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Tangent-point self-avoidance energies for curves

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Abstract

We study a two-point self-avoidance energy $\mathscr{E}_q$ which is defined for all rectifiable curves in $\mathbb{R}^n$ as the double integral along the curve of $1/r^q$. Here r stands for the radius of the (smallest) circle that is tangent to the curve at one point and passes through another point on the curve, with obvious natural modifications of this definition in the exceptional, non-generic cases. It turns out that finiteness of $\mathscr{E}_q(\gamma)$ for $q \geq 2$ guarantees that γ has no self-intersections or triple junctions and therefore must be homeomorphic to the unit circle $\mathbb{S}^1$ or to a closed interval I. For $q > 2$ the energy $\mathscr{E}_q$ evaluated on curves in $\mathbb{R}^3$ turns out to be a knot energy separating different knot types by infinite energy barriers and bounding the number of knot types below a given energy value. We also establish an explicit upper bound on the Hausdorff-distance of two curves in $\mathbb{R}^3$ with finite $\mathscr{E}_q$-energy that guarantees that these curves are ambient isotopic. This bound depends only on q and the energy values of the curves. Moreover, for all q that are larger than the critical exponent $q_{\rm crit} = 2$, the arclength parametrization of γ is of class $C^{1,1-2/q}$, with Hölder norm of the unit tangent depending only on q, the length of γ, and the local energy. The exponent $1-2/q$ is optimal.

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1 Introduction

Imagine a space craft travelling with constant speed along an unknown and possibly quite irregular closed path Γ in an unexplored territory of the universe. After some time $L > 0$ the loop is completed at least once, and the only data the astronauts can measure at time t are the ratios of the squared distance from any previous position $\Gamma(s)$, to the distance of the current line of direction $\ell(t)$ from that previous position $\Gamma(s)$, i.e., the quotients

$$2r(\Gamma(t),\Gamma(s)) := \frac{|\Gamma(t)-\Gamma(s)|^2}{\mathrm{dist}(\ell(t),\Gamma(s))} \in [0,\infty] \quad \text{for } s < t. \tag{1.1}$$

What can the astronauts say about their path of travel? In other words, how much information about a closed curve of finite length in Euclidean space is encoded in the relative tangent-point data (1.1)? The answer is: If the astronauts obtain a finite integral mean of some inverse power of all these data (after time $2L$) they can extract essential topological information as well as explicit smoothness properties of their path of travel!

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To make this precise we assume from now on that the path $\Gamma \subset \mathbb{R}^n$ is a rectifiable curve of finite length, parametrized by arclength on the circle $S_L \cong \mathbb{R}/(L\mathbb{Z})$ of perimeter L. Hence, Γ is a (not necessarily injective) Lipschitz continuous mapping with $|\Gamma'| = 1$ a.e. on S_L. Geometrically, the tangent-point function

$$r(\Gamma(t),\Gamma(s)) = \frac{|\Gamma(t)-\Gamma(s)|^2}{2\,\mathrm{dist}(\ell(t),\Gamma(s))} = \frac{|\Gamma(t)-\Gamma(s)|}{2\sin\sphericalangle(\Gamma'(t),\Gamma(s)-\Gamma(t))},$$

involving the tangent line $\ell(t) := \{\Gamma(t)+\mu\Gamma'(t) : \mu \in \mathbb{R}\}$ and defined for all $s \in S_L$ and almost all $t \in S_L$, determines the radius of the unique circle that is tangent to Γ at the position $\Gamma(t)$ and passes through $\Gamma(s)$. (This radius is set to be zero if $\Gamma(t) = \Gamma(s)$, and is infinite if the vector $\Gamma(s)-\Gamma(t) \neq 0$ is parallel to the tangent $\Gamma'(t)$).

The only assumption in the result indicated above is finiteness of the *tangent-point potential*

$$\mathscr{E}_q(\Gamma) := \int_0^L \int_0^L \frac{ds\,dt}{r^q(\Gamma(t),\Gamma(s))} \quad \text{for some } q \geq 2. \tag{1.2}$$

Theorem 1.1 (Finite energy path is a manifold)**.** *If $\mathscr{E}_q(\Gamma) < \infty$ for some $q \geq 2$ then the image $\Gamma(S_L)$ is a one-dimensional topological manifold (possibly with boundary), embedded in $\mathbb{R}^n$.*

In particular, the image curve has no self-intersections, although there is no chance to deduce injectivity of the arclength parametrization Γ itself, since the integrand depends only on the image $\Gamma(S_L)$. Take, for example, a k-times covered circle of length L/k, for which the integrand is constant, $r(\Gamma(t),\Gamma(s)) \equiv r_0$ for all $s,t \in S_L$, so that the energy amounts to

$$\mathscr{E}_q(k\text{-times covered circle}) = \frac{L^2}{r_0^q} = k^2 \int_0^{L/k} \int_0^{L/k} \frac{ds\,dt}{r_0^q} = k^2 \mathscr{E}_q(\text{once-covered circle}) < \infty.$$

So the space craft's course cannot be too wild, since it traces a one-dimensional manifold without any non-tangential self-crossings. But without further input the astronauts have no clue of how often they have completed that course. Moreover, in case their path forms a manifold with boundary, say, a circular arc, there would be an abrupt (and for the crew probably quite noticeable) change of direction at the endpoints of that arc. Mathematically, one can easily reparametrize the manifold to obtain a new injective arclength parametrization, which translates to the additional information that the spacecraft does not pass by any previous position at all, $\Gamma(t) \neq \Gamma(s)$ for all $t \neq s$, which we will assume from now on.

In light of Theorem 1.1 the tangent-point potential $\mathscr{E}_q$ evaluated on closed curves in $\mathbb{R}^3$ may serve as a valid *knot energy* as suggested by Gonzalez and Maddocks in [12, Section 6], that is, as a functional separating different knot types by infinite energy barriers. It was shown by Sullivan [26, Prop. 2.2] that for $q > 2$ the energy $\mathscr{E}_q$ blows up on a sequence of smooth knots converging smoothly to a smooth curve with self-crossings. (His proof uses the Taylor formula up to order two for the converging curves, and a uniform bound for the remainders.) As a consequence of our analysis we generalize this result to continuous curves replacing smooth convergence by uniform convergence (see Proposition 5.1). Thus $\mathscr{E}_q$ for $q > 2$ is indeed *self-repulsive* or *charge*, and hence a knot energy according to the definition given by O'Hara [16, Def. 1.1], which provides an affirmative answer to an open question posed in [16, Problem 8.1]. It also turns out that $\mathscr{E}_q$ is *strong* for $q > 2$: among all continuous closed curves γ of fixed length L and $\mathscr{E}_q(\gamma) < E$ there are only finitely many knot types, see Proposition 5.2. This gives a partial answer to a conjecture expressed by Sullivan in [26, p. 184] (leaving open the case $q = 2$, and we do not consider links with more than one component). Both these knot-theoretic results are based on a priori $C^{1,\alpha}$-estimates for curves of finite $\mathscr{E}_q$-energy, discussed in more detail later on.

We will show in addition that two curves, whose Hausdorff-distance is bounded above by an explicit small constant depending only on the energy values, are in fact in the same knot class. A qualitative version of such an isotopy result is well-known in the smooth category; see e.g. [14, Chapter 8], or [2]. Here, however, we have explicit quantitative bounds. Notice also that Hausdorff-distance alone, no matter how small, does not suffice to separate knot classes;[1] bounded $\mathscr{E}_q$-energy is crucial here.

Theorem 1.2 (Isotopy). *For any $q > 2$ there is an explicit constant $\delta(q) > 0$ depending only on q such that any two closed rectifiable curves with injective arclength parametrizations Γ_1, Γ_2, with finite $\mathscr{E}_q$-energy, are ambient isotopic if their Hausdorff-distance is less than*

$$\delta(q)\max\{\mathscr{E}_q(\Gamma_1),\mathscr{E}_q(\Gamma_2)\}^{-\frac{1}{q-2}}.$$

Our proof of Theorem 1.2 follows closely the arguments of Marta Szumańska who proved in her Ph.D. thesis a similar result [27, Chapter 5] for a related three-point potential, the *integral Menger curvature*

$$\mathscr{M}_p(\Gamma) := \int_0^L\int_0^L\int_0^L \frac{ds\,dt\,d\sigma}{R^p(\Gamma(s),\Gamma(t),\Gamma(\sigma))}, \quad p > 3, \tag{1.3}$$

where $R(x,y,z)$ denotes the circumcircle radius of three points x,y,z in Euclidean space. Essentially one reduces the isotopy question to that between polygons inscribed in Γ_1 and Γ_2, whose edge lengths are solely controlled in terms of the energy. For polygonal knots a similar result is contained in the work of Millet, Piatek, and Rawdon [15, Theorem 4.2], where instead of (1.3), the polygonal thickness of the polygons together with their edge length determines the smallness condition on the Hausdorff distance that guarantees isotopy of two polygonal knots. For general curves, thickness was defined by Gonzalez and Maddocks in [12] as the smallest possible circumcircle radius $R(\cdot,\cdot,\cdot)$ when evaluated on all triples of distinct curve points. This concept of thickness was used as a tool in variational applications involving curves and elastic rods subject to various topological constraints; see e.g. [13], [6], [18]–[20], [10], [11], and has been studied numerically, [7], [8], [1].

The inverse of thickness of a curve Γ can be obtained as limits $\mathscr{M}_p^{1/p}(\Gamma)$ for $p \to \infty$, or $\mathscr{E}_q^{1/q}(\Gamma)$ for $q \to \infty$. In our papers [23, 21, 22] we have studied regularizing, self-avoidance and compactness effects of several integral energies, including $\mathscr{M}_p$, which involve, vaguely speaking, various bounds for $1/R$ understood as a function of three variables, including bounds in L^p, in $L^p(X_1, L^\infty(X_2))$ where $X_1 = S_L$ and $X_2 = S_L \times S_L$ (or vice versa), and in spaces that resemble the classic Morrey spaces $L^{p,\lambda}$. In each case we were able to detect similar phenomena: there is a certain limiting exponent for which an appropriate functional is scale invariant, and above this exponent three sorts of effects take place. First, curves with finite energy have no self-intersections. Second, these energies serve well as knot energies allowing for valuable compactness results for equibounded families of loops in fixed isotopy classes, which is due to the third, the regularizing effect: Curves with finite energy are more regular than initially assumed.

For the present tangent-point potential $\mathscr{E}_q$ we obtain the following regularity theorem, which shows that the astronauts would not experience any sudden change of direction during their travel.

Theorem 1.3 (Regularity). *If $q > 2$ and the arclength parametrization $\Gamma : S_L \to \mathbb{R}^n$ is chosen to be injective, then Γ is continuously differentiable with a Hölder continuous tangent, i.e., Γ is of class*

[1] Consider for example two different torus knots on the surface of a very thin rotational torus; for the classification of torus knots see e.g. Burde and Zieschang [5, Chapter 3.E].

$C^{1,1-(2/q)}$. *More precisely, for each* $q>2$ *there exist two constants* $\delta(q)>0$ *and* $c(q)<\infty$ *depending only on* q *such that each injective arclength parametrization* Γ *with* $\mathscr{E}_q(\Gamma)<\infty$ *satisfies*

$$|\Gamma'(u)-\Gamma'(v)|\le c(q)\left(\int_u^v\int_u^v \frac{ds\,dt}{r(\Gamma(s),\Gamma(t))^q}\right)^{1/q}|u-v|^{1-2/q} \tag{1.4}$$

for all $u,v\in S_L$ *with* $|u-v|\le\min\big(\delta(q)\mathscr{E}_q(\Gamma)^{-1/(q-2)},\frac12\operatorname{diam}\gamma\big)$.

