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How averaged Menger curvatures control regularity and topology of curves and surfaces

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Abstract. We demonstrate how integrated Menger curvatures can replace the nonsmooth ropelength functional to control regularity and topology of knotted curves and surfaces.

1. Introduction

Classic local curvature κ_γ of a closed smooth curve $\gamma \subset \mathbb{R}^3$ is a function that is defined pointwise: at the point $x = \gamma(s)$ one has $\kappa_\gamma(x) = |\gamma''(s)|$ via the arclength or unit-speed parametrization $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^3$ with $|\gamma'(t)| = 1$ for all $t \in \mathbb{R}/\mathbb{Z}$. Alternatively and in more geometric terms, the best local approximation of γ by circles near the curve point x is given by the osculating circle that touches γ at the point x . The radius $r = r(x)$ of this osculating circle is the *local radius of curvature* whose inverse equals the local curvature $\kappa_\gamma(x)$ of γ at x . Likewise for surfaces or higher-dimensional submanifolds, classic curvature functions such as mean or Gauß curvature are local, that is, they take single surface points as arguments. In contrast to that, Menger (1930) considered functions on curves that depend on more than one curve point. The most prominent example of such a *multipoint function* is the circumcircle radius $R(x, y, z)$ of three points $x, y, z \in \gamma$, which tends to the local radius of curvature $r(x)$ in the coalescent limit $y, z \rightarrow x$ if γ is smooth. This and the fact that $R(x, y, z)$ may be expressed solely in terms of the mutual distances of the three points x, y , and z , led Menger to the idea of developing a purely metric geometry based on such multipoint functions in contrast to classic differential geometry. Instead of following that thread here let us add one further simple observation: in order to define $R(x, y, z)$ along γ it is by no means necessary to assume any smoothness of γ , which turns out to be useful from the variational point of view. Indeed, quite often one has to enlarge the classic function spaces to a wider class of nonsmooth functions, such as Sobolev spaces, to actually find energy minimizers in the calculus of variations, or to find weak solutions of partial differential equations.

With that in mind and motivated by computational issues in the numerical simulation of macromolecules with self-contact, Gonzalez & Maddocks (1999) reconsidered the circumcircle radius as a means to define an analytically tractable notion of *thickness* for curves. Indeed, their idea was to search for the smallest possible circumcircle radius among all triples of curve points. This way one may end up with an osculating circle at a point of highest local curvature of the curve, or, on



the other hand, with a limiting circle whose radius is much smaller than the local curvature at any of the curve strands it touches, thereby reflecting the fact that two (or more) strands of the same curve are close to each other in the ambient space; see Figure 1. This interplay between local and global

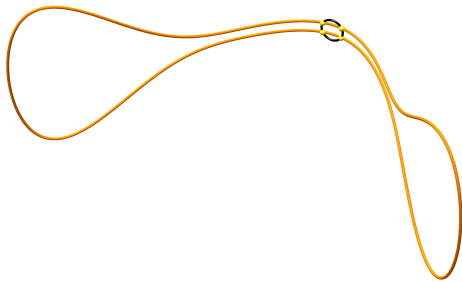


Figure 1. $1/R$ can be high at many locations where classic local curvature remains moderate.

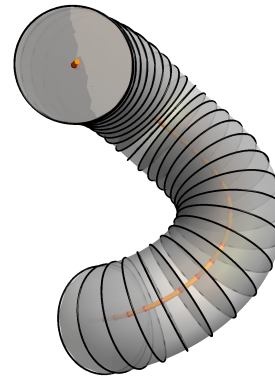


Figure 2. The tubular neighbourhood of a $C^{1,1}$ -curve

control by means of a pointwise minimization of $R(x, y, z)$ is reflected in the following theorem proved independently by Cantarella *et al.* (2002) and Gonzalez *et al.* (2002).

Theorem 1.1. *An arclength parametrized closed curve γ with*

$$\Delta[\gamma] := \inf_{x \neq y \neq z \neq x} R(x, y, z) > 0 \quad (1)$$

is embedded and continuously differentiable, with a Lipschitz continuous tangent vector, and local curvature κ_γ defined and bounded almost everywhere by $1/\Delta[\gamma]$.

Even more is true: the quantity $\Delta[\gamma]$ can justifiably be called *thickness* of γ since there is a uniform tube around the curve, consisting of the disjoint union of normal disks centered on the curve and with uniform radius $\Delta[\gamma]$, so that in particular the nearest-point projection onto γ is well-defined and unique within that tubular neighbourhood; see Figure 2. In that sense, positive thickness $\Delta[\gamma]$ serves as an *exact excluded volume constraint* for γ . Theorem 1.1 may be viewed as the foundation for the analytic investigation of *ideal knots* as minimizers of *ropelength*, i.e. the quotient of length and thickness, see Cantarella *et al.* (2002, 2006, 2011, 2012); Gonzalez *et al.* (2002); Gonzalez & de la Llave (2003); Schuricht & von der Mosel (2004), as well as for elastic rods with self-contact Schuricht & von der Mosel (2003), or packing problems as in Gerlach & von der Mosel (2011*b,a*). One possible generalization of thickness to surfaces including an existence theory for area-minimizing thick surfaces with prescribed genus or in given isotopy classes was published by Strzelecki & von der Mosel (2005, 2006).

