{1,0}(\Omega\times(0,\infty))$.

Definition 3.2. *We say that β is a global in time weak solution to problem (3.9), iff $\beta \in V^{1,0}(\Omega\times(0,T))$ and the following identity*

$$\begin{array}{c} -\int_0^T\int_\Omega \beta\varphi_t dxdt + \int_\Omega \beta(x,T)\varphi(x,T)dx + \\ \nu\int_0^T\int_\Omega \nabla\beta\cdot\nabla\varphi dxdt - \int_0^T\int_\Omega v\beta\cdot\nabla\varphi dxdt = \int_0^T\int_\Omega v\bar{b}\cdot\nabla\varphi dxdt \end{array} \tag{3.10}$$

holds for any $\varphi \in H_0^{1,1}(\Omega\times(0,T))$ for any $T>0$.

Lemma 3.1. *Let $\nu > 0$. There exists a weak solution of problem (3.9) in the sense of Definition 3.2 such that any $T > 0$*

$$||\beta||_{B(0,T;L_2(\Omega))} + ||\nabla\beta||_{L_2(\Omega\times(0,T))} \le c(T,\nu)S; \tag{3.11}$$

moreover $\alpha \in B(0,\infty;L_\infty(\Omega))$ *with the following independent of* ν *estimate*

$$||\alpha||_{B(0,\infty;L_\infty(\Omega))} \leq ||2\chi||_{L_\infty(\partial\Omega)}||v||_{B(0,\infty;C(\partial\Omega))} + S. \tag{3.12}$$

Proof. To prove the existence for formulation (3.10) it is enough to apply the standard Galerkin method which follows from the energy estimate

$$\sup_{0\leq t\leq T} ||\beta(\cdot,t)||^2_{L_2(\Omega)} + \nu \int_0^T \int_\Omega |\nabla\beta|^2 dxdt \leq c\int_0^T \int_\Omega |v\bar{b}|^2 dxdt. \tag{3.13}$$

Estimate (3.13) gives (3.11).

Prove (3.12). We should note that to be formal we should make these estimations for the approximations from the Galerkin method to get the solution in the L_∞-space. However below formal estimates will imply (3.12) for the limit given by the Galerkin method, since the technique is based on the weak formulation given by Definition 3.2. We omit this standard procedure to avoid unnecessary complications.

By properties of function $\bar{b}$ and (3.8) we write (3.10) in the form

$$\int_0^T \int_\Omega \alpha_t \varphi dx + \nu \int_0^T \int_\Omega \nabla\alpha \cdot \nabla\varphi dxdt + \int_0^T \int_\Omega \alpha v \cdot \nabla\varphi dxdt = 0 \tag{3.14}$$

Take $\varphi = (\alpha - k^*)_+ = \max\{\alpha - k^*, 0\}$, where

$$k^* = \max\left\{ \sup_{(x,t)\in\partial\Omega\times(0,\infty)} 2\chi v\cdot\tau + \alpha_{in/out}, \sup_{x\in\Omega} \alpha_0(x) \right\}.$$

Inserting $\varphi = (\alpha - k)_+$ into (3.14) we get

$$\frac{1}{2}\int_\Omega (\alpha - k^*)^2_+(x,T)dx + \nu\int_0^T \int_\Omega |\nabla(\alpha - k^*)_+|^2 dxdt = 0$$

for all $T > 0$, hence we conclude

$$\alpha \leq k^* \quad \text{in the sense of } \; B(0,\infty;L_\infty(\Omega)). \tag{3.15}$$

The same result we get for $\varphi = (\alpha - k_*)_- = \min\{\alpha - k_*, 0\}$ with

$$k_* = \min\left\{ \inf_{(x,t)\in\partial\Omega\times(0,\infty)} 2\chi v\cdot\tau + \alpha_{in/out}, \inf_{x\in\Omega} \alpha_0(x) \right\},$$

i.e.

$$\alpha \geq k_* \quad \text{in the sense of } \; B(0,\infty;L_\infty(\Omega)). \tag{3.16}$$

Relations (3.15) and (3.16) imply inequality (3.12). Lemma 3.1 is proved.

Since estimate (3.12) is time independent we would like to have a sense of solutions suitable to this bound.