The exponent $q=2$ is a limiting one here. It is relatively easy to use scaling arguments and check that $\mathscr{E}_q(\Gamma)=\infty$ for each $q\ge 2$ when Γ parametrizes a closed polygonal curve, but polygons have finite energy for all $q<2$. The resulting Hölder exponent $1-2/q$ is reminiscent of the classic Sobolev imbedding theorem in the supercritical case: the domain of integration is two-dimensional, and the integrand is related to curvature. For C^2-curves the behaviour of $1/r$ close to the diagonal of $S_L\times S_L$ (where $1/r$ might blow up for curves with low regularity) encodes some information about curvature, i.e. about second derivatives of the arclength parametrization Γ. The point is that we need no information about the existence of Γ'' in order to prove Theorem 1.3. A priori, we deal with curves that are rectifiable only, and even the existence of Γ' at all parameters cannot be taken for granted.

Note that inequality (1.4) is qualitatively optimal: for curves of class $C^{1,1}$ the integrand $1/r$ is bounded, and (1.4) yields then $|\Gamma'(u)-\Gamma'(v)|\lesssim|u-v|\sup(1/r)$ for u,v sufficiently close; nothing stronger can be expected as the familiar example of a stadium curve shows. We discuss other examples briefly at the end of Section 6.

Before describing the main ideas of the proof and the structure of the paper we would like to mention that while working on generalizations of self-avoidance energies to surfaces in $\mathbb{R}^3$, see [24], which involved a search for suitable integrands, we have realized that $\mathscr{E}_q$ is a model energy that might be the easiest one to extend to the fully general case, i.e. to submanifolds of arbitrary dimension and co-dimension [25]. This was one of the motivations to write the present note: to lay out in a simple, relatively easily tractable case all the arguments that should be applicable in much greater generality.

Theorem 1.1 is obtained as a corollary of a slightly more general result, see Theorem 1.4 below. We first prove a technical lemma (see Section 2) which shows how $\mathscr{E}_q$ can be used to control the behaviour of the so-called P. Jones' *β-numbers*,

$$\beta_\gamma(x,r):=\inf\left\{\sup_{y\in\gamma\cap B(x,r)}\frac{\operatorname{dist}(y,G)}{r}\quad:\quad G\text{ is a straight line through }x\right\}, \tag{1.5}$$

for small $r>0$ and closed balls $B(x,r)$ of radius r with center x. It turns out that if $\mathscr{E}_2(\Gamma)<\infty$ then $\beta_\gamma(x,r)\to 0$ as $r\to 0$ uniformly with respect to x, see Lemma 2.3 and the remark at the end of Section 2. And this is the key point to prove that $\gamma=\Gamma(S_L)$ is a topological manifold, as we have the following.

Theorem 1.4. *If* $\Gamma\colon S_L\to\mathbb{R}^n$ *is arclength, and the image* $\gamma=\Gamma(S_L)$ *satisfies*

$$\sup_{x\in\gamma}\beta_\gamma(x,d)\le\omega(d) \tag{1.6}$$

where $\omega\colon[0,L]\to\mathbb{R}$ *is a continuous nondecreasing function with* $\omega(0)=0$*, then* γ *is a one-dimensional submanifold of* $\mathbb{R}^n$ *(possibly with boundary).*

The main idea behind the proof of Theorem 1.4 is simple: if the result were false, then we could find a point x in γ where a *triple junction* occurs; in a small ball B centered at x we would have (at least) three disjoint arcs of γ in a long narrow tube. Two of them would then be very close (i.e., would leave B crossing ∂B in the same spherical cap at one end of the tube). Observing points of those two arcs, and using (1.6) on smaller and smaller scales, we are able to obtain a contradiction and eventually show that there could be no triple junction at x. For details, see Section 3.

By the preliminary results of Section 2, if $\mathscr{E}_q(\Gamma) < \infty$ for some $q > 2$, then the control of β numbers is much better than just (1.6). Namely,

$$\sup_{x\in\gamma} \beta_\gamma(x,r) \lesssim r^\kappa \tag{1.7}$$

for $\kappa = (q-2)/(q+4) < \lambda = 1-2/q$; the constant in (1.7) depends on $\mathscr{E}_q(\Gamma)$. Applying (1.7) iteratively, we find in Section 4 suitably defined cones that contain short arcs of γ and obtain an estimate for their opening angles, proving that Γ' exists everywhere and is of class[2] C^κ.

Section 5 contains the proof of the isotopy result, Theorem 1.2. In the last section we show how to bootstrap the initial gain of $C^{1,\kappa}$-regularity obtained in Section 4, to the optimal regularity $\Gamma \in C^{1,1-(2/q)}$, and we will establish (1.4). We stress the fact that Inequality (1.4) in Theorem 1.3 provides a uniform a priori estimate. This can be used in variational applications and to ensure compactness for infinite families of curves with uniformly bounded energy. Some results of that type have been stated in [21, 22]; we do not follow that thread here.

Finally, let us say that, at the moment, we have no clue how Γ' behaves in the limiting case $q = 2$ (we do not even know if it is defined everywhere for curves with finite $\mathscr{E}_2$-energy) but we are tempted to think that Γ' has vanishing mean oscillation for $q = 2$ and that local oscillations of the tangent can be controlled by the local energy of the curve.

Notation

We write $G(x,y)$ to denote the straight line through two distinct points $x,y \in \mathbb{R}^n$. If $x = \Gamma(s), y = \Gamma(t) \in \gamma := \Gamma(S_L) \subset \mathbb{R}^n$, then, abusing the notation slightly, we write sometimes $G(s,t)$ instead of $G(\Gamma(s),\Gamma(t))$.

For a closed set F in $\mathbb{R}^n$ we set

$$U_\delta(F) := \{x \in \mathbb{R}^n : \operatorname{dist}(x,F) < \delta\}, \qquad \delta > 0.$$

In some places, it will be more convenient to work directly with the slabs U_δ around appropriately selected lines than to deal with the information expressed only in the language of β-numbers. Finally, in Section 4 we work with cones. For $x \neq y \in \mathbb{R}^n$ and $\varepsilon \in (0, \frac{\pi}{2})$ we denote by

$$C_\varepsilon(x;y) := \{z \in \mathbb{R}^n : \exists\, t \neq 0 \text{ such that } \sphericalangle(t(z-x), y-x) < \frac{\varepsilon}{2}\} \tag{1.8}$$

the double cone whose vertex is at the point x, with cone axis passing through y, and with opening angle ε. All balls $B(x,r)$ with radius $r > 0$ and center $x \in \mathbb{R}^n$ are closed balls throughout the paper.

[2] Let us remark that for an m-dimensional set $\Sigma \subset \mathbb{R}^n$ that is *Reifenberg flat with vanishing constant* uniform estimates of β-numbers imply that Σ is a $C^{1,\kappa}$-manifold, see David, Kenig and Toro [9] and Preiss, Tolsa and Toro [17]. Here, we have no Reifenberg flatness a priori – in general rectifiable curves do not have to be Reifenberg flat; in fact we prove it by hand, using energy bounds leading to (1.7).

2 Decay of beta numbers

Lemma 2.1. *Let $\mathscr{E}_q(\Gamma)$ be finite. There exists a constant $c_0 = c_0(q) > 0$ such that if $\varepsilon < 1/200$ and $d < \operatorname{diam}\gamma$ satisfy*

$$\varepsilon^{4+q} d^{2-q} \geq c_0(q)\mathscr{E}_q(\Gamma)\,, \tag{2.9}$$

then for every two points of the curve such that $|\Gamma(s)-\Gamma(t)| = d$ we have

$$\gamma \cap B_{2d}\big(\Gamma(s)\big) \subset U_{20\varepsilon d}\big(G(s,t)\big)\,.$$

In particular,

$$\beta_\gamma(\Gamma(s), 2d) \leq 10\varepsilon.$$

For $q > 2$ we set $\kappa = (q-2)/(q+4)$.

Corollary 2.2. *There exists a $\delta_1 = \delta_1(q) > 0$ such that if $\mathscr{E}_q(\Gamma)^{1/(q+4)} d^\kappa < \delta_1$, then*

$$\beta_\gamma(\Gamma(s), 2d) \leq c_1(q)\mathscr{E}_q(\Gamma)^{1/(q+4)} d^\kappa\,.$$

Proof of Lemma 2.1. For $s,t \in S_L$, $d = |\Gamma(s)-\Gamma(t)| > 0$ and $\varepsilon > 0$ small, we set

$$\begin{aligned}
A_d(s,\varepsilon) &:= \Gamma^{-1}(B_{\varepsilon^2 d}(\Gamma(s)) = \{\tau \in S_L :\ \Gamma(\tau) \in B_{\varepsilon^2 d}(\Gamma(s))\}\,,\\
X_d(s,t,\varepsilon) &:= \{\sigma \in A_d(s,\varepsilon) :\ \Gamma'(\sigma) \text{ exists with } \sphericalangle(\Gamma'(\sigma), \Gamma(t)-\Gamma(s)) \in \big[\tfrac{\varepsilon}{10}, \pi - \tfrac{\varepsilon}{10}\big]\}\,,\\
N_d(s,t,\varepsilon) &:= A_d(s,\varepsilon) \setminus X_d(s,t,\varepsilon)\,.
\end{aligned}$$

Note that $|A_d(s,\varepsilon)| \geq 2\varepsilon^2 d$. The proof has two steps:

- we use the inequality
$$\mathscr{E}_q(\Gamma) \geq \int_{X_d(s,t,\varepsilon)} \int_{A_d(t,\varepsilon)} r^{-q}\, d\sigma\, d\tau$$
to show that $X_d(s,t,\varepsilon)$ must be a small subset of $A_d(s,\varepsilon)$, so that $|N_d(s,t,\varepsilon)| \gtrsim \varepsilon^2 d$;
- we argue by contradiction, using energy estimates again, and show the desired inclusion.

Step 1. Fix $\sigma \in X_d(s,t,\varepsilon)$ and $\tau \in A_d(t,\varepsilon)$. We shall show that $1/r((\Gamma(\sigma),\Gamma(\tau)) \gtrsim \varepsilon/d$.

Since $|\Gamma(s)-\Gamma(t)| = d$, the triangle inequality yields

$$2d > d(1+2\varepsilon^2) \geq |\Gamma(\sigma)-\Gamma(\tau)| \geq d(1-2\varepsilon^2)\,. \tag{2.10}$$

Let

$$x := \Gamma(\sigma) + d\frac{\Gamma(t)-\Gamma(s)}{|\Gamma(t)-\Gamma(s)|} = \Gamma(\sigma) + \big(\Gamma(t)-\Gamma(s)\big) \in \mathbb{R}^n\,.$$

Then $|x-\Gamma(\sigma)| = d$ and $|\Gamma(\tau)-x| \leq 2\varepsilon^2 d$ by the triangle inequality. By definition of $X_d(s,t,\varepsilon)$, the angle $\alpha = \sphericalangle(x-\Gamma(\sigma), \Gamma'(\sigma))$ is contained between $\varepsilon/10$ and $\pi - \varepsilon/10$. Therefore

$$\operatorname{dist}(x, \ell(\sigma)) = d\sin\alpha \geq d\sin\frac{\varepsilon}{10} \geq \frac{d\varepsilon}{20},$$

and

$$\operatorname{dist}(\Gamma(\tau), \ell(\sigma)) \geq \frac{d\varepsilon}{20} - 2\varepsilon^2 d \geq \frac{d\varepsilon}{25} \tag{2.11}$$

(here we use $\varepsilon < 1/200$). Combining (2.10) and (2.11), we obtain

$$\frac{1}{2r(\Gamma(\sigma),\Gamma(\tau))} \geq \frac{d\varepsilon}{25}(2d)^{-2} = \frac{\varepsilon}{100d}\,.$$

Integration gives

$$\mathscr{E}_q(\Gamma) \geq \int_{X_d(s,t,\varepsilon)} \int_{A_d(t,\varepsilon)} r^{-q}\, d\tau\, d\sigma \geq \text{const} \cdot |X_d(s,t,\varepsilon)|\varepsilon^{2+q} d^{1-q},$$

as $A_d(t,\varepsilon) \geq 2\varepsilon^2 d$. If $|X_d(s,t,\varepsilon)| \geq \frac{1}{2}\varepsilon^2 d$, then

$$\mathscr{E}_q(\Gamma) \geq \text{const}(q) \cdot \varepsilon^{4+q} d^{2-q},$$

which gives a contradiction for an appropriate choice of $c_0(q) > 0$ in the lemma.