The analytic drawback of thickness or ropelength is that it is a nonsmooth functional due to the minimization over triples of points; see (1). Consequently, techniques from nonsmooth analysis are required to derive variational equations as a set of necessary conditions for minimizers of ropelength. In the light of this, it is not too surprising that up to now the optimal regularity of ideal knots is not known¹, and there is no rigorous analytic treatment of a suitable gradient flow for ropelength. So, it is

¹ Cantarella *et al.* (2002) have constructed ideal links with more than one component that are only $C^{1,1}$, and there is numerical evidence that the ideal trefoil knot may have jumps in local curvature.

natural to ask for suitable relaxations of the ropelength functional without losing too many features that qualify ropelength as a *knot energy* separating different knot classes by infinite energy barriers.

In this short note we will address one possible relaxation of ropelength $1/\Delta[\gamma]$ (keeping unit length of all curves γ fixed), by replacing each pointwise maximization by averaging. That is, we consider the *integral Menger curvature*²

$$\mathcal{M}_p(\gamma) := \int_{\gamma} \int_{\gamma} \int_{\gamma} \frac{1}{R(x, y, z)^p} d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z) \quad (2)$$

for a suitable range of integrabilities $p > 0$. It is clear that one may lose a lot of control over the shape and regularity of a finite energy curve, since even for large positive p the circumcircle R may be arbitrarily small at certain triples of curve points. The only hope is that such sets of triples cannot have large measure. We discuss in this note which analytic and topological effects can still be observed for curves with finite integral Menger curvature – at least for p above the scale invariant case $p = 3$. To address a wider audience we will make an effort to survey our results in a more informal way avoiding rigorous mathematical proofs, but we will try to explain some of the underlying (mostly) geometric ideas and principles. We would like to invite the reader who is interested in the detailed proofs to consult our respective papers (partly joint work with S. Kolasiński or M. Szumańska), explicit references will be given.

In Section 2 we discuss integral Menger curvature and its regularizing and topological effects on closed curves (Theorem 2.1), including a short sketch on how to obtain initial higher regularity and a weak form of an excluded volume constraint, which we refer to as the *diamond property*. This suffices to identify \mathcal{M}_p as a valuable knot energy: in each knot class we find minimizing representatives, the energy bounds the stick number and the average crossing number, thus the number of isotopy types, and the energy separates knot space in terms of Hausdorff-distance; see Theorem 2.2. In Section 3 we describe how to generalize integral Menger curvature to higher dimensions and co-dimensions and present some of our more recent results such as a geometric variant of the Morrey-Sobolev embedding theorem (Theorem 3.1), the existence of \mathcal{M}_p -minimizing submanifolds in given isotopy classes, and a finiteness theorem on the isotopy types of Lipschitz submanifolds with unit area with uniformly bounded integral Menger curvature (Theorem 3.2). Apart from the technically more demanding proofs in higher dimensions that cannot even be sketched in the present note, there is one crucial conceptual difference to one-dimensional sets, the so-called *Ahlfors regularity*, that we do address at the end of Section 3, for simplicity for two-dimensional surfaces in \mathbb{R}^3 : there is absolutely no control how much measure of a closed surface is contained in a small ball that is centered on the surface, it depends heavily on the individual surface; see Figure 3. For closed curves, on the other hand, it is clear that the length of the curve within a small ball centered on the curve exceeds the ball's radius. We present a new algorithm to find “voluminous” tetrahedra on a general closed surface, which may be of independent interest; see Theorem 3.3. These voluminous tetrahedra can then be used to control Ahlfors regularity of finite energy surfaces purely in terms of the energy value, which is the essential first step to even start regularity considerations for integral Menger curvature in higher dimensions.

To conclude this introduction let us point out an interesting connection to measure theoretic results in harmonic analysis where integral Menger curvature was used to prove Vitushkin's conjecture on the solution of the famous Painlevé problem on the removability of singularities for bounded analytic functions in the complex plane³. Léger (1999) proved that one dimensional Borel sets $E \subset \mathbb{R}^n$

² As already suggested in Gonzalez & Maddocks (1999) as a potential knot energy; see also Sullivan (2002). Banavar *et al.* (2003) discuss its possible relevance for modeling self-avoidance since no regularization is necessary in contrast to repulsive potentials such as O'Hara's energies, in particular the Möbius energy; see O'Hara (1991, 1992a,b).