Definition 3.3. *We say that $\alpha \in B(0,\infty;L_\infty(\Omega))$ is a weak-$*$ solution to problem (3.5), iff the following identity*

$$\begin{array}{c} -\int_0^T\int_\Omega \alpha\varphi_t dxdt + \int_\Omega \alpha(x,T)\varphi(x,T)dx - \int_\Omega \alpha_0(x)\varphi(x,0)dx \\ +\nu\int_0^T\int_\Omega \alpha\Delta\varphi dxdt - 2\nu\int_0^T\int_{\partial\Omega}\chi b(v)\frac{\partial\varphi}{\partial n} - \int_0^T\int_\Omega v\alpha\cdot\nabla\varphi dxdt = 0 \end{array} \tag{3.17}$$

holds for every $\varphi \in W_1^{2,1}(\Omega\times(0,\infty))\cap\{\varphi|_{\partial\Omega}=0\}$ and any $T>0$.

Since $\alpha \in B(0,\infty;L_\infty(\Omega))$ and boundary data $(3.4)_3$ are well posed, we can find a regular solution to problem (3.4). This type of elliptic systems are ill posed in L_∞-spaces, but since domain Ω is bounded, so we can embed $L_\infty(\Omega)$ into $L_p(\Omega)$ for any $p<\infty$. Then by the elementary theory the solution satisfies

$$||\tilde v||_{B(0,\infty;W_p^1(\Omega))} \le c(\Omega)||\alpha||_{B(0,\infty;L_\infty(\Omega))} + S. \tag{3.18}$$

In particular, the following lemma holds.

Lemma 3.2. *Let $0\le a<1$. There exists a regular solution to problem (3.4) such that $\tilde v \in B(0,\infty;W_p^1(\Omega))$ for any $p<\infty$ such that*

$$\begin{array}{c} ||\tilde v||_{B(0,\infty;C(\Omega))} \le A(\Omega)||\alpha||_{B(0,\infty;L_\infty(\Omega))} + S, \\ ||\tilde v||_{B(0,\infty;C^a(\Omega))} \le c(\Omega)||\alpha||_{B(0,\infty;L_\infty(\Omega))} + S. \end{array} \tag{3.19}$$

We combine Lemmas 3.1 and 3.2, getting the following result.

Lemma 3.3. *Let $2A(\Omega)||\chi||_{L_\infty(\partial\Omega)}<1$, then*

$$\text{if} \quad ||v||_{B(0,\infty;C(\Omega))} \le M_0, \quad \text{then} \quad ||\tilde v||_{B(0,\infty;C(\Omega))} \le M_0, \tag{3.20}$$

where $M_0 = (1-2A(\Omega)||\chi||_{L_\infty(\partial\Omega)})^{-1}S$.

Note that since relation (3.20) follows from $(3.19)_1$ and (3.12) then $M_0 > \max\{||\alpha_0||_{L_\infty(\Omega)}, ||d||_{B(0,\infty;C(\partial\Omega))}\}$, see also the sketch of the proof and estimate (1.25).

To prove the existence to nonlinear system (1.1) we use the Banach lemma. We show that map Φ is a contraction.

Lemma 3.4. *For any $\nu>0$, map Φ is the contraction in $B(0,\infty;C(\Omega))\cap\Xi$, i.e. for any $v_1,v_2\in B(0,\infty;C(\Omega))\cap\Xi$*

$$||\Phi(v_1)-\Phi(v_2)||_{B(0,\infty;C(\Omega))} \le \Theta||v_1-v_2||_{B(0,\infty;C(\Omega))}, \tag{3.21}$$

where $\Theta < 1$.

Proof. Let functions α_1 and α_2 be solutions of system (3.5) for v_1 and v_2, respectively. Introduce

$$w = \alpha_1 - \alpha_2, \tag{3.22}$$

then new function w satisfies the following problem

$$\begin{array}{lll} w_t + v_1 \cdot \nabla w - \nu \Delta w = -(v_1 - v_2) \cdot \nabla \alpha_2 & \text{in} & \Omega \times (0,T), \\ w = b(v_1) - b(v_2) & \text{in} & \partial\Omega \times (0,T), \\ w|_{t=0} = 0 & \text{in} & \Omega. \end{array} \tag{3.23}$$

Put

$$k^* = \sup_{(x,t)\in\partial\Omega\times(0,T)} b(v_1) - b(v_2).$$

Then we define

$$(w - k^*)_+ = \max\{w - k^*, 0\}.$$

Multiplying $(3.23)_1$ by $(w - k^*)_+^2$, integrating over Ω, we get

$$\begin{array}{c} \frac{1}{3}\frac{d}{dt}\int_\Omega (w - k^*)_+^3 dx + 2\nu \int_\Omega (w - k^*)_+ |\nabla (w - k^*)_+|^2 dx = \\ 2\int_\Omega \alpha_2 (w - k^*)_+ (v_1 - v_2) \cdot \nabla (w - k^*)_+ dx. \end{array}$$