Thus, we have

$$|X_d(s,t,\varepsilon)| < \frac{1}{2}\varepsilon^2 d \qquad \text{and} \qquad |N_d(s,t,\varepsilon)| > \frac{3}{2}\varepsilon^2 d.$$

Step 2. Suppose now that $\Gamma(\tau) \in B_{2d}(\Gamma(s)) \setminus U_{20\varepsilon d}\big(G(s,t)\big)$. Fix a $\sigma \in N_d(s,t,\varepsilon)$.

Since then $|\Gamma(\sigma) - \Gamma(s)| < \varepsilon^2 d$ and the (acute) angle between the vectors $\Gamma'(\sigma)$ and $\Gamma(t) - \Gamma(s)$ is very close to 0 or π (the difference is at most $\varepsilon/10$), one can check that in fact

$$\ell(\sigma) \cap B_{2d}(\Gamma(s)) \ \subset \ U_{\varepsilon d}(G(s,t)) \cap B_{2d}(\Gamma(s))\,.$$

Therefore the distance from $\Gamma(\tau)$ to $\ell(\sigma)$ is at least $19\varepsilon d$. If $\tau_1 \in A_d(\tau,\varepsilon)$, then

$$\operatorname{dist}(\Gamma(\tau_1),\ell(\sigma)) \geq 19\varepsilon d - \varepsilon^2 d \geq 18\varepsilon d\,,$$

and

$$\frac{1}{r(\Gamma(\sigma),\Gamma(\tau_1))} \geq \frac{18\varepsilon d}{(3d)^2} > \frac{\varepsilon}{d}\,.$$

Integrating this inequality, we obtain

$$\mathscr{E}_q(\Gamma) \geq \int_{N_d(s,t,\varepsilon)} \int_{A_d(\tau,\varepsilon)} r^{-q}\, d\tau_1\, d\sigma \geq \frac{3}{2}\varepsilon^2 d \cdot 2\varepsilon^2 d \cdot \left(\frac{\varepsilon}{d}\right)^q = 3\varepsilon^{4+q} d^{2-q}\,.$$

Again, for an appropriate choice of $c_0(q)$ this gives a contradiction with (2.9). □

Since the assumption $q > 2$ was not used at all in the proof of the lemma, it is easy to check that the same reasoning that was used to obtain (2.9) gives in fact the following

Lemma 2.3. *Assume that $q = 2$ and $\mathscr{E}_2(\Gamma) < \infty$. Then there exists a constant $c > 0$ such that*

$$\sup_{x\in\gamma} \beta_\gamma(x,d) \leq c\,\omega_E(d), \qquad d \leq \operatorname{diam}\gamma, \tag{2.12}$$

where

$$\omega_E(d) := \sup\left(\int_A \int_B \frac{ds\,dt}{r\big(\Gamma(s),\Gamma(t)\big)^2}\right)^{1/6}, \tag{2.13}$$

the supremum being taken over all pairs of subsets $A, B \subset S_L$ with $\mathscr{H}^1(A), \mathscr{H}^1(B) \leq \frac{1}{100}d$.

Remark. By the absolute continuity of the integral, this lemma implies that every curve with finite $\mathscr{E}_2$ energy satisfies the assumptions of Theorem 1.4.

3 The image of Γ is a manifold

This section is devoted to the proof of Theorem 1.4. We will argue by contradiction. The proof has two steps; one of them has preparatory topological character and the second one shows how to use the assumption on the uniform decay of β's.

Proof of Theorem 1.4. We recall the assumption of the theorem that the arclength parametrization $\Gamma : S_L \to \mathbb{R}^n$ with image $\gamma = \Gamma(S_L)$ satisfies (1.6) for some continuous nondecreasing function $\omega : [0,L] \to \mathbb{R}$ with $\omega(0) = 0$. In addition, however, we assume that γ is neither homeomorphic to the unit circle $\mathbb{S}^1$ nor to the unit interval $I = [0,1]$. Our goal is to show that this leads to a contradiction.

Step 1. Triple junctions.
Claim: *There exists a triple junction $x \in \gamma$, i.e. there are three closed sets $\alpha_i \subset \gamma$, $i = 1,2,3$, such that α_i is a continuous image of the unit interval with* $\operatorname{diam}\alpha_i > 0$ *for $i = 1,2,3$, and such that*

$$\alpha_i \cap \alpha_j = \{x\} \qquad \textit{whenever } i \neq j,\ i,j = 1,2,3. \tag{3.1}$$

Remark. We allow the α_i to have self-intersections, i.e. we do not require α_i to be a homeomorphic image of the interval. Moreover, more than three arcs of the curve may meet at x; we just need three of them to obtain the desired contradiction in Step 2 in order to complete the proof of Theorem 1.4.

Proof of the claim. We consider two distinct cases.

Case 1. Suppose that γ contains a proper closed subset γ_1 that is homeomorphic to $\mathbb{S}^1$. Take a point $y \in \gamma \setminus \gamma_1$, $y = \Gamma(s)$. Suppose w.l.o.g. that $\Gamma(s_1) \in \gamma_1$ for some $s_1 > s$, $s_1 \in [0,L]$ (otherwise just reverse the parametrization). Let

$$\sigma_0 := \inf\{\sigma > s \colon \Gamma(\sigma) \in \gamma_1\}$$

It is easy to see that $x = \Gamma(\sigma_0) \in \gamma_1$ is a triple junction; two of the arcs α_i of γ are contained in γ_1 and the third one joins $y \notin \gamma_1$ to x.

Case 2. Suppose that Case 1 fails and γ contains no proper closed subset homeomorphic to $\mathbb{S}^1$. Consider the family of all proper subarcs of γ,

$$\mathscr{A} = \{\tilde{\gamma} \subset \gamma \colon\ \tilde{\gamma} \text{ is homeomorphic to } I\},$$

which is partially ordered by inclusion. We will prove in detail below that every chain in $\mathscr{A}$ has an upper bound in $\mathscr{A}$, so that by the Kuratowski–Zorn Lemma $\mathscr{A}$ has a maximal element, $\gamma_{\max}$. We have $\gamma_{\max} \neq \gamma$, as γ is not homeomorphic to I by assumption. Now, take a point $y \in \gamma \setminus \gamma_{\max}$, $y = \Gamma(s)$, and proceed like in Case 1 joining y with an arc to a point $x \in \gamma_{\max}$. Notice that x cannot be an endpoint of $\gamma_{\max}$, since this would contradict the maximality of $\gamma_{\max}$.

It remains to be shown that every chain in $\mathscr{A}$ indeed has an upper bound in $\mathscr{A}$, which is obvious for any finite chain. For an infinite chain $\mathscr{C} := \{\gamma_l\}_{l\in\Sigma} \subset \mathscr{A}$ where the index may be chosen to coincide with the length of the respective arc, $l = \mathscr{L}(\gamma_l) \leq \mathscr{H}^1(\gamma)$ for $\gamma_l \in \mathscr{C}$, i.e. where the index set Σ is a (in general uncountable) subset of $[0, \mathscr{H}^1(\gamma)]$, we can choose a nondecreasing sequence of indices l_i with $\gamma_i \equiv \gamma_{l_i} \in \mathscr{C}$, $\gamma_i \subset \gamma_{i+1}$ and $l_i \to l^* := \sup\Sigma \in (0, \mathscr{H}^1(\gamma)]$.[3] Now continuously extend the corresponding nested injective arclength parametrizations

$$\Gamma_i : [-l_i/2, l_i/2] \to \mathbb{R}^n \quad \text{with } \Gamma_i([-l_i/2, l_i/2]) = \gamma_i \text{ and } \Gamma_{i+1}|_{[-l_i/2,l_i/2]} = \Gamma_i \text{ for all } i \in \mathbb{N} \tag{3.2}$$

[3] Assuming that at least one member of $\mathscr{C}$ has positive diameter, otherwise the claim is trivially true.

by virtue of

$$\Gamma_i(t) := \begin{cases} \Gamma_i(-l_i/2) & \text{for } t \in [-l^*/2, -l_i/2) \\ \Gamma_i(l_i/2) & \text{for } t \in (l_i/2, l^*/2] \end{cases}$$

to all of $[-l^*/2, l^*/2]$. Since $|\Gamma_i'(t)| \le 1$ for all $t \in [l^*/2, l^*/2]$, $i \in \mathbb{N}$, we obtain the uniform bound $\|\Gamma_i\|_{C^{0,1}([-l^*/2,l^*/2],\mathbb{R}^n)} \le C$ for all $i \in \mathbb{N}$, which implies by the Theorem of Arzela-Ascoli that some subsequence Γ_j converges to some curve $\Gamma \in C^{0,1}([-l^*/2, l^*/2], \mathbb{R}^n)$ uniformly on $[-l^*/2, l^*/2]$. For distinct parameters $s, t \in (-l^*/2, l^*/2)$ one can find $j_0 \in \mathbb{N}$ such that for all $j \ge j_0$ we have $s, t \in (-l_j/2, l_j/2)$, so that by (3.2)

$$|\Gamma(s) - \Gamma(t)| = \lim_{j\to\infty} |\Gamma_j(s) - \Gamma(t)| \overset{(3.2)}{=} |\Gamma_{j_0}(s) - \Gamma_{j_0}(t)| \ne 0,$$

which means that Γ is injective, hence a homeomorphism on the open interval $(-l^*/2, l^*/2)$. But if $\Gamma(l^*/2)$ were equal to $\Gamma(\tau)$ for some $\tau \in [-l^*/2, l^*/2)$ then the arc $\Gamma([\tau, l^*/2])$ would be homeomorphic to $\mathbb{S}^1$ which would contradict our assumption that γ is neither homeomorphic to $\mathbb{S}^1$ nor contains a proper closed subset homeomorphic to $\mathbb{S}^1$. The same contradiction would occur if $\Gamma(-l^*/2) = \Gamma(\tau)$ for some $\tau \in (-l^*/2, l^*/2]$. Hence $\gamma^* := \Gamma([-l^*/2, l^*/2])$ is homeomorphic to the unit interval I, that is $\gamma^* \in \mathscr{A}$. Finally γ^* is maximal for the chain $\mathscr{C}$. Indeed, if $l^* = \sup \Sigma \in \Sigma$ then γ^* is the desired upper bound because for $l < l^*$ it cannot be that γ_{l^*} is contained in γ_l, so that total ordering in the chain implies that $\gamma_l \subset \gamma_{l^*}$. If $l^* \notin \Sigma$, on the other hand, we have $l < l^*$ for any $l \in \Sigma$, which implies that the corresponding arc γ_l is contained in one of the γ_i for i sufficiently large, and hence also $\gamma_l \subset \gamma^*$.

The proof of our claim on the existence of (at least one) triple junction is complete now.

Step 2. Tilting tubes. We now fix a point $x \in \gamma$ that is a triple junction, and a small distance d_0,

$$0 < d_0 < \frac{1}{2} \min_{i=1,2,3} \left(\operatorname{diam} \alpha_i \right),$$

where α_i denote the closed, connected subsets of γ satisfying (3.1) above.