³ For more details in that context; see, e.g., the surveys David (1999), Tolsa (2006) or Mattila (2004).

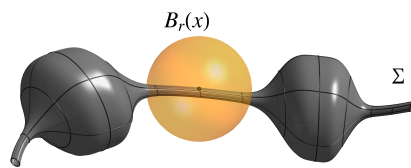


Figure 3. For fixed radius r , the measure of the surface in $B_r(x)$ can be made arbitrarily small just by making the tube thinner.

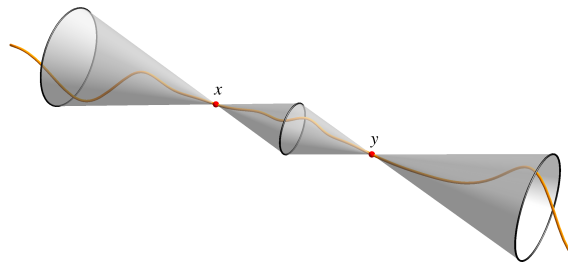


Figure 4. In a small scale, all arcs of a finite energy curve are trapped in double cones.

with $\mathcal{M}_2(E) < \infty$ are 1-rectifiable in the sense of geometric measure theory. That is, such sets can (apart from a set of measure zero) be covered by a countable union of graphs of Lipschitz continuous functions on the real line. Notice that the set can still consist of many disconnected little bits and pieces, nevertheless this result yields a dramatic increase of regularity even though $p = 2$ is below the scale-invariant exponent. This, on the one hand, serves as additional motivation to prove classic higher regularity, starting with closed rectifiable curves already on a higher regularity level and with $p \geq 3$ as we describe in the present note. On the other hand, it opens more terrain for future research possibly bridging the gap between the different ranges of integrability: polygons are shown to have finite integral Menger curvature \mathcal{M}_p exactly for $p \in (0, 3)$ (Scholtes (2011)), so classic differentiability is not to be expected in this range, but the situation remains unsettled for the scale-invariant case $p = 3$.

2. Integral Menger curvature for curves

To fix notation, we denote by \mathcal{C} the class of all *unit loops*, i.e., closed curves $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^3$ parametrized by arclength with $\gamma(0) = 0$, and with $\mathcal{H}^1(\gamma) = 1$ to exclude undesirable parametrizations that cover subarcs or all of γ more than once. Since we are mainly interested in the various shapes of different representatives of knot classes this is not a serious restriction. Moreover, integral Menger curvature does not penalize multiple coverings as can be seen, e.g., for the twice covered semicircle: the integrand of \mathcal{M}_p is simply constant along the semicircle which gives finite energy, the image is a one-dimensional manifold with boundary but the parametrization is not smooth at the two endpoints since the tangent vector performs an abrupt turn by 180° .

The first theorem is fundamental, it reflects how much control on topology and regularity finite integral Menger curvature imposes on a unit loop.

Theorem 2.1 (Strzelecki *et al.* (2010)). *All unit loops $\gamma \in \mathcal{C}$ with $\mathcal{M}_p(\gamma) \leq E < \infty$ for $p \geq 3$ are homeomorphic to a circle. Moreover, if $p > 3$, then γ is of class $C^{1,1-(3/p)}$ with a uniform estimate of the corresponding $C^{1,1-(3/p)}$ -norm solely in terms of E and p .*

So, self-intersections are excluded for $p \geq 3$, and for $p > 3$ one has a Hölder continuous tangent vector with the Hölder exponent $1 - (3/p)$. This exponent is optimal as one can either see by specific examples constructed in Szumańska (2009); Kolasiński & Szumańska (2013), or by the more recent characterization of finite energy curves in terms of fractional Sobolev spaces which exactly embed into $C^{1,1-(3/p)}$ and not into any better classic function space; see Blatt (2013); Blatt & Kolasiński (2012). For the reader who is more familiar with classic Sobolev spaces we would like to offer an interpretation of Theorem 2.1 as a *geometric variant of the Morrey-Sobolev embedding theorem*: the integrand $1/R$ is a very weak form of curvature integrated on a three-dimensional domain $\gamma \times \gamma \times \gamma$,

so it is comparable to the classic Morrey-Sobolev embedding theorem that guarantees $C^{1,1-(3/p)}$ -regularity for functions whose second derivatives are p -integrable on a three-dimensional domain. Theorem 2.1 is a generalization of Theorem 1.1 for curves of finite ropelength. Notice that ropelength $1/\Delta[\gamma]$ is the limit of $\mathcal{M}_p^{1/p}(\gamma)$ as $p \rightarrow \infty$ for any unit loop $\gamma \in \mathcal{C}$, so that one might think of taking the formal limit in the regularity statement as well, to obtain the limiting $C^{1,1}$ -regularity for finite ropelength.