Hence

$$\sup_{0\le t\le T} ||(w - k^*)_+||^3_{L_3(\Omega)} \le c(\nu) \int_0^T \int_\Omega |v_1 - v_2|^2 (w - k^*)_+ dx dt,$$

yields

$$||(w - k^*)_+||_{B(0,T;L_3(\Omega))} \le c(\nu) T ||v_1 - v_2||_{B(0,T;L_\infty(\Omega))}. \tag{3.24}$$

The same bound we obtain for $(w - k_*)_- = \min\{w - k_*, 0\}$ with

$$k_* = \inf_{(x,t)\in\partial\Omega\times(0,T)} b(v_1) - b(v_2),$$

i.e.

$$||(w - k_*)_-||_{B(0,T;L_3(\Omega))} \le c(\nu) T ||v_1 - v_2||_{B(0,T;L_\infty(\Omega))}. \tag{3.25}$$

Since $\Phi(v_1) - \Phi(v_2)$ satisfies the system

$$\begin{array}{lll} \text{rot } (\tilde v_1 - \tilde v_2) = w & \text{in} & \Omega \times (0,T), \\ \text{div } (\tilde v_1 - \tilde v_2) = 0 & \text{in} & \Omega \times (0,T), \\ n \cdot (\tilde v_1 - \tilde v_2) = 0 & \text{on} & \partial\Omega \times (0,T). \end{array} \tag{3.26}$$

Hence by the elementary theory we conclude the following estimate

$$||\tilde{v}_1 - \tilde{v}_2||_{B(0,T;C(\Omega))} \leq c(\Omega)||w||_{B(0,T;L_3(\Omega))}. \tag{3.27}$$

Applying (3.24) and (3.25) to (3.27) we get

$$||\tilde{v}_1-\tilde{v}_2||_{B(0,T;C(\Omega))} \leq c(\nu)T||v_1-v_2||_{B(0,T;C(\Omega))}+A(\Omega)\max\{|k^*|,|k_*|\}, \tag{3.28}$$

where $A(\Omega)$ is the same constant as in (1.7). By definitions of k^*, k_* and b we conclude

$$\max\{|k^*|,|k_*|\} \leq ||2\chi||_{L_\infty(\partial\Omega)}||v_1 - v_2||_{B(0,T;C(\Omega))}. \tag{3.29}$$

Hence (3.28) yields

$$||\tilde{v}_1 - \tilde{v}_2||_{B(0,T;C(\Omega))} \leq \Big(c(\nu)T + A(\Omega)||2\chi||_{L_\infty(\Omega)}\Big)\,||v_1 - v_2||_{B(0,T;C(\Omega))}. \tag{3.30}$$

We have assumed that $A(\Omega)||2\chi||_{L_\infty(\partial\Omega)} < 1$, thus if $\nu > 0$ we can find a positive T such that

$$c(\nu)T + A(\Omega)||2\chi||_{L_\infty(\partial\Omega)} < 1. \tag{3.31}$$

Thus, we get the contraction on time interval $[0,T]$, but since bound (3.20) holds for $(0,\infty)$ and the constant in (3.31) depends on ν and domain Ω we get (3.31) for time interval $[T,2T]$, and this procedure can be repeated for any next interval, hence we obtain (3.21). Lemma 3.4 is proved.

Note that in the proof of Lemma 3.4 we used estimate (3.27) in $L_3(\Omega)$, but it would be enough to consider the $L_{2+\varepsilon}(\Omega)$-bound to control the $C(\Omega)$-norm of the velocity.

