The Cauchy Problem for the Compressible Navier-Stokes Equations in the L_p-framework*

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Abstract. The global in time existence of regular solutions to the compressible Navier-Stokes equations in the whole space is obtained. The solutions are close to nontrivial equilibrium solutions. Moreover, the result is sharp in the L_p-framework, the velocity of the fluid belongs to $W_r^{2,1}$ with $r > 3$.

1 Introduction

We consider motion of a viscous isentropic fluid in the whole space described by the compressible Navier-Stokes equations

$$\begin{aligned} &\varrho(v_t + v\nabla v) - \mu\Delta v - \nu\nabla \mathrm{div}\, v + \nabla p(\varrho) = \varrho f, \\ &\varrho_t + \mathrm{div}(\varrho v) = 0, \\ &\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0, \end{aligned} \tag{1.1}$$

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where $v(x,t)$ is the velocity of the fluid, ϱ the density, $p(\varrho) = a\varrho^\gamma$, f the external force, μ and ν constant positive viscous coefficients. Our considerations we restrict to the case when $\gamma > 1$ and the external force is potential ($f = \nabla\varphi$).

The aim of this paper is to show stability of the static solution of $(1.1)_{1,2}$

$$\nabla a\bar{\varrho}^\gamma = \bar{\varrho}\nabla\varphi, \tag{1.2}$$

where

$$\bar{\varrho} \in C^\alpha(\mathbf{R}^3), \ \ 0 < \bar{\varrho}_* \le \bar{\varrho} \le \bar{\varrho}^* < \infty \ \text{ and } \ \varphi = \frac{a\gamma}{\gamma-1}\bar{\varrho}^{\gamma-1}.$$

On function $\bar{\varrho}$ there is no condition of smallness.

The technique applied in the paper requires to introduce the Lagrangian coordinates which are defined as initial data to the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x,t), \quad x|_{t=t_0} = \xi. \tag{1.3}$$

Solving (1.3) we get

$$x = \xi + \int_{t_0}^t u(\xi,t')dt', \tag{1.4}$$

where $u(\xi,t) = v(x(\xi,t),t)$. Equation (1.4) gives the relation between the Eulerian x and the Lagrangian ξ coordinates. The transformation given by (1.4) is denoted by $x = \mathcal{T}_{t_0}(\xi,t)$.

After this transformation problem (1.1) with initial time at t_0 reads

$$\begin{aligned} &\eta u_t - \mu\Delta_u u - \nu\nabla\mathrm{div}_u u + \nabla_u p = \eta f, \\ &\eta_t + \eta\mathrm{div}_u u = 0, \\ &u|_{t=t_0} = v(\cdot,t_0), \quad \eta|_{t=t_0} = \varrho(\cdot,t_0), \end{aligned} \tag{1.5}$$

where $\eta(\xi,t) = \varrho(x(\xi,t),t)$, $\nabla_u = \frac{\partial\xi_i}{\partial x}\partial_{\xi_i}$, $\mathrm{div}_u = \nabla_u\cdot$ and the summations convention over the repeating indices is used.

System (1.1) is treated as a perturbation of the equilibrium state (1.2). The perturbation of the density of the fluid in the Eulerian coordinates we denote by

$$\sigma(x,t) = \varrho(x,t) - \bar{\varrho}(x), \tag{1.6}$$

then problem (1.1) takes in the form

$$\begin{aligned} &\varrho(v_t + v\nabla v) - \mu\Delta v - \nu\nabla\mathrm{div}\, v + p_1\nabla\sigma = \sigma f - \sigma\nabla p_1, \\ &\sigma_t + \mathrm{div}(\varrho v) = 0, \\ &\sigma|_{t=t_0} = \varrho(x,t_0), \quad v|_{t=t_0} = v(x,t_0), \end{aligned} \tag{1.7}$$

where p_1 is defined by the relation

$$p(\varrho) - p(\bar{\varrho}) = \sigma \int_0^1 p'(\varrho + s(\bar{\varrho} - \varrho)ds = \sigma p_1. \tag{1.8}$$

And the same we consider in the Lagrangian coordinates

$$\chi(\xi, t) = \eta(\xi, t) - \bar{\eta}(\xi, t), \tag{1.9}$$

with (1.5) in the form

$$\begin{array}{l} \eta u_t - \mu \Delta_u u - \nu \nabla \mathrm{div}_u u + p_1 \nabla_u \chi = \chi f - \chi \nabla_u p_1, \\ \chi_t + \eta \mathrm{div}_u u = -\bar{\eta}_t, \\ u|_{t=t_0} = v(\xi, t_0), \quad \chi|_{t=t_0} = \varrho(\xi, t_0) - \bar{\varrho}(\xi), \end{array} \tag{1.10}$$

where $\bar{\eta}(\xi, t) = \bar{\varrho}(\mathcal{T}_{t_0}(\xi, t))$.

To formulate the result of the paper we recall two results for system (1.1). First one is the almost global in time of existence of solutions to (1.10) - see [6].

