Simon Blatt, Philipp Reiter, Armin Schikorra (Eds.)
New Directions in Geometric and Applied Knot Theory

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Contents

1 Introduction

Is the "cable spaghetti" on the floor really knotted
or is it enough to pull on both ends of the wire
to completely unfold it?

Since its invention, knot theory has always been a place of interdisciplinarity. The first knot tables composed by Tait have been motivated by Lord Kelvin's theory of atoms. But it was not until a century ago that tools for rigorously distinguishing the trefoil and its mirror image as members of two *different* knot classes were derived. In the first half of the twentieth century, knot theory seemed to be a place mainly driven by algebraic and combinatorial arguments, mentioning the contributions of Alexander, Reidemeister, Seifert, Schubert, and many others. Besides the development of higher dimensional knot theory, the search for new knot invariants has been a major endeavour since about 1960.

At the same time, connections to applications in DNA biology and statistical physics have surfaced. DNA biology had made a huge progress and it was well understood that the topology of DNA strands matters: modeling of the interplay between molecules and enzymes such as topoisomerases necessarily involves notions of 'knottedness'.

Any configuration involving long strands or flexible ropes with a relatively small diameter leads to a mathematical model in terms of curves. Therefore knots appear almost naturally in this context and require techniques from algebra, (differential) geometry, analysis, combinatorics, and computational mathematics.

The discovery of the Jones polynomial in 1984 has led to the great popularity of knot theory not only amongst mathematicians and stimulated many activities in this direction. Many deep connections between the Jones polynomial and quantum physics were found. In the 1990s, monographs of Adams, Kauffman, and Livingston addressed a wide audience of mathematicians, from undergraduate students to research professionals, disseminated new results and ideas and thereby added to the perception that knot theory is a vivid discipline having impact far beyond its roots.

Today we find that knots appear in almost all mathematical disciplines, having important applications in the sciences and, most importantly, that concepts from one field have impact on problems in other areas. Even more, we will see that applications do not only make use of existing theories developed in entirely theoretical frameworks, but that questions from the sciences also stimulate theoretical developments in turn.

In this edition we focus on four aspects that thematically contour its basis and background and these will be outlined below. Of course, this choice is not meant to be exhaustive, e.g., we do not cover recent developments in the theory of low-dimensional

Paweł Strzelecki and Heiko von der Mosel

Geometric curvature energies: facts, trends, and open problems

Abstract: This survey focuses on geometric curvature functionals, that is, geometrically defined self-avoidance energies for curves, surfaces, or more general k-dimensional sets in $\mathbb{R}^d$. Previous investigations of the authors and collaborators concentrated on the regularising effects of such energies, with a priori estimates in the regime above scale-invariance that allowed for compactness and variational applications for knotted curves and surfaces under topological restrictions. We briefly describe the impact of geometric curvature energies on geometric knot theory. Currently, various attempts are being made to obtain a deeper understanding of the energy landscape of these highly singular and nonlinear nonlocal interaction energies. Moreover, a regularity theory for critical points is being developed in the setting of fractional Sobolev spaces. We describe some of these current trends and present a list of open problems.

Keywords: geometric curvature energies, singular integrals, critical points, regularity theory, geometric knot theory, elastic knots, rectifiability

2.1 Facts

Energies. Geometric curvature functionals are characterised as geometrically defined energies on a priorily non-smooth k-dimensional subsets Σ^k of $\mathbb{R}^d$, and these functionals are designed to penalise self-intersections. In addition, there are regularising effects: finite energy implies some higher degree of smoothness of Σ. One of the first examples is that of the *Möbius energy* on rectifiable curves $\gamma \subset \mathbb{R}^d$, introduced by J. O'Hara [82] and investigated analytically by M. H. Freedman et al. [41, 50],

$$\mathscr{E}_{\text{Möb}}(\gamma) := \int_\gamma \int_\gamma \left[\frac{1}{|x-y|^2} - \frac{1}{d_\gamma(x,y)^2} \right] d\mathscr{H}^1(x) d\mathscr{H}^1(y), \tag{2.1.1}$$

where $d_\gamma(x,y)$ denotes the intrinsic distance between the points x, y on the curve γ, and $\mathscr{H}^1$ stands for the one-dimensional Hausdorff-measure. The first summand in the integrand resembles a Coulomb-type *repulsive potential* suitably regularised by the second term so as to obtain finite energy for smooth embedded curves. Another exam-

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ple is that of *ropelength*[2.1] defined as the quotient of length $\mathscr{L}(\gamma)$ and thickness $\triangle[\gamma]$ of curves γ, where the latter is a non-smooth functional introduced by O. Gonzalez and J. H. Maddocks [48], defined as

$$\triangle[\gamma] := \inf_{\substack{x,y,z\in\gamma \\ x\neq y\neq z\neq x}} R(x,y,z). \tag{2.1.2}$$

Here, $R(x,y,z)$ denotes the circumcircle radius of the three curve points x, y, and z. Thickness can be regarded as a hard core potential or steric constraint, in contrast to repulsive potentials.

Both functionals have had their impact on the modeling of macromolecules such as DNA and proteins [25, 76, 77], and on *geometric knot theory*, where one studies relations between the geometry of space curves and the knot types they represent. Several geometric curvature energies can be minimised within given knot classes to obtain particularly nice representatives of that knot, for example *ideal knots* as ropelength minimisers [29, 49]. Bounds on the energy sometimes imply bounds on knot invariants like stick number or crossing number, hence bounds on the number of knot classes that possess representatives below these energy thresholds; see, e.g., [24, 73].

However, both these extreme forms of energies have serious drawbacks. The highly singular integrals involved in the definition of any kind of repulsive potential like (2.1.1) need some sort of regularisation, and – besides the ambiguity in the choice of such a regularisation – it is by no means clear how to generalise this concept to higher dimensional objects. The steric constraint of given thickness (2.1.2) or the ropelength functional, on the other hand, is a non-smooth quantity imposing challenging technical problems, e.g., for the derivation and analysis of variational equations. This led to our systematic research between 2007 and 2012, devoted to a whole range of intermediate energies on curves and surfaces interpolating in some sense between hard steric constraints and "soft" repulsive potentials. Examples of such energies on one-dimensional sets include [119]

$$\mathscr{U}_p(\gamma) := \int_\gamma \sup_{\substack{y,z\in\gamma \\ z\neq y\neq x\neq z}} \frac{1}{R^p(x,y,z)}\, d\mathscr{H}^1(x), \qquad p \geq 1, \tag{2.1.3}$$

or the double integral [114]

$$\mathscr{I}_p(\gamma) := \int_\gamma \int_\gamma \sup_{\substack{z\in\gamma \\ z\neq x\neq y\neq z}} \frac{1}{R^p(x,y,z)}\, d\mathscr{H}^1(x) d\mathscr{H}^1(y), \qquad p \geq 2, \tag{2.1.4}$$

2.1 The name of that functional is coined after the following geometric variational problem: given a rope of fixed constant thickness, what is the minimum length of this rope required to tie a given knot?

and also *integral Menger*[2.2] *curvature* [115]

$$\mathscr{M}_p(\gamma) := \int_\gamma \int_\gamma \int_\gamma \frac{1}{R^p(x,y,z)}\, d\mathscr{H}^1(x) d\mathscr{H}^1(y) d\mathscr{H}^1(z), \quad p \geq 3. \tag{2.1.5}$$

On a fixed loop γ of unit length, these energies are ordered as

$$\mathscr{M}_p^{1/p}(\gamma) \leq \mathscr{I}_p^{1/p}(\gamma) \leq \mathscr{U}_p^{1/p}(\gamma) \leq \frac{1}{\triangle[\gamma]}, \tag{2.1.6}$$

where the last term is the ropelength of γ. Moreover, the p-th root of $\mathscr{M}_p$, $\mathscr{I}_p$, and of $\mathscr{U}_p$ tends to ropelength as $p \to \infty$ both on fixed conformations of knots and in the sense of Γ-convergence.

