A COMPACTNESS THEOREM FOR WEAK SOLUTIONS OF HIGHER-DIMENSIONAL H-SYSTEMS

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Abstract

We prove that any weak limit of weak solutions u_k of the degenerate nonlinear elliptic system, the so-called (perturbed) H-system

$$\operatorname{div}(|\nabla u_k|^{n-2}\nabla u_k) = H(u_k)\frac{\partial u_k}{\partial x_1} \wedge \cdots \wedge \frac{\partial u_k}{\partial x_n} + \Phi_k,$$

where $\Phi_k \to 0$ in $(W^{1,n})^*$ as $k \to \infty$, solves the limiting H-system

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = H(u)\frac{\partial u}{\partial x_1} \wedge \cdots \wedge \frac{\partial u}{\partial x_n}.$$

(Hence, in particular, the space of weak solutions of the latter system is closed with respect to weak convergence in $W^{1,n}$.)

Sequences of that type arise naturally as Palais-Smale sequences for the n-Dirichlet integral plus a volume term. Maps that are critical points of this functional and satisfy an additional conformality condition parametrize hypersurfaces of prescribed mean curvature H. This was part of our main motivation.

1. Introduction

In this note, we consider weak solutions $u = (u^1, \ldots, u^{n+1}) \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$, where $\mathbb{B}^n : = \{x \in \mathbb{R}^n : |x| < 1\}$ denotes the unit n-dimensional ball, of the so-called H-system

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = H(u)\frac{\partial u}{\partial x_1} \wedge \cdots \wedge \frac{\partial u}{\partial x_n}. \tag{1.1}$$

Here, $H : \mathbb{R}^{n+1} \to \mathbb{R}$ is a C^1-function that satisfies the estimate

$$\sup_{y\in\mathbb{R}^{n+1}} \big(|H(y)| + |\nabla H(y)|\big) \le C, \tag{1.2}$$

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and $\frac{\partial u}{\partial x_1} \wedge \cdots \wedge \frac{\partial u}{\partial x_n}$ denotes the cross product of vectors $\frac{\partial u}{\partial x_i} \in \mathbb{R}^{n+1}$ $(i = 1, \ldots, n)$. A map $u \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$ is a weak solution of (1.1) if and only if

$$\int_{\mathbb{B}^n} |\nabla u|^{n-2} \nabla u \cdot \nabla \psi \, dx = - \int_{\mathbb{B}^n} H(u) \psi \cdot \frac{\partial u}{\partial x_1} \wedge \cdots \wedge \frac{\partial u}{\partial x_n} \, dx \tag{1.3}$$

for all test maps $\psi \in C_0^\infty(\mathbb{B}^n, \mathbb{R}^{n+1})$.

Systems of the general form (1.1) appear in many places in differential geometry and in the calculus of variations. For $n = 2$, conformal solutions of (1.1) parametrize, away from branch points, surfaces of prescribed mean curvature. In this case, existence of so-called small solutions under sharp geometric conditions has been established by Hildebrandt [16].

For all $n \geq 2$ and for $H \equiv$ const., weak solutions of (1.1) correspond to critical points of the functional

$$I[u] = \int_{\mathbb{B}^n} |\nabla u|^n \, dx$$

in the class of admissible functions

$$\mathscr{A} = \left\{ u \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1}) : \, u\big|_{\partial \mathbb{B}^n} = \eta, \; V(u) = c \right\},$$

where $\eta \colon \partial \mathbb{B}^n \to \mathbb{R}^{n+1}$ is a fixed map and

$$V(u) := \frac{1}{n+1} \int_{\mathbb{B}^n} u \cdot \frac{\partial u}{\partial x_1} \wedge \cdots \wedge \frac{\partial u}{\partial x_n} \, dx$$

denotes the *volume* of the cone in $\mathbb{R}^{n+1}$ generated by the image $u(\mathbb{B}^n)$. For variable H, *conformal* solutions of (1.1) represent hypersurfaces of prescribed mean curvature, equal to $n^{-n/2} H(u(x))$ at $u(x)$. (A map $u \colon \mathbb{B}^n \to \mathbb{R}^{n+1}$ is called conformal if $u_{x_i} \cdot u_{x_j} = \lambda(x) \delta_{ij}$ a.e., for some real-valued function λ and all i, j.) A theory of existence and regularity of minimizing solutions to (1.1) has been set forth by Duzaar and Grotowski in [8].