Theorem A (see [6]). *Let $r > 3$, $f \in W^1_\infty(\mathbf{R}^3)$, $\sigma_0 \in W^1_r(\mathbf{R}^3) \cap L_2(\mathbf{R}^3)$, $u_0 \in W_r^{2-\frac{2}{r}}(\mathbf{R}^3) \cap L_2(\mathbf{R}^3)$,*

$$\varrho_0 \geq \frac{\bar{\varrho}_*}{2},$$

then for any $T > 0$ and $M_1 \leq \bar{M}_1$ the solution of (1.10) exists such that $u \in W^{2,1}_r(\mathbf{R}^3 \times [0, T])$ and $\chi \in V_r(\mathbf{R}^3 \times [0, T])$ and the following estimate holds

$$||u||_{W^{2,1}_r(R^3 \times [0,T])} + ||\chi||_{V_r(R^3 \times [0,T])} \leq M_1(T), \tag{1.11}$$

if

$$||v_0||_{W_r^{2-\frac{2}{r}}(R^3)} + ||\sigma_0||_{W^1_r(R^3)} + ||v_0||_{L_2(R^3)} + ||\sigma_0||_{L_2(R^3)} \leq M_2(T, M_1),$$

where $M_2(T, M_1) \to 0$ with $T \to \infty$, $M_1 \to 0$ and $\bar{M}_1(T)$ ensures the following boundedness

$$\eta(\xi, t) \geq \frac{\bar{\varrho}_*}{4}$$

for $(\xi, t) \in \mathbf{R}^3 \times [0, T]$, where $||\chi||_{V^r} = ||\chi||_{W^{1,0}_r} + ||\chi_t||_{W^{1,0}_r}$.

The second one is a L_r-estimate for the linearized system (see [5])

$$\begin{array}{c} u_t - \mu \Delta u - \nu \nabla \mathrm{div}\, u + a \nabla \eta = f, \\ \eta_t + b\, \mathrm{div}\, u = g, \end{array} \tag{1.12}$$

moreover we assume that

$$supp\, u, \eta \subset B(0,1) \times (0,\infty), \tag{1.13}$$

where $B(0,1) = \{x \in \mathbf{R}^3 : |x| \le 1\}$.

Theorem B (see [5]). *Let $r \ge 2$, $f \in L_r(\mathbf{R}^4) \cap L_2(\mathbf{R}^4)$, $g \in W_r^{1,0}(\mathbf{R}^4) \cap W_2^{1,0}(\mathbf{R}^4)$ then for $0 < T \le \infty$ the solution of (1.12)-(1.13) satisfies the following estimate*

$$\begin{array}{c} ||u||_{W_r^{2,1}(R^3\times[0,T])} + ||\eta||_{W_r^{1,0}(R^3\times[0,T])} + ||\eta_t||_{W_r^{1,0}(R^3\times[0,T])} + \\ ||u||_{W_2^{2,1}(R^3\times[0,T])} + ||\eta||_{W_2^{1,0}(R^3\times[0,T])} + ||\eta_t||_{W_2^{1,0}(R^3\times[0,T])} \le \\ A_0 \Big(||f||_{L_r(R^3\times[0,T])} + ||g||_{W_r^{1,0}(R^3\times[0,T])} + \\ ||f||_{L_2(R^3\times[0,T])} + ||g||_{W_2^{1,0}(R^3\times[0,T])} \Big), \end{array} \tag{1.14}$$

where A_0 is independent of T.

And finally we define

Smallness Assumption

Here we define some quantities which will be necessary and fixed during our considerations. First we fix $T > 10^4 A_0$, where A_0 comes from Theorem B, inequality (1.14). Next we define $M_0 > 0$ such that if

$$||u|_{t=t_0}||_{L_2(R^3)} + ||\chi|_{t=t_0}||_{L_2(R^3)} + ||u|_{t=t_0}||_{W_r^{2-\frac{1}{r}}(R^3)} + ||\chi|_{t=t_0}||_{W_r^1(R^3)} \le M_0$$

then by Theorem A solutions of (1.10) exists on interval $[t_0, t_0 + 2T]$ and M_0 is so small that the Jacobian of transformation (1.4) satisfies

$$\frac{1}{2} \le ||\mathcal{T}_{t_0}|| \le \frac{3}{2}$$

for $\mathbf{R}^3 \times [t_0, t_0 + 2T]$, moreover quantities $||\chi||_{V_r}$ and $T^{\frac{r-1}{r}} ||u||_{W_r^{2,1}}$ are small enough, where smallness depends only on μ, ν and φ.

Let us define the construction of global solutions to system (1.1). First we solve (1.5) on interval $[0,T]$, next we return to (1.1) and get v and ϱ on $[0,2T]$, hence we can solve (1.5) with initial time $t_0 = T$, this leads to existence on $[T,3T]$, repeating we try to obtain $[2T,4T]$, etc.