Besides averaging and maximising over the *multi-point interactions* in the circumradius we investigated *tangent-point interactions* such as [121, 57]

$$\mathscr{E}_p(\gamma) := \int_\gamma \int_\gamma \frac{1}{r_{\mathrm{tp}}(x,y)^p}\, d\mathscr{H}^1(x) d\mathscr{H}^1(y), \quad p \geq 2, \tag{2.1.7}$$

as well, where $r_{\mathrm{tp}}(x,y)$ is the radius of the unique circle through two given curve points x and y that is additionally tangent to γ in x.

We also introduced and studied geometric curvature energies on higher-dimensional sets such as *thickness for surfaces* $\Sigma \subset \mathbb{R}^d$ [118, 117], where one minimises over all pairs of points $x, y \in \Sigma$ the *tangent-point radius* $R_{\mathrm{tp}}(x,y)$ of the smallest sphere through x and y that is tangent to Σ in x. Later, we investigated *integral Menger curvature for surfaces* $\Sigma \subset \mathbb{R}^3$ [120],

$$\mathscr{M}_p(\Sigma) := \int_\Sigma \int_\Sigma \int_\Sigma \int_\Sigma \mathcal{K}^p(x,y,z,\xi)\, d\mathscr{H}^2(x) d\mathscr{H}^2(y) d\mathscr{H}^2(z) d\mathscr{H}^2(\xi), \quad p \geq 8, \tag{2.1.8}$$

where the integrand $\mathcal{K}$ is defined[2.3] on tetrahedra $T = (x,y,z,\xi)$ with vertices $x, y, z, \xi \in \Sigma$, as

$$\mathcal{K}(T) = \mathcal{K}(x,y,z,\xi) := \frac{\mathscr{H}^3(T)}{\mathrm{area}\,(T)\mathrm{diam}^2(T)}. \tag{2.1.9}$$

2.2 Karl Menger considered in the 1930's the circumradius $R(x,y,z)$ of three curve points $x, y, z \in \gamma$ knowing that the coalescent limit of $R(x,y,z)$ as x and y tend to z coincides with the local radius of curvature if the curve γ is sufficiently smooth. Menger was also aware of the fact that there is an elementary formula for the circumradius solely in terms of the mutual distances of the points x, y, and z. By means of *multipoint functions* such as $R(\cdot,\cdot,\cdot)$ Menger indeed intended to develop a purely metric geometry in contrast to classic differential geometry. The idea of using Menger curvature as a tool – both in harmonic analysis and in modeling – has been re-discovered in the last 20 years.

2.3 The most obvious choice to take as integrand in (2.1.8) a negative power of the *circumsphere radius* of a tetrahedron does *not* serve our purposes since there are smooth embedded surfaces for which such an integrand would not be bounded; see our detailed discussion on various integrands in [120, Appendix B].

Here $\mathscr{H}^3(T)$ is the volume of the tetrahedron T and area (T) the sum of its facet areas. For k-dimensional submanifolds $\Sigma^k \subset \mathbb{R}^d$ we looked at the *tangent-point energy* [123]

$$\mathscr{E}_p(\Sigma) := \int_\Sigma \int_\Sigma \frac{1}{R_{\mathrm{tp}}^p(x,y)}\, d\mathscr{H}^k(x) d\mathscr{H}^k(y), \tag{2.1.10}$$

and S. Kolasiński investigated in his Ph.D. thesis *integral Menger curvature for* $\Sigma^k \subset \mathbb{R}^d$ [61, 62]

$$\mathscr{M}_p(\Sigma) := \underbrace{\int_\Sigma \cdots \int_\Sigma}_{k+2 \text{ integrals}} \mathcal{K}^p(x_0, \dots, x_{k+1})\, d\mathscr{H}^k(x_0) \cdots d\mathscr{H}^k(x_{k+1}) \tag{2.1.11}$$

for $p > k(k+2)$, where the integrand generalises (and simplifies) the one given in (2.1.9) to

$$\mathcal{K}(T) = \mathcal{K}(x_0, \dots, x_{k+1}) := \frac{\mathscr{H}^{k+1}(T)}{(\operatorname{diam}(T))^{k+2}}, \tag{2.1.12}$$

for $(k+1)$-dimensional simplices $T = (x_0, \dots, x_{k+1})$ with each vertex x_i on Σ.

Regularising effects. Summarising the essential results of this systematic research (which is well documented in a number of publications [119, 114, 115, 121, 117, 118, 120, 123, 61, 62], we can say the following: we have a pretty clear understanding of the topological and regularising effects of each of these energies, with sharp regularity statements and uniform a priori estimates. For example, a rectifiable curve γ with finite integral Menger curvature $\mathscr{M}_p(\gamma)$ for some $p > 3$ (i.e., above the scale-invariant case $p = 3$) is homeomorphic to the unit-circle or unit-interval, and the arclength parametrisation of that curve satisfies the uniform a priori estimate

$$|\gamma'(s) - \gamma'(t)| \lesssim \mathscr{M}_p(\gamma[s,t])^{1/p} |s-t|^{1-3/p} \qquad \text{for all } s, t. \tag{2.1.13}$$

In other words, $\gamma \in C^{1,1-(3/p)}$, so although $\mathscr{M}_p$ does not capture the pointwise value of local curvature (which may be simply undefined even if $\mathscr{M}_p(\gamma)$ is finite), it does capture the average oscillation of the unit tangent vector; see [115, Theorem 1.2]. We may interpret this result as a *geometric Morrey-Sobolev embedding*: the integrand corresponds to a very weak form of curvature integrated to some power $p > 3$, and the total domain of integration is three-dimensional; the classic Morrey-Sobolev theorem applied to second derivatives (instead of curvature) would give exactly the optimal Hölder exponent $1 - (3/p)$ for the first derivatives.

Likewise for higher-dimensional subsets of $\mathbb{R}^d$, exemplified by integral Menger curvature $\mathscr{M}_p(\Sigma)$ for two-dimensional surfaces in Euclidean 3-space, as defined in (2.1.8); see [120, Theorem 1.4]: if an admissible two-dimensional set $\Sigma \subset \mathbb{R}^3$ satisfies $\mathscr{M}_p(\Sigma) < \infty$ for some $p > 8$ (again above the scale-invariant case $p = 8$), then Σ is actually an orientable $C^{1,1-(8/p)}$-submanifold with a controlled local graph representation: There is a uniform radius $R > 0$ depending only on p and the energy

value $\mathscr{M}_p(\Sigma)$, such that for each $x \in \Sigma$ the intersection $B_R(x) \cap \Sigma$ equals the graph of a $C^{1,1-(8/p)}$-function with uniform estimates on this function solely depending on p and $\mathscr{M}_p(\Sigma)$. Again, this is a geometric variant of the Morrey-Sobolev embedding theorem with optimal Hölder exponent $1-(8/p)$ for the oscillation of tangent planes. Similar results hold for k-dimensional admissible sets $\Sigma^k \subset \mathbb{R}^d$ with finite integral Menger curvature $\mathscr{M}_p(\Sigma)$ for $p > k(k+2)$, or finite tangent point energy $\mathscr{E}_p(\Sigma)$ for $p > 2k$, both in the regime above scale-invariance; see [61, 62, 123]. A thorough discussion of the respective admissibility class of sets can be found, e.g., in [64]. At this point, we may roughly describe our mild requirements on the set Σ^k as a certain degree of local flatness around many (but not all) points, together with an amount of connectivity to allow for some degree-theoretic arguments.

