Hardy Space Estimates for Higher-Order Differential Operators

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ABSTRACT. We generalize a theorem of Coifman, Lions, Meyer and Semmes to higher-order differential operators (also with variable coefficients). Moreover, we present a direct, elementary proof of a duality inequality which is a special case of the duality of Hardy spaces and BMO. The tools used in this proof come from standard Sobolev space theory; no knowledge of Hardy spaces is required.

1. INTRODUCTION

Coifman, Lions, Meyer and Semmes in their celebrated paper [5] have proved that if two vector fields in conjugate Lebesgue spaces,

$$B \in L^q(\mathbb{R}^n)^n, \quad E \in L^{q/(q-1)}(\mathbb{R}^n)^n, \quad \text{where } q \in (1, \infty),$$

satisfy the equations $\operatorname{rot} B = \operatorname{div} E = 0$ in the sense of distributions, then their scalar product $E \cdot B$—which a priori, by Hölder inequality, is just integrable—belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, a strict subspace of $L^1(\mathbb{R}^n)$. It would be impossible to quote here all applications of this result in the theory of nonlinear PDE, so let us just mention the fundamental work of Hélein [11], [12] on the regularity of weakly harmonic maps from surfaces into arbitrary compact Riemannian manifolds, which gave rise to a stream of generalizations. The reader can consult [5], [12], [17], and their lists of references for more details (see also [10]).

The paper of Coifman, Lions, Meyer and Semmes contains a lot of examples of various nonlinear quantities which—due to some cancellations—have slightly better regularity than one might a priori expect (typically, $\mathcal{H}^1(\mathbb{R}^n)$ instead of $L^1(\mathbb{R}^n)$). The aim of this note is twofold. First of all, in Theorem 1 below, we give another straightforward example of this type, with higher-order differential operators replacing the divergence and rotation. This result follows in an implicit way from earlier work of Coifman and Grafakos [4, 8], but we give a direct proof, avoiding the theory of singular integrals and pseudodifferential operators. Second, we present a direct and relatively simple proof of an inequality which can be viewed as a special case of the duality of Hardy space and BMO. This proof is modelled on the proof of a duality lemma in [10], completely bypasses the theory of Hardy spaces, and—although our tools are definitely not new—seems to be of some independent interest even for $k = 1$.

To obtain these results, we use a mixture of ideas from [5], [17], and [10]. All necessary tools come from fairly standard Sobolev space theory. We hope that the theorems presented in this note can find applications in the theory of higher-order nonlinear PDE. The search for such applications (like, possibly, interpretation and extensions of the result of Chang, Wang, and Yang [3] on the regularity of biharmonic maps) shall be object of a future study.

2. Statement of Results

2.1. ***The notation.*** Throughout this note the notation is fairly standard. For open sets $\Omega \subset \mathbb{R}^n$, the Sobolev space of functions having all their distributional derivatives up to order m in L^s is denoted by $W^{m,s}(\Omega)$. Greek letters α, β and γ denote multiindices in $\mathbb{R}^n$. We employ the commonly used abbreviations: $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$ is the length of a multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where all α_i are nonnegative integers; we write $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$ and $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$. For a measurable set A and a function v, the barred integral denotes the mean value of v on A, i.e. $⨍_A v(x)\,dx = |A|^{-1}\int_A v(x)\,dx$; sometimes we also write $v_A = ⨍_A v\,dx$. For $v \in W^{m,1}_{\text{loc}}$ we write

$$T_z^m v(y) = \sum_{|\beta|\le m} D^\beta v(z)\frac{(y-z)^\beta}{\beta!}$$

to denote the Taylor polynomial of v; moreover,

$$T_A^m v(y) :\,= ⨍_A T_z^m v(y)\,dz$$

denotes the averaged Taylor polynomial of v. We often write p', q', r', etc. to denote Hölder conjugates of various exponents p, q, r, etc. $\in (1, +\infty)$. In all computations, C denotes a general constant whose value is not really important (and may change even in one string of inequalities). If $A, B \ge 0$, then $A \approx B$ means that $c(n)^{-1}A \le B \le c(n)A$ for some positive constant $c(n)$ which depends *only on the dimension*, n.

2.2. ***Hardy spaces.*** Let us recall first that a measurable function $f \in L^1(\mathbb{R}^n)$ belongs to the *Hardy space* $\mathcal{H}^1(\mathbb{R}^n)$ if and only if

$$f_* :\,= \sup_{\varepsilon>0}|\varphi_\varepsilon * f| \in L^1(\mathbb{R}^n).$$

Here, $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$, and φ is a fixed function of class $C_0^\infty(B(0,1))$ with $\int \varphi(y)\,dy = 1$. It turns out that the definition does not depend on the choice of φ (see [7]).