Now, we prove Theorem 3.1. Follows the Banach approximation we construct a sequence $v_{N+1} = \Phi(v_N)$ such that

$$||v_N||_{B(0,\infty;C(\Omega))} \leq M_0 \tag{3.32}$$

which follows from Lemma 3.3, if we choose $||v_0||_{B(0,\infty;C(\Omega))} \leq M_0$. Moreover, by Lemma 3.4 vorticity α_N given as a solution of (3.5) with v_N satisfies

$$||\alpha_N||_{B(0,\infty;L_\infty(\Omega))} \leq M_1. \tag{3.33}$$

We want to find a solution satisfying (3.1). Take $T > 0$. Then we choose a subsequence from $\{\alpha_N\}_{N=0}^{\infty}$ such that

$$\begin{array}{rll} \alpha_{N_k} \rightharpoonup \alpha_* & \text{weakly-}* \text{ in} & L_\infty(\Omega\times(0,T)), \\ \alpha_{N_k}(\cdot,T) \rightharpoonup \alpha_*(\cdot,T) & \text{weakly-}* \text{ in} & L_\infty(\Omega). \end{array} \tag{3.34}$$

By Lemma 3.4 sequence $\{v_N\}_{N=0}^{\infty}$ has a strong unique limit, hence in particular

$$v_{N_k+1} \to v_* \ \text{ strongly in } \ B(0,\infty;C(\Omega)). \tag{3.35}$$

Note that rot $v_* = \alpha_*$ which follows from the fact that rot $v_* = \alpha_*$. By (3.18) also we obtain that $v_* \in B(0,\infty;W_p^1(\Omega))$ for any $p<\infty$ which defined us function $\alpha_* \in B(0,\infty;L_p(\Omega))$. Hence $\alpha_*(\cdot,t)$ is well defined for all $t\in[0,\infty)$.

Thus, inserting pair $\{v_{N_k},\alpha_{N_k}\}$ into formulation (3.17), going with $k\to\infty$, remembering about (3.34) and $||v_{N_k+1}-v_{N_k}||_{B(0,\infty;C(\Omega))}\to 0$ as $k\to\infty$, we deduce that (v_*,α_*) satisfies (3.1) for T as at the beginning. Note that the convergence holds, because to control behavior of the nonlinear term we use (3.35).

The same consideration we repeat for another $T>0$. But since the limit of sequence $\{v_N\}_{N=0}^{\infty}$ is strong and unique, also the vorticity is well defined and we obtain the same solution. Hence pair $\{v_*,\alpha_*\}$ is a unique solution in the sense of Definition 3.1. Theorem 3.1 is proved.

To find the inviscid limit of solutions there is a need of estimates which are independent of ν as $\nu\to 0$.

Lemma 3.5. *The following bounds hold*

$$\begin{gathered} ||\alpha^\nu||_{B(0,\infty;L_\infty(\Omega))} \le S, \\ ||v^\nu||_{B(0,\infty;W_p^1(\Omega))} + ||v^\nu||_{B(0,\infty;C^a(\Omega))} \le c(p,a)S, \\ ||\alpha_t^\nu||_{L_p(0,T;(W_q^2(\Omega))^*)} \le T^{1/p}S(S+\nu), \\ ||v^\nu||_{C^{a,a/2}(\Omega\times(0,\infty))} \le c(a)S(S+\nu) \end{gathered} \tag{3.36}$$

for any $T>0$, $1<p,q<\infty$ and a as in Theorem 1.1.

Proof. Bounds $(3.36)_{1,2}$ follow from (3.18) and Theorem 3.1. We have to prove only $(3.36)_{3,4}$.

From Definition 3.1 we conclude that the solution satisfies the following identity

$$\begin{gathered} \int_0^T\int_\Omega \alpha_t\varphi dxdt = \int_0^T\int_\Omega \alpha v\cdot\nabla\varphi dxdt \\ -\nu\int_0^T\int_\Omega \alpha\Delta\varphi dxdt + 2\nu\int_0^T\int_{\partial\Omega}\chi b(v)\frac{\partial\varphi}{\partial n}d\sigma dt \end{gathered} \tag{3.37}$$

for any $\varphi\in W_1^{2,1}(\Omega\times(0,\infty))\cap\{\varphi|_{\partial\Omega}=0\}$. But it is easily seen that the r.h.s. of (3.37) is well defined for any $\varphi\in W_1^{2,0}(\Omega\times(0,\infty))\cap\{\varphi|_{\partial\Omega}=0\}$. In particular, since $|\Omega|<\infty$, for any finite $T>0$ the r.h.s. is well defined for any $\varphi\in L_{p^*}(0,T;W_q^2(\Omega))\cap\{\varphi|_{\partial\Omega}=0\}$ for $1<p^*,q<\infty$, too. Hence by the standard definition of the norm in the L_p-space we deduce that $\alpha_t\in L_p(0,T;(W_q^2(\Omega))^*)$.