That these regularity estimates are indeed sharp, can be seen either by explicit examples constructed in [66, 57], or by the complete *characterisation of energy spaces* for all these energies in the work of S. Blatt and Kolasiński [17, 13, 11]: based on our results that finite energy implies that the admissible sets are already C^1-submanifolds in $\mathbb{R}^d$, they use the explicit structure of the energies to estimate locally the seminorms of fractional Sobolev spaces to find that Σ has finite energy *if and only if* Σ is embedded and has local graph representations of exactly that Sobolev regularity. Recall, e.g., from [128, Section 2.2.2], that a function $u \in L^p(\mathbb{R}^k)$ belongs to the *Sobolev-Slobodeckiĭ space* $W^{m+s,p}(\mathbb{R}^k)$ for some $m \in \mathbb{N}$, $s \in (0,1)$, and $p \in [1,\infty)$ if u belongs to the classic Sobolev space $W^{m,p}$ and satisfies, in addition,

$$\|u\|^p_{m+s,p} := \|u\|^p_{W^{m,p}(\mathbb{R}^k)} + \sum_{|\alpha|=m} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{k+sp}}\, dy dx < \infty. \tag{2.1.14}$$

As an example, let us mention Blatt's and Kolasiński's characterisation of the energy space for integral Menger curvature as defined in (2.1.11); see [17, Corollary 1.2]: If $p > k(k+2)$ and $\Sigma^k \subset \mathbb{R}^d$ is an admissible set, then its integral Menger curvature $\mathscr{M}_p(\Sigma)$ is finite if and only if Σ is a submanifold with local graph representation of class $W^{1+s,p}(\mathbb{R}^k, \mathbb{R}^{d-k})$, where $s = 1-(k(k+1)/p) \in (0,1)$. Blatt and Kolasiński treated also all intermediate energies where up to all but two integrations in (2.1.11) are replaced by maximisations, obtaining corresponding fractional Sobolev spaces as the exact energy spaces with suitably adapted differentiability and integrability. The missing cases where only one integration is left, i.e.,

$$\int_\Sigma \sup_{x_1,\dots,x_{k+1}\in\Sigma} \mathcal{K}^p(x_0, x_1, \dots, x_{k+1})\, d\mathscr{H}^k(x_0), \tag{2.1.15}$$

generalising (2.1.3) to k-dimensional sets $\Sigma^k \subset \mathbb{R}^d$, and the *global tangent-point energy* (cf. (2.1.10)),

$$\int_\Sigma \sup_{y\in\Sigma} \frac{1}{R^p_{\mathrm{tp}}(x,y)}\, d\mathscr{H}^k(x), \tag{2.1.16}$$

were treated in cooperation with Kolasiński in [64] leading to the theorem that finite energy *characterises* embedded submanifolds of classic Sobolev regularity $W^{2,p}$ if $p > k$; see [64, Theorem 1.4]. This result may be compared to Allard's famous $C^{1,\alpha}$-regularity theorem [3, 37] for k-dimensional varifolds whose generalised mean curvature is p-integrable, where, again, $p > k$.

Connections to geometric knot theory. The uniform estimates obtained in [119, 114, 115] for finite energy curves γ (like the one in (2.1.13)) together with a uniform geometric rigidity of these curves (replacing the excluded volume constraint of thickness) was used to connect the respective energies to *geometric knot theory*, as described in detail in [116, Section 4]; see also the recent surveys [122, 124]. This geometric rigidity[2.4] means, roughly speaking, that the curve may be equipped with a necklace of consecutive double-cones whose size and opening angle are determined purely in terms of the respective energy [116, Proposition 4.7]. The circular cross-sections of each piece of this necklace, i.e., of each such double cone (with its two tips located on the curve γ), are intersected by γ transversally and exactly in one point; see Figure 2.1. Once this necklace is established one can fairly easily construct an ambient isotopy from γ to the inscribed polygon made of the consecutive double cones' axes.

Thus, any one of the geometric curvature energies for curves (2.1.3)–(2.1.5), (2.1.7), bounds the *stick number*, which is the minimal number of straight segments you need to build a polygonal representative of the same knot type. Since stick number is a knot invariant, any such energy bounds the number of knot types: given any constant $E \geq 0$ there is a nonnegative integer $N(E)$ depending only on E, such that at most $N(E)$ knot types can be represented by curves of geometric curvature energy below the energy threshold E.

On the other hand, the double-cone property described above also serves as a substitute of the excluded volume constraint given by finite ropelength. This allows us to control the *average crossing number* $\operatorname{acn}(\gamma)$, where you count the number of self-intersections of every planar projection of the given curve γ and then average over all directions of projections. Indeed, Freedman et al. derived in [41, Section 3] a double integral formula for $\operatorname{acn}(\gamma)$,

$$\operatorname{acn}(\gamma) := \frac{1}{4\pi} \int_0^1 \int_0^1 \frac{|(\gamma'(s) \times \gamma'(t)) \cdot (\gamma(t) - \gamma(s))|}{|\gamma(t) - \gamma(s)|^3} \, ds \, dt, \tag{2.1.17}$$

where $\times$ denotes the usual cross-product in $\mathbb{R}^3$. While the local interaction terms in that formula may be estimated by the local smoothness properties of a finite energy curve γ, one can follow the strategy of G. Buck and J. Simon [24] for curves of finite ropelength, to estimate the global interaction terms by estimating the volume of a spatial region necessary to fit in a maximally compactified curve γ. Only here, one

2.4 Referred to as *diamond property* in [116].

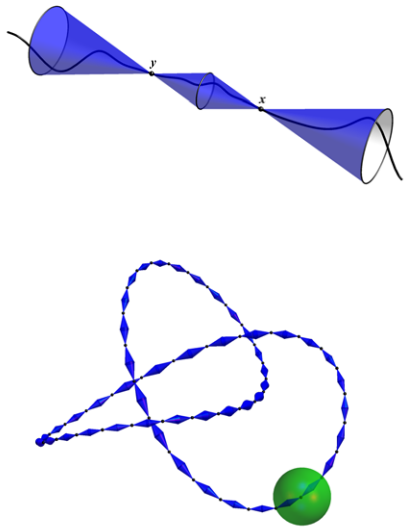


Fig. 2.1: Top: The curve γ is trapped in a conical region with two tips at $x, y \in \gamma$ and does not meander back and forth: each cross section of the double-cones contain exactly one point of the curve. Bottom: Necklace made of such small double cones with vertices along the curve γ that have pairwise disjoint interiors. The polygonal curve joining the consecutive vertices of the cones is ambient isotopic to γ.

has to replace the excluded volume constraint by the double-cone condition, so that our constants are far from being optimal; see [116, Proposition 4.13]. Since the average crossing number bounds the classic knot invariant *crossing number*, we thus have established another means to control the number of knot types below given energy thresholds.