Equivalently, one can define $\mathcal{H}^1(\mathbb{R}^n)$ as the space of those elements of $L^1(\mathbb{R}^n)$, for which all the Riesz transforms $R_j f$, $j = 1, 2, \dots, n$, are also of class $L^1(\mathbb{R}^n)$. The reader is referred to [17] and [18, Chapters 3 and 4] for more details. Let us just mention here that $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_{\mathcal{H}^1} = \|f\|_{L^1} + \|f_*\|_{L^1}.$$

Moreover, the condition $f \in \mathcal{H}^1(\mathbb{R}^n)$ implies $\int f(y)\,dy = 0$.

C. Fefferman [6], [7] proved that the dual of $\mathcal{H}^1(\mathbb{R}^n)$ is equal to the space of functions of bounded mean oscillation, $BMO(\mathbb{R}^n)$. More precisely, there exists a constant C such that

$$\left| \int_{\mathbb{R}^n} h(y)\psi(y)\,dy \right| \le C\|h\|_{\mathcal{H}^1}\|\psi\|_{BMO} \tag{2.1}$$

for all $h \in \mathcal{H}^1(\mathbb{R}^n)$ and $\psi \in BMO(\mathbb{R}^n)$. This inequality is highly nontrivial since in general the integral on the left-hand side does not converge absolutely.

2.3. *Assumptions and results.* For sake of clarity, we state first the results is their simplest form. Some rather straightforward generalizations (e.g. to operators with uniformly bounded coefficients and to subelliptic calculus) are briefly discussed at the end of the paper.

Let $u \in W^{k,q}(\mathbb{R}^n)$ for some $k = 1, 2, \dots$ and $q \in (1, \infty)$, and let

$$E = (E_\alpha)_{|\alpha|=k} \in L^{q/(q-1)}(\mathbb{R}^n)^N,$$

where N denotes the number of all multiindices α of length k in $\mathbb{R}^n$. We also assume that $\nabla^k \cdot E = 0$ in the sense of distributions, i.e.

$$\sum_{|\alpha|=k} \int_{\mathbb{R}^n} E_\alpha(x) D^\alpha \varphi(x)\,dx = 0 \tag{2.2}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Note that by a standard limit argument, identity (2.2) holds in fact for all functions $\varphi \in W^{k,q}(\mathbb{R}^n)$.

Theorem 2.1. *Under all assumptions listed above, the function*

$$h = \sum_{|\alpha|=k} E_\alpha D^\alpha u$$

belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ and

$$\|h\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\|E\|_{L^{q'}(\mathbb{R}^n)}\|\nabla^k u\|_{L^q(\mathbb{R}^n)}, \tag{2.3}$$

where the constant C depends only on n, q and k.

As we have already noted in the Introduction, this result follows in an implicit way from the work of Coifman and Grafakos (see, in particular, [8, Theorem II A (2)]) who considered multilinear expressions formed by scalar products of two vectors of Calderon–Zygmund operators satisfying an additional 'vanishing moments' condition. To translate the assumptions above to the somewhat different setting of [4, 8], one may use the isomorphism of Sobolev spaces and Bessel potential spaces (or other representation formulae for Sobolev functions). This is not immediate, and since the explicit motivation of Coifman and Grafakos in [4, 8] was to investigate expressions involving only second order derivatives, we give an independent, direct proof in Section 3.

Theorem 2.2. *Assume that $n \geq 2$, u and E satisfy the above assumptions with $\mathbb{R}^n$ replaced by a ball $B(a, 10\varrho)$. There exists a constant $C = C(n, q, k)$ such that for all functions $\psi \in W_0^{1,n}(B(a, \varrho))$ we have*

$$\left| \int_{B(a,\varrho)} \psi \sum_{|\alpha|=k} E_\alpha D^\alpha u \, dx \right| \leq C \|\nabla \psi\|_{L^n} \|E\|_{L^{q'}} \|\nabla^k u\|_{L^q} ; \tag{2.4}$$

the norm of $\nabla \psi$ is taken on the smaller ball $B(a, \varrho)$, and two other norms, of E and $\nabla^k u$, on the larger ball $B(a, 10\varrho)$.

Note that in dimension $n = 1$ such inequality follows immediately from the imbedding of $W^{1,1}(\mathbb{R})$ into absolutely continuous functions.

For $k = 1$ this Theorem has been applied to prove regularity of p-harmonic maps in the borderline case, when p is equal to the dimension of the domain (see [12] and [10]).

For $n \geq 2$, as we have already mentioned, Theorem 2 follows easily from Theorem 1 and Fefferman's inequality (2.1): any function $\psi \in W_0^{1,n}(B(a, \varrho))$ belongs to $BMO(\mathbb{R}^n)$, since by Hölder and Poincaré inequalities

$$⨍_Q |\psi - \psi_Q| \, dx \leq \left(⨍_Q |\psi - \psi_Q|^n \, dx \right)^{1/n} \leq C(n) \left(\int_Q |\nabla \psi|^n \, dx \right)^{1/n}$$

for any cube $Q \subset \mathbb{R}^n$, and hence

$$\|\psi\|_{BMO} \equiv \sup_{Q \subset \mathbb{R}^n} ⨍_Q |\psi - \psi_Q| \, dx \leq C(n) \|\nabla \psi\|_{L^n(B(a,\varrho))} .$$

However, our direct proof presented in Section 4 bypasses the burden of the full proof of the duality of Hardy space and BMO and uses just the theory of Sobolev spaces. This road seems to be shorter and simpler to follow.