Putting $\varphi \in L_{p^*}(0,T;W_q^2(\Omega))$ such that $||\varphi||_{L_{p^*}(0,T;W_q^2(\Omega))} = 1$ into (3.37), using $(3.36)_{1,2}$ to estimate the r.h.s. of (3.37) we obtain the following bound

$$\begin{array}{c} ||\alpha_t^\nu||_{L_p(0,T;(W_q^2(\Omega))^*)} = \sup_\varphi \int_0^T \int_\Omega \alpha_t \varphi dxdt \leq \\ T^{1/p}(|\Omega|^{1/q^*}||\alpha^\nu||_{B(0,\infty;L_\infty(\Omega))}(\nu + ||v||_{B(0,\infty;C(\Omega))}) \\ +2\nu|\partial\Omega|^{1/q^*}||v||_{B(0,\infty;C(\Omega))}), \end{array} \tag{3.38}$$

where $p^*, q^* : 1/p + 1/p^* = 1, 1/q + 1/q^* = 1$. From (3.38) we get $(3.36)_3$.

As a corollary we obtain the following bound

$$\sup_{T\in[0,\infty)} ||\alpha_t^\nu||_{L_p(T,T+1;(W_q^2(\Omega))^*)} \leq S(S+\nu), \tag{3.39}$$

since the method to get $(3.36)_3$ depends only on length of the time interval.

To prove $(3.36)_4$, we consider elliptic problem (1.23). Solution v can be examined as a sum of two functions v_1 and v_2 which satisfy the following problems

$$\begin{array}{llll} \text{rot } v_1 = 0, & \text{rot } v_2 = \alpha & \text{in} & \Omega \times (0,\infty), \\ \text{div } v_1 = 0, & \text{div } v_2 = 0 & \text{in} & \Omega \times (0,\infty), \\ n \cdot v_1 = d, & n \cdot v_2 = 0 & \text{on} & \partial\Omega \times (0,\infty). \end{array} \tag{3.40}$$

By the theory of the elliptic systems, we easily find $v_1 \in C^{a,a/2}(\Omega\times(0,\infty))$, since boundary data $d \in C^{a,a/2}(\partial\Omega \times (0,\infty))$.

For the second problem we obtain $v_2 \in W_q^{1,1/2}(\Omega \times (T,T+1))$ for any $T > 0$ and $q < \infty$. The regularity with respect to the space coordinates is obtained from $(3.36)_1$ and for the time regularity we apply (3.39) with $p = q$.

Hence if q is sufficiently large, by the imbedding theorem, we get $v_2 \in C^{a,a/2}(\Omega \times (T,T+1))$. Since bound (3.39) is independent of T, we conclude that $v_2 \in C^{a,a/2}(\Omega \times (0,\infty))$ which gives estimate $(3.36)_4$. Lemma 3.5 is proved.

From Lemma 3.5 and Theorem 3.1 we deduce Theorem 1.1.

By Lemma 3.5 we conclude the following convergence for a subsequence of solutions with ν_k going to zero.

Lemma 3.6. *There exists a subsequence $\{v^{\nu_k}, \alpha^{\nu_k}\}_{k=1}^\infty$ such that $\nu_k \to 0$ as $k \to \infty$ satisfying the following conditions*

$$\begin{array}{rcr} v^{\nu_k} \to v_E & \textit{strongly in} & C^{a,a/2}(\Omega \times (0,T)), \\ v^{\nu_k} \rightharpoonup v_E & \textit{weakly in} & L_p(0,T;W_{p'}^1(\Omega)), \\ \alpha^{\nu_k} \rightharpoonup \alpha_E & \textit{weakly-* in} & L_\infty(0,\infty;L_\infty(\Omega)), \\ \alpha^{\nu_k}(\cdot,t) \rightharpoonup \alpha_E(\cdot,t) & \textit{weakly-* in} & L_\infty(\Omega) \ \textit{for} \ t \in (0,\infty) \end{array} \tag{3.41}$$

for any $T > 0$, $1 < p, p', q, q' < \infty$ and a is as in Theorem 1.1.

Moreover rot $v_E = \alpha_E$ and the following estimate is valid

$$||v_E||_{C^{a,a/2}(\Omega\times(0,\infty))} + ||v_E||_{B(0,\infty;W^1_p(\Omega))} + ||\alpha_E||_{B(0,\infty;L_\infty(\Omega))} \leq S. \tag{3.42}$$

Proof. Convergences $(3.41)_{2,3,4}$ follow from estimates $(3.36)_{1,2}$. To get $(3.41)_1$, it is enough to use $(3.36)_4$. Bound (3.42) is a consequence of (3.36).