In addition, we could show that all these energies are *charge* and *tight*, which means that they blow up along sequences that converge to curves with self-intersections and also along sequences where one small knotted subarc pulls tight, i.e., vanishes in the limit. Being tight distinguishes these geometric curvature energies from the Möbius energy (2.1.1): O'Hara showed in [83, Theorem 3.1] that the Möbius energy does *not* prevent the pull-tight phenomenon. Moreover, $\mathscr{U}_p$ and $\mathscr{I}_p$ could be shown to distinguish the knot from the unknot: there is a gap between the infimum over unknots and the infimum over non-trivially knotted curves. The infima of all these energies (in contrast, e.g., to the Möbius[2.5] energy) are attained on each given knot

2.5 The Möbius energy can, however, be minimised within prescribed *prime* (or *irreducible*) knot classes; see [41, Theorem 4.3].

class. Freedman et al. [41] showed that the Möbius energy $\mathscr{E}_{\text{Möb}}$ is uniquely minimised by the round circle, a knot energy with that property is called *basic*. A. Abrams et al. [1] extended that uniqueness result to a larger family of energies. Also ropelength is basic, and more generally, $\mathscr{U}_p$ as well; see [119, Lemma 7]. A more recent monotonicity formula[2.6] for compact free boundary surfaces by A. Volkmann [129, Section 5] implies the same for the tangent-point energy $\mathscr{E}_p$ (see (2.1.7)), and hence also $\mathscr{I}_p$ by the arguments in [116, Proof of Cor. 3.7]. All these results resolve some of the open problems formulated in geometric knot theory, e.g. in [125, Section 2], or in [86, Chapter 8], and give some first insights into the presumably complicated energy landscape of these energies on knot space. Almost nothing is known about the actual shape of knotted energy minimisers, apart from the explicit continuous family of ideal links (minimising ropelength in fixed link classes) presented by J. Cantarella, R. B. Kusner, and J. M. Sullivan in [29], and necessary criticality conditions for ropelength minimisers [110, 26, 27]. Moreover, studying ideal knots in $\mathbb{R}^4$ lead to the discovery of unique explicit solution families of longest (thick) ropes on the two-sphere by H. Gerlach and the second author; see [45] and the popular account in [44].

2.2 Trends and open problems

Regularity. Higher regularity of local minimisers or critical points is only known in a few cases. Freedman et al. [41] used the Möbius-invariance of the Möbius energy $\mathscr{E}_{\text{Möb}}$ defined in (2.1.1) to apply reflection arguments to show that local minimisers are of class $C^{1,1}$, and they derived the Euler-Lagrange equation

$$\begin{aligned}\delta\mathscr{E}_{\text{Möb}}(\gamma, h) &:= \lim_{\tau\to 0} \frac{\mathscr{E}_{\text{Möb}}(\gamma+\tau h) - \mathscr{E}_{\text{Möb}}(\gamma)}{\tau} \\ &= 2\lim_{\varepsilon\searrow 0} \iint\limits_{|u-v|\geq\varepsilon} \left(\frac{\gamma'(u)\cdot h'(u)}{|\gamma'(u)|^2} - \frac{(\gamma(u)-\gamma(v))\cdot(h(u)-h(v))}{|\gamma(u)-\gamma(v)|^2} \right) \\ &\quad \times \frac{|\gamma'(v)||\gamma'(u)|}{|\gamma(u)-\gamma(v)|^2}\, du\, dv\end{aligned} \tag{2.2.1}$$

for injective and regular curves γ and perturbations h both of class $C^{1,1}$. Later Zh.-X. He [50] used this Euler-Lagrange equation to improve the regularity of local $\mathscr{E}_{\text{Möb}}$-minimisers to C^∞-smoothness; see also [98]. Quite recently, there has been considerable progress through the work of Blatt, P. Reiter, and A. Schikorra, who established the following deep regularity result [22, Theorem I] in the correct fractional Sobolev space $W^{3/2,2}$ corresponding – according to Blatt's earlier work [11] – to finite Möbius energy:

2.6 Or alternatively, a secant map approach of Blatt reminiscent of an argument to prove the classic Fenchel inequality; see [129, Section 5].

Theorem 2.2.1 ($\mathscr{E}_{\text{Möb}}$-critical points are smooth)**.** *Any arclength parametrised critical point $\gamma \in W^{3/2,2}$ of the Möbius energy is C^∞-smooth.*

It is remarkable that no use at all is made of the Möbius-invariance of $\mathscr{E}_{\text{Möb}}$ to prove Theorem 2.2.1 in contrast to the previous work of Freedman et al. Here, one uses the Euler-Lagrange equation (suitably extended to the correct fractional Sobolev spaces) to first gain some additional regularity, i.e., a slightly higher integrability of the tangent [22, Theorem III], before a bootstrapping process can be started. One should point out that, similarly to other geometric equations like the variational equation for the Willmore functional, the Euler-Lagrange equation for $\mathscr{E}_{\text{Möb}}$ is in a sense critical, which requires some very intricate techniques that were developed in the context of fractional harmonic mappings [34, 33, 102, 101].

Somewhat less involved is the regularity proof for other members of O'Hara's families of repulsive potentials [82, 84, 85], namely for the energies $\mathscr{E}^\alpha$, where a power $\alpha \in (2,3)$ replaces the quadratic power in the denominators of (2.1.1). Blatt and Reiter established the Fréchet-differentiability of $\mathscr{E}^\alpha$ on the space of regular curves of finite energy, and proved C^∞-smoothness for arclength parametrised critical points [19, Theorems 1.1 & 1.2].

To carry over this regularity program to critical points of geometric curvature energies such as the tangent-point energy (2.1.7) or integral Menger curvature (2.1.5), Blatt and Reiter embedded those energies into larger two-parameter families of energies by decoupling the integrability exponent p into different powers for numerator and denominator of the integrands. In this way they obtain, for instance, *modified tangent-point energies*

$$\mathscr{TP}^{(p,q)} := \int_\gamma \int_\gamma \frac{1}{r^{(p,q)}(x,y)}\, d\mathscr{H}^1(x) d\mathscr{H}^1(y), \tag{2.2.2}$$

by replacing the p-th power of the *inverse tangent-point radius*

$$\frac{1}{r_{\text{tp}}(x,y)} = \frac{2\text{dist}\,(T_x, y)}{|x-y|^2} \text{ by the less geometric expression } \frac{1}{r^{(p,q)}(x,y)} := \frac{\text{dist}\,(T_x,y)^q}{|x-y|^p}.$$

(Here, T_x denotes the tangent-line to γ through the point $x \in \gamma$.) In the parameter regime $q > 1$ and $p \in (q+2, 2q+1)$ the modified tangent-point energies turn out to be well-behaved knot energies that are minimisable in every knot class. The fractional Sobolev regularity $W^{(p-1)/q,q}$ characterises finite energy (see[20, Theorems 1.1 & 1.3]), and allows a first variation formula even without Cauchy principal values [20, Theorem 1.4] in contrast to the variational equations of O'Hara's repulsive energies. Blatt and Reiter then identify a *non-degenerate* parameter range $q = 2$, $p \in (4,5)$ that permits a regularity result[20, Theorem 1.5]. Unfortunately this range excludes the original tangent-point energy $\mathscr{E}_p = \mathscr{TP}^{2p,p}$ (cf. (2.1.7)).

Theorem 2.2.2 ($\mathscr{TP}^{(p,2)}$-critical points are smooth)**.** *For $p \in (4,5)$ any $\mathscr{TP}^{(p,2)}$-critical arclength parametrised injective curve of class $W^{(p-1)/2,2}$ is C^∞-smooth.*

A similar result holds also for a subfamily of *modified integral Menger energies*; see [21], but apart from that non-degenerate parameter regime, in particular for (the original) integral Menger curvature (2.1.5) and tangent point energies (2.1.7) the regularity of critical points, even that of local minimisers remains open.