For the proof of Theorem 2.1 one establishes first a geometric rigidity of the curve that serves as a weak form of an excluded volume constraint, and provides an intermediate Hölder regularity of the tangent with non-optimal exponent. After that, one proves in a more technical second step the optimal regularity by slicing and iteration arguments. Let us very briefly sketch the more geometric idea behind the first step, in which one tries to control the *beta numbers* introduced in the context of harmonic analysis in Jones (1990) and defined for $x \in \gamma$ and $d > 0$ as

$$\beta_\gamma(x, d) := \inf \left\{ \sup_{y \in \gamma \cap B_d(x)} \frac{\text{dist}(y, L)}{d} : L \text{ is a straight line through } x \right\}. \quad (3)$$

The beta numbers measure how well the curve can be approximated locally by straight lines, or in other words, $\beta_\gamma(x, d) \cdot d$ is the width of the thinnest cylinder centered at $x \in \gamma$ that contains the portion of the curve γ in the ball $B_d(x)$. If the beta number happens to be small then the set is locally fairly flat. In the present situation the following holds:

Claim. There is a universal constant $c_0(p) > 0$ depending only on p such that for any $\epsilon \leq 0.001$ and $d < \text{diam } \gamma$ satisfying the balance condition

$$\epsilon^{6+p} d^{3-p} \geq c_0(p) \mathcal{M}_p(\gamma) \quad (4)$$

one has $\beta_\gamma(x, d) \leq \epsilon$.

Applying this for ϵ with equality in the balance condition one observes that $\beta_\gamma(x, d) \lesssim d^\kappa$ for $\kappa = (p-3)/(p+6)$ which means that the height of the cylinder confining γ within the ball $B_d(x)$ is bounded by $d^{\kappa+1}$. Keeping the curve point x fixed, one can repeat the same reasoning on every scale $d/2^n$ for $n = 1, 2, \dots$ to obtain a sequence of thinner and thinner cylinders centered at $x \in \gamma$ and containing the curve in the balls $B_{d/2^n}(x)$. The height of the previous cylinder bounds the maximal tilt angle of the following cylinder, and all these angles add up like a geometric series to a total tilt angle $\lesssim d^\kappa$. Consequently, taking the limit $n \rightarrow \infty$, one has established the following geometric rigidity: $\gamma \cap B_d(x)$ is contained in a double cone with vertex x , and cone axis containing the segment $x-y$ for any curve point $y \in \partial B_{d/2}(x)$, and with cone angle $\lesssim d^\kappa$. Reversing the roles of x and y one actually finds locally near x and y the curve γ being trapped in the intersection of two such double cones; see Figure 4. This immediately implies the intermediate Hölder regularity of the tangent γ' with the non-optimal exponent κ , since one can argue first at all points x and y of differentiability, so that the corresponding unit tangents differ by at most the cone angle $\lesssim d^\kappa$. This uniform estimate on the oscillation of tangents, that exist almost everywhere, can then be uniquely extended to all of the parameter domain to find that the tangent exists everywhere and satisfies the same oscillation estimate. In order to establish the claim one assumes to the contrary that any cone with axis through $x, y \in \gamma$ with $|x-y| < d$ misses some point $z \in B_d(x)$, so that the triangle with vertices x, y, z has height greater than ϵd ; see Figures 5 and 6. Elementary geometry yields the inequality

$$\frac{1}{R(x, y, z)} = 2 \frac{\text{height}}{|x-z| \cdot |y-z|} \gtrsim \frac{\epsilon d}{d^2} = \frac{\epsilon}{d}. \quad (5)$$

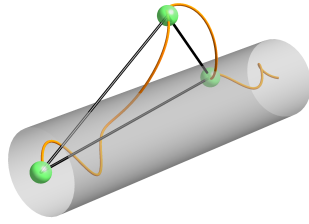


Figure 5. The beta number is large: the curve does not fit into a narrow cylinder.

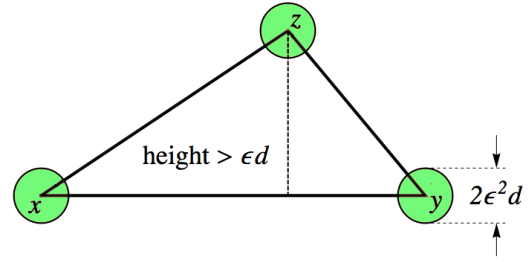


Figure 6. Small perturbations of vertices do not change $1/R$ significantly.