Let $h(s) := s\omega(s)$ for $s \in [0, L]$. Shrinking d_0 if necessary, we can ensure the initial smallness condition

$$h(d_0) < \frac{1}{20} d_0. \tag{3.3}$$

Rotating and translating the coordinate system in $\mathbb{R}^n$, we can assume without loss of generality that $x = 0 \in \mathbb{R}^n$ and select the three distinct points

$$y_i \in \alpha_i \cap \partial B(0, d_0), \qquad i = 1, 2, 3$$

where $y_1 = (d_0, 0, \dots, 0)$. Assumption (1.6) implies now

$$\gamma \cap B(0, d_0) \ \subset \ U_{2h(d_0)}(G(x, y_1)). \tag{3.4}$$

The intersection of the sphere $\partial B(0, d_0)$ with the tube $U_{2h(d_0)}(G(x, y_1))$ consists of two symmetric spherical caps; by Dirichlet's pigeon-hole principle, one of these caps must contain two of the three distinct points y_i. Renumbering the α_i and y_i if necessary, we may assume that y_1 is as above and $y_2 = (a, y_2') \in \alpha_2 \cap \partial B(0, d_0)$ with $a > 0$ and $y_2' \in \mathbb{R}^{n-1}$, $|y_2'| \le 2h(d_0)$.

Let $v_0 = (-1, 0, \dots, 0)$ and $H_0 = (v_0)^\perp$. Fix a point $z \in \alpha_1 \cap \left(\frac{1}{2} y_1 + H_0\right)$.

From now on, we will work only with α_1 and α_2. Proceeding inductively, we shall define a sequence of distances $d_m \to 0$, unit vectors v_m, linear $(n-1)$-dimensional subspaces $H_m = (v_m)^\perp$ and points $x_m \in \alpha_2$ such that

$$|z - x_m| \le 2h(d_m), \qquad m = 1, 2, \dots \tag{3.5}$$

As $d_m \to 0$ and $h(s) \to 0$ as $s \to 0$, this will yield $z = \lim x_m \in \alpha_1 \cap \alpha_2$, a contradiction.

The distances d_m, auxiliary vectors v_m and hyperplanes $H_m = (v_m)^\perp$ will be defined in such a way that for all $m = 1, 2, \dots$

$$4h_{m-1} \le d_m \le 6h_{m-1} \qquad \text{where } h_m := h(d_m), \tag{3.6}$$

$$\sphericalangle(v_m, v_{m-1}) \le \frac{\pi}{4}, \tag{3.7}$$

$$z_m = z + d_m v_m \in \alpha_1, \tag{3.8}$$

$$\gamma \cap B(z, d_m) \subset U_{2h_m}(G_m) \qquad \text{where } G_m = G(z, z_m). \tag{3.9}$$

For $P_m(t) = z + tv_m + H_m$ we shall also show that

$$P_m(t) \cap \alpha_i \cap U_{2h_m}(G_m) \ne \emptyset \qquad \text{for all } |t| \le \tfrac{1}{2} d_m \text{ and } i = 1, 2, \tag{3.10}$$

for each $m = 1, 2, \dots$. Notice that (3.6) in connection with the initial smallness condition (3.3) will yield $d_m \to 0$ as $m \to \infty$.

We begin the construction for $m = 1$. Select $z_1 \in P_0(4h_0) \cap \alpha_1$, $h_0 = h(d_0)$. Such a point exists since α_1 joins z to $x = 0$ and by continuity must intersect all planes $z + tv_0 + H_0$, $|t| \le \frac{1}{2} d_0$, while staying in the tube $U_{2h(d_0)}(G(x, y_1))$. Let $v_1 = (z_1 - z)/|z_1 - z|$, $H_1 := (v_1)^\perp$, and $P_1(t) := z + tv_1 + H_1$. Note that $\sphericalangle(v_1, v_0) \le \pi/4$ by construction. Set $d_1 = |z_1 - z|$.

We already have (3.6)–(3.8) for $m = 1$; condition (3.9) for $m = 1$ follows directly from (1.6). To obtain (3.10) for $m = 1$, we just use (3.7) and continuity.

Assume now that d_m, v_m, H_m, z_m, and P_m have already been defined for $m = 1, \dots, N$ so that (3.6)–(3.10) are satisfied for all $1 \le m \le N$. We use (3.10) for $m = N$ to select a point z_{N+1},

$$z_{N+1} \in U_{2h_N}(G_N) \cap P_N(2h_N) \cap \alpha_1 .$$

Clearly, $4h_N \le |z_{N+1} - z| \le 6h_N$ (the second estimate is a simple application of the triangle inequality). Thus, $d_{N+1} := |z_{N+1} - z|$ satisfies (3.6) for $m = N+1$, and choosing $v_{N+1} := (z_{N+1} - z)/|z_{N+1} - z|$ we also have (3.7)–(3.8) for $m = N+1$.

Again, (3.9) for $m = N+1$ follows from the assumption on the decay of β's. Thus, the intersection $\alpha_i \cap B(z, d_{N+1}) \subset U_{2h_{N+1}}(G_{N+1})$, $i = 1, 2$; combining these inclusions with (3.7) and with continuity, we obtain (3.10) for $m = N+1$.

This completes the inductive construction. Now, using (3.10) for $i = 2$, we select for each m a point

$$x_m \in U_{2h_m}(G_m) \cap P_m(0) \cap \alpha_2 .$$

By definition of $U_{2h_m}(G_m)$, (3.5) does hold. This completes the whole proof of Theorem 1.4.

4 Differentiability

Throughout this section, we fix $q > 2$ and consider a rectifiable curve $\gamma = \Gamma(S_L)$ whose arclength parametrization Γ is injective on S_L. The first step towards the proof of Theorem 1.3 is to establish the following.

Proposition 4.1. *Let $q > 2$. Assume that $\Gamma\colon S_L \to \mathbb{R}^n$ is injective and $\mathscr{E}_q(\Gamma) < E < \infty$. Then Γ' is well defined everywhere and $\Gamma' \in C^\kappa$ for $\kappa := \frac{q-2}{q+4} \in (0,1)$.*

Moreover there exist two positive constants $\delta_2(q)$, $c_2(q)$ such that whenever $x=\Gamma(s)$ and $y=\Gamma(t)$ satisfy $|x-y| = d < \delta_2(q)E^{-1/(q-2)}$, then

$$\phi := c_2(q)E^{1/(q+4)}d^{(q-2)/(q+4)} < \frac{1}{4} \tag{4.1}$$

and we have

$$|\Gamma'(s)-\Gamma'(t)| \le c_2(q)\,E^{1/(q+4)}|\Gamma(s)-\Gamma(t)|^{\kappa}, \tag{4.2}$$

$$\frac{3}{4}|s-t| \le |\Gamma(s)-\Gamma(t)| \le |s-t|, \tag{4.3}$$

$$\gamma\cap B(x,2d)\cap B(y,2d) \ \subset\ C_\phi(x,y)\cap C_\phi(y,x). \tag{4.4}$$

Proof. The argument is in fact similar to the proof of Corollary 2.6 and Theorem 2.10 in [22]. We just sketch the main points, leaving (relatively easy) computational details as an exercise.

Fix $x,y\in\gamma$ with $0<|x-y|=d$.

Step 1. For $N=0,1,2\ldots$ set $d_N = d/2^N$, and select points $y_N\in\partial B(x,d_N)\cap\gamma$ so that $y_0=y$. Let

$$\varepsilon_N := \big(c_0(q)E\big)^{1/(q+4)}{d_N}^{\kappa} \tag{4.5}$$

so that condition (2.9) of Lemma 2.1 is satisfied for ε_N and d_N. The lemma yields

$$\gamma\cap B(x,2d_N)\subset U_{20\varepsilon_N d_N}(G(x,y_N)), \qquad N=0,1,2,\ldots \tag{4.6}$$

so that the lines $G_N := G(x,y_N)$ satisfy

$$\sin\sphericalangle(G_N,G_{N+1}) \le \frac{20\varepsilon_N d_N}{d_{N+1}} = 40\varepsilon_N\,. \tag{4.7}$$

Thus, $\phi_N := \sphericalangle(G_N,G_{N+1}) \le 80\varepsilon_N$. Using (4.5) and summing a geometric series (here the assumption $q>2$ is crucial!), we obtain

$$\sum_{N=0}^{\infty}\phi_N \le \phi := c_2(q)E^{1/(q+4)}d^{\kappa} \tag{4.8}$$

where $c_2(q) = 80\,c_0(q)^{1/(q+4)}\sum_{N=0}^{\infty}2^{-N\kappa}$. Now, to guarantee $\phi<1/4$, one just assumes that d is sufficiently small, i.e. $d<\delta_2(q)E^{-1/(q-2)}$ with $\delta_2(q) := (4c_2(q))^{-1/\kappa}$. By induction,

$$\gamma\cap B(x,2d) \ \subset\ C_{\phi_0+\cdots+\phi_N}(x,y)\cup\left(U_{20\varepsilon_N d_N}(G_N)\cap B(x,2d_N)\right). \tag{4.9}$$

Passing to the limit $N\to\infty$, we obtain

$$\gamma\cap B(x,2d)\subset C_\phi(x,y) \tag{4.10}$$

with $\phi\equiv\phi(q,E,d)$ defined by (4.8).

Step 2. Reversing the roles of x and y we obtain

$$\gamma\cap B(x,2d)\cap B(y,2d) \ \subset\ C_\phi(x,y)\cap C_\phi(y,x)$$

where ϕ is defined by (4.8); this is the desired condition (4.4).

Step 3. Assume now that Γ is differentiable at s and t and recall that Γ was supposed to be injective. Condition (4.4) yields then

$$\sphericalangle(\Gamma'(s),\Gamma'(t)) \leq \phi = c_2(q)E^{1/(q+4)}d^{\kappa} = c_2(q)E^{1/(q+4)}|\Gamma(s)-\Gamma(t)|^{\kappa} \tag{4.11}$$

(note that the difference quotients of Γ at s and t must belong to cones with vertices at 0, axis parallel to $y-x$ and opening angle given by (4.1)).

Step 4. Since Γ is differentiable everywhere, and $|\Gamma'| = 1$ a.e., (4.11) gives (4.2) on a (dense) set of full measure. Thus, Γ' has a continuous extension F to all of S_L; one easily checks that in fact $F = \Gamma'$ *everywhere*. Finally, assuming without loss of generality that $t > s$, we estimate

$$\begin{aligned} |\Gamma(t)-\Gamma(s)| &\geq \langle \Gamma(t)-\Gamma(s),\Gamma'(s)\rangle \\ &= \left\langle \int_s^t \big(\Gamma'(\tau)-\Gamma'(s)+\Gamma'(s)\big)\,d\tau,\,\Gamma'(s)\right\rangle \\ &\geq (t-s)\left(1-\sup_{\tau\in[s,t]}|\Gamma'(\tau)-\Gamma'(s)|\right) \geq \frac{3}{4}(t-s)\,. \end{aligned}$$

(To check the last inequality, let S be the closed slab bounded by two planes passing through x and y, and perpendicular to $x-y$, i.e. to the common axis of the two cones, and note that for each $\tau \in [s,t]$ we have in fact $\Gamma(\tau) \in C_\phi(x,y)\cap C_\phi(y,x)\cap S$. This follows from the bound $|\Gamma'(s)-\Gamma'(t)| < 1/4$, injectivity of Γ and (4.4). Thus, for each such τ we also have $|\Gamma'(\tau)-\Gamma'(s)| < 1/4$.) The bi-Lipschitz condition (4.3) follows.

The proof of Proposition 4.1 is complete now. (See also [22, Proof of Thm. 2.10] where a similar scheme of reasoning is used.) □

5 Energy bounds and knot classes

We start this section with the observation that $\mathscr{E}_q$ is repulsive (or charge), that is, $\mathscr{E}_q$ blows up on a sequence of knots converging uniformly to a limit curve with self-crossings.