Open questions 2.2.3. *Despite degeneracies in the non-local Euler-Lagrange equations, is there any chance to prove additional regularity (beyond the fractional Sobolev regularity characterising energy spaces) for local minimisers or critical points of tangent-point energies* (2.1.7)*, or of integral Menger curvature* (2.1.5)*? Does the optimal regularity depend on p, and if yes, what happens with that p-dependent regularity in the limit $p \to \infty$? Does that lead to new insights into the still open optimal regularity of ideal knots?*

We know that the geometric curvature energies in higher dimensions such as integral Menger curvature (2.1.8), (2.1.11), or the tangent-point energy (2.1.10) can be minimised within given isotopy classes of submanifolds [117, Theorem 7.1], [120, Theorem 1.7], [65, Corollary 1], but nothing is known about higher regularity of these minimisers, not to speak of a regularity statement about possible critical points. Not even a variational equation has been derived so far in higher dimensions. In case of the non-smooth energies (2.1.15), (2.1.16), and also for the one-dimensional prototype (2.1.3), non-smooth analysis tools such as Clarke gradients would have to be applied to derive the variational differential inclusion, similar to the analysis performed for the ropelength functional for curves involving the non-smooth expression (2.1.2) for thickness; see [109, 110, 26, 27].

Open questions 2.2.4. *What are the Euler-Lagrange-equations for higher-dimensional geometric curvature energies like integral Menger curvature* (2.1.5) *or tangent-point energies* (2.1.10)*? Is there any chance to prove higher regularity of local minimisers or critical points of these energies? What form do the expected variational inclusions have for the non-smooth geometric energies* (2.1.3), (2.1.15), (2.1.16)*? What can be said about the regularity of thick knotted surfaces minimising area?*

Below or in the scale-invariant regime. Integral Menger curvature $\mathscr{M}_p$ on one-dimensional sets $E \subset \mathbb{C}$ with integrability exponent $p = 2$ (well below the scale-invariant exponent $p = 3$) has played a fundamental rôle in harmonic analysis, e.g., in the proof of the famous Vitushkin conjecture on the removability of compact subsets of the complex plane for complex analytic functions; see, for instance, X. Tolsa's quite recent excellent monograph [126]. Motivated by some of G. David's methods [35] for his final proof of this conjecture, J.-C. Léger [70] proved the following remarkable rectifiability result:

Theorem 2.2.5 (Rectifiability for sets of finite integral Menger curvature $\mathscr{M}_2$). *Any Borel set $E \subset \mathbb{R}^n$ with $0 < \mathscr{H}^1(E) < \infty$ satisfying*

$$\mathscr{M}_2(E) = \int_E \int_E \int_E \frac{1}{R^2(x,y,z)}\, d\mathscr{H}^1(x) d\mathscr{H}^1(y) d\mathscr{H}^1(z) < \infty \tag{2.2.3}$$

is 1-rectifiable, i.e., there exists a countable family of Lipschitz functions $f_i : \mathbb{R} \to \mathbb{R}^n$ such that $\mathscr{H}^1(E \setminus \bigcup_i f_i(\mathbb{R})) = 0$.

So, even below scale-invariance, integral Menger curvature has regularising effects on sets. In our context of geometric curvature energies one is naturally lead to the question if one can generalise Leger's deep result to sets of higher dimensions? What are suitable generalisations of the integrand in (2.2.3) which is defined on point triples forming two-dimensional simplices. Already for our generalisations to surfaces and submanifolds as given in (2.1.9) and (2.1.12) we had discussed several variants of integrands defined on general $(k+1)$-dimensional simplices; see, e.g., the introduction and appendix of [120]. Recently, M. Meurer [75] presented a collection of integrands for which Léger's result could indeed be extended to arbitrary dimensions and codimensions, including, e.g., the integrand, defined on $(k+1)$-dimensional simplices $T = (x_0, x_1, \dots, x_{k+1})$; see [75, Section 3.2],

$$\mathcal{K}_M(T) = \mathcal{K}_M(x_0, \dots, x_{k+1}) := \frac{\mathscr{H}^{k+1}(T)}{\operatorname{diam}(T)^{(k+1)(k+2)/2}}, \tag{2.2.4}$$

which is one out of several possible generalisations of Léger's integrand $1/R(x,y,z)$ in (2.2.3). Meurer could prove the following rectifiability theorem [75, Theorem 1.1].

Theorem 2.2.6 (Rectifiability in arbitrary dimensions). *Any Borel set $E \subset \mathbb{R}^n$ with $0 < \mathscr{H}^k(E) < \infty$ satisfying*

$$\mathscr{M}_2(E) := \underbrace{\int_E \cdots \int_E}_{k+2 \text{ integrals}} \mathcal{K}_M^2(x_0, \dots, x_{k+1})\, d\mathscr{H}^k(x_0) \cdots d\mathscr{H}^k(x_{k+1}) < \infty \tag{2.2.5}$$

is k-rectifiable, i.e., can be covered (up to sets of $\mathscr{H}^k$-measure zero) by a countable union of Lipschitz images of $\mathbb{R}^k$.

Meurer's class of admissible integrands includes also the discrete curvatures used by G. Lerman and J. T. Whitehouse in [71, 72] to give a characterisation of David's and S. Semmes' concept of uniform rectifiability; cf [36, Theorem 1.57]. J. Azzam and Tolsa [7] recently established a new rectifiability criterion in terms of P. Jones's β-numbers [56] which are fundamentally related to integral Menger curvature as shown in [70, 75]. Interestingly, however, and somewhat surprising is the fact, that the integrands (2.1.9) and (2.1.12) we studied in the integrability regime above scale-invariance, are

not included in Meurer's class of integrands; they do scale differently. At this point it remains open, if one can replace $\mathcal{K}_M$ in (2.2.5) by the expression $\mathcal{K}$ defined in (2.1.12).

In the definition of rectifiability one covers the set (up to a set of measure zero) by Lipschitz images, and one might think about improving the regularity of the covering images. The step from Lipschitz to C^1-images is immediate by Whitney's extension theorem; see, e.g. [112, Section 3, Lemma 11.1], but improving that to $C^{1,\alpha}$ (as in the regime above scale-invariance) is highly non-trivial. This was recently accomplished by Kolasiński [63] also for a large class of discrete curvatures with a certain overlap with Meurer's class including (2.2.4), so that, e.g., the following *higher order rectifiability result* holds true and can be deduced from [63, Theorem 1.1].

Theorem 2.2.7 ($C^{1,\alpha}$-rectifiability)**.** *Any Borel set $E \subset \mathbb{R}^n$ with $0 < \mathscr{H}^k(E) < \infty$ and a.e. positive lower density, satisfying*

$$\mathscr{M}_p(E) := \underbrace{\int_E \cdots \int_E}_{k+2 \text{ integrals}} \mathcal{K}_M^p(x_0, \dots, x_{k+1})\, d\mathscr{H}^k(x_0) \cdots d\mathscr{H}^k(x_{k+1}) < \infty \tag{2.2.6}$$

for some $p > 2$ is k-rectifiable of class $C^{1,\alpha}$ for some positive Hölder exponent $\alpha = \alpha(p)$, i.e., the set E can be covered (up to sets of $\mathscr{H}^k$-measure zero) by a countable union of k-dimensional $C^{1,\alpha}$-submanifolds of $\mathbb{R}^n$.