Let us prove that function v_E given by Lemma 3.6 is a solution of the Euler system. But first we need a sense of solutions to the inviscid system which is compatible with Definition 3.1.

To formulate a suitable weak formulation of problem (1.10) there is a need of an extension of boundary data to obtain homogeneous conditions for a modification of the solution of (1.10). Since by (1.4) we have already one such a field we consider the solution in the following form

$$v = u + \mathcal{D}, \tag{3.43}$$

where u satisfies the following system

$$\begin{array}{lll} u_t + v\cdot\nabla u + \nabla p = -\mathcal{D}_t - v\cdot\nabla\mathcal{D} & \text{in} & \Omega\times(0,T), \\ \text{div } u = 0 & \text{in} & \Omega\times(0,T), \\ n\cdot u = 0 & \text{on} & \partial\Omega\times(0,T), \\ u|_{t=0} = v_0 - \mathcal{D}|_{t=0} & \text{on} & \Omega. \end{array} \tag{3.44}$$

Now we formulate a definition of a weak solution to problem (3.44).

Definition 3.4. *We say that u is a weak solution to problem (3.44), iff $u \in C(\Omega\times(0,\infty))$ and*

$$\begin{array}{c} -\int_0^T\int_\Omega u\cdot\Phi_t dxdt + \int_\Omega u(x,T)\Phi(x,T)dx - \int_\Omega u_0(x)\Phi(x,0)dx \\ -\int_0^T\int_\Omega v\cdot\nabla\Phi u dxdt + \int_0^T\int_{\partial\Omega} n\cdot(v\otimes\Phi)\cdot ud\sigma dt = \\ \int_0^T\int_\Omega v\cdot\nabla\Phi\cdot\mathcal{D}dxdt - \int_0^T\int_{\partial\Omega} n\cdot(v\otimes\Phi)\cdot\mathcal{D}d\sigma dt \\ -\int_0^T\int_\Omega \mathcal{D}\Phi_t - \int_\Omega\mathcal{D}(x,T)\Phi(x,T)dx + \int_\Omega\mathcal{D}(x,0)\Phi(x,0)dx \end{array} \tag{3.45}$$

holds for any $T > 0$ and for any $\Phi \in W^{1,1}_1(\Omega\times(0,T);\mathbf{R}^2)$ such that div $\Phi = 0$ and $n\cdot\Phi|_{\partial\Omega} = 0$.

Theorem 3.2. *Limit v_E given by Lemma 3.6 is a solution of the Euler system in the sense of Definition 3.4.*

Proof. First we rewrite (3.45) in a suitable distributional form

$$\begin{gathered} -\int_0^T\int_\Omega v\Phi_t dxdt + \int_\Omega v(x,T)\Phi(x,T)dx - \int_\Omega v_0(x)\Phi(x,0)dx \\ +\int_0^T\int_\Omega v\cdot\nabla v\Phi dxdt = 0. \end{gathered} \tag{3.46}$$

Since the dimension of the domain is two and test functions have to satisfy: div $\Phi = 0$ and $\Phi\cdot n|_{\partial\Omega} = 0$, we can represent Φ in the following way

$$\Phi = \nabla^\perp\varphi = (-\partial_{x_2}\varphi, \partial_{x_1}\varphi) \text{ and } \varphi|_{\partial\Omega} = 0 \tag{3.47}$$

for a scalar function φ.

Then, inserting from (3.47) into (3.46), integrating by parts, we obtain

$$\begin{gathered} \int_0^T\int_\Omega \text{rot } v\varphi_t dxdt - \int_\Omega \text{rot } v(x,T)\varphi(x,T)dx + \int_\Omega \text{rot } v_0(x)\varphi dx \\ +\int_0^T\int_\Omega (\text{rot } v)v\cdot\nabla\varphi dxdt = 0 \end{gathered} \tag{3.48}$$

for all $\varphi\in C_0^\infty(\Omega\times(0,\infty))$; we restrict our attention to smooth test functions, since C^∞ is dense in each test function spaces which are considered here. To get (3.48) we used a fact that in 2D the following relation rot $(v\cdot\nabla v) = v\cdot\nabla(\text{rot } v)$ holds for divergence free fields v.