For $n = 2$, it is well known that (1.1), that is, $\Delta u = H(u) u_x \wedge u_y$ in this case, also has unstable solutions for both constant and nonconstant H. This is true for both Dirichlet and Plateau boundary problems. The existence of such solutions, which correspond to geometrically different surfaces spanning a given contour in $\mathbb{R}^3$, has been established by various authors (see, e.g., Brezis and Coron [4], Steffen [25], Struwe [27], [28], [29], Bethuel and Rey [3] and the references therein).

For higher dimensions $n \geq 3$, much less is known. Mou and Yang [21] obtain existence of unstable solutions for sufficiently small constant H, with an estimate far from optimal. Nothing is known for nonconstant H.

Existence of unstable solutions is usually proved via applications of the mountain pass lemma. To proceed that way, one must be able to analyze the behaviour of weakly convergent Palais-Smale sequences for a corresponding variational functional. This is a delicate task (see Bethuel's paper [2] for example!) since the equation is highly nonlinear, and the right-hand side is not continuous with respect to weak convergence. A well-known phenomenon of bubbling, which appears in various related problems, especially in the investigation of harmonic and p-harmonic maps, leads to defects of strong convergence: typically, some part of (n-)Dirichlet energy is lost in the limit passage, and the sequence does not have to converge strongly. Thus, one is forced to use subtle tools coming from compensated compactness theory and harmonic analysis (or to work with perturbed functionals, as Sacks and Uhlenbeck do in their pioneering paper [23]).

Our main result is the following theorem. We hope that it may be applied in a proof of existence of unstable solutions of (1.1) for nonconstant H.

THEOREM 1.1
Assume that $u_k \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$ are weak solutions of the system

$$\operatorname{div}(|\nabla u_k|^{n-2}\nabla u_k) = H(u_k)\frac{\partial u_k}{\partial x_1} \wedge \cdots \wedge \frac{\partial u_k}{\partial x_n} + \Phi_k, \tag{1.4}$$

where $\Phi_k \to 0$ in $(W^{1,n})^*$, and $u_k \rightharpoonup u$ weakly in $W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$. Then u is a weak solution of (1.1).

This result has an immediate corollary: the limit of any weakly convergent sequence of weak solutions of (1.1) is again a weak solution of (1.1).

THEOREM 1.2
Assume that $u_k \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$ are weak solutions of (1.1), $k = 1, 2, \ldots,$ and $u_k \rightharpoonup u$ weakly in $W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$. Then u is a weak solution of (1.1).

To prove the first theorem, we employ the following strategy, inspired by the results of Freire, Müller, and Struwe [12] on weak compactness of wave maps and harmonic maps in dimensions 3 and 2, respectively. First, we generalize slightly a result of Hardt, Lin, and Mou [14] (see also Courilleau [6]) on compactness of p-harmonic maps so that it may be applied to (1.4). This yields convergence of the gradients a.e. and allows one to pass to the limit on the left-hand side of (1.4). The next step forms the core of the whole proof; we use the results of Coifman, Lions, Meyers, and Semmes [5], the duality of Hardy space $\mathscr{H}^1$ and bounded mean oscillation (BMO), and a theorem of Jones and Journé [18] on weak-$*$ convergence in $\mathscr{H}^1$ to prove a lemma modeled on the famous paper of Lions [20] (see his Lemma 4.3, designed to analyze large solutions of the equation of surfaces with constant mean curvature in $\mathbb{R}^3$). A direct, simple application of the Jones-Journé theorem is not possible since the right-hand side of (1.1) is not in the Hardy space, and the u_k need not converge in $W^{1,n}$.