So there are defined sequences u^k, η^k as solutions of following problems

$$\begin{array}{l}\eta^k u^k_t - \mu\Delta_{u^k}u^k - \nu\nabla\mathrm{div}_{u^k}u^k + \nabla_{u^k}p^k = \eta^k f,\\ \eta^k_t + \eta^k\mathrm{div}_{u^k}u^k = 0,\\ u^k|_{t=(k-1)T} = v(\mathcal{T}_{(k-2)T}(\xi,(k-1)T),(k-1)T),\\ \eta^k|_{t=(k-1)T} = \varrho(\mathcal{T}_{(k-2)T}(\xi,(k-1)T),(k-1)T).\end{array} \tag{1.15}$$

If we obtain the solutions of (1.15) for all $k > 0$, we get global in time existence of solutions of (1.1).

The main result of the paper is the following.

Theorem C. *Let* $r > 3$, $\varrho_0 - \bar\varrho \in W^1_r(\mathbf{R}^3) \cap L_2(\mathbf{R}^3)$, $v_0 \in W_r^{2-\frac{2}{r}}(\mathbf{R}^3) \cap L_2(\mathbf{R}^3)$ *and*

$$||v_0||_{L_2(R^3)} + ||\varrho_0 - \bar\varrho||_{L_2(R^3)} + ||v_0||_{W_r^{2-\frac{2}{r}}(R^3)} + ||\varrho_0 - \bar\varrho||_{W^1_r(R^3)} \le M_0,$$

then for all $k > 0$ *system (1.15) has a unique solution such that*

$$u^k \in W^{2,1}_r(\mathbf{R}^3 \times [(k-1)T, kT]), \quad \eta^k - \bar\eta \in V_r(\mathbf{R}^3 \times [(k-1)T, kT])$$

and the following estimate holds

$$\begin{array}{c}||u^k||_{W^{2,1}_r(R^3\times[(k-1)T,kT])} + ||\eta^k - \bar\eta||_{V_r(R^3\times[(k-1)T,kT])} +\\ ||u^k||_{W^{2,1}_2(R^3\times[(k-1)T,kT])} + ||\eta^k - \bar\eta||_{V_2(R^3\times[(k-1)T,kT])} \le \delta(M_0),\end{array} \tag{1.16}$$

if $||v_0||_{L_2(R^3)} + ||\varrho_0 - \bar\varrho||_{L_2(R^3)}$ *is small enough.*

Theorem C gives a global in time solution of (1.1) with sharp regularity in the L_p-framework such that the velocity $u \in W^{2,1}_r$ with $r > 3$. Index r cannot be smaller or equal to 3, because we require transformation (1.4) to be well defined, so we need

$$\left|\left|\int_0^t |\nabla u| ds\right|\right|_{L_\infty} \le c||u||_{W^{2,1}_r}$$

which is satisfied, by the imbedding theorem - Proposition 2.1, if $r > 3$. Moreover we obtain stability of any nonconstant regular static state. To prove our result in the L_p-framework we have to apply the Lagrange coordinates, because we can not treat equation $(1.1)_2$ using this approach. Considering problem (1.10) we can localize it near an equilibrium state. Theorems A and

B enable to prove existence of solutions to problems (1.15) for all k. The crucial point of the proof of Theorem C is the independence of time of the constant form estimate (1.14) from Theorem B.

The global in time existence of regular solutions to problem (1.1) first has been obtained by Matsumura and Nishida in [3]. The generalizations of their results one can find in [4,8]. The proofs in these papers base on the L_2-approach (energy estimates). The first sharp result in the L_2-framework dues to Zajączkowski and Kobayashi [2]. They have shown that $u \in W_2^{2+\alpha,1+\alpha/2}$ with $\alpha > \frac{1}{2}$. In the L_p-framework global existence of our equations with the Dirichlet boundary conditions has been proved by Ströhmer in [7], but his results are not sharp. Applying the theory of semigroups he obtained $u \in W_r^{3,1}$. In these papers there were examined only stability states which are close to constant solutions.

2 Notation

In our considerations we will need the anisotropic Sobolev spaces $W_r^{m,n}(Q_T)$ where $m, n \in R_+ \cup \{0\}$, $r \geq 1$ and $Q_T = Q \times (0,T)$ with the norm

$$
\begin{aligned}
||u||^r_{W_r^{m,n}(Q_T)} &= \int_0^T \int_Q |u(x,t)|^r dxdt \\
&+ \sum_{0\leq |m'| \leq [|m|]} \int_0^T \int_Q |D_x^{m'} u(x,t)|^r dxdt \\
&+ \sum_{|m'|=[|m|]} \int_0^T dt \int_Q \int_Q \frac{|D_x^{m'} u(x,t) - D_x^{m'} u(x',t)|^r}{|x-x'|^{s+r(|m|-[|m|])}} dxdx' \\
&+ \sum_{0\leq |n'| \leq [|n|]} \int_0^T \int_Q |D_t^{n'} u(x,t)|^r dxdt \\
&+ \int_Q dx \int_0^T \int_0^T \frac{|D_t^{[n]} u(x,t) - D_t^{[n]}(x,t')|^r}{|t-t'|^{1+r(n-[n])}} dtdt',
\end{aligned}
\tag{2.1}
$$

where $s = dim\, Q$, $[\alpha]$ - the integral part of α, $D_x^l = \partial_{x_1}^{l_1} ... \partial_{x_s}^{l_2}$, where $l = (l_1, ..., l_s)$ a multiindex.