Open questions 2.2.8. *Can one extend Meurer's rectifiability result to the integrands* (2.1.9) *or* (2.1.12) *of integral Menger curvature or to the tangent-point energies* (2.1.10) *defined on a suitable wide class of non-smooth sets? How does Meurer's result relate to other recent rectifiability results like [127, 7]?*

Not much is known about geometric curvature energies in the scale-invariant regime, but simple scaling arguments reveal the fact that cone-type singularities do lead to infinite geometric curvature energies; see Figure 2.2. S. Scholtes could indeed demonstrate that embedded polygons have finite integral Menger curvature $\mathscr{M}_p$ if and only if $p \in (0, 3)$; see [103]. Recall that $p = 3$ is the scale-invariant exponent for integral Menger curvature for curves. In addition, Scholtes established certain weak tangential properties of arbitrary (a priori fairly wild) sets at *every* point if the one-dimensional set $E \subset \mathbb{R}^n$ has finite integral Menger curvature $\mathscr{M}_3(E)$ [104].

So, one can indeed hope for mild regularising effects, like for the energy $\mathscr{U}_p$ for curves for $p = 1$, where we proved in [119, Theorem 1] that finite $\mathscr{U}_1$-energy implies that the curve is embedded and in the Sobolev class $W^{2,1}$. However, not every embedded $W^{2,1}$-curve has finite U_1-energy; see [119, Example pp. 120–121]. Finiteness of the tangent-point energy $\mathscr{E}_p$ in the scale-invariant case $p = 2$ (see definition (2.1.7)) yields at least a topological one-dimensional manifold – possibly with boundary; see [121, Theorem 1.1]. Only for the Möbius energy (2.1.1), whose Möbius-invariance implies scale-invariance, one has Blatt's [11] characterisation of the appropriate energy spaces

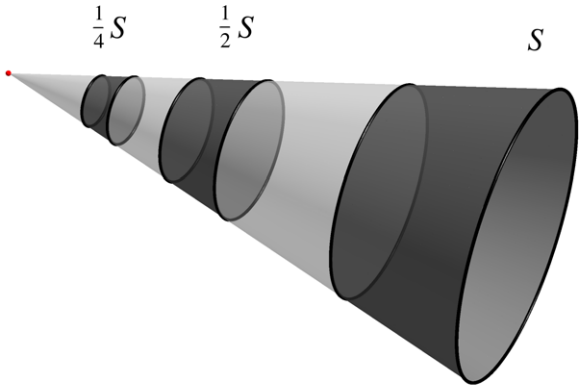


Fig. 2.2: A cone has infinite scale-invariant geometric curvature $\mathscr{E}$, since scaling of a fixed portion S of the cone leads to the same quantum of energy $\mathscr{E}(S) = \mathscr{E}(S/2) = \mathscr{E}(S/4) = \ldots$. Adding up these infinitely many contributions $\mathscr{E}(S/2^i)$, $i \in \mathbb{N}$ leads to a divergent series of positive real numbers as a lower bound for the cone's energy if $(S/2^i) \cap (S/2^j) = \emptyset$ for $i \neq j$.

as the fractional Sobolev space $W^{3/2,2}$ (assuming injective arclength parametrised curves), and already earlier Blatt and Reiter used an idea of He to construct a closed bi-Lipschitz curve with finite Möbius energy that is not differentiable [18, Corollary 4.2]. But very recently, Blatt has established a nice approximation result on convolutions of curves whose tangents have vanishing mean oscillations which in particular implies that arclength parametrised curves of finite Möbius energy can be approximated in the $W^{3/2,2}$-norm and in energy[2.7] by smooth curves; see [14, Theorem 1.3]. At present there are a few suggestions how to generalise the Möbius energy to higher-dimensional submanifolds – we are aware of Kusner and Sullivan [67, 68] and D. Auckly and L. Sadun [5] (see also the very recent contribution by O'Hara and G. Solanes [88], [87]) – but no satisfactory analysis regarding regularity or variational issues has been performed yet.

2.7 This has various consequences in geometric knot theory, for instance, it completes Scholtes' recent investigations on a discrete version of the Möbius energy for polygons with n edges, that can now be shown to Γ-converge to the Möbius energy (2.1.1) as $n \to \infty$; see [105, Theorem 1.1] and [14, Theorem 3.8]. We do not address the very interesting questions regarding suitable discretisations and merely refer to the work of Rawdon et al. [92, 93, 95, 78, 94, 96, 97, 106] on discretised versions of ropelength, and to [108, 105, 107] for discretisations of a few other geometric curvature energies.

The scale-invariant exponent for integral Menger curvature $\mathscr{M}_p(\Sigma)$ on two-dimensional surfaces $\Sigma \subset \mathbb{R}^3$ is $p = 8$, and we proved a Fenchel-type theorem [120, Theorem 1.2].

Theorem 2.2.9 (Fenchel-type theorem)**.** *There is an absolute constant $\gamma_0 > 0$ such that $\mathscr{M}_8(\Sigma) \geq \gamma_0$ for any closed compact connected two-dimensional Lipschitz surface $\Sigma \subset \mathbb{R}^3$.*

Due to our rather rough estimates the constant γ_0 is far from being optimal.

Open questions 2.2.10. *What is the optimal regularity of finite energy Lipschitz curves or submanifolds for any of the geometric curvature energies in the scale-invariant case? How regular are submanifolds of finite energy for a suitable generalisation of the Möbius energy to higher dimensional objects? How much geometric curvature energy does one really need to close a curve or a surface, in other words what are the optimal constants in Fenchel-type theorems like Theorem 2.2.9? Is every k-dimensional manifold $\mathcal{M}^k$ immersed in $\mathbb{R}^n$ automatically embedded if its image has finite scale-invariant tangent-point energy $\mathscr{E}_{2k}$ (see* (2.1.10)*), or finite scale-invariant integral Menger curvature $\mathscr{M}_{k(k+2)}$ (see* (2.1.11)*)?*

Existence of critical points. For all geometric curvature energies above scale-invariance one can find (at least one) minimising knot in a given isotopy class. This even works for higher-dimensional geometric curvature energies such as integral Menger curvature or tangent-point energies for submanifolds as described above. But are there other critical points, and how can one prove their existence? One of the first attempts in that direction is the work of D. Kim and Kusner [59] on the Möbius energy. They applied R. S. Palais' *principle of symmetric criticality* [89] to obtain $\mathscr{E}_{\text{Möb}}$-critical torus knots by minimising the Möbius energy within the appropriate subclass of torus knots enjoying particular symmetries . In addition, together with G. Stengle [59, p. 4] they used classic residue calculus from complex analysis to calculate their energy values. Further numerical experiments lead them to conjecture that most of these $\mathscr{E}_{\text{Möb}}$-critical torus knots are not local minimisers. For the non-smooth ropelength functional Cantarella et al. [28] successfully modified Palais' symmetric criticality principle to find new critical points in several symmetry classes of knots and links, e.g. in the non-trivial (a, b)-torus knots. They used their numerical ropelength minimising algorithm RIDGE RUNNER to compute their respective values for ropelength; see Figure 2.3.