Proposition 5.1. *If $\Gamma : S_L \to \mathbb{R}^n$ is a closed arclength parametrized curve of length $0 < L < \infty$ with $\Gamma(s) = \Gamma(t)$ for different arclength parameters $s \neq t$, $s,t \in S_L$, and if there is a sequence of rectifiable closed injective curves $\gamma_k : S_L \to \mathbb{R}^n$ converging uniformly to Γ, then $\mathscr{E}_q(\gamma_k) \to \infty$ as $k \to \infty$ for any $q > 2$.*

PROOF: Assume to the contrary that (for a suitable subsequence) $\lim_{k\to\infty}\mathscr{E}_q(\gamma_k) < E < \infty$. We set

$$\varepsilon := \frac{1}{2}\min\left\{\operatorname{diam}\Gamma([s,t]),\operatorname{diam}\Gamma(S_L\setminus[s,t]),\delta_2(q)E^{\frac{-1}{q-2}}\right\} > 0, \tag{5.12}$$

where $\delta_2(q)$ is the constant of Proposition 4.1, and choose $\tau \in (s,t)$ and $\sigma \in S_L \setminus [s,t]$ such that

$$|\Gamma(\tau)-\Gamma(t)| = \frac{1}{2}\operatorname{diam}\Gamma([s,t]) \quad \text{and} \quad |\Gamma(\sigma)-\Gamma(s)| = \frac{1}{2}\operatorname{diam}\Gamma(S_L\setminus[s,t]).$$

For sufficiently large $k_0 = k_0(\varepsilon) \in \mathbb{N}$ we find $\|\gamma_k - \Gamma\|_{C^0(S_L,\mathbb{R}^n)} < \varepsilon/10$ for all $k \geq k_0$. In particular, by (5.12),

$$\begin{aligned} |\gamma_k(\tau)-\gamma_k(t)| &\geq |\Gamma(\tau)-\Gamma(t)| - \frac{2\varepsilon}{10} = \frac{1}{2}\operatorname{diam}\Gamma([s,t]) - \frac{\varepsilon}{5} \overset{(5.12)}{\geq} \frac{4}{5}\varepsilon, \\ \text{and, analagously,} \quad & |\gamma_k(\sigma)-\gamma_k(s)| \geq \frac{4}{5}\varepsilon, \end{aligned} \tag{5.13}$$

but

$$\delta_k := |\gamma_k(t) - \gamma_k(s)| \le \frac{\varepsilon}{5} \overset{(5.12)}{<} \delta_2(q) E^{\frac{-1}{q-2}} \quad \text{for all } k \ge k_0.$$

Hence, we can apply (4.4) of Proposition 4.1 to obtain the inclusion

$$\gamma_k \cap B(\gamma_k(t), 2\delta_k) \cap B(\gamma_k(s), 2\delta_k) \overset{(4.4)}{\subset} C_\phi(\gamma_k(t), \gamma_k(s)) \cap C_\phi(\gamma_k(s), \gamma_k(t)).$$

Since there is an integer $k_1 \ge k_0$ such that $\mathscr{E}_q < E$ for all $k \ge k_1$ we know that the corresponding injective arclength parametrizations Γ_k are continuously differentiable according to Proposition 4.1, so that the points $\gamma_k(t)$ and $\gamma_k(s)$ must be connected by a subarc of γ_k that is completely contained in the doubly conical region

$$D_k := C_\phi(\gamma_k(t), \gamma_k(s)) \cap C_\phi(\gamma_k(s), \gamma_k(t)) \cap B(\frac{1}{2}(\gamma_k(t) + \gamma_k(s)), \frac{\delta_k}{2})$$

of diameter $\delta_k \le \varepsilon/5$. (Otherwise, the unit tangent vector of the arclength parametrization Γ_k would jump at $\gamma_k(t)$ and $\gamma_k(s)$ contradicting C^1-smoothness for $k \ge k_1$.) Since all γ_k are simple, either the point $\gamma_k(\tau)$, or $\gamma_k(\sigma)$ lies on that connecting arc within D_k, thus contradicting the lower bound $4\varepsilon/5$ in (5.13). □

Proposition 5.2. *If $q > 2$, then the $\mathscr{E}_q$-energy is* strong *in the following sense: For each $E > 0$ and $L > 0$ there are at most finitely many knot types which have a representative γ such that*

$$\mathscr{E}_q(\gamma) < E, \qquad \mathscr{H}^1(\gamma) = L.$$

Remark. The length constraint $\mathscr{H}^1(\gamma) = L$ is necessary here, since by rescaling an arbitrary smooth simple curve we can make its $\mathscr{E}_q$-energy as small as one wishes. An alternative would be to consider $\tilde{\mathscr{E}}_q(\gamma) := (\mathscr{H}^1(\gamma))^{q-2}\mathscr{E}_q(\Gamma)$. This is a scale invariant energy.

Proof. We argue by contradiction. Assume there are infinitely many knot types of length L with the same energy bound, and by translational invariance we can assume moreover that all these knots contain the origin. Take their arclength representatives Γ_j, $j = 1, 2, \dots$, and use inequality (4.2) of Proposition 4.1 to conclude that the family

$$\{\Gamma_j'\}_{j=1,2,\dots} \subset C^0(S_L, \mathbb{S}^2)$$

is eqicontinuous. Invoking the Arzela–Ascoli compactness theorem and passing to a subsequence, we may assume that Γ_j converges in the C^1-topology to some limit $\Gamma \in C^1(S_L, \mathbb{R}^3)$. Let γ be the curve parametrized by Γ.

We next check that γ is simple, i.e. Γ is injective on $S_L \equiv \mathbb{R}/L\mathbb{Z}$. To this end, we shall rely on Proposition 4.1 to prove that there exists an $\varepsilon_0 = \varepsilon_0(q, E) > 0$ such that all Γ_j satisfy

$$|\Gamma_j(s) - \Gamma_j(t)| \ge \min\Big(\varepsilon_0, \frac{|s-t|}{2}\Big) \qquad \text{for all } j \text{ and all } s, t \in S_L. \tag{5.14}$$

Upon passing to the limit $j \to \infty$, this implies the injectivity of Γ. All γ_j with j sufficiently large are contained in a small C^1 neighbourhood of γ. Thus, according to a known isotopy result, see e.g. [14, Chapter 8] or [2], they would all have to be of the same knot type, thereby contradicting the assumption that each γ_j is in a different isotopy class.

To complete the proof, it is now enough to prove (5.14). Consider $g_j \in C^1(S_L \times S_L)$ given by

$$g_j(s,t) :\, = |\Gamma_j(s) - \Gamma_j(t)|^2 .$$

By Proposition 4.1 the Γ_j are uniformly bounded in $C^{1,\kappa}$, where $\kappa = (q-2)/(q+4)$. Thus, it is easy to show that there is a constant $\varepsilon_1 = \varepsilon_1(q,E) > 0$ such that

$$g_j(s,t) \geq \frac{|s-t|^2}{4} \qquad \text{for all } j \text{ and all } s,t \text{ such that } |s-t| \leq \varepsilon_1(q,E). \tag{5.15}$$

Since $\Sigma = S_L \times S_L \setminus \{(s,t) \colon |s-t| < \varepsilon_1(q,E)\}$ is compact, for each j there is a pair $(s_j,t_j) \in \Sigma$ such that

$$g_j(s_j,t_j) \leq g_j(s,t) \qquad \text{for all } (s,t) \in \Sigma.$$

Now, we either have $|s_j - t_j| = \varepsilon_1(q,E)$ in which case (5.15) implies

$$g_j(s,t) \geq \frac{\varepsilon_1(q,E)^2}{4} \qquad \text{for all } s,t \in \Sigma, \tag{5.16}$$

or, by minimality, we have $\nabla g_j(s_j,t_j) = 0$, which is equivalent to

$$\Gamma_j'(s_j) \perp \big(\Gamma_j(s_j) - \Gamma_j(t_j)\big) \qquad \text{and} \qquad \Gamma_j'(t_j) \perp \big(\Gamma_j(s_j) - \Gamma_j(t_j)\big). \tag{5.17}$$

Fix j. Let $d_j := |\Gamma_j(s_j) - \Gamma_j(t_j)|$. If $d_j < \delta_2(q)E^{-1/q-2}$, where $\delta_2(q)$ stands for the constant from Proposition 4.1, then, by (4.1) and (4.4) of that Proposition, we have

$$\phi_j := c_2(q)E^{1/(q+4)} d_j^{\kappa} < \frac{1}{4}$$

and

$$\gamma_j \cap B(\Gamma_j(s_j), 2d_j) \cap B(\Gamma_j(t_j), 2d_j) \ \subset\ C_{1/4}(\Gamma_j(s_j), \Gamma_j(t_j)) \cap C_{1/4}(\Gamma_j(t_j), \Gamma_j(s_j)).$$

The last condition, however, clearly contradicts (5.17). Hence,

$$\begin{aligned} d_j = |\Gamma_j(s_j) - \Gamma_j(t_j)| &= \inf_{(s,t)\in\Sigma} |\Gamma_j(s) - \Gamma_j(t)| \\ &\geq \varepsilon_2(q,E) :\, = \delta_2(q)E^{-1/q-2} \qquad \text{for each } j = 1,2,\ldots \end{aligned} \tag{5.18}$$

Summarizing (5.15), (5.16) and (5.18), we obtain (5.14) with $\varepsilon_0 :\, = \min\big\{\varepsilon_1(q,E)/2, \varepsilon_2(q,E)\big\}$. □

Now we present the proof of the isotopy result, Theorem 1.2. The proof consists of two steps. The first one, see Proposition 5.4 below, is preparatory: we use Proposition 4.1 to show that a curve γ of length L and finite energy at most E is ambient isotopic to a polygonal line that has roughly $LE^{1/(q-2)}$ vertices, all of them belonging to γ. In the second step, we replace two curves that are close in Hausdorff distance by polygonal curves (staying in the same knot class) and exhibit a series of Δ and Δ^{-1}-moves[4] transforming one of them into the other one. (The proof that we present gives a value of δ_3 which is far from optimal; we do not know how to obtain a sharp result of that type.)

Before passing to the details, let us recall a definition, see e.g. [5, Chapter 1].

[4] These are *not* the so-called Reidemeister moves; see [5, Chapter 1] for the distinction.

Definition 5.3. Let u be one of the segments of a polygonal knot γ in $\mathbb{R}^3$ and let $T = \mathrm{conv}(u,v,w)$ be a triangular surface bounded by the segments u,v,w such that $T \cap \gamma = u$. We say that

$$\gamma' = (\gamma \setminus u) \cup v \cup w$$

results from γ *by a* Δ*-move*. The inverse operation is called a Δ^{-1}*-move*.

Let γ_1 and γ_2 be two polygonal knots in $\mathbb{R}^3$. If γ_1 can be obtained from γ_2 by a finite sequence of Δ and Δ^{-1}-moves, then one says that γ_1 and γ_2 are *combinatorially equivalent*. Two polygonal knots γ_1 and γ_2 are ambient isotopic if and only if they are combinatorially equivalent, see [5, Chapter 1].

Proposition 5.4. *Let* $q > 2$*. Assume that* $\Gamma\colon S_L \to \mathbb{R}^3$ *is injective and* $\mathscr{E}_q(\Gamma) < E$*. Let* $\delta_2(q) > 0$ *be the constant defined in Proposition 4.1. Then* $\gamma = \Gamma(S_L)$ *is ambient isotopic to the polygonal curve*

$$P_\gamma = \bigcup_{i=1}^{N} [x_i, x_{i+1}]$$

with N *vertices* $x_i = \Gamma(t_i) \in \gamma$*, whenever the parameters* $0 = t_1 < \ldots < t_N < L$ *and* $t_{N+1} = t_1$ *are chosen on* S_L *so that*

$$|x_i - x_{i+1}| < \delta_2(q) E^{-1/(q-2)} . \tag{5.19}$$

Proof. We follow [27, Prop. 5.2] with minor technical changes. For $x \neq y \in \mathbb{R}^3$ we denote the closed halfspace

$$H^+(x,y) \colon = \{z \in \mathbb{R}^3 \colon \langle z - x, y - x\rangle \geq 0\} .$$

We shall work with 'double cones'

$$K(x,y) \colon = C_{1/4}(x,y) \cap C_{1/4}(y,x) \cap H^+(x,y) \cap H^+(y,x) .$$

For sake of brevity, set $K_i := K(x_i, x_{i+1})$ and $v_i := x_{i+1} - x_i$. We are going to use Proposition 4.1 to verify two properties of K_i.