We also define space $V_r(Q_T)$ as the closure

$$
V_r(Q_T) = \overline{C^\infty(Q_T)}^{||\cdot||_{V_r(Q_T)}},
$$

where

$$||u||_{V_r(Q_T)} = ||u||_{W_r^{1,0}(Q_T)} + ||u_t||_{W_r^{1,0}(Q_T)}. \tag{2.2}$$

In the proof we use the following results.

Proposition 2.1 (see [1]). *Let* $u \in W_r^{m,n}(\Omega_T)$, $m, n \in \mathbf{R}_+$ *then if*

$$\sum_{i=1}^{3}(\alpha_i + \frac{1}{r} - \frac{1}{q})\frac{1}{m} + (\beta + \frac{1}{r} - \frac{1}{q})\frac{1}{n} < 1$$

the following estimate holds

$$||D_t^\beta D_x^\alpha u||_{L_q(\Omega_T)} \le \varepsilon ||u||_{W_r^{m,n}(\Omega_T)} + c(\varepsilon)||u||_{L_2(\Omega_T)}, \tag{2.3}$$

where $q \ge r \ge 2$, $\varepsilon \in (0,1)$ *and* $c(\varepsilon) \to \infty$ *with* $\varepsilon \to 0$.

Proposition 2.2. *Let* $r > 3$ *then* $V_r(\Omega^T) \subset C^\alpha(\Omega^T)$, *where* $0 < \alpha < 1 - \frac{3}{r}$ *and the following estimate holds*

$$||f||_{C^\alpha(\Omega^T)} \le c||f||_{V_r(\Omega^T)}.$$

During our considerations will use well known results like the imbedding theorems for Sobolev spaces. All constants are denoted by c.

3 Energy estimate

To prove Theorem C we need a global in time estimate which gives smallness of the $L_2(\mathbf{R}^3)$-norm of solutions of problem (1.1) uniformly in time. Such a knowledge will be very useful to estimate nonlinear terms in the proof of the main result.

Lemma 3.1. *If* $||\sigma||_{L_\infty(R^3\times[0,t])}$ *is sufficiently small, we have*

$$||v(\cdot,t)||_{L_2(R^3)} + ||\sigma(\cdot,t)||_{L_2(R^3)} \le B(||v_0||_{L_2(R^3)} + ||\sigma_0||_{L_2(R^3)}) \tag{3.1}$$

for $t > 0$, *where* B *is independent of time.*

Proof. Multiplying $(1.1)_1$ by v, integrating over $\mathbf{R}^3$ we get

$$\begin{gathered} \int \varrho(v_t + v\nabla v)\cdot v dx - \mu\int \Delta v \cdot v dx - \nu \int \nabla \mathrm{div}\, v \cdot v dx + \\ \int \nabla p(\varrho)\cdot v dx = \int \varrho f \cdot v dx. \end{gathered} \tag{3.2}$$

By $(1.1)_2$ we obtain

$$\begin{gathered}\int\varrho(v_t+v\nabla v)\cdot vdx=\tfrac{1}{2}\tfrac{d}{dt}\int\varrho v^2dx,\quad -\mu\int\Delta v\cdot vdx=\mu\int|\nabla v|^2dx,\\ -\nu\int\nabla\mathrm{div}\,v\cdot vdx=\nu\int|\mathrm{div}\,v|^2dx,\quad \int\nabla p(\varrho)\cdot vdx=\tfrac{d}{dt}\int\tfrac{a}{\gamma-1}\varrho^\gamma dx,\\ \int\nabla\varphi(x)\cdot\varrho vdx=\tfrac{d}{dt}\int\varrho\cdot\varphi dx.\end{gathered}$$

Hence we have

$$\frac{d}{dt}\int\left[\frac{1}{2}\varrho v^2+\frac{a}{\gamma-1}\varrho^\gamma-\varrho\varphi\right]dx+\mu\int|\nabla v|^2dx+\nu\int|\mathrm{div}\,v|^2dx=0. \quad (3.3)$$

Next we examine properties of equality (3.3), we consider the following functional which appear in the r.h.s.

$$I(\sigma)=\int\left[\frac{a}{\gamma-1}(\bar\varrho+\sigma)^\gamma-(\bar\varrho+\sigma)\varphi\right]dx-\int\left[\frac{a}{\gamma-1}\bar\varrho^\gamma-\bar\varrho\varphi\right]dx.$$

We show that if $||\sigma||_{W^1_r(R^3)}$ is small enough then

$$a_1||\sigma||^2_{L_2(R^3)}\le I(\sigma)\le a_2||\sigma||^2_{L_2(R^3)}. \quad (3.4)$$

To prove (3.4) we note

$$\frac{d}{dt}I(t\sigma)|_{t=0}=\int\left[\frac{a\gamma}{\gamma-1}\bar\varrho^{\gamma-1}-\varphi\right]\sigma dx=0,$$

which follows from (1.2).

$$\frac{d^2}{dt^2}I(t\sigma)|_{t=0}=\int a\gamma\bar\varrho^{\gamma-2}\sigma^2dx\ge c||\sigma||^2_{L_2(R^3)},$$

$$\frac{d^3}{dt^3}I(t\sigma)|_{t=0}=\int a\gamma(\gamma-2)\bar\varrho^{\gamma-3}\sigma^3dx.$$

From this we get immediately (3.4).