In ongoing cooperative work with A. Gilsbach we apply Palais' principle to O'Hara's repulsive energies, integral Menger curvature, and tangent-point energies to produce symmetric critical configurations in every prescribed knot class. Specifically, in non-trivial (a, b)-torus knot classes we even obtain two distinct symmetric critical knots with this method [46], [47]. Very helpful in that context is the knowledge of the respec-

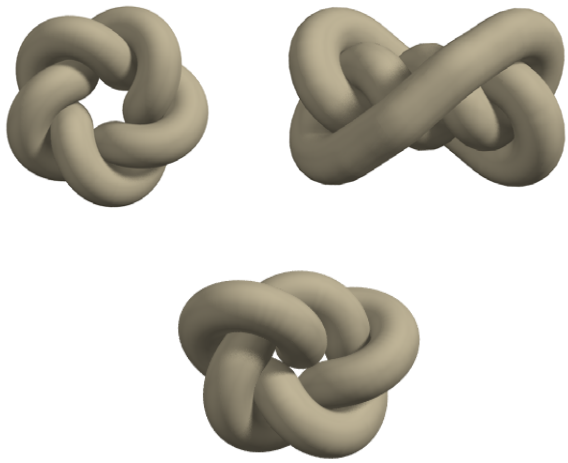
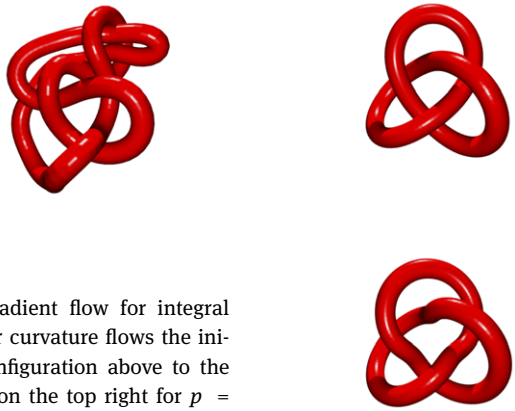


Fig. 2.3: A 5-fold and a 2-fold symmetric ropelength-critical (2, 5)-torus knot, both with ropelength values distinctly larger than the unsymmetric global ropelength minimiser on the right according to the computations with RIDGE RUNNER of Cantarella et al. in [28]. (Images by courtesy of J. Cantarella.)

tive correct energy spaces described in Section 2.1. Gilsbach also uses Γ-convergence arguments to show that her symmetric critical points of integral Menger curvature $\mathscr{M}_p$ do converge to ropelength-critical points as $p \to \infty$. Recently, Gilsbach has modified T. Hermes' numerical code [52] to actually compute the energy values of the symmetric critical points of integral Menger curvature. Hermes had rigorously derived the first variation formula for integral Menger curvature in the suitable fractional Sobolev space, and could prove that the round circle is a critical point. He created a numerical tool to explore the presumably quite complicated energy landscape of integral Menger curvature. His numerical experiments exhibit among other things the ability of the Menger gradient flow to untangle complicated unknots to the round circle after fairly short time, as well as varying features as p approaches infinity. For p only slightly above the scale-invariant exponent one finds smoothing as the predominant feature (while keeping the curves embedded in contrast to, e.g., the classic mean curvature flow on space curves), whereas for large p, say $p \geq 50$, the similarity to Cantarella's RIDGE RUNNER (corresponding to the case $p = \infty$) is striking [4]: both flows try to embed the curves as nicely as possible.

A second variation formula has been derived and analysed in detail for the Möbius energy by A. Ishizeki and T. Nagasawa [54]. They used a very interesting decomposition theorem for the Möbius energy itself [53], and studied recently the Möbius-invariance of the various parts of that decomposition [55]. Also quite recently J. Knapp-



The gradient flow for integral Menger curvature flows the initial configuration above to the trefoil on the top right for $p = 3.5$, and to the bottom trefoil on the right as $p = 50$.

Fig. 2.4: Different parameters $p > 3$ lead to different final configurations for the gradient flow of the rescaled integral Menger curvature but the knot type is preserved. (Images by courtesy of T. Hermes.)

mann [60] succeeded in deriving rigorously a second variation formula for integral Menger curvature $\mathscr{M}_p$ on curves in the appropriate fractional Sobolev spaces.

The only approach to deal with higher-dimensional critical points for geometric curvature energies is the ingenious paper by A. Nabutovsky [80] who combined complexity theory with real algebraic geometry to prove the existence of infinitely many critical unknotted hyperspheres in $\mathbb{R}^n$ for $n \geq 6$ for a higher-dimensional variant of ropelength.

Open questions 2.2.11. *Are there critical points for geometric curvature energies in every prescribed knot class other than the known global minimisers? In particular, are there critical unknots different*[2.8] *from the round circle? And if so, how many are there? Can we relate critical points of different geometric curvature energies with each other, e.g., via Γ-convergence? What can be said about the stability of such critical points? Is it possible to find critical configurations in higher dimensions?*

2.8 Energies that allow such critical unknots would therefore not be suitable to give an alternative proof of the Smale conjecture by means of a gradient flow as, e.g., suggested by Freedman et al. in [41, p. 41] for the Möbius energy. We do not address here the very challenging topic of gradient flows for geometric curvature energies and just refer to the pioneering work of Blatt on the gradient flow for the Möbius energy and other O'Hara energies [12, 15, 16]

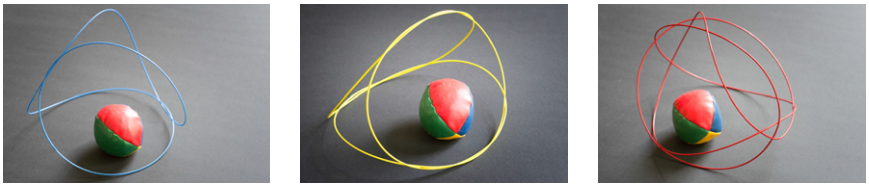


Fig. 2.5: Springy knots: figure-eight knot, mathematician's loop, and Chinese button knot. Wire models manufactured by WHY KNOTS, Aptos, in 1980; photographs by B. Bollwerk, Aachen.

Further implications on geometric knot theory. The round circle minimises many of the known geometric knot energies [41, Corollary 2.2],[1], [119, Lemma 7],[129, Corollary 5.12], and we expect the same for integral Menger curvature due to strong numerical evidence based on Hermes' numerical experiments with his gradient flow algorithm [52, Section 4.3]. In addition, we mentioned the explicit continuous families of ropelength-minimising links constructed by Cantarella et al. [29]. More recently, I. Agol, F. C. Marques, and A. Nèves applied their ingenious min-max-theory for minimal currents to resolve not only the famous Willmore conjecture [74] but also a conjecture by Freedman et al. by proving that the stereographic projection of the standard Hopf-link minimises the Möbius energy; see [2]. Apart from these results nothing is known analytically about the shape of non-trivially knotted minimising curves. For the ropelength-minimising trefoil, the so-called *ideal trefoil* one has presumably fairly accurate numerical solutions [8, 30, 9, 4, 91] and some local analytic information on the possible shape of general ideal knots [110, 26, 39, 40, 27] extracted from the complicated necessary conditions.

If one combines geometric curvature energies with (higher order) local energies like the classic bending energy $\mathscr{E}_{\mathrm{bend}}(\gamma) := \int_\gamma \kappa^2\, ds$, one can study minimal configurations for such energies under topological constraints on the competing curves or surfaces. This leads to the concept of *elastic knots* that can be obtained as limits of minimisers γ_ϑ of the total energy

$$\mathscr{E}_\vartheta(\gamma) := \mathscr{E}_{\mathrm{bend}}(\gamma) + \vartheta \frac{1}{\triangle[\gamma]} \tag{2.2.7}$$

as $\vartheta \to 0$. Recall that $\triangle[\gamma]$ denotes the thickness as defined in (2.1.2), so that the particular geometric curvature energy chosen in (2.2.7) is the ropelength functional if one restricts to curves of length one. Indeed, it can be shown [43, Theorem 2.2] that in every given knot class one finds such a limiting curve γ_0, which has smaller bending energy than any knotted competitor. However, as one would expect from the simple toy models of knotted wires designed by J. C. Langer, see Figure 2.5, γ_0 has self-intersections unless the given knot class is the unknot in which case γ_0 is the round circle [43, Proposition 3.1].