At the end of the paper, in Section 5, we discuss some extensions of these results to more general k-th order differential operators in $\mathbb{R}^n$, and to a subelliptic setting.

3. Proof of Theorem 1

We follow closely the proof of Theorem II.1 in the paper of Coifman, Lions, Meyer and Semmes, with some more or less obvious modifications which are neccessary to take higher order derivatives into account.

Fix $\varepsilon > 0$. As $\nabla^k \cdot E = 0$ in the sense of distributions, we have

$$\sum_{|\alpha|=k} \int_{\mathbb{R}^n} E_\alpha(y) D_y^\alpha \left[\varphi_\varepsilon(x-y) \left(u(y) - T_{B_\varepsilon}^{k-1} u(y) \right) \right] dy = 0 . \tag{3.1}$$

Here, $x \in \mathbb{R}^n$ is arbitrary, and $B_\varepsilon \equiv B(x,\varepsilon)$. By the Leibniz formula,

$$\begin{aligned}
&D_y^\alpha \left[\varphi_\varepsilon(x-y) \left(u(y) - T_{B_\varepsilon}^{k-1} u(y) \right) \right] \\
&= \varphi_\varepsilon(x-y) D^\alpha u(y) + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<m}} \binom{\alpha}{\beta} D_y^\beta \varphi_\varepsilon(x-y) D^\gamma \left(u(y) - T_{B_\varepsilon}^{k-1} u(y) \right) \\
&= \varphi_\varepsilon(x-y) D^\alpha u(y) \\
&\qquad + \sum_{\substack{\beta+\gamma=\alpha \\ |\gamma|<m}} \binom{\alpha}{\beta} D_y^\beta \varphi_\varepsilon(x-y) \left(D^\gamma u(y) - T_{B_\varepsilon}^{k-|\gamma|-1} D^\gamma u(y) \right) .
\end{aligned}$$

Therefore, identity (3.1) implies that $h = \sum_{|\alpha|=k|} E_\alpha D^\alpha u$ satisfies the equation

$$h * \varphi_\varepsilon(x) = - \sum_{|\alpha|=k} \sum_{\substack{\beta\le\alpha \\ \beta\neq 0}} \binom{\alpha}{\beta} S_{\alpha\beta}^\varepsilon(x) , \tag{3.2}$$

where

$$S_{\alpha\beta}^\varepsilon(x) = \int_{B_\varepsilon} E_\alpha(y) D_y^\beta \varphi_\varepsilon(x-y) \left(D^{\alpha-\beta} u(y) - T_{B_\varepsilon}^{k-|\alpha-\beta|-1} D^{\alpha-\beta} u(y) \right) dy .$$

We shall estimate each term $S_{\alpha\beta}^\varepsilon(x)$ separately. To this end, fix α with $|\alpha| = k$ and $\beta \le \alpha$ with $|\beta| = m \ge 1$. Denoting $D^{\alpha-\beta} u = v$, and applying triangle inequality, we obtain the following estimate:

$$|S_{\alpha\beta}^\varepsilon(x)| \le \|\varphi\|_{C^k} \varepsilon^{-m} \fint_{B_\varepsilon} |E_\alpha| \left| v - T_{B_\varepsilon}^{m-1} v \right| dy$$

By the Hölder inequality, the right-hand side does not exceed

$$C\varepsilon^{-m} \left(\fint_{B_\varepsilon} |E_\alpha|^{s'} dy \right)^{1/s'} \left(\fint_{B_\varepsilon} \left| v - T_{B_\varepsilon}^{m-1} v \right|^{s} dy \right)^{1/s} .$$

To estimate the second integral, we apply the classical higher-order Sobolev–Poincaré inequality,

$$(3.3)\qquad \left(\fint_{B_\varepsilon} \left|v - T^{m-1}_{B_\varepsilon} v\right|^s dy\right)^{1/s} \le C\varepsilon^m \left(\fint_{B_\varepsilon} |\nabla^m v|^{s_{*,m}}\, dy\right)^{1/s_{*,m}},$$

where $C = C(n,m,s)$ and $s_{*,m} = ns/(n+ms)$ is an exponent whose m-th Sobolev conjugate is equal to s. As $|\nabla^m v| = |\nabla^m (D^{\alpha-\beta} u)| \le |\nabla^k u|$, this gives

$$(3.4)\qquad \begin{aligned} |S^\varepsilon_{\alpha\beta}(x)| &\le C\left(\fint_{B_\varepsilon} |E_\alpha|^{s'}\, dy\right)^{1/s'} \left(\fint_{B_\varepsilon} |\nabla^k u|^{s_{*,m}}\, dy\right)^{1/s_{*,m}} \\ &\le C\left(\fint_{B_\varepsilon} |E_\alpha|^{s'}\, dy\right)^{1/s'} \left(\fint_{B_\varepsilon} |\nabla^k u|^{s_*}\, dy\right)^{1/s_*} \end{aligned}$$