Next, we consider the Navier-Stokes equations. Take (3.1) with v^{ν_k} and α^{ν_k}

$$\begin{gathered} \int_0^T\int_\Omega \alpha^{\nu_k}\varphi_t dxdt - \int_\Omega \alpha^{\nu_k}(x,T)\varphi(x,T)dxdt + \int_\Omega \alpha_0(x)\varphi(x,0)dx \\ +\int_0^T\int_\Omega \alpha^{\nu_k}v^{\nu_k}\cdot\nabla\varphi dxdt - \nu_k\int_0^T\int_\Omega \alpha^{\nu_k}\Delta\varphi dxdt \\ +2\nu_k\int_0^T\int_\Omega \chi b(v^{\nu_k})\frac{\partial\varphi}{\partial n}d\sigma dt = 0. \end{gathered} \tag{3.49}$$

Going to the limit with $\nu_k\to 0$, we get

$$\begin{gathered} \int_0^T\int_\Omega \alpha_E\varphi_t dxdt - \int_\Omega \alpha_E(x,T)\varphi(x,T)dxdt + \int_\Omega \alpha_0(x)\varphi(x,0)dx \\ +\int_0^T\int_\Omega \alpha_E v_E\cdot\nabla\varphi dxdt = 0. \end{gathered} \tag{3.50}$$

Convergence of two first terms of (3.49) follows from $(3.41)_{3,4}$, of the forth one from $(3.41)_{1,3}$. Vanishing of two last terms follows from bound (1.8).

Thus, limit v_E satisfies (3.48), hence is a solution in the sense of Definition 3.4. Theorem 3.2 is proved.

To prove Theorem 1.2 we need to improve the sense of solutions given by Theorem 3.2. From (3.50) we get in a distributional sense the following formulation of the Euler system

$$\int_0^T\int_\Omega v_{E,t}\Phi dxdt = -\int_0^T\int_\Omega v_E\cdot\nabla v_E\Phi dxdt. \tag{3.51}$$

But from (3.42) limit $v_E \in L_q(0,T;W^1_q(\Omega))$ for any $T>0$ and $1<q<\infty$; hence for any $1<p<\infty$

$$v_E \cdot \nabla v_E \in L_p(\Omega \times (0,T)). \tag{3.52}$$

This follows that $v_{E,t} \in (L_{p^*}(\Omega \times (0,T)))^*$, where $p^* : 1/p + 1/p^* = 1$, since by (3.52) the r.h.s. of (3.51) is well defined for $\Phi \in L_{p^*}(\Omega \times (0,T))$. Hence we conclude

$$v_{E,t} \in L_p(\Omega \times (0,T)). \tag{3.53}$$

To finish the proof we note that since test functions are divergence free, formulation (3.51) is a projection on the divergence free part of the L_q-space. Thus there exists a scalar function $p_E : \Omega \times (0,T) \to \mathbf{R}$ such that $\nabla p_E \in L_q(\Omega \times (0,T)$ and

$$v_{E,t} + v_E \cdot \nabla v_E + \nabla p_E = 0 \quad \text{a.e. in } \ \Omega \times (0,T)). \tag{3.54}$$

But (3.54) holds for any $T>0$. Theorem 1.2 is proved.

4 The steady system

In this section we prove Theorem 1.4. Since Theorem 1.3 has been shown in [7, Theorem 3.1], we recall only the weak formulation for problem (1.12).

Definition 4.1. *We say that $v \in \Xi$ is a weak-$*$ solution to problem (1.12), iff $v \in C(\Omega)$, rot $v \in L_\infty(\Omega)$ and the following identity*

$$\nu \int_\Omega \alpha \Delta \varphi dx - 2\nu \int_{\partial\Omega} \chi b(v) \frac{\partial \varphi}{\partial n} d\sigma + \int_\Omega \alpha v \cdot \nabla \varphi dx = 0 \tag{4.1}$$

holds for any $\varphi \in W^2_1(\Omega) \cap \{\varphi|_{\partial\Omega} = 0\}$, where Ξ is defined by (2.10) and $b(\cdot)$ by (1.20).

Bound (1.13) from Theorem 1.3 establishes the following result. We again use index ν to underline dependence of the viscous coefficient.