Since $\bar\varrho$ is independent of t we get

$$\frac{d}{dt}\left[\frac{1}{2}\int\varrho v^2dx+I(\sigma)\right]\le 0 \quad (3.5)$$

From (3.5) and smallness of $||\sigma||_{L_\infty}$, we get (3.1).

4 Proof of Theorem C

To show Theorem C we adapt a method from the theory of parabolic equations. This technique is based on applying a function which cuts off initial data. This way we get more information from the energy estimate. As we see this approach can be done having the estimate from Theorem B.

We introduce smooth functions $\zeta_k : \mathbf{R}_+ \to \mathbf{R}$ such that

$$\zeta_k = \begin{cases} 1 & \text{for} \quad t \geq Tk \\ 0 & \text{for} \quad t \leq (k-1)T \end{cases}$$

and $0 \leq \zeta_k \leq 1$, $|\nabla \zeta_k| \leq \frac{2}{T}$, where $T \geq 10^4 A_0$ (A_0 is defined as in Theorem B in inequality (1.14) see also Smallness Assumption).

Using functions ζ_k we define new searched functions

$$U^k = \zeta_k u \quad \text{and} \quad X^k = \zeta_k \chi, \tag{4.1}$$

where u and χ are solutions of (1.10) with initial time $t_0 = (k-1)T$, then we get the following system on new functions

$$\begin{array}{l} \eta U^k_t - \mu \Delta U^k - \nu \nabla \mathrm{div}\, U^k + p_1 \nabla X^k = K' - \eta \zeta'_k U^{k-1}, \\ X^k_t + \eta \mathrm{div}\, U^k = L' - \zeta'_k X^{k-1}, \\ U^k|_{t=(k-1)T} = 0, \quad X^k|_{t=(k-1)T} = 0, \end{array} \tag{4.2}$$

where

$$\begin{array}{c} K' = \mu(\Delta_u - \Delta)U^k + \nu(\nabla_u \mathrm{div}_u - \nabla \mathrm{div}\,)U^k + \\ X^k \nabla p_1 + (\nabla - \nabla_u)(p_1 X^k) + X^k f, \\ L' = \eta(\mathrm{div} - \mathrm{div}_u)U^k - \bar{\eta}_t. \end{array} \tag{4.3}$$

In $(4.2)_2$ we use U^{k-1} and X^{k-1} which are defined by

$$\begin{array}{l} U^{k-1}(\xi,t) = v(\mathcal{T}_{(k-1)T}(\xi,t),t), \\ X^{k-1}(\xi,t) = \sigma(\mathcal{T}_{(k-1)T}(\xi,t),t), \end{array} \tag{4.4}$$

where $\mathcal{T}_{t_0}$ is transformation defined by (1.4) with initial time t_0.

To apply Theorem B to (4.2) we localize the problem. We fix $y_0 \in \mathbf{R}^3$ and define a smooth function $\pi_{y_0} : \mathbf{R}_+ \to \mathbf{R}$ such that

$$\pi_{y_0}(\xi) = \begin{cases} 1 & \text{for} \quad |\xi - y_0| \leq \lambda \\ 0 & \text{for} \quad |\xi - y_0| \geq 2\lambda \end{cases}$$

and $0 \leq \pi_{y_0} \leq 1$, $|\nabla \pi_{y_0}| \leq \frac{c}{\lambda}$, $\lambda \leq 1$ is a parameter which will be specified later. And also we define sets

$$O_m = O_m(y_0) = B(y_0, 2\lambda) \times [mT, (m+1)T]. \tag{4.5}$$

Next we multiply (4.2) by $\pi = \pi_{y_0}$ (we will omit index y_0 if it causes no misunderstanding) and obtain

$$\begin{array}{l} \bar{\eta}(y_0)(\pi U^k)_t - \mu\Delta(\pi U^k) - \nu\nabla \mathrm{div}\,(\pi U^k) + p'(\bar{\eta}(y_0))\nabla(\pi X^k) = \\ K'' + \pi K' - \pi\eta\zeta_k' U^{k-1} \equiv K, \\ (\pi X^k)_t + \bar{\eta}(y_0)\mathrm{div}\,(\pi U^k) = L'' + \pi L' - \pi\zeta_k' X^{k-1} \equiv L, \\ (\pi U^k)|_{t=(k-1)T} = 0, \quad (\pi X^k)|_{t=(k-1)T} = 0, \end{array} \tag{4.6}$$

where

$$\begin{array}{c} K'' = (\bar{\eta}(y_0) - \eta)(\pi U^k)_t + \mu(\pi\Delta U^k - \Delta(\pi U^k)) + \\ \nu(\pi\nabla \mathrm{div}\, U^k - \nabla \mathrm{div}\,(\pi U^k)) + \\ p_1(\pi\nabla X^k - \nabla(\pi X^k)) + (p'(\bar{\eta}(y_0) - p_1)\nabla(\pi X^k), \\ L'' = \pi(\bar{\eta}(y_0) - \eta)\mathrm{div}\, U^k + \bar{\eta}(y_0)(\mathrm{div}\,(\pi U^k) - \pi \mathrm{div}\, U^k). \end{array} \tag{4.7}$$