Here, $s_* \equiv s_{*,1} : = ns/(n+s) \ge s_{*,m}$ for all $m \ge 1$, and to obtain the second line in (3.4), where both exponents do not depend on m any more, we simply apply Hölder inequality. In order to deal with quantities which are integrable with powers greater than 1, we want to have here $s' < q/(q-1) = q'$ and $s_* < p$, or, equivalently, $s > q$ and $1/s + m/n > 1/q$ for all $m = 1, 2, \ldots, k$. To this end, it is enough to choose

$$\frac{1}{s} = \frac{1}{q} - \delta \quad \text{for any fixed } \delta \in \left(0, \min\left(\frac{1}{q}, \frac{1}{n}\right)\right).$$

For such s, inequality (3.4) implies

$$\left|\sup_{\varepsilon>0} S^\varepsilon_{\alpha\beta}(x)\right| \le C[M(|E_\alpha|^{s'})(x)]^{1/s'} [M(|\nabla^k u|^{s_*})(x)]^{1/s_*},$$

where $M(\ldots)$ denotes the classical Hardy–Littlewood maximal function. By the Hardy–Littlewood maximal theorem,

$$[M(|E_\alpha|^{s'})]^{1/s'} \in L^{q/(q-1)}(\mathbb{R}^n), \quad [M(|\nabla^k u|^{s_*})]^{1/s_*} \in L^q(\mathbb{R}^n)$$

and their norms can be estimated by a constant times $\|E\|_{L^{q/(q-1)}(\mathbb{R}^n)}$ and $\|\nabla^k u\|_{L^q(\mathbb{R}^n)}$, respectively. Hence, we obtain

$$\int_{\mathbb{R}^n} \left|\sup_{\varepsilon>0} S^\varepsilon_{\alpha\beta}(x)\right| dx \le C\|E\|_{L^{q/(q-1)}(\mathbb{R}^n)} \|\nabla^k u\|_{L^q(\mathbb{R}^n)}.$$

Recall that, by (3.2), $h * \varphi_\varepsilon$ is a linear combination of finitely many terms $S^\varepsilon_{\alpha\beta}$. Therefore, we have

$$\int_{\mathbb{R}^n} \left|\sup_{\varepsilon>0} h * \varphi_\varepsilon(x)\right| dx \le C\|E\|_{L^{q/(q-1)}(\mathbb{R}^n)} \|\nabla^k u\|_{L^q(\mathbb{R}^n)}.$$

This completes the proof of Theorem 1. $\square$

4. Proof of Theorem 2

In this Section, we basically follow the proof of the duality Lemma 3.2 in [10], introducing some necessary changes to cope with higher-order derivatives. Apart from these, there is one major difference: we do not assume that $|E|^{q'} \le \text{const} \cdot |\nabla^k u|^q$, and the proof from [10], which yields an estimate of a suitable integrand by a single Riesz potential, simply does not work in this case. Therefore, we have to use a product of two different, well-chosen Riesz potentials in the estimates—and the proof contains a bit of novelty even in the case $k = 1$.

The overall strategy is as follows: first, we use a standard representation formula to express ψ in terms of its gradient and to write the integral $\int \psi h$, where $h = \sum_{|\alpha|=k} E_\alpha D^\alpha u$, in the form $\int A \nabla \psi$. Here, roughly speaking, A is a convolution of h times a cutoff function with the gradient of the fundamental solution of Laplace's operator. Next, employing the so-called Whitney decomposition and integrating by parts, we carefully estimate $|A|$ by a product of two generalized Riesz potentials (all these notions are defined below). Finally, we apply a version of fractional integration theorem to obtain higher integrability of these two potentials and to conclude that $|A|$ belongs to $L^{n/(n-1)}$. Since $|\nabla \psi|$ is in L^n, the theorem follows from the Hölder inequality.

Here are the details.

Step 1. It is enough to prove the theorem for $\psi \in C_0^\infty(B(a, \varrho))$, and then to use a density argument. Take a cutoff function $\eta \in C_0^\infty(B(a, \varrho + 2\lambda\varrho); [0, 1])$ with $\eta \equiv 1$ on $B(a, \varrho + \lambda\varrho)$ and

$$|\nabla^m \eta(x)| \le \frac{\text{const}}{\varrho^m} \quad \text{for all } x \text{ and all } m = 1, \dots, k. \tag{4.1}$$

Here, $\lambda \in (\frac{1}{100}, \frac{1}{3})$ is a number which shall be fixed later; because of the lower bound $\frac{1}{100}$ we may assume that the constant in the above inequality depends only on m and k.