This lemma (see Lemma 3.1) allows us to pass to the limit on the right-hand side—and to obtain the desired expression, plus an additional error term that is an at most countable linear combination of Dirac measures. Finally, it is very simple to see that all the coefficients of that combination are in fact zero. This is because $W^{1,n}(\mathbb{R}^n)$ is *not* embedded in L^∞, and thus a single point has zero Cap_n (or $B_{1,n}$) capacity.

One might also think about a different proof: the energy of all u_k can concentrate only on a finite "bad set" $\Sigma = \mathbb{B}^n \setminus G$, where the "good set" G consists of those a for which

$$\liminf_{k=\infty} \int_{B(a,r)} |\nabla u_k|^n \, dx < \eta_0 \quad \text{for some } r > 0, \tag{1.5}$$

with some sufficiently small but otherwise fixed $\eta_0 > 0$. It is easy to see that G is open and Σ is finite, and one may hope to have good uniform regularity estimates on G. Such estimates would allow one to pass to the limit on G, and then one would be left with finitely many singularities in Σ. Alas, for $n \geq 3$ one needs stronger assumptions than just (1.2) to obtain local regularity of weak solutions. (In the general case, regularity remains open. Duzaar and Fuchs [7] obtain regularity of *bounded* weak solutions; Mou and Yang [21] obtain regularity of *conformal* solutions; finally, Wang [32], using Hardy space methods originating in Hélein's work [15], proves that all weak solutions are of class $C^{1,\alpha}$ *if* $|\nabla H|$ decays at infinity like $|y|^{-1}$. Using this additional assumption and Wang's theorem, one may in fact give another proof of Theorem 1.2. Since the details are rather tedious and the result is less general than Theorem 1.1, we do not pursue that point further.)

2. Analytic tools

Convergence a.e. *of the gradients*

Let us begin with a result that allows us to control the gradients of a sequence of weak solutions in spaces that are slightly larger than $W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$.

In the case of mappings solving linear equations $\Delta u_k = f_k$, with $(f_k)_{k=1}^\infty$ bounded in L^1, this is the so-called Murat's lemma. The generalization to sequences of maps with p-Laplacians bounded in L^1 is due to Hardt, Lin, and Mou [14] and independently to Courilleau [6].

THEOREM 2.1

Assume the sequence $(u_k)_{k=1}^\infty \subset W^{1,n}(\mathbb{B}^n, \mathbb{R}^m)$ *to be bounded, assume*

$$\mathrm{div}(|\nabla u_k|^{n-2} \nabla u_k) = f_k \in L^1(\mathbb{B}^n, \mathbb{R}^m),$$

and assume also

$$\sup_{k\in\mathbb{N}} \|f_k\|_{L^1} < +\infty.$$

Then one can select a subsequence $(u_{j_k})_{k=1}^{\infty}$ *such that* $u_{j_k} \rightharpoonup u$ *weakly in* $W^{1,n}(\mathbb{B}^n, \mathbb{R}^m)$ *and strongly in* $W^{1,q}(\mathbb{B}^n, \mathbb{R}^m)$ *for every* $q \in [1, n)$.

The above theorem can be easily improved.

THEOREM 2.2
Assume the sequence $(u_k)_{k=1}^{\infty} \subset W^{1,n}(\mathbb{B}^n, \mathbb{R}^m)$ *to be bounded, assume*

$$\operatorname{div}(|\nabla u_k|^{n-2}\nabla u_k) = f_k + \Phi_k,$$

where $f_k \in L^1(\mathbb{B}^n, \mathbb{R}^m)$ *and* $\Phi_k \to 0$ *in* $(W^{1,n})^*$*, and assume also*

$$\sup_{k\in\mathbb{N}} \|f_k\|_{L^1} < +\infty.$$

Then one can select a subsequence $(u_{j_k})_{k=1}^{\infty}$ *such that* $u_{j_k} \rightharpoonup u$ *weakly in* $W^{1,n}(\mathbb{B}^n, \mathbb{R}^m)$ *and strongly in* $W^{1,q}(\mathbb{B}^n, \mathbb{R}^m)$ *for every* $q \in [1, n)$.