Claim 1. *For each* $z \in K_i$ *the intersection of* γ *and the two-dimensional disk*

$$D_i(z) := K_i \cap (z + v_i^{\perp})$$

contains precisely one point. If $\mathrm{diam}\, D_i(z) > 0$*, then this point of* γ *is in the interior of* K_i*.*

Indeed, note first that $\gamma \cap D_i(z)$ is nonempty, as an arc of γ joining x_i with x_{i+1} must be contained in K_i since if this were not the case, then (4.4) of Proposition 4.1 would be impossible for an injective and differentiable Γ. If there were two distinct points $y_1, y_2 \in \gamma \cap D_i(z)$, then (4.4) could not hold both for the couple $x = x_i, y = y_1$, and for the couple $x = x_i, y = y_2$, simultaneously. Finally, the second statement of Claim 1 follows from the fact that Inequality (4.1) is strict.

Claim 2. *Whenever* $i \neq j \pmod N$ *we find that the sets* $K_i \setminus \{x_i, x_{i+1}\}$ *and* $K_j \setminus \{x_j, x_{j+1}\}$ *are disjoint.*

Suppose to the contrary that

$$(K_i \setminus \{x_i, x_{i+1}\}) \cap (K_j \setminus \{x_j, x_{j+1}\}) \neq \emptyset, \tag{5.20}$$

and assume without loss of generality

$$\mathrm{diam}\, K_j \leq \mathrm{diam}\, K_i. \tag{5.21}$$

If $x_j = \Gamma(t_j)$ were contained in $K_i \setminus \{x_i, x_{i+1}\}$ then we would either find that the disk $D_i(x_j)$ contains two distinct curve points contradicting Claim 1, or that there is a parameter $\tau \in (t_i, t_{i+1})$ such that $\Gamma(\tau) = \Gamma(t_j)$ although Γ is injective, which is absurd. The same reasoning can be applied to $x_{j+1} = \Gamma(t_{j+1})$, so that we conclude from (4.4) and Assumptions (5.20) and (5.21) that the two tips x_j, x_{j+1} of K_j are contained in the set Z_i defined as

$$Z_i := C_{\frac{1}{4}}(x_i, x_{i+1}) \cap C_{\frac{1}{4}}(x_{i+1}, x_i) \cap B(x_i, 2|v_i|) \cap B(x_{i+1}, 2|v_i|) \setminus \Big[K_i \setminus \{x_i, x_{i+1}\}\Big], \tag{5.22}$$

which is just the intersection of the two cones within the balls centered in x_i and x_{i+1} but without the slab bounded by the two parallel planes $\partial H^+(x_i, x_{i+1})$ and $\partial H^+(x_{i+1}, x_i)$.

We know that $x_j \neq x_i$ since $i \neq j \pmod N$ and Γ is injective. If $x_j \neq x_{i+1}$ then (5.20), (5.21), and (5.22) enforce

$$|v_i| \overset{(5.21)}{\geq} |v_j| \overset{(5.20)}{>} \min\{|x_j - x_{i+1}|, |x_j - x_i|\},$$

and

$$x_{j+1} \in \operatorname{int}(H^+(x_i, x_{i+1})) \cap \operatorname{int}(H^+(x_{i+1}, x_i)), \tag{5.23}$$

which by (4.4) leads to $x_{j+1} \in K_i$ contradicting (5.22) unless $x_{j+1} = x_i$. If in the latter case x_j is contained in $\mathbb{R}^3 \setminus H^+(x_{i+1}, x_i)$ then we obtain $|v_j| = |x_{j+1} - x_j| > |v_i|$ contradicting our assumption (5.21). If, on the other hand, x_j is in $H^+(x_{i+1}, x_i)$, it is by (5.22) actually contained in $\mathbb{R}^3 \setminus H^+(x_i, x_{i+1})$, but then (5.20) cannot hold.

Finally, $x_j = x_{i+1}$ in combination with (5.20) also leading to (5.23) is a contradictory statement, since $|v_j| \leq |v_i|$ by (5.21).

We are now in the position to define the ambient isotopy between γ and P_γ. Note that $F\colon S_L \to \mathbb{R}^3$ given by

$$F(t) := [x_i, x_{i+1}] \cap D_i(\Gamma(t)) \qquad \text{for} \quad t \in [t_i, t_{i+1}),\, i = 1, \ldots, N$$

is a well defined homeomorphism, parametrizing P_γ. The desired isotopy

$$H\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

is equal to the identity on $\mathbb{R}^3 \setminus \bigcup_{i=1}^N K_i$, and on each 'double cone' K_i it maps each two-dimensional slice $D_i(z)$, $z \in K_i$, homeomorphically to itself, keeping the boundary circle of $D_i(z)$ fixed and moving the point $\Gamma(s)$ along a straight segment on $D_i(\Gamma(s))$ until it hits $[x_i, x_{i+1}]$. □

Proof of Theorem 1.2. Abbreviate the maximal energy value $E := \max\{\mathscr{E}_q(\Gamma_1), \mathscr{E}_q(\Gamma_2)\}$ of the two simple arclength parametrized curves $\Gamma_i : S_{L_i} \to \mathbb{R}^3$ of respective (and a priori possibly quite different) lengths L_i, $i = 1, 2$. Recall the assumption that the two curves are close in Hausdorff-distance: $\operatorname{dist}_H(\Gamma_1, \Gamma_2) < \delta(q) E^{-1(q-2)}$.

Fix $N = N(q, E)$ so that $L_1/N =: \eta < \frac{1}{3}\delta_2(q) E^{-1/(q-2)}$, set $\varepsilon := \eta/50$ and let $t_i := (i-1)\eta \in S_{L_1}$ for $i = 1, \ldots, N$, and $t_{N+1} := t_1$. By Proposition 5.4, γ_1 is ambient isotopic to the polygonal line

$$P_{\gamma_1} := \sum_{i=1}^N [x_i, x_{i+1}]$$

where $x_i := \Gamma_1(t_i)$. Now, for $i = 1, \ldots, N$ we set $w_i := \Gamma_1'(t_i)$, $\alpha_i := \Gamma_1\big([t_i, t_{i+1}]\big) \subset \gamma_1$, and introduce the half-spaces $H_i^+ := H^+(x_i, x_i + w_i)$ and $H_i^- := \mathbb{R}^3 \setminus H_i^+$, which are bounded by affine planes $P_i := x_i + w_i^\perp$. Consider the tubular regions

$$T_i := H_i^+ \cap H_{i+1}^- \cap B_\varepsilon(\alpha_i).$$

Their union contains $\gamma_1 = \bigcup \alpha_i$; we clearly have $T_i \cap T_{i+1} = \emptyset$ as $\alpha_{i+1} \subset H^+_{i+1}$. Moreover, $T_i \cap T_j = \emptyset$ also when $|i-j| > 1$. To see this, we will use Proposition 4.1 to prove

$$\inf\{|\Gamma_1(\tau) - \Gamma_1(\sigma)| : (\sigma,\tau) \in S_{L_1} \times S_{L_1},\ |\sigma - \tau| \geq \eta\} \geq \frac{3}{4}\eta. \tag{5.24}$$

Before doing so, let us conclude from (5.24): If there existed a point $z \in T_i \cap T_j$ with $|i-j| > 1$, we could find $\sigma \in [t_i, t_{i+1})$ and $\tau \in [t_j, t_{j+1})$ such that $|\Gamma(\sigma) - \Gamma(\tau)| \leq 2\varepsilon = \eta/25$ by triangle inequality, a contradiction to (5.24).

To verify (5.24), we repeat the trick that has already been used in the proof of Proposition5.2. Notice that (4.3) applied to Γ_1 implies

$$|\Gamma_1(\tau) - \Gamma_1(\sigma)| \geq \frac{3}{4}|\tau - \sigma| \geq \frac{3}{4}\eta \quad \text{for all } \eta \leq |\tau - \sigma| \leq 3\eta, \tag{5.25}$$

so that the continuously differentiable function $g : S_{L_1} \times S_{L_1} \to \mathbb{R}$ given by $g(s,t) := |\Gamma_1(s) - \Gamma_1(t)|^2$ attains a positive minimum $g_0 > 0$ on the compact set $K_{3\eta}$, where we set $K_\rho := S_{L_1} \times S_{L_1} \setminus \{|s-t| < \rho\}$, i.e., there is a pair of parameters $(s^*, t^*) \in K_{3\eta}$ such that $g(s,t) \geq g(s^*,t^*) = g_0$ for all $(s,t) \in K_{3\eta}$. If $|s^* - t^*| = 3\eta$ we can apply (5.25) to find

$$|\Gamma_1(\tau) - \Gamma_1(\sigma)| = \sqrt{g(\tau,\sigma)} \geq \sqrt{g(s^*,t^*)} = |\Gamma_1(s^*) - \Gamma_1(t^*)| \overset{(5.25)}{\geq} \frac{3}{4}\eta \quad \text{for all } (\tau,\sigma) \in K_{3\eta}.$$

If, on the other hand, $|s^* - t^*| > 3\eta$ then by minimality $\nabla g(s^*,t^*) = 0$, which implies that both tangents $\Gamma_1'(s^*)$ and $\Gamma_1'(t^*)$ are perpendicular to the segment $\Gamma_1(s^*) - \Gamma_1(t^*)$. Thus the intersection

$$\Gamma_1(S_{L_1}) \cap B(\Gamma_1(s^*), 2\sqrt{g_0}) \cap B(\Gamma_1(t^*), 2\sqrt{g_0})$$

cannot be contained in the intersection $C_\phi(\Gamma_1(s^*), \Gamma_1(t^*)) \cap C_\phi(\Gamma_1(t^*), \Gamma_1(s^*))$, which according to (4.4) means that

$$|\Gamma_1(s^*) - \Gamma_1(t^*)| \geq \delta_2(q) E^{\frac{-1}{q-2}} > 3\eta,$$

establishing (5.24) also in this case.

Assume now that $\mathrm{dist}_H(\gamma_1, \gamma_2) < \varepsilon$. We shall prove that γ_2 is ambient isotopic to γ_1; by the choice of ε, this will mean that Theorem 1.2 holds with $\delta_3(q) = \delta_2(q)/150$.

Claim. *For each $i = 1, \ldots, N$ there is a point*

$$y_i \in P_i \cap \gamma_2 \cap B(x_i, 2\varepsilon).$$

Without loss of generality we can assume that the curve Γ_1 is oriented in such a way that

$$\sphericalangle(\Gamma_1'(t_i), v_i) < \frac{1}{8} \quad \text{and} \quad \sphericalangle(\Gamma_1'(t_i), v_{i-1}) < \frac{1}{8} \quad \text{for all } i = 1, \ldots, N, \tag{5.26}$$

that is, each tangent $\Gamma_1'(t_i)$ points into the set $K_i := K(x_i, x_{i+1}) = K(\Gamma_1(t_i), \Gamma_1(t_{i+1}))$, which readily implies for the hyperplanes $P_i \perp \Gamma_1'(t_i)$, $i = 1, \ldots, N$,

$$\sphericalangle(P_i, v_i) \geq \sphericalangle(P_i, \Gamma_1'(t_i)) - \sphericalangle(\Gamma_1'(t_i), v_i) > \frac{\pi}{2} - \frac{1}{8},$$

and similarly $\sphericalangle(P_i, v_{i-1}) > \frac{\pi}{2} - \frac{1}{8}$.