Since $\psi \in C_0^\infty$, we have

$$\psi(x) = \int_{\mathbb{R}^n} \nabla \psi(y) K(x, y)\, dy\,,$$

where

$$K(x, y) = \frac{1}{n\omega_n} \frac{x - y}{|x - y|^n}.$$

Using this formula and the Fubini theorem, we rewrite the left hand side of inequality (2.4) as

$$\int_{B(a,\varrho)} \psi(x)h(x)\,dx = \int_{\mathbb{R}^n} \eta(x)\psi(x)h(x)\,dx \tag{4.2}$$
$$= \int_{B(a,\varrho)} A(y)\nabla\psi(y)\,dy\,,$$

where, as before, $h = \sum_{|\alpha|=k} E_\alpha D^\alpha u$, and

$$A(y) := \sum_{|\alpha|=k} \int_{\mathbb{R}^n} \eta(x)K(x,y)E_\alpha(x)D^\alpha u(x)\,dx \tag{4.3}$$

for $y \in B(a,\varrho)$.

We claim that

$$A \in L^{n/(n-1)}(B(a,\varrho)) \tag{4.4}$$

and

$$\|A\|_{L^{n/(n-1)}(B(a,\varrho))} \le C\|E\|_{L^{q/(q-1)}(B(a,10\varrho))}\|\nabla^k u\|_{L^q(B(a,10\varrho))}\,. \tag{4.5}$$

Once these two statements are proved, Theorem 2 follows from formula (4.2) and the Hölder inequality. We therefore proceed now to prove (4.4) and (4.5).

Step 2. Estimates of A, part 1: Whitney decomposition. Fix $y \in B(a,\varrho)$. We take the Whitney decomposition of $\mathbb{R}^n \setminus \{y\}$, i.e. a family of balls $B_i \equiv B(x_i, r_i)$, $i \in I$, such that

- (i) $\mathbb{R}^n \setminus \{y\} = \bigcup_{i\in I} B(x_i, r_i)\,,$
- (ii) $B(x_i, r_i/3)$ are pairwise disjoint,
- (iii) $r_i = \frac{1}{1000}\mathrm{dist}\,(x_i, y)$ and $B(x_i, 2r_i) \subset \mathbb{R}^n \setminus \{y\}$ for every $i \in I$,
- (iv) each point of $\mathbb{R}^n \setminus \{y\}$ belongs to at most $M = M(n)$ different balls $B(x_i, 2r_i)$.

With such a family of balls one can associate a smooth partition of unity $(\vartheta_i)_{i\in I}$ which satisfies the following conditions:

- (v) $\vartheta_i \in C_0^\infty(B(x_i, 2r_i); [0,1])$ for each $i \in I$,
- (vi) $\sum_{i\in I} \vartheta_i \equiv 1$ on $\mathbb{R}^n \setminus \{y\}$,
- (vii) $|\nabla^m \vartheta_i| \le \mathrm{const} \cdot r_i^{-m}$ for all $m = 1, \ldots, k$.

The construction of B_i and ϑ_i is standard and can be traced back to Whitney; see [18, pages 14–16] and [15, Chapter 1, Lemma 3.1 and its proof] for more details.

We use the identity $\sum \vartheta_i \equiv 1$ to write

$$A(y) = \sum_{i\in I} \int_{2B_i} \Phi_i(x,y)h(x)\,dx\,,$$

where $2B_i := B(x_i, 2r_i)$ and $\Phi_i(x,y) = \eta(x)K(x,y)\vartheta_i(x)$. Moreover, we can assume that for all indices $i \in I$ the ball $2B_i$ intersects the support of η. By triangle inequality, this implies $\bigcup_{i\in I} 2B_i \subset B(a, \varrho + 3\lambda\varrho)$. Since $\nabla^k \cdot E = 0$, we have, for each $i \in I$,

$$\sum_{|\alpha|=k} \int_{2B_i} E_\alpha(x) D_x^\alpha[\Phi_i(x,y)(u(x) - T_{2B_i}^{k-1}u(x))]\,dx = 0\,. \tag{4.6}$$

(Note that $y \notin 2B_i$ and therefore Φ_i is smooth with respect to x on $2B_i$.) As in the proof of Theorem 1, we use the Leibniz formula to obtain

$$\begin{aligned} D_x^\alpha[\Phi_i(x,y)(u(x) - T_{2B_i}^{k-1}u(x))] &= \Phi_i(x,y)D^\alpha u(x) \\ &+ \sum_{\substack{\beta\le\alpha\\ \beta\ne 0}} \binom{\alpha}{\beta} D_x^\beta \Phi_i(x,y)(D^{\alpha-\beta}u(x) - T_{2B_i}^{k-|\alpha-\beta|-1}D^{\alpha-\beta}u(x))\,. \end{aligned}$$