This estimate alone is useless since it is an estimate for just *one* triple of points which could be ignored completely by the energy (in contrast to ropelength!), but there are enough triples of points close by with roughly the same estimate, as long as those triples are in sufficiently small balls (say of radius $\epsilon^2 d$) about x, y , and z , respectively; see Figure 6. Indeed, any curve that hits the center of a ball and that leaves the ball has one dimensional Hausdorff measure at least as large as the radius of the ball:

$$\mathcal{H}^1(\gamma \cap B_{\epsilon^2 d}) \geq \epsilon^2 d. \quad (6)$$

This property is commonly referred to as *uniform (lower) Ahlfors regularity*, a fact not shared by higher dimensional objects such as two-dimensional surfaces, and we will make an effort in Section 3 to explain an algorithm which implies uniform Ahlfors regularity for general finite energy surfaces.

Returning to the proof of the claim we can now see how to obtain the desired contradiction by estimating the total energy from below by the contribution of the three little portions of γ within the balls of radius $\epsilon^2 d$ about x, y , and z , and finally using the balance condition (4):

$$\mathcal{M}_p(\gamma) > \int_{B_{\epsilon^2 d}(x) \cap \gamma} \int_{B_{\epsilon^2 d}(y) \cap \gamma} \int_{B_{\epsilon^2 d}(z) \cap \gamma} \frac{1}{R^p} \stackrel{(5)(6)}{\gtrsim} (\epsilon^2 d)^3 \cdot \left(\frac{\epsilon}{d}\right)^p = \epsilon^{6+p} d^{3-p} \stackrel{(4)}{\geq} c_0(p) \mathcal{M}_p(\gamma), \quad (7)$$

which is absurd for an appropriately chosen constant $c_0(p)$. Thus, the initial step in the proof of Theorem 2.1 is completed.

Let us recall that the curve is actually locally trapped in the intersection of two double cones with vertices at curve points on a sufficiently small scale $d(E)$ that depends only on the energy value $\mathcal{M}_p(\gamma) \leq E$. Since this property holds true on this and every smaller scale, no matter where the two points lie on the curve, we have established what we call the *diamond property*:

For each two points $x, y \in \gamma$ with $|x - y| \leq d(E)$ the portion of γ within the intersection of balls, $B_{2|x-y|}(x) \cap B_{2|x-y|}(y)$ is contained in the intersection of the two double cones with opening angle $\lesssim |x - y|^\kappa$, common axes through x and y , and with vertices at x and y , respectively; see Figure 4. In addition, in each cross section perpendicular to the common cone axis, one finds exactly one curve point γ .

One can use this property to discover that γ carries along a whole necklace of such double cones, in our terms *diamonds*, that have mutually disjoint interiors. This necklace serves as a weak form of an excluded volume constraint, since the curve is restricted to the diamonds in a whole neighbourhood of the curve. This holds for every partition of the curve into small arcs, as long as each such arc is sufficiently small in diameter, i.e. smaller than the scale $d(E)$. However, let us point out that there is

no unique nearest-point projection, and we do not have the disjoint union of uniformly sized normal disks about the curve, as finite ropelength would imply. Still, the diamond property gives the curve enough rigidity to control local bending, and to allow for topological results that qualify integral Menger curvature as a valuable knot energy.

Theorem 2.2 (Strzelecki *et al.* (2013)). *Let $p > 3$, then \mathcal{M}_p satisfies the following conditions:*

- (i) \mathcal{M}_p is charge, i.e., it blows up along sequences $\{\gamma_i\} \subset \mathcal{C}$ that converge uniformly to some curve γ with self-intersections. Moreover, \mathcal{M}_p is tight in the sense that it blows up along sequences for which small nontrivially knotted parts pull tight in the limit.
- (ii) In each knot class $[K]$ one finds a representative $\gamma_K \in \mathcal{C} \cap [K]$ that minimizes \mathcal{M}_p , i.e., $\inf_{\gamma \in [K] \cap \mathcal{C}} \mathcal{M}_p(\gamma) = \mathcal{M}_p(\gamma_K)$.
- (iii) \mathcal{M}_p bounds the stick number and the crossing number, hence it bounds the number of knot classes that can be represented under a given energy level.
- (iv) Any two finite energy curves $\gamma_1, \gamma_2 \in \mathcal{C}$ that are closer in Hausdorff-distance than a constant depending only on p and on $\max\{\mathcal{M}_p(\gamma_1), \mathcal{M}_p(\gamma_2)\}$ represent the same knot class.