By Theorem B we obtain the following estimate

$$\begin{array}{c} ||\pi U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + ||\pi X^k||_{V_r(O_{k-1}\cup O_k)} + ||\pi U^k||_{W_2^{2,1}(O_{k-1}\cup O_k)} \\ +||\pi X^k||_{V_2(O_{k-1}\cup O_k)} \leq A_0 \Big(||K||_{L_r(O_{k-1}\cup O_k)} + ||L||_{W_r^{1,0}(O_{k-1}\cup O_k)} + \\ ||K||_{L_2(O_{k-1}\cup O_k)} + ||L||_{W_2^{1,0}(O_{k-1}\cup O_k)} \Big). \end{array} \tag{4.8}$$

Here we estimate only the last terms of K and L; K', K'', L' and L'' we examine in Appendix using the interpolation theorems. For the last terms we have

$$\begin{array}{c} ||\pi\eta\zeta_k' U^{k-1}||_{L_r(O_{k-1}\cup O_k)} + ||\pi\eta\zeta_k' U^{k-1}||_{L_2(O_{k-1}\cup O_k)} \leq \\ \varepsilon ||U^{k-1}||_{W_r^{2,1}(O_{k-1})} + c(\varepsilon)||u||_{L_2(O_{k-1})}, \\ \\ ||\pi\zeta_k' X^{k-1}||_{W_2^{1,0}(O_{k-1}\cup O_k)} + ||\pi\zeta_k' X^{k-1}||_{W_r^{1,0}(O_{k-1}\cup O_k)} \leq \\ \frac{4}{T}||X^{k-1}||_{W_2^{1,0}(O_{k-1})} + \frac{4}{T}||X^{k-1}||_{W_r^{1,0}(O_{k-1})} + c(T,\lambda)||\chi||_{L_2(O_{k-1})}. \end{array} \tag{4.9}$$

Note that to estimate the term $||X^{k-1}||_{W_r^{1,0}(O_{k-1})}$ we can not apply any interpolation theorem. By by the definition of functions ζ_k we conclude

$$||\zeta' \nabla X^{k-1}||_{L_r(O_{k-1})} \leq \frac{2}{T}||\nabla X^{k-1}||_{L_r(O_{k-1})}.$$

Hence we need the estimate with the constant independent of T which is guaranteed by Theorem B. It enables to take T so large that $\frac{1}{T}$ is sufficiently small.

Putting our estimates to the (4.8) we get

$$\begin{aligned} &||\pi U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + ||\pi X^k||_{V_r(O_{k-1}\cup O_k)} + ||\pi U^k||_{W_2^{2,1}(O_{k-1}\cup O_k)} \\ &\quad + ||\pi X^k||_{V_2(O_{k-1}\cup O_k)} \le A_0\left(G_1 + G_2 + G_3 + G_4\right). \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} &G_1 = c(\lambda^\alpha + ||\chi||_{V_r(O_{k-1}\cup O_k)})||\pi U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)}, \\ &G_2 = c(\varepsilon + ||\chi||_{V_r(O_{k-1}\cup O_k)} + T^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_{k-1}\cup O_k)})(||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + \\ &||X^k||_{V_r(O_{k-1}\cup O_k)} + ||U^k||_{W_2^{2,1}(O_{k-1}\cup O_k)} + ||X^k||_{V_2(O_{k-1}\cup O_k)}), \\ &G_3 = c(\varepsilon, \lambda)(||u||_{L_2(O_{k-1}\cup O_k)} + ||\chi||_{L_2(O_{k-1}\cup O_k)}), \\ &G_4 = (\varepsilon + \tfrac{4}{T})(||U^{k-1}||_{W_r^{2,1}(O_{k-1})} + ||X^{k-1}||_{V_r(O_{k-1})} \\ &+ ||U^{k-1}||_{W_2^{2,1}(O_{k-1})} + ||X^{k-1}||_{V_2(O_{k-1})}). \end{aligned}$$