Therefore,

$$A(y) = -\sum_{i\in I} A_i(y)$$

with

$$A_i(y) := \sum_{|\alpha|=k} \sum_{\substack{\beta\le\alpha\\ \beta\ne 0}} \binom{\alpha}{\beta} A_{\alpha\beta i}(y)$$

and

$$A_{\alpha\beta i}(y) := \int_{2B_i} E_\alpha(x) D_x^\beta \Phi_i(x,y)(D^{\alpha-\beta}u(x) - T_{2B_i}^{k-|\alpha-\beta|-1}D^{\alpha-\beta}u(x))\,dx\,.$$

We now estimate $A_{\alpha\beta i}(y)$ for fixed $i \in I$, and fixed α and β with $|\alpha| = k$, $|\beta| = m$. Note that for all $x \in 2B_i$ we have $|x - y| \approx r_i$. Therefore, for any multiindex γ of length $|\gamma| \le k$ and for all points $x \in 2B_i$, we have the following estimates:

$$\begin{aligned} |D_x^\gamma K(x,y)| &\le \frac{C(\gamma,n)}{|x-y|^{n+|\gamma|-1}} \le \frac{C(\gamma,n)}{r_i^{n+|\gamma|-1}}\,, \\ |D^\gamma \eta(x)| &\le \frac{C(\gamma,n)}{\varrho^{|\gamma|}} \le \frac{C(\gamma,n)}{|x-y|^{|\gamma|}} \le \frac{C(\gamma,n)}{r_i^{|\gamma|}}\,. \end{aligned}$$

(To check the second one, notice that the derivatives of η do not vanish only on the annulus $\varrho + \lambda\varrho < |x-a| < \varrho + 2\lambda\varrho$; this yields an estimate $|x-y|$ from below.) Hence, applying the Leibniz formula again and using the estimates (vii) for derivatives of ϑ_i, we obtain

$$|D_x^\beta \Phi_i(x,y)| \le \frac{C(n,k)}{r_i^{n+m-1}} \quad \text{for all } x \in 2B_i \text{ and all } |\beta| = m. \tag{4.7}$$

This implies the estimate

$$|A_{\alpha\beta i}(y)| \le \frac{C(n,k)}{r_i^{m-1}} \fint_{2B_i} |E_\alpha(x)| \left| D^{\alpha-\beta}u(x) - T_{2B_i}^{k-|\alpha-\beta|-1} D^{\alpha-\beta}u(x) \right| \, dx\,.$$

Now, in order to obtain estimates below the natural exponents, we proceed as in the proof of inequality (3.4) in the last section, in the proof of Theorem 1. Applying the Hölder inequality with exponents $s' = s/(s-1)$ and s, where

$$\frac{1}{s} + \frac{1}{n} = \frac{n+q-1}{nq}, \tag{4.8}$$

we separate here $|E_\alpha|$ from the terms depending on u. Next, we apply the Sobolev–Poincaré inequality (3.3) to estimate the second integral, containing the s-th power of the difference of $D^{\alpha-\beta}u(x)$ and its averaged Taylor polynomial. Finally, we again apply Hölder inequality to this integral, to obtain an exponent which does not depend on m. Summation with respect to α, β and i yields

$$\begin{aligned} |A(y)| \le C(n,k,q) \sum_{i\in I} r_i \left(\fint_{2B_i} |E|^{p_1}\, dx \right)^{1/p_1} \\ \times \left(\fint_{2B_i} \left| \nabla^k u \right|^{p_2} dx \right)^{1/p_2}, \end{aligned} \tag{4.9}$$

where, since s was defined by (4.8), we have

$$p_1 = s' = \frac{nq}{nq-n+1} < q' \quad \text{and} \quad p_2 = \frac{ns}{n+s} = \frac{nq}{n+q-1} < q\,. \tag{4.10}$$

Step 3. Estimates of A, part 2: generalized Riesz potentials. In order to obtain an estimate of $|A|$ by a product of two potential operators, we next write

$$|A(y)| \le C(n,k,q) \cdot \Sigma_1 \cdot \Sigma_2\,,$$

where

$$\Sigma_1 = \sum_{i\in I} r_i^\nu \left(\fint_{2B_i} |E|^{p_1}\, dx \right)^{1/p_1}, \tag{4.11}$$

$$\Sigma_2 = \sum_{i\in I} r_i^{1-\nu} \left(\fint_{2B_i} \left| \nabla^k u \right|^{p_2} dx \right)^{1/p_2}. \tag{4.12}$$

Here, ν is some number in $(0,1)$ which shall be fixed later. Note that the product of Σ_1 and Σ_2 gives the whole sum on the right-hand side of (4.9), plus many extra positive terms. Before proceeding further, let us pause for a while and recall one

4.1. *Definition.* For $g \in L^p(B(a,10\varrho))$, the *generalized Riesz potential* $J_{\nu,p}g(y)$, where $\nu > 0$, is given by

$$J_{\nu,p}g(y) = \sum_{\ell=-\infty}^{[\log_2 9\varrho]} 2^{\ell\nu} \left(⨍_{B(y,2^\ell)} |g(x)|^p \, dx \right)^{1/p} .$$

Here, y is an arbitrary point in $B(a,\varrho)$. For such y and for $\ell \le [\log_2 9\varrho]$, we have $B(y,2^\ell) \subset B(a,10\varrho)$ and all the integrals in the above sum are well defined.