Lemma 4.1. *There exists a subsequence $\{v^{\nu_k}, \alpha^{\nu_k}\}_{k=0}^{\infty}$ such that $\nu_k \to 0$ as $k \to \infty$ satisfying the following conditions*

$$\begin{array}{lll} v^{\nu_k} \to v_E & \textit{strongly in} & C^a(\Omega), \\ \alpha^{\nu_k} \to \alpha_E & \textit{weakly-}* \textit{ in} & L_\infty(\Omega). \end{array} \tag{4.2}$$

Moreover rot $v_E = \alpha_E$ *and the following estimate is valid*

$$||v_E||_{C^a(\Omega)} + ||\alpha_E||_{L_\infty(\Omega)} \leq S. \tag{4.3}$$

Let us show that limit v_E is a solution to problem (1.15). The same as for the evolutionary case there is a need of a weak formulation for the Euler system. Since (1.4) gives us an extension of boundary data we investigate the solution of (1.15) in the following form

$$v = u + \mathcal{D}, \tag{4.4}$$

where u is satisfies the following system

$$\begin{array}{lll} v \cdot \nabla u + \nabla p = -v \cdot \nabla \mathcal{D} & \text{in} & \Omega, \\ \text{div } u = 0 & \text{in} & \Omega, \\ n \cdot u = 0 & \text{on} & \partial\Omega. \end{array} \tag{4.5}$$

A weak formulation to (4.5) has the following form.

Definition 4.2. *We say that* $u \in \Xi$ *is a weak solution to problem (4.5), iff* $u \in C(\Omega)$ *and*

$$\begin{array}{c} -\int_\Omega v \cdot \nabla \Phi u dx + \int_{\partial\Omega} n \cdot (v \otimes \Phi) \cdot u d\sigma = \\ \int_\Omega v \cdot \nabla \Phi \mathcal{D} dx - \int_{\partial\Omega} n \cdot (v \otimes \Phi) \cdot \mathcal{D} d\sigma \end{array} \tag{4.6}$$

for $\Phi \in W_1^1(\Omega)$ *such that div* $\Phi = 0$ *and* $n \cdot \Phi|_{\partial\Omega} = 0$.

Theorem 4.1. *Limit* v_E *given by Lemma 4.1 is a solution of the Euler system in the sense of Definition 4.2.*

Proof. Formulation (4.6) may be written in the following distributional form

$$\int_\Omega v \cdot \nabla v \Phi dx = 0. \tag{4.7}$$

Analogically to (3.47) we consider test functions Φ as follows

$$\Phi = \nabla^\perp \varphi \quad \text{and} \quad \varphi|_{\partial\Omega} = 0. \tag{4.8}$$

Inserting (4.8) into (4.7), integrating by parts we get

$$\int_\Omega (\text{rot } v) v \cdot \nabla \varphi dx = 0. \tag{4.9}$$

Now, take (4.1) with v^{ν_k} and α^{ν_k}

$$\nu_k \int_\Omega \alpha^{\nu^k} \Delta\varphi dx - 2\nu_k \int_{\partial\Omega} \chi b(v^{\nu_k}) \frac{\partial\varphi}{\partial n} d\sigma + \int_\Omega \alpha^{\nu_k} v^{\nu_k} \cdot \nabla\varphi dx = 0. \qquad (4.10)$$

Taking the limit in (4.10) yields

$$\int_\Omega \alpha_E v_E \cdot \nabla\varphi dx = 0. \qquad (4.11)$$

Convergence of the third term follows from the strong limit $(4.2)_1$ and weak-$*$ one $(4.2)_2$. Vanishing of two first terms is a consequence of estimate (1.13).

Thus, limit v_E satisfies (4.9), hence also Definition 4.2. Theorem 4.1 is proved.

Prove Theorem 1.4. Since rot $v_E \in L_\infty(\Omega)$, then the same as for (3.18) we conclude that

$$\nabla v_E \in L_q(\Omega) \text{ for } \; q < \infty. \qquad (4.12)$$

This follows that

$$v_E \cdot \nabla v_E \in L_q(\Omega) \text{ for } \; q < \infty \qquad (4.13)$$

as well. Thus, (4.9) is well defined for any $\Phi \in L_{q^*}(\Omega)$ with $q^* : 1/q + 1/q^* = 1$.

But we put a constrain on the test functions div $\Phi = 0$. This follow that

$$v_E \cdot \nabla v_E + \nabla p_E = 0 \;\text{ a.e. in } \Omega \qquad (4.14)$$

for a scalar function $p_E : \Omega \to \mathbf{R}$ such that $\nabla p_E \in L_q(\Omega)$. Theorem 1.4 is proved.

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