Taking small λ and using Smallness Assumption we can put terms of G_1 into l.h.s. of (4.10). Since λ has been already fixed we construct a cover of $\mathbf{R}^3$. Define a set of balls

$$B(k\lambda, l\lambda, m\lambda; 2\lambda) \quad k, l, m \in \mathbf{Z}$$

The multiplicity of the cover is less then 125. We take (4.10) with $\pi = \pi_{(k\lambda, l\lambda, m\lambda)}$. Summing over k, l, m we obtain

$$\begin{aligned} &||U^k||_{W_r^{2,1}(R_{k-1}\cup R_k)} + ||X^k||_{V_r(R_{k-1}\cup R_k)} + ||U^k||_{W_2^{2,1}(R_{k-1}\cup R_k)} + \\ &\quad ||X^k||_{V_2(R_{k-1}\cup R_k)} \le 250A_0\left(H_1 + H_2 + H_3)\right), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} &H_1 = c(\varepsilon + ||\chi||_{V_r(R_{k-1}\cup R_k)} + T^{\frac{r-1}{r}}||u||_{W_r^{2,1}(R_{k-1}\cup R_k)})(||U^k||_{W_r^{2,1}(R_{k-1}\cup R_k)} \\ &+ ||X^k||_{V_r(R_{k-1}\cup R_k)} + ||U^k||_{W_2^{2,1}(R_{k-1}\cup R_k)} + ||X^k||_{V_2(R_{k-1}\cup R_k)}), \\ &H_2 = c(\varepsilon, \lambda)(||u||_{L_2(R_{k-1}\cup R_k)} + ||\chi||_{L_2(R_{k-1}\cup R_k)}), \\ &H_3 = (\varepsilon + \tfrac{4}{T})(||U^{k-1}||_{W_r^{2,1}(R_{k-1})} + ||X^{k-1}||_{V_r(R_{k-1})} \\ &+ ||U^{k-1}||_{W_2^{2,1}(R_{k-1})} + ||X^{k-1}||_{V_2(R_{k-1})}). \end{aligned}$$

where $R_m = \mathbf{R}^3 \times [mT, (m+1)T]$.

Taking small $\varepsilon = T/4$, by Smallness Assumption we get

$$\begin{gathered}||U^k||_{W_r^{2,1}(R_k)} + ||X^k||_{V_r(R_k)} + ||U^k||_{W_2^{2,1}(R_k)} + ||X^k||_{V_2(R_k)} \le \\ Ac(\varepsilon,\lambda)(||u||_{L_2(R_{k-1}\cup R_k)} + ||\chi||_{L_2(R_{k-1}\cup R_k)}) + \tfrac{1000}{T}A_0(||X^{k-1}||_{V_r(R_{k-1})} \\ +||X^{k-1}||_{V_2(R_{k-1})} + ||U^{k-1}||_{W_r^{2,1}(R_{k-1})} + ||U^{k-1}||_{W_2^{2,1}(R_{k-1})}).\end{gathered} \tag{4.12}$$

By (1.4), considering problem (1.16) for u^{k-1} and η^{k-1} we conclude

$$||U^{k-1}||_{W_2^{2,1}(R_{k-1})} + ||X^{k-1}||_{V_2(R_{k-1})} + ||U^{k-1}||_{W_r^{2,1}(R_{k-1})} + ||X^{k-1}||_{V_r(R_{k-1})} \le 5\delta. \tag{4.13}$$

And because $5000A_0 < \frac{1}{2}T$, from (4.12) and (4.13) we get

$$\begin{gathered}||U^k||_{W_r^{2,1}(R_k)} + ||X^k||_{V_r(R_k)} + ||U^k||_{W_2^{2,1}(R_k)} + ||X^k||_{V_2(R_k)} \le \\ \tfrac{1}{2}\delta + c(\varepsilon,\lambda,T)(||u_0||_{L_2(R^3)} + ||\sigma_0||_{L_2(R^3)}).\end{gathered} \tag{4.14}$$

Thus if $||u_0||_{L_2(R^3)} + ||\sigma_0||_{L_2(R^3)}$ is small enough, from (4.14) we obtain

$$||U^k||_{W_r^{2,1}(R_k)} + ||X^k||_{V_r(R_k)} + ||U^k||_{W_2^{2,1}(R_k)} + ||X^k||_{V_2(R_k)} \le \delta \tag{4.15}$$

which gives (1.16) for u^k and η^k.

Since $r > 3$ then transformation (1.4) is well defined, thus from (4.15) we get estimate on traces

$$||v(\cdot,kT)||_{W_r^{2-\frac{2}{r}}(R^3)} + ||\sigma(\cdot,kT)||_{W_r^1(R^3)} \le A_1\delta \tag{4.16}$$

which ensure the existence on interval $[kT,(k+2)T]$. By the induction Theorem C has been proved.