Indeed, according to (4.4)

$$\Big[\gamma_1 \cap B(x_i, 2|v_i|) \cap B(x_{i+1}, 2|v_i|) \cap H^+(x_i, x_{i+1}) \cap H^+(x_{i+1}, x_i)\Big] \subset K_i,$$

which implies that the tangent direction of the curve Γ_1 at x_i cannot deviate too much from the straight line through x_i and x_{i+1}. The inequalities in (5.26) provide a quantified version of this fact.

Since $\operatorname{dist}_H(\gamma_1, \gamma_2) < \varepsilon$ we find three points

$$z_i \in \gamma_2 \cap B(x_i, \varepsilon), \quad z_{i+1} \in \gamma_2 \cap B(x_{i+1}, \varepsilon) \quad \text{and} \quad z_{i-1} \in \gamma_2 \cap B(x_{i-1}, \varepsilon) \quad \text{for all } i = 1, \ldots, N.$$

If $z_i \in P_i$ we set $y_i := z_i$, and we are done. Else we know that $z_i \in H_i^+ \setminus P_i$ or that $z_i \in H_i^-$. In the first case we will work with the two points z_i and z_{i-1}, in the second with z_i and z_{i+1} in the same way, so let us assume the second situation $z_i \in H_i^-$. We know that $z_{i+1} \in H^+ \setminus P_i$ since by (4.3)

$$\operatorname{dist}(z_{i+1}, H_i^-) \geq \operatorname{dist}(x_{i+1}, H_i^-) - \varepsilon \overset{(4.3)}{\geq} \left(\frac{3}{4} - \frac{1}{50}\right)\eta > 0.$$

On the other hand, z_i and z_{i+1} are not too far apart,

$$\rho_i := |z_i - z_{i+1}| \leq |z_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - z_{i+1}| < 2\varepsilon + \eta < \delta_2(q) E^{-\frac{1}{q-2}},$$

so that we can infer from (4.4) applied to the points $x := z_i$ and $y := z_{i+1}$ that

$$\gamma_2 \cap B(z_i, 2\rho_i) \cap B(z_{i+1}, 2\rho_i) \cap H^+(z_i, z_{i+1}) \cap H^+(z_{i+1}, z_i) \subset K(z_i, z_{i+1}). \tag{5.27}$$

We will show that

$$\Big[K(z_i, z_{i+1}) \cap P_i\Big] \subset B(x_i, 2\varepsilon). \tag{5.28}$$

Notice that $K(z_i, z_{i+1}) \setminus P_i$ consists of two components, one containing $z_i \in \gamma_2$, and the other one containing $z_{i+1} \in \gamma_2$, which implies that the intersection in (5.28) is not empty. Since γ_2 connects z_i and z_{i+1} by (5.27) within the set $K(z_i, z_{i+1})$, the inclusion in (5.28) yields the desired curve point

$$y_i \in P_i \cap \gamma_2 \cap B(x_i, 2\varepsilon) \quad \text{for all } i = 1, \ldots, N,$$

thus proving the claim.

To prove (5.28) we first estimate the angle $\sphericalangle(z_{i+1} - z_i, v_i)$ by the largest possible angle between a line tangent to both $B(x_i, \varepsilon)$ and $B(x_{i+1}, \varepsilon)$ and the line connecting the centers x_i, x_{i+1}:

$$\sphericalangle(z_{i+1} - z_i, v_i) \leq \arcsin \frac{\varepsilon}{|v_i|/2},$$

so that

$$\begin{aligned} \sphericalangle(z_{i+1} - z_i, \Gamma_1'(t_i)) &< \frac{1}{8} + \arcsin \frac{2\varepsilon}{|v_i|} \\ &\overset{(4.3)}{<} \frac{1}{8} + \arcsin \frac{2\eta/50}{3\eta/4} < \frac{1}{5}. \end{aligned}$$

Now, let $\tilde{z}_i$ be the orthogonal projection of z_i onto P_i. Since $\sphericalangle(\tilde{z}_i - z_i, z_{i+1} - z_i) = \sphericalangle(\Gamma'(t_i), z_{i+1} - z_i) < \frac{1}{5}$, it is easy to see that $K(z_i, z_{i+1}) \cap P_i \subset B(\tilde{z}_i, \tilde{h}) \cap P_i$ where

$$\tilde{h} \leq |z_i - \tilde{z}_i| \tan\left(\frac{1}{5} + \frac{1}{8}\right) \leq \varepsilon \tan\left(\frac{8+5}{40}\right) < \frac{\varepsilon}{2}$$

(see Figure 1 below), which establishes $K(z_i, z_{i+1}) \cap P_i \subset B(x_i, 2\varepsilon)$ and hence (5.28).

Since $|y_i - y_{i+1}| < \eta + 4\varepsilon < 3\eta < \delta_2(q)E^{-1/(q-2)}$, the curve γ_2 is ambient isotopic to the polygonal curve $P_{\gamma_2} = \bigcup_{i=1}^{N}[y_i, y_{i+1}]$. To finish the proof of Theorem 1.2, it is now sufficient to check that P_{γ_1} and P_{γ_2} are combinatorially equivalent.

Since the sets T_i are pairwise disjoint, we have

$$\mathrm{conv}(x_i, x_{i+1}, y_i, y_{i+1}) \cap P_{\gamma_1} = [x_i, x_{i+1}].$$

This guarantees that all steps in the construction that follows involve legitimate Δ and Δ^{-1}-moves. The first step, taking place in $\overline{T}_1$, is to replace $[x_1, x_2]$ by the union of $[x_1, y_1]$ and $[y_1, x_2]$, and then to replace $[y_1, x_2]$ by the union of $[y_1, y_2]$ and $[y_2, x_2]$. Next we perform one Δ^{-1} and one Δ-move in each of the $\overline{T}_j$ for $j = 2, \ldots, N-1$, replacing first $[y_j, x_j]$ and $[x_j, x_{j+1}]$ by $[y_j, x_{j+1}]$, and next trading $[y_j, x_{j+1}]$ for the union of $[y_j, y_{j+1}]$ and $[y_{j+1}, x_{j+1}]$. Finally, for $j = N$ we perform two Δ^{-1}-moves: first replace $[y_N, x_N]$ and $[x_N, x_1]$ by $[y_N, x_1]$, and then replace $[y_N, x_1]$ and $[x_1, y_1]$ (which has been added at the very beginning of the construction) by $[y_N, y_1]$. This concludes the whole proof. □

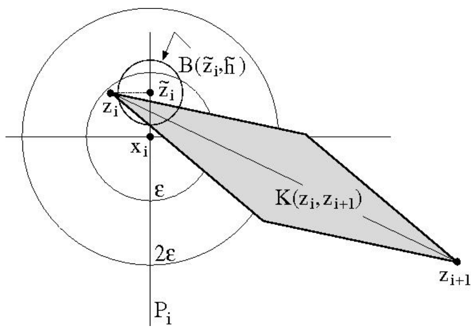

Figure 1. The intersection of the doubly conical region $K(z_i, z_{i+1})$ with the plane P_i is contained in the ball $B(\tilde{z}_i, \tilde{h}) \subset B(x_i, 2\varepsilon)$.

6 Bootstrap: optimal regularity of Γ'

In this section, we show how to derive Theorem 1.3. The overall idea is similar to the one in [22, Section 6] but here the proof is a little bit less involved.

Assume that Γ is 1–1, $\Gamma' \in C^{\kappa}$, $\kappa = (q-2)/(q+4)$. Restricting Γ to a sufficiently short interval I in $[0, L]$, and rotating the coordinate system if necessary, we may assume that the first component Γ_1' of the tangent vector satisfies $\Gamma_1' \geq 0.99$ on I and $|\Gamma_i'| \approx 0$ on I for all $i = 2, \ldots, n$. In fact, to achieve such control of Γ' on I it is enough to assume that

$$|I| \leq \delta_4(q)\mathscr{E}_q(\Gamma)^{-1/(q-2)}$$

for some $\delta_4(q) > 0$ sufficiently small; the desired control of Γ' follows then from Proposition 4.1.

Let

$$\Phi(t) := \sup_{\substack{J \subset I \\ \mathscr{L}^1(J) \leq t}} \left(\operatorname*{osc}_J \Gamma' \right) \qquad \text{for } |t| \leq \mathscr{L}^1(I) \tag{6.1}$$

(here, J denotes an arbitrary subinterval of I). We shall show that for every $u, v \in I$, $u < v$,

$$|\Gamma'(u) - \Gamma'(v)| \leq 2\Phi\Big(\frac{|u-v|}{N}\Big) + 100K_0|u-v|^{\lambda}, \tag{6.2}$$

where $\lambda = 1 - 2/q$, $N = N(q) > 8$ is a large number such that $2/N^{\kappa} < 1/2$, and

$$K_0 := \left(N^2 \int_u^v \int_u^v r^{-q}\, ds\, dt \right)^{1/q}.$$

Once (6.2) is established, we can iterate it to get rid of the first term on the right hand side of (6.2) and prove that

$$|\Gamma'(u)-\Gamma'(v)| \le c_3(q)\left(\int_u^v\int_u^v r^{-q}\,ds\,dt\right)^{1/q}|u-v|^{1-2/q}. \tag{6.3}$$

The argument that shows that (6.2) yields (6.3) is technical but relatively easy; similar reasonings are well known in the theory of PDE (e.g. when one deals with various Campanato–Morrey estimates). Similar arguments are described in more detail in our papers [22, Section 6] (see the Remark that follows the statement of Lemma 6.1 there) and [24, Section 6]. The reader is invited to fill in the computational details or to consult [22, 24].

Proof of (6.2)**.** We fix $u<v\in I$ and set

$$\begin{aligned} Y_0 &:= \{s\in[u,v] : \mathscr{H}^1(Y_1(s)) \ge 2|u-v|/N\},\\ Y_1(s) &:= \{t\in[u,v] : 1/r(\Gamma(s),\Gamma(t)) \ge K_0|u-v|^{-2/q}\}. \end{aligned}$$

The reader should think of the parameters in Y_0 and $Y_1(s)$ as 'bad' ones. Here is a word of informal explanation. Suppose that a curve is just $C^{1,\lambda}$ for $\lambda=1-2/q$ and not smoother, say like the graph of $x\mapsto |x|^{2-2/q}$ near zero. We would then expect that a typical point $\Gamma(v)$ can be roughly at the distance $d^{1+\lambda}$ from the tangent line at $\Gamma(u)$ when $|\Gamma(u)-\Gamma(v)|\approx d$ or, equivalently for a flat graph over some interval, $|u-v|\approx d$. But then $1/r$ at these two points would not exceed a constant multiple of $d^{1+\lambda}/d^2\approx|u-v|^{-2/q}$ by the explicit formula (1.1) for the radius r. As we know nothing about the existence of Γ'', there are no a priori upper bounds for $1/r$ that we might use. However, it is illustrative to look at the sets of points where the model bound $1/r\lesssim|u-v|^{-2/q}$ is violated. It will turn out that there are 'not too many' such points at all scales, and this will be enough to conclude.