Is there anything one can say about the actual shape of the elastic knot γ_0 for non-trivial knot classes? This is indeed the case for any $(2, b)$-torus knot as shown in [43, Corollary 6.5]:

Theorem 2.2.12 (Elastic $(2, b)$-torus knots)**.** *For any odd integer $|b| \geq 3$ the unique elastic $(2, b)$-torus knot is the doubly covered circle. In particular, the elastic trefoil is the doubly covered circle.*

This result confirms mechanical and numerical experiments (see Figure 2.6), as well as the heuristics and Metropolis Monte Carlo simulations of R. Gallotti and O. Pierre-Louis [42, 90], and the numerical gradient-descent results by S. Avvakumov and A. Sossinsky [6]. However, adding twist changes the geometry of the springy wire drastically; see bottom right of Figure 2.6. And there is no theory yet, describing these twist effects for knotted elastic wires.

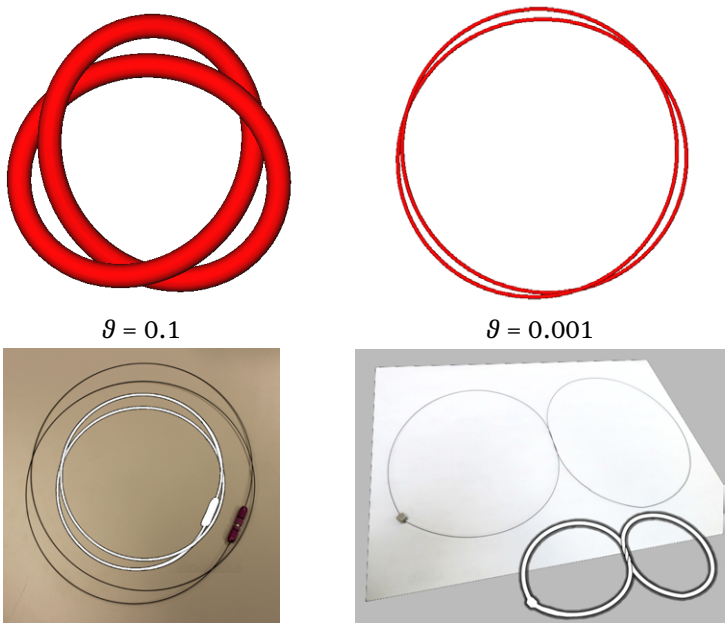


Fig. 2.6: Top: Numerically computed minimisers of the total energy $\mathscr{E}_\vartheta$ in the class of trefoils approaching the doubly covered circle as ϑ tends to zero. Bottom left: Mechanical experiments: The springy trefoil knot is close to the doubly covered circle. Bottom right: Adding twist leads to a stable flat trefoil configuration close to a planar figure-eight. (Wire models by courtesy of J. H. Maddocks, Lausanne.)

Open questions 2.2.13. *Does the round circle minimise integral Menger curvature* (2.1.5)*? Can one prove more about the actual shape of ideal knots? What is the shape of links with more than two components minimising the Möbius energy? Can one identify the shapes of global minimisers of other geometric curvature energies for curves? What can be said about elastic knots for non-trivial knot classes different from* $(2, b)$*-torus knots? What is the shape of twisted elastic knots?*

A higher-dimensional branch of geometric knot theory is naturally much less developed yet. The shape of possible minimising configurations for higher-dimensional geometric curvature energies is wide open and seems currently out of reach. There is one exception, however: We proved in [64, Theorem 1.5] with the isoperimetric inequality and a simple measure-theoretic argument the following uniqueness result for the global tangent-point-energy (2.1.16):

Theorem 2.2.14 (Spheres are unique minimisers)**.** *The round sphere uniquely minimises the global tangent-point energy* (2.1.16) *among all compact embedded* C^1*-hypersurfaces in* $\mathbb{R}^n$.

Recently, we proved in [65] that many higher-dimensional geometric curvature energies including integral Menger curvature (2.1.8), (2.1.11) or tangent-point energies (2.1.10), or their more singular variants (2.1.15), (2.1.16), are valuable knot energies. All these energies are self-repulsive (on the scale above scale-invariance), lower-semicontinuous on sublevel sets with respect to Hausdorff-convergence, they enjoy nice compactness properties and can thus be minimised in given isotopy classes; see [65, Theorem 2, Corollary 1]. They also bound the number of isotopy types with explicit constants only depending on the energy level and the integrability exponent, on a diameter bound, and on the dimensions [65, Theorem 1 & Remark 1.1]. In particular, one has the following boundedness result.

Theorem 2.2.15 (Isotopy finiteness)**.** *Let* $E, d > 0$ *be given. Then there are at most* $K = K(E, d, k, n, p)$ *different ambient isotopy types among all* k*-dimensional Lipschitz submanifolds* $\Sigma \subset \mathbb{R}^n$ *with integral Menger curvature* $\mathscr{M}_p(\Sigma) \leq E$ *and* $\operatorname{diam} \Sigma \leq d$*. This constant may be estimated as*

$$\log\log K \leq c(k, n, p)\Big(|\log d| + \log(E^{1/p} + 1) + 1\Big). \tag{2.2.8}$$

This result can be compared to a whole series of finiteness theorems of diffeomorphism types under given bounds on classic curvatures, beginning with the work of J. Cheeger [31], and extended by many others, see, e.g, Cheeger's exhaustive survey [32] and the references therein. The notable difference is here that we deal with embedded submanifolds of lower regularity, whose Riemannian metrics are just Hölder continuous so that the classic notion of curvature does not make sense. The geometric curvature energies in the regime above scale-invariance turn out to be valuable substitutes. The

only comparable result with this emphasis is the work of O. Durumeric [38] who, however, works in the context of $C^{1,1}$-submanifolds with positive thickness.

Higher-dimensional variants of elastic knots have not been discussed explicitly yet, but L. Simon's pioneering work [113] solves the problem of minimising the Willmore energy

$$\int_{\Sigma} H^2(x)\, d\mathscr{H}^2(x) \tag{2.2.9}$$

in the class of two-dimensional embedded surfaces with prescribed genus or under alternative constraints; see also [10, 100, 111, 69, 58, 99, 79]. But minimising the Willmore energy or related functionals such as the Helfrich functional [51, 81] on given isotopy classes has to the best of our knowledge not been investigated yet — with the exception of recent work of P. Breuning, J. Hirsch, and E. Mäder-Baumdicker [23] on Willmore minimising Klein bottles.

Open questions 2.2.16. *Is it possible to identify the shape of global minimisers of higher-dimensional geometric curvature energies? The explicit constant in Theorem 2.2.15 is far from being optimal, what is the best constant bounding the number of isotopy types under given energy values? For curves the energies could often be related to knot invariants or to quantities like the average crossing number* (2.1.17) *controlling knot invariants. Are there meaningful topological invariants or geometric quantities for higher-dimensional knots that could be controlled by means of higher-dimensional geometric curvature energies? Are there higher-dimensional elastic knots, for instance minimisers of the Willmore functional in arbitrary prescribed isotopy classes? And can one say anything about their shapes?*

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