Let us comment on some of the aspects and on typical arguments that enter the proof of this result. For example, as to part (i), if $\gamma_i \rightarrow \gamma$ for some sequence with $\mathcal{M}_p(\gamma_i) \leq E$ for all i , then we may extract a subsequence converging to γ in C^1 because of the uniform $C^{1,1-(3/p)}$ -a-priori estimates depending only on E . These estimates for the embedded curves γ_i yield a uniform bi-Lipschitz estimate which carries over to the limit curve γ . Hence γ cannot have a double point. A similar compactness argument, or alternatively, the rigidity imposed by the diamond property, shows that \mathcal{M}_p is also tight, since each cross section of any diamond contains only one curve point, which prevents small knotted subarcs to pull tight. The improvement to C^1 -convergence guarantees the same knot class in the limit, which yields the existence of energy minimizers in each given knot class in part (ii) by the direct method in the calculus of variations. The stick number is the minimal amount of straight segments (sticks) needed to construct a polygonal representative of a given knot class. To bound that number which is a knot invariant we can show that for a given curve $\gamma \in \mathcal{C}$ with $\mathcal{M}_p(\gamma) \leq E < \infty$, any inscribed polygon with edge length smaller than a constant $C = C(p, E)$ is automatically isotopic to γ . To that end we cover γ with a necklace of diamonds connecting the polygonal vertices, so that the axial segments of the diamonds coincide with the polygonal edges. Then we construct a suitable ambient isotopy between γ and the polygon by keeping everything constant outside the diamonds, while suitably mapping homeomorphically each of the distinct cross section of each diamond onto itself (keeping the boundaries fixed), whilst moving in each cross section the one and only curve point towards the axis of the double cone, i.e., onto the respective polygonal edge. The *minimal crossing number* of a representative γ of a knot class $[K]$ is the minimal number of self-crossings that one obtains when looking at all planar projections of γ , and the *crossing number* of $[K]$ is then the smallest possible such minimal crossing number if one looks at all representatives of $[K]$. To bound the latter we bound the minimal crossing number of a given representative $\gamma \in [K]$ by bounding its *average crossing number* $\text{acn}(\gamma)$, which is the average value of crossing numbers upon taking all possible planar projections into account. Due to the practical integral representation; see Freedman *et al.* (1994),

$$\text{acn}(\gamma) = \frac{1}{4\pi} \int \int_{[0,1]^2} \frac{|\det(\gamma'(s), \gamma'(t), \gamma(s) - \gamma(t))|}{|\gamma(s) - \gamma(t)|^3} ds dt, \quad (8)$$

and following ideas of Buck & Simon (1997) for thick knots, we can use the diamond property as a weaker form of excluded volume constraint to estimate this double integral. The local estimate of

this integral on pairs of parameters that are sufficiently close to each other is a result of the geometry in one diamond: the respective tangents and the chord $\gamma(s) - \gamma(t)$ for such parameters span a fairly flat parallelepiped, so that the integrand is sufficiently small. For mutually distant parameters, on the other hand, one can use the necklace of diamonds as an excluded volume constraint to compute a worst case scenario of closest possible packing since the numerator in (8) as the dominating term needs to be controlled. Both, stick number and the crossing number, are knot invariants, and one knows from knot theory how bounds on either of these transfer to bounds on the number of possible knot types representable under these bounds, which concludes part (iii). For part (iv) one uses inscribed polygons of the same knot type as in part (iii) for each of the two curves, and shows by virtue of so called Δ -moves the combinatorial equivalence of these inscribed polygons to guarantee the same knot type of the two curves.

3. Integral Menger curvature in higher dimensions

Léger (1999) proposed to replace the integrand $1/R$ in \mathcal{M}_p by the expression $\mathcal{L}(\xi, x, y, z) := \frac{\text{dist}(\xi, \langle x, y, z \rangle)}{|\xi - x||\xi - y||\xi - z|}$ for two-dimensional sets, where $\langle x, y, z \rangle$ denotes the 2-plane spanned by x, y , and z . This suggestion might or might not work to generalize his proof of rectifiability, but it is certainly not the right choice in our context, since even the unit sphere \mathbb{S}^2 would have infinite energy

$$\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \mathcal{L}^p(\xi, x, y, z) d\mathcal{H}^2(\xi) d\mathcal{H}^2(x) d\mathcal{H}^2(y) d\mathcal{H}^2(z) \quad (9)$$

for sufficiently large p . The most obvious generalization of the circumcircle radius of three curve points would be the circumsphere radius of four points on a surface, which would provide a constant integrand equal to 1 for the unit sphere \mathbb{S}^2 . But even that choice is too singular for our purposes, since there are smooth surfaces, e.g. with straight nodal lines such as the graph of the smooth function $f(x, y) := xy$, where the inverse of the circumsphere radius as a possible integrand is not bounded. This led us to the idea to use a very similar but less singular integrand defined on tetrahedra $T = (\xi, x, y, z)$ of surface points, namely $\text{Volume}(T)/(\text{Area}(T) \cdot (\text{diam } T)^2)$ (Strzelecki & von der Mosel (2011)), where $\text{Area}(T)$ is the sum of all facet areas of T , or even simpler (Kolasinski (2011, 2012)), and for all dimensions $k < n$ and $(k + 1)$ -dimensional simplices $T = (x_1, \dots, x_{k+2}) \in \mathbb{R}^n$:

$$\mathcal{K}(T) := \text{Volume}(T)/(\text{diam } T)^{k+2} \quad \text{if } T \text{ is non-flat, } \mathcal{K}(T) := 0 \text{ otherwise.} \quad (10)$$