Set also

$$E(u,v) := \int_u^v\int_u^v r^{-q}\,ds\,dt.$$

We have

$$\begin{aligned} E(u,v) &\ge \int_{Y_0}\int_{Y_1(s)} r^{-q}\,dt\,ds\\ &\ge \mathscr{H}^1(Y_0)\cdot\frac{2|u-v|}{N}\cdot K_0^q|u-v|^{-2} = \mathscr{H}^1(Y_0)\cdot\frac{2N}{|u-v|}\cdot E(u,v), \end{aligned}$$

so that

$$\mathscr{H}^1(Y_0) \le \frac{|u-v|}{2N}.$$

Now, select $s\in[u,v]\setminus Y_0$ and $t\in[u,v]\setminus Y_0$ such that

$$\max(|u-s|,|t-v|) < \frac{|u-v|}{N}.$$

By the triangle inequality,

$$\begin{aligned} |\Gamma'(u)-\Gamma'(v)| &\le |\Gamma'(u)-\Gamma'(s)|+|\Gamma'(s)-\Gamma'(t)|+|\Gamma'(t)-\Gamma'(v)|\\ &\le 2\Phi\Big(\frac{|u-v|}{N}\Big)+|\Gamma'(s)-\Gamma'(t)|. \end{aligned}$$

If the tangent lines $\ell(s)$ and $\ell(t)$ are parallel, we have $\Gamma'(s)=\Gamma'(t)$ and there is nothing more to prove. Thus, let us assume that $\ell(s)$ and $\ell(t)$ are not parallel and proceed to estimate $|\Gamma'(s)-\Gamma'(t)|$.

Let $G := [u,v] \setminus \big(Y_1(s) \cup Y_1(t)\big)$. By definition of $Y_1(\cdot)$ and choice of s,t, we have

$$\mathscr{H}^1(G) > |u-v| \left(1 - \frac{4}{N}\right) > \frac{|u-v|}{2}\,. \tag{6.4}$$

If $\sigma \in G$, then by definition of $Y_1(s)$ and of the tangent-point radius (see (1.1)) we obtain

$$\operatorname{dist}(\Gamma(\sigma), \ell(s)) < \frac{1}{2} K_0 |u-v|^{-2/q} |\Gamma(\sigma) - \Gamma(s)|^2 \le \frac{1}{2} K_0 |u-v|^{2-2/q} =: h_0\,. \tag{6.5}$$

A similar inequality is satisfied by the distance of $\Gamma(\sigma)$ to the other line, $\ell(t)$.

Now, let $H = \operatorname{span}(\Gamma'(s), \Gamma'(t)) \subset \mathbb{R}^n$ be the two-dimensional plane spanned by the two tangent vectors $\Gamma'(s)$ and $\Gamma'(t)$. Choose two points $p_1 \in \ell(s)$ and $p_2 \in \ell(t)$ such that $|p_1 - p_2| = \operatorname{dist}(\ell(s), \ell(t))$ and let $x := (p_1 + p_2)/2$. (If $\ell(s)$ and $\ell(t)$ intersect, $x = p_1 = p_2$ is their common point; otherwise, the segment $J(s,t) := [p_1, p_2]$ is perpendicular to each of these two lines and x is its midpoint.)

Let $P = x + H$. Then $\operatorname{dist}(\ell(s), P) = \operatorname{dist}(\ell(t), P) = |p_1 - p_2|/2$. Let π_P be the orthogonal projection onto P and let

$$l_1 := \pi_P(\ell(s))\,, \qquad l_2 := \pi_P(\ell(t))\,.$$

The lines l_1, l_2 intersect at $x \in P$. Note that since G is nonempty by (6.4), we have in fact by virtue of (6.5)

$$|p_1 - p_2| = 2|x - p_1| \le 2h_0,$$

and

$$\operatorname{dist}(\Gamma(\sigma), l_i) \le 2h_0, \qquad i = 1,2, \quad \sigma \in G.$$

Thus,

$$Z := \Gamma(G) \ \subset \ U_{3h_0}(l_1) \cap U_{3h_0}(l_2)\,. \tag{6.6}$$

Therefore, the projection $\pi_P(Z)$ of Z onto P is contained in a rhombus R in P. The center of symmetry of R is at x; the sides of R are parallel to l_1 and l_2; its height equals $6h_0$ and its acute angle

$$\gamma_0 := \sphericalangle(l_1, l_2) = \sphericalangle(\Gamma'(s), \Gamma'(t))$$

(since $\Gamma_1' \ge 0.99$ on I, the angle $\sphericalangle(\Gamma'(s), \Gamma'(t))$ *is* acute). The longer half-diagonal D of R is given by

$$D = \frac{6h_0}{\sin(\gamma_0/2)}\,, \tag{6.7}$$

and

$$\pi_P(Z) \ \subset \ R \subset B_D(x) \cap P.$$

Since $D \ge 6h_0$, invoking (6.6) and the triangle inequality we conclude that

$$Z = \Gamma(G) \subset B_{2D}(x)\,.$$

Now, recall that $\Gamma_1' \ge 0.99$ on I. Let $t_2 = \sup G$ and $t_1 = \inf G$. We then have

$$\begin{aligned} 4D &= \operatorname{diam} B_{2D}(x) \ge |\Gamma(t_2) - \Gamma(t_1)| \ge \Gamma_1(t_2) - \Gamma_1(t_1) \\ &= \int_{t_1}^{t_2} \Gamma_1'(\sigma)\, d\sigma \\ &\ge 0.99 \mathscr{H}^1(G)\,. \end{aligned}$$

Thus

$$\begin{aligned}\mathscr{H}^1(G) &< 5D = \frac{30h_0}{\sin(\gamma_0/2)} \qquad \text{by (6.7)}\\ &\le \frac{30\pi h_0}{\gamma_0} \qquad (6.8)\end{aligned}$$

Combining two estimates of $\mathscr{H}^1(G)$, (6.4) and (6.8), we obtain

$$\sphericalangle(\Gamma'(s),\Gamma'(t)) = \gamma_0 \le \frac{60\pi h_0}{|u-v|} \overset{(6.5)}{\equiv} 30\pi K_0|u-v|^{1-2/q} < 100K_0|u-v|^{1-2/q}.$$

This yields the desired estimate of $|\Gamma'(t)-\Gamma'(s)|$. The proof of the second part of Theorem 1.3 is now complete. □

Remark. To see that the exponent $1-2/q$ is indeed optimal and cannot be replaced by any larger exponent, we follow the idea given by M. Szumańska in her PhD thesis [27]. One has to fix an arbitrary $a \in (2-2/q, 2]$ and consider γ that is the graph of $f(x) = x^a$ say on $[0,1]$. It is possible to check that $\mathscr{E}_q(\gamma)$ is finite; however, the derivative of the arclength parametrization of γ is not Hölder continuous with any exponent larger than $\beta = a-1$. Since β can be an arbitrary number in $(1-2/q, 1]$, the exponent $1-2/q$ is indeed optimal. We do not give here the computational details which are somewhat tedious but routine; one just has to pass from the graph description of γ to the arclength parametrization and use Taylor's formula in estimates. The key point is that $f'(x) = ax^{a-1}$ is not Hölder continuous with any exponent larger than $\beta = a-1$, due to its behaviour near to 0.

References

[1] Ashton, T.; Cantarella, J.; Piatek, M.; Rawdon, E. Knot tightening by constrained gradient descent. - arXiv:1002.1723v1 [math.DG], 2010.

[2] Blatt, S. Note on continuously differentiable isotopies. Preprint Nr. 34, Institut f. Mathematik, RWTH Aachen University (2009).

[3] Buck, G.; Simon, J. Energy and length of knots. *Lectures at KNOTS '96 (Tokyo)*, pp. 219–234, Ser. Knots Everything, 15, World Sci. Publ., River Edge, NJ, 1997.

[4] Buck, G.; Simon, J. Thickness and crossing number of knots. *Topology Appl.* **91** (1999), no. 3, 245–257.

[5] Burde, G.; Zieschang, H. *Knots*. Walter de Gruyter, Berlin, New York 2003.

[6] Cantarella, J.; Kusner, R.B.; Sullivan, J.M. On the minimum ropelength of knots and links. Inv. math. **150** (2002), 257–286.

[7] Cantarella, J.; Piatek, M.; Rawdon, E. Visualizing the tightening of knots. In: *VIS'05: Proc. of the 16th IEEE Visualization 2005,* pp 575–582, IEEE Computer Society, Washington, DC, 2005.

[8] Carlen, M.; Laurie, B.; Maddocks, J.H.; Smutny, J. Biarcs, global radius of curvature, and the computation of ideal knot shapes. In: Calvo, Millett, Rawdon, Stasiak (eds.) *Physical and Numerical Models in Knot Theory*, pp. 75–108. Series on Knots and Everything – Vol. 36, World Scientific, Singapore 2005.

[9] David, G.;, Kenig, C.; Toro, T. Asymptotically optimally doubling measures and Reifenberg flat sets with vanishing constant. *Comm. Pure Appl. Math.* **54** (2001), no. 4, 385–449.

[10] Gerlach, H.; von der Mosel, H. What are the longest ropes on the unit sphere? Preprint Nr. 32, Institut f. Mathematik, RWTH Aachen University (2009).

[11] Gerlach, H., von der Mosel, H. On sphere-filling ropes. Preprint Nr. 44, Institut f. Mathematik, RWTH Aachen University (2010).

[12] Gonzalez, O.; Maddocks, J. H. Global curvature, thickness, and the ideal shapes of knots. *Proc. Natl. Acad. Sci. USA* **96** (1999), no. 9, 4769–4773

[13] Gonzalez, O.; Maddocks, J.H.; Schuricht, F.; von der Mosel, H. Global curvature and self-contact of nonlinearly elastic curves and rods. Calc. Var. Partial Differential Equations **14** (2002), 29–68.

[14] Hirsch, M.W. *Differential topology*. Corrected reprint of the 1976 original, Graduate Texts in Mathematics 33, Springer, New York, Berlin, Heidelberg 1997.

[15] Millett, K.; Piatek, M.; Rawdon, E. Polygonal space near ropelength-minimized knots. J. Knot Theory Ramifications **17** (2008), 601–631.

[16] O'Hara, J. *Energy of knots and conformal geometry.* Series on Knots and Everything – Vol. 33, World Scientific, Singapore, 2003.

[17] Preiss, D.; Tolsa, X.; Toro, T. On the smoothness of Hölder doubling measures. *Calc. Var.* **35** (2009), 339–363.

[18] Schuricht, F.; von der Mosel, H. Global curvature for rectifiable loops. Math. Z. **243** (2003), 37–77.

[19] Schuricht, F.; von der Mosel, H. Euler-Lagrange equations for nonlinearly elastic rods with self-contact. Arch. Rat. Mech. Anal. **168** (2003), 35–82.

[20] Schuricht, F.; von der Mosel, H. Characterization of ideal knots. Calc. Var. Partial Differential Equations **19** (2004), 281–305.

[21] Strzelecki, P.; Szumańska, M.; von der Mosel, H. A geometric curvature double integral of Menger type for space curves. Ann. Acad. Sci. Fenn. **34** (2009), 195–214.

[22] Strzelecki, P.; Szumańska, M.; von der Mosel, H. Regularizing and self-avoidance effects of integral Menger curvature. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **9** (2010), 145–187.

[23] Strzelecki, P.; von der Mosel, H. On rectifiable curves with L^p-bounds on global curvature: Self-avoidance, regularity, and minimizing knots. Math. Z. **257** (2007), 107–130.

[24] Strzelecki, P.; von der Mosel, H. Integral Menger curvature for surfaces. Preprint Nr. 36 Institut f. Mathematik, RWTH Aachen University (2009), arXiv:0911.2095v2[math.CA], 2009.

[25] Strzelecki, P.; von der Mosel, H. Tangent-point repulsive potentials for a class of non-smooth m-dimensional surfaces in $\mathbb{R}^n$. Part I: Smoothing and self-avoidance effects. In preparation.

[26] Sullivan, J.M. Approximating ropelength by energy functions. In: Calvo, Millet, Rawdon (eds.) *Physical Knots: Knotting, Linking, and Folding Geometric Objects in* $\mathbb{R}^3$, pp- 181–186. Contemp. Math. 304, AMS, Providence RI 2002.

[27] Szumańska, M. Integral versions of Menger curvature: smoothing potentials for rectifiable curves. (In Polish.) Ph.D. thesis, Technical University of Warsaw, 2009.

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