4.2. *Remark.* In [9], Hajłasz and Koskela have considered more sophisticated variants of this definition. Lemma 4.1 below is a simplified version of Theorem 5.3 (part 2) from their paper. It can be obtained in a rather straightforward manner, by mimicking Hedberg's proof of the classical Hardy–Littlewood–Sobolev fractional integration theorem. An interested reader may check that it is also possible to estimate $|A(y)|$ by a product of appropriate powers of two the *classical* local Riesz potential, and obtain Theorem 2 via an application of the classical fractional integration theorem. This, however, would require a bit more care in the last piece of the reasoning below.

Lemma 4.1. *Assume that* $g \in L^q(B(a,10\varrho))$ *and that* $0 < p < q < n/\nu$. *Then,* $J_{\nu,p}g \in L^{q^*}(B(a,\varrho))$, *where* $q^* = nq/(n-\nu q)$, *and we have the estimate*

$$\|J_{\nu,p}g\|_{L^{q^*}(B(a,\varrho))} \le C(n,p,q,\nu)\|g\|_{L^q(B(a,10\varrho))} .$$

To estimate each of the sums Σ_1 and Σ_2 by an appropriate generalized Riesz potential, we proceed as follows. Fix an integer ℓ and set

$$I_\ell : = \{ i \in I : \, 2^{\ell-2} \le \operatorname{dist}(x_i, y) < 2^{\ell-1} \} .$$

As $x_i \in B(a,\varrho + 3\lambda\varrho)$ and $y \in B(a,\varrho)$, we only have to consider $\ell \le \ell_{\max}$, where $\ell_{\max}$ is defined by the inequalities

$$2^{\ell_{\max}-2} < 2\varrho + 3\lambda\varrho \le 2^{\ell_{\max}-1} .$$

(For $\ell > \ell_{\max}$ the set I_ℓ is just empty.) If $i \in I_\ell$, then $2B_i \subset B(y,2^\ell)$ and $r_i \approx 2^\ell$; hence, fixing $\lambda = \frac{1}{12}$, we see that for each ℓ and each $i \in I_\ell$,

$$2B_i \subset B(y,2^\ell) \subset B(y,2^{\ell_{\max}}) \subset B(a,10\varrho) .$$

Moreover, we have $\operatorname{card} I_\ell \le K(n)$, where $K(n)$ is some universal constant depending only on n. This follows from the properties (i)–(iv) of the family $\{B(x_i, r_i) : i \in I\}$, as

$$\begin{aligned} \omega_n 2^{\ell n} &= |B(y, 2^\ell)| \ge \Big| \bigcup_{i \in I_\ell} B(x_i, r_i/3) \Big| \\ &= \sum_{i \in I_\ell} |B(x_i, r_i/3)| \ge \omega_n \operatorname{card} I_\ell \left(\frac{2^{\ell-2}}{3000} \right)^n . \end{aligned}$$

Therefore,

$$\begin{aligned} \Sigma_1 &= \sum_{i \in I} r_i^\nu \left(\fint_{2B_i} |E|^{p_1}\, dx \right)^{1/p_1}, \\ &\le \sum_{\ell=-\infty}^{[\log_2 9\varrho]} \sum_{i \in I_\ell} 2^{\ell\nu} \left(\frac{2^\ell}{r_i} \right)^{n/p_1} \left(\fint_{B(y,2^\ell)} |E|^{p_1}\, dx \right)^{1/p_1}, \\ &\le C(n,q) \sum_{\ell=-\infty}^{[\log_2 9\varrho]} 2^{\ell\nu} \left(\fint_{B(y,2^\ell)} |E|^{p_1}\, dx \right)^{1/p_1} = C(n,q) J_{\nu,p_1} |E|(y) . \end{aligned}$$

Similarly, $\Sigma_2 \le C(n,q) J_{1-\nu,p_2} |\nabla^k u|(y)$. Thus

$$A(y) \le C(n,q,k) \cdot J_{\nu,p_1} |E|(y) \cdot J_{1-\nu,p_2} |\nabla^k u|(y) . \tag{4.13}$$

Applying now Lemma 4.1, we see that

$$F_1 : = J_{\nu,p_1} |E| \in L^{q_1^*}(B(a,\varrho)) , \qquad q_1^* = \frac{nq'}{n - \nu q'} , \tag{4.14}$$

$$F_2 : = J_{1-\nu,p_2} |\nabla^k u| \in L^{q_2^*}(B(a,\varrho)) , \quad q_2^* = \frac{nq}{n - (1-\nu) q} , \tag{4.15}$$

provided that $n/\nu > q' = q/(q-1)$ and a$n/(1-\nu) > q$, or, equivalently,

$$1 - \frac{n}{q} < \nu < n - \frac{n}{q} . \tag{4.16}$$

It is clear that for $n \ge 2$ we can choose some $\nu \in (0,1)$ which satisfies these inequalities.