The scale invariant case for integral Menger curvature $\mathcal{M}_p(\Sigma)$ on k -dimensional subsets $\Sigma^k \subset \mathbb{R}^n$,

$$\mathcal{M}_p(\Sigma) := \underbrace{\int_{\Sigma} \dots \int_{\Sigma}}_{k+2 \text{ integrations}} \mathcal{K}^p(T) d\mathcal{H}^k \dots d\mathcal{H}^k, \quad (11)$$

would be $p = k(k + 2)$, which corresponds to $k + 2$ integrations with respect to the k -dimensional Hausdorff measure \mathcal{H}^k . Again, for exponents above the scale invariant case we obtain a *geometric Morrey-Sobolev embedding theorem* (Strzelecki & von der Mosel (2011); Kolasinski (2011, 2012)):

Theorem 3.1. *If an admissible k -dimensional set $\Sigma^k \subset \mathbb{R}^n$ satisfies $\mathcal{M}_p(\Sigma) \leq E < \infty$ for some $p > k(k + 2)$, then Σ is a submanifold of class $C^{1,1-(k(k+2))/p}$. In addition, there is a length scale $R_0 = R_0(E, p)$ depending only on E and p , such that for each $x \in \Sigma$ and $0 < r < R_0$ the intersection $\Sigma \cap B_r(x)$ equals, up to isometry, $\text{Graph} f \cap B_r(0)$, where $\text{Graph} f$ is the graph of a function $f \in C^{1,1-(k(k+2))/p}(\mathbb{R}^k, \mathbb{R}^{n-k})$ with $\|f\|_{C^{1,1-(k(k+2))/p}} \lesssim \mathcal{M}_p(\Sigma)^{1/p}$.*

We will briefly comment on the notion of “admissible sets” later on, for the moment one may assume that Σ is a compact, connected Lipschitz submanifold of \mathbb{R}^n without boundary, but much more general sets are in fact admissible. As for curves, the integrand $\mathcal{K}(T)$ is a form of weak discrete curvature, and it is integrated on a $k(k+2)$ dimensional domain, so that Theorem 3.1 corresponds to the classic Morrey-Sobolev theorem for functions whose second weak derivatives are p -integrable on a $k(k+2)$ -dimensional domain. But our result says more: the uniform control over the “patch size” of the local graph representation purely in terms of the integrability p and the energy value E allows for nice topological applications, stated here for two-dimensional⁴ surfaces in \mathbb{R}^3 (Strzelecki & von der Mosel (2011)):

Theorem 3.2. *Let $\mathcal{C}_A(M_g)$ be the class of all closed, compact and connected Lipschitz surfaces $\Sigma^2 \subset \mathbb{R}^3$ without boundary, ambiently isotopic to a fixed reference manifold $M_g \subset \mathbb{R}^n$ of genus g with the uniform area bound $\mathcal{H}^2(\Sigma) \leq A$. Then, for any $p > 2(2+2) = 8$, there is an \mathcal{M}_p -minimizing surface $\Sigma_g \in \mathcal{C}_A(M_g)$, i.e., $\inf_{\mathcal{C}_A(M_g)} \mathcal{M}_p = \mathcal{M}_p(\Sigma_g)$. Moreover, there are only finitely many isotopy types of surfaces that can be represented under a given \mathcal{M}_p -energy level.*

The last statement should be compared to finiteness theorems in smooth Riemannian geometry by Anderson & Cheeger (1991), but in the present setting we are below the C^2 -category.

The general strategy to prove Theorem 3.1 is, albeit much more technical, the same as for curves. One tries to estimate the beta numbers, say in the two-dimensional case ($k = 2$),

$$\beta_\Sigma(x, d) := \inf \left\{ \sup_{y \in \Sigma \cap B_d(x)} \frac{\text{dist}(y, P)}{d} : P \text{ is a 2-dimensional affine plane through } x \right\}. \quad (12)$$

for $x \in \Sigma$ and $d > 0$, to show local flatness in a first step, which leads to an initial (non-optimal) oscillation estimate for tangent planes. A second technical step, as for curves, improves the Hölder exponent via slicing and iteration.