5 Appendix

In this part we estimate quantities from (4.3) and (4.7). Main tools which apply here are Propositions 2.1 and 2.2. First we find estimate on $K'' = \sum_{i=1}^5 K_i$ (see $((4.7)_1)$;

$$\begin{gathered}||K_1||_{L_r(O_{k-1}\cup O_k)} = ||(\bar\eta(y_0) - \eta)(\pi U^k)_t||_{L_r(O_{k-1}\cup O_k)} \le \\ c(||\bar\eta(y_0) - \bar\eta||_{L_\infty(O_{k-1}\cup O_k)} + ||\chi||_{L_\infty(O_{k-1}\cup O_k)})||\pi U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} \le \\ c(\lambda^\alpha + ||\chi||_{V(O_{k-1}\cup O_k)})||\pi U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)}\end{gathered}$$

where we have used $\bar{\eta} \in C^\alpha$.

$$||K_2||_{L_r(O_{k-1}\cup O_k)} = ||\mu(\pi\Delta U^k - \Delta(\pi U^k))||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$c||\Delta\pi U^k||_{L_r(O_{k-1}\cup O_k)} + c||\nabla\pi \cdot \nabla U^k||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$\varepsilon||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon,\lambda)||u||_{L_2(O_{k-1}\cup O_k)}$$

The same we have for K_3 (see $(4.3)_1$)

$$||K_3||_{L_r(O_{k-1}\cup O_k)} = ||\nu\pi(\nabla_u \mathrm{div}_u - \nabla \mathrm{div}\,)U^k||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$\varepsilon||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon,\lambda)||u||_{L_2(O_{k-1}\cup O_k)},$$

$$||K_4||_{L_r(O_{k-1}\cup O_k)} = ||p_1(\pi\nabla X^k - \nabla(\pi X^k))||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$c||X^k\nabla\pi||_{L_r(O_{k-1}\cup O_k)} \leq \varepsilon||X^k||_{W_r^{1,1}(O_{k-1}\cup O_k)} + c(\varepsilon,\lambda)||X^k||_{L_2(O_{k-1}\cup O_k)},$$

$$||K_5||_{L_r(O_{k-1}\cup O_k)} = ||(p'(\bar{\eta}(y_0) - p_1)\nabla(\pi X^k)||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$c||\chi||_{V(O_{k-1}\cup O_k)}||X^k||_{V(O_{k-1}\cup O_k)},$$

Next we consider $K' = \sum_6^{10} K_i$

$$||K_6||_{L_r(O_{k-1}\cup O_k)} = ||\mu\pi(\Delta_u - \Delta)U^k||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$cT^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_{k-1}\cup O_k)}||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)},$$

the same for K_7 (see $(4.3)_2$ and $(4.7)_2$)

$$||K_7||_{L_r(O_{k-1}\cup O_k)} = ||\nu(\pi\nabla \mathrm{div}\, U^k - \nabla \mathrm{div}\,(\pi U^k))||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$cT^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_{k-1}\cup O_k)}||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)},$$

$$||K_8||_{L_r(O_{k-1}\cup O_k)} = ||\pi X^k\nabla p_1||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$\varepsilon||X^k||_{W_r^{1,1}(O_{k-1}\cup O_k)} + c(\varepsilon)||\chi||_{L_2(O_{k-1}\cup O_k)},$$

$$||K_9||_{L_r(O_{k-1}\cup O_k)} = ||\pi(\nabla - \nabla_u)(p_1X^k)||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$c^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_{k-1}\cup O_k)} + ||X^k||_{V(O_{k-1}\cup O_k)},$$

$$||K_{10}||_{L_r(O_{k-1}\cup O_k)} = ||\pi X^{k-1}f||_{L_r(O_{k-1}\cup O_k)} \leq$$
$$\varepsilon||X^{k-1}||_{V(O_{k-1}\cup O_k)} + c(\varepsilon)||X^{k-1}||_{L_2(O_{k-1}\cup O_k)}.$$

Next we examine $L'' + \pi L' = \sum_{i=1}^4 L_i$

$$||L_1||_{W_r^{1,0}(O_{k-1}\cup O_k)} = ||\pi(\bar{\eta}(y_0) - \eta)\mathrm{div}\, U^k||_{W_r^{1,0}(O_{k-1}\cup O_k)} \leq$$
$$c(\lambda^\alpha + ||\chi||_{V(O_{k-1}\cup O_k)})||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)},$$

The same like K_2 we do with L_2

$$||L_2||_{W_r^{1,0}(O_{k-1}\cup O_k)} = ||\bar{\eta}(y_0)(\mathrm{div}\,(\pi U^k) - \pi \mathrm{div}\, U^k)||_{W_r^{1,0}(O_{k-1}\cup O_k)} \leq \\ \varepsilon ||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon, \lambda)||u||_{L_2(O_{k-1}\cup O_k)}.$$

The same as for K_5 we have

$$||L_3||_{W_r^{1,0}(O_{k-1}\cup O_k)} = ||\pi\eta(\mathrm{div} - \mathrm{div}_u)U^k||_{W_r^{1,0}(O_{k-1}\cup O_k)} \leq \\ cT^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_{k-1}\cup O_k)}||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + \\ \frac{c}{\lambda}T^{\frac{r-1}{r}}||u||_{W_r^{2,1}(O_{k-1}\cup O_k)}(\varepsilon||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon)||u||_{L_2(O_{k-1}\cup O_k)}),$$

$$||L_4||_{W_r^{1,0}} = ||\zeta\pi\bar{\eta}_t||_{W_r^{1,0}} = ||\zeta\pi\nabla\bar{\eta}\cdot u||_{W_r^{1,0}} \leq \\ \varepsilon||U^k||_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon)||u||_{L_2(O_{k-1}\cup O_k)}.$$

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