The proof is exactly the same as in [14, Theorem 1]. We test the equation with the same function $\psi = \xi\eta \circ (u_k - u)$, where

$$\xi(x) = \min\left\{\frac{\operatorname{dist}(x, \partial\mathbb{B}^n)}{\delta}, 1\right\}, \quad x \in \mathbb{B}^n; \qquad \eta(y) = \min\left\{\frac{\delta}{|y|}, 1\right\} y, \quad y \in \mathbb{R}^{n+1}.$$

The crucial thing (cf. [14, inequality (3)]) is to estimate the integral

$$\left|\int_{\mathbb{B}^n} \xi |\nabla u_k|^{n-2}\nabla u_k \cdot \nabla\big(\eta \circ (u_k - u)\big)\, dx\right|;$$

here, in addition to all the terms listed and estimated in [14], we have an extra term $\langle \Phi_k, \xi\eta \circ (u_k - u)\rangle$. As the Φ_k tend to zero in $(W^{1,n})^*$, and the $\xi\eta \circ (u_k - u)$ are bounded in $W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$, this term tends to zero as $k \to +\infty$.

Hardy spaces
Recall that a measurable function $f \in L^1(\mathbb{R}^n)$ belongs to the *Hardy space* $\mathscr{H}^1(\mathbb{R}^n)$ if and only if

$$f_* := \sup_{\varepsilon>0} |\varphi_\varepsilon * f| \in L^1(\mathbb{R}^n).$$

Here, $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ for a fixed nonnegative function φ of class $C_0^\infty(\mathbb{B}^n)$ with $\int \varphi(y)\, dy = 1$. The definition does not depend on the choice of φ (see [11]).

Equivalently, one can define $\mathscr{H}^1(\mathbb{R}^n)$ as the space of those elements of $L^1(\mathbb{R}^n)$ for which all the Riesz transforms $R_j f$, $j = 1, 2, \dots, n$, are also of class $L^1(\mathbb{R}^n)$. The reader is referred to [24] and [26, Chapters 3 and 4] for more details. We just mention here that $\mathscr{H}^1(\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_{\mathscr{H}^1} = \|f\|_{L^1} + \|f_*\|_{L^1}.$$

Moreover, the condition $f \in \mathscr{H}^1(\mathbb{R}^n)$ implies $\int f(y)\,dy = 0$. This is the primary reason for diverse cancellation phenomena.

C. Fefferman [10], [11] proved that the dual of $\mathscr{H}^1(\mathbb{R}^n)$ is equal to the space of functions of bounded mean oscillation, BMO($\mathbb{R}^n$). More precisely, there exists a constant C such that

$$\Big| \int_{\mathbb{R}^n} h(y)\psi(y)\,dy \Big| \le C\|h\|_{\mathscr{H}^1}\|\psi\|_{\mathrm{BMO}} \tag{2.1}$$

for all $h \in \mathscr{H}^1(\mathbb{R}^n)$ and $\psi \in \mathrm{BMO}(\mathbb{R}^n)$. We do not need the full strength of his result. A particular case that is stated below as Lemma 2.4 is sufficient.

In their celebrated paper [5], Coifman, Lions, Meyer, and Semmes proved, among lots of other results, that the Jacobian determinant of a map $v \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ is not just integrable (this follows trivially from Hölder inequality) but belongs to the Hardy space. For the sake of further reference, we record here their result.

PROPOSITION 2.3
If $v = (v^1, \dots, v^n) \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$, *then* $\det Dv \in \mathscr{H}^1(\mathbb{R}^n)$. *Moreover,*

$$\|\det Dv\|_{\mathscr{H}^1(\mathbb{R}^n)} \le C \prod_{j=1}^{n} \|\nabla v^j\|_{L^n(\mathbb{R}^n)}. \tag{2.2}$$

The constant C *depends only on the dimension* n.