To obtain in the first step such beta number estimates under a suitable balance condition one uses elementary geometry in a contradiction argument: assuming a fourth surface point $\xi \in \Sigma \cap B_d(x)$ too far away from a flat slab around the plane $\langle x, y, z \rangle$ spanned by $x, y, z \in \Sigma$, one can deduce a lower bound on the integrand $\mathcal{K}(T)$ for that particular tetrahedron $T = (\xi, x, y, z)$. But here comes the catch: to derive a contradiction one needs similar bounds on sufficiently many tetrahedra T' close to T . But are there “sufficiently many” such tetrahedra with vertices in small balls around the four vertices of T ? What was automatic for curves (cf. (6)) is not at all true for arbitrary surfaces. Even smooth surfaces $\Sigma \subset \mathbb{R}^3$ may exhibit very slender fingers so that there might be points $x \in \Sigma$ such that the surface area $\mathcal{H}^2(\Sigma \cap B_r(x))$ is not all comparable to r^2 ; see Figure 3 again. So, apart from the technical complications in higher dimensions, one needs an additional step to prove *uniform Ahlfors regularity* for finite energy surfaces: *there is a universal constant $R = R(E, p)$ depending only on p and the energy level E such that*

$$\mathcal{H}^2(\Sigma \cap B_r(x)) \geq \frac{\pi}{2} r^2 \quad \text{for all } r \leq R. \quad (13)$$

The idea to prove this can be described as follows. Assume, that the tangent plane $T_x \Sigma$ of Σ at $x \in \Sigma$ exists. Then one can imagine this point x as a “seed” from which we can grow double cones $C(x)$ with opening angle, say, $\pi/4$ and cone axis perpendicular to $T_x \Sigma$, until we hit another part of the

⁴ This result extends to higher dimensions and co-dimensions due to the uniform estimates on the graph patches; see Kolasinski *et al.* (2013b).

surface after some positive distance, the “stopping distance” $d(x)$ depending on the starting point x . (Notice that one half of the cone is in the interior, the other in the exterior of Σ , which, as a closed surface in \mathbb{R}^3 , separates space into a bounded interior and an unbounded exterior. So eventually, this increasing cone must hit the compact surface Σ at some surface point $\xi \in \Sigma \setminus \{x\}$.) As long as there is no hit one has large orthogonal projections of Σ onto the tangent plane $T_x \Sigma$, which in turn gives the (not yet uniform) lower Ahlfors estimate

$$\mathcal{H}^2(\Sigma \cap B_r(x)) \geq \frac{\pi}{2} r^2 \quad \text{for all } r \leq d(x). \quad (14)$$

If the first hit yields a surface point $\xi \in \Sigma$ fairly close to the cone axis, then it is easy to find two other points $y, z \in \Sigma$ to generate a “voluminous” tetrahedron $T = (\xi, x, y, z)$ whose smallest height (of one point over the plane spanned by the remaining three points) is comparable to the diameter of T . Indeed, any segment parallel to the cone axis connecting the two components of $C(x) \setminus \{x\}$ must hit the (connected) surface Σ since they connect the surface’s interior to its exterior. Just choose two such appropriately spaced segments so that their intersection points y, z with Σ together with x and ξ span a voluminous tetrahedron; see Figure 7.

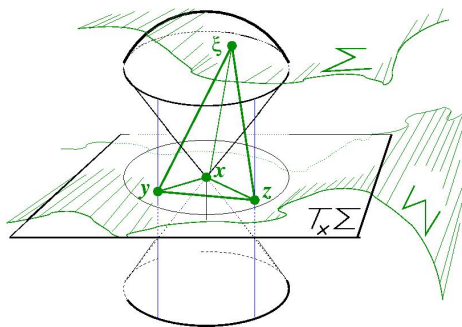


Figure 7. Voluminous tetrahedra.

The answer is “no”, but in that case a subtle algorithm can unfold: the growing cone can be turned suitably depending on the geometric situation, and one can grow further by doubling the cone’s size, possibly ignoring some parts closer to x , but always keeping a connected set with one portion in the interior and the other in the exterior of Σ . One continues that algorithm by growing and turning (possibly neglecting some subsets closer to x). After a finite number of steps one has to hit the compact surface Σ in a suitable point to find the voluminous tetrahedron; otherwise Σ would be unbounded. The quantitative reasoning to make this algorithm work is quite involved, but the outcome is valid for arbitrary surfaces and may be of independent interest (Strzelecki & von der Mosel (2011)):

Theorem 3.3. *For any closed, compact, connected Lipschitz surface Σ and every point $x \in \Sigma$ where the tangent plane $T_x \Sigma$ exists, one finds a positive stopping distance $d(x)$ and a voluminous tetrahedron $T = (\xi, x, y, z)$. In addition, the lower Ahlfors estimate (14) holds at such x .*

After this informal description it becomes clear that one does not need Lipschitz surfaces (guaranteeing tangent planes at almost every point). One simply needs a dense set of starting points $x \in \Sigma$, and the corresponding planes $P(x)$ orthogonal to which one can start growing cones for some positive distance. The second ingredient is topology: in co-dimension one, i.e., for hypersurfaces Σ , this is granted by a bounded interior and an unbounded exterior of Σ . In higher co-dimensions one has to add suitable linking conditions to define the class of admissible surfaces; see Strzelecki & von der Mosel (2013); Kolasiński *et al.* (2013a); Kolasiński (2011).

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