Finally, by the Hölder inequality and Lemma 1,

$$\begin{aligned} &\int_{B(a,\varrho)} |A(y)|^{n/n-1}\, dy \\ &\quad \le C \left(\int_{B(a,\varrho)} |F_1(y)|^{bn/(n-1)}\, dy \right)^{1/b} \left(\int_{B(a,\varrho)} |F_2(y)|^{b'n/(n-1)}\, dy \right)^{1/b'} \end{aligned}$$

if there exists a pair of Hölder conjugate exponents b and b' such that $bn/(n-1) = q_1^*$ and $b'n/(n-1) = q_2^*$. However, a trivial check shows that if q_1^* and q_2^* are as in (4.14), (4.15) above, then the exponents $b = (n-1)q_1^*/n$ and $b' = (n-1)q_2^*/n$ are indeed Hölder conjugate for *every* ν which satisfies condition (4.16). This observation completes the proof of claims (4.4) and (4.5), and of the whole theorem. □

5. Some generalizations and remarks

5.1. In the case of first order derivatives, $k = 1$, Chanillo [1] has given another direct proof, also bypassing the theory of Hardy spaces, of a slightly different duality inequality. His conditions on ψ are slightly weaker than $\psi \in W^{1,n}$. He assumes that $\psi \in W^{1,2}$ and $\sup_{x,r} r^{2-n} \int_{B(x,r)} |\nabla\psi|^2\, dy \le C$ for some constant C—which obviously implies that ψ belongs to BMO—and obtains also a weighted variant of the inequality, with a weight in the Muckenhoupt class A_2. On the other hand, he treats only the case $q = q' = 2$. Later, Chanillo and Li have noted [2, Lemma 2.2] that Chanillo's proof extends also, via standard Littlewood–Paley theory, to the case $\sup_{x,r} r^{p-n} \int_{B(x,r)} |\nabla\psi|^p\, dy \le C$.

5.2. It is clear that with practically the same proofs both theorems can be extended to the following situation. Let

$$L = \mathcal{A}(x)\nabla^k ,$$

where $\mathcal{A}(x) = (\mathcal{A}_{\alpha\beta}(x))$ is a symmetric $N \times N$ matrix, with entries indexed by multiindices of length k in $\mathbb{R}^n$. Assume that $\mathcal{A}_{\alpha\beta} \in L^\infty(\mathbb{R}^n)$ and set $\mu_0 = \max_{\alpha,\beta} \|\mathcal{A}_{\alpha\beta}\|_{L^\infty}$. Moreover, replace the assumption $\nabla^k \cdot E = 0$ by

$$L^* \cdot E = 0 \qquad \text{in the sense of distributions,}$$

where L^* denotes the formal adjoint of L. Then, for $h = E \cdot Lu$, where $u \in W^{k,q}$ and $E \in L^{q'}$ on appropriate sets, analogues of Theorem 1 and Theorem 2 hold. One just has to introduce obvious changes in the integral identities (**??**), (4.6) and to replace the constants in the inequalities (2.3) and (2.4) by $C(n,k,q) \cdot \mu_0$. (No smoothness of the coefficients $\mathcal{A}_{\alpha\beta}$ is required.)

5.3. With natural modifications of the assumptions, in particular upon replacement of n by the so-called homogeneous dimension, Theorem 2 extends to other, more general situations, e.g. to calculus on Carnot groups. All important ingredients of the proof (Whitney decomposition, a suitable representation formula for smooth compactly supported ψ, estimates of all derivatives of the fundamental solution of the subelliptic laplacian, and fractional integration theorem for generalized Riesz potentials) are available in this general setting. The key is to be able to use polynomials and variants of higher-order *local* Sobolev inequalities. These have been obtained by G. Lu, and G. Lu and R.L. Wheeden in [13], [14].

Our computations heavily rely on the fact that derivatives of Taylor polynomials are equal to Taylor polynomials of derivatives. A suitable replacement for this property is provided by Lu in [13, Theorem 5.3]. Using this result, one may obtain a subelliptic version of Theorem 2 (generalizing the duality lemma of [10] to higher-order derivatives along suitable vector fields). Details are left to the reader.

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ACKNOWLEDGMENT: The author is partially supported by a KBN grant no. PO3A–055–14. He also gratefully acknowledges the generous support of Alexander von Humboldt Foundation, and the warm hospitality of Mathematisches Institut der Universität Bonn, where this research has been carried out. In particular, many thanks are due to Professor Stefan Hildebrandt.

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