Estimate (2.2) is not explicitly stated in [5] but follows from the proof presented there. One has to combine the pointwise estimate of $(\det Dv)_* = \sup_\varepsilon (\varphi_\varepsilon * \det Dv)$ given in [5, Section 2] with the Hardy-Littlewood maximal theorem.

In Section 3, we find it convenient to use the language of differential forms. Thus,

$$dv^1 \wedge \cdots \wedge dv^n = \det Dv\, dx_1 \wedge \cdots \wedge dx_n$$

whenever the map v is of class $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$, and we interpret $\|dv^1 \wedge \cdots \wedge dv^n\|_{\mathscr{H}^1}$ as $\|\det Dv\|_{\mathscr{H}^1}$. (The wedge symbol $\wedge$ is used to denote two operations: the exterior product of differential forms and the cross product in $\mathbb{R}^n$; the context should always be clear.) Combining Proposition 2.3 with the imbedding $W^{1,n} \subset \mathrm{BMO}$ (which follows easily from Poincaré inequality), one obtains the following.

LEMMA 2.4
Let B be a ball in $\mathbb{R}^n$. Assume that the functions w, $v^1, v^2, \dots, v^n$ belong to $W_0^{1,n}(B)$, and assume that $w \in L^\infty(B)$. There exists a constant C that depends only on n such that

$$\Big| \int_{\mathbb{R}^n} w\, dv^1 \wedge \cdots \wedge dv^n \Big| \le C \|\nabla w\|_{L^n(B)} \prod_{j=1}^{n} \|\nabla v^j\|_{L^n(B)}. \tag{2.3}$$

A detailed explanation can be found, for example, in [30, Section 2]. Direct proofs that bypass the theory of Hardy spaces are also available; see, for example, [13, proof of Lemma 3.2] or [31, Theorem 2] (both results are more general than the above lemma).

We also use the following result on weak-$*$ convergence in the Hardy space.

THEOREM 2.5 (Jones, Journé)
Let $(g_k) \subset \mathscr{H}^1(\mathbb{R}^n)$ be a bounded sequence such that $g_k \to g$ a.e., and let $g \in L^1(\mathbb{R}^n)$. Then $g \in \mathscr{H}^1(\mathbb{R}^n)$ and $g_k \overset{*}{\rightharpoonup} g$ in $\mathscr{H}^1(\mathbb{R}^n)$; that is,

$$\lim_{k=\infty} \int_{\mathbb{R}^n} g_k \phi\, dx = \int_{\mathbb{R}^n} g\phi\, dx \tag{2.4}$$

for all $\phi \in$ VMO($\mathbb{R}^n$).

Recall that VMO is the space of functions having vanishing mean oscillation, that is, the closure of C_0^∞ in the BMO norm. A particular case of Theorem 2.5, ascertaining the continuity of Jacobian determinants in the sense of distributions, is well known and dates back at least to Reshetnyak [22].

LEMMA 2.6
Assume that $(u_k)_{k=1}^\infty \subset W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$, and assume that $u_k \rightharpoonup u$ weakly in $W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$. Then $\frac{\partial u_k}{\partial x_1} \wedge \cdots \wedge \frac{\partial u_k}{\partial x_n} \to \frac{\partial u}{\partial x_1} \wedge \cdots \wedge \frac{\partial u}{\partial x_n}$ in the sense of distributions; that is,

$$\lim_{k=\infty} \int_{\mathbb{B}^n} \psi \cdot \frac{\partial u_k}{\partial x_1} \wedge \cdots \wedge \frac{\partial u_k}{\partial x_n}\, dx = \int_{\mathbb{B}^n} \psi \cdot \frac{\partial u}{\partial x_1} \wedge \cdots \wedge \frac{\partial u}{\partial x_n}\, dx$$

for all smooth, compactly supported test maps ψ.

See also Ball [1] and Iwaniec [17] for related results.

3. Proof of the main result

In this section, we present a proof of Theorem 1.1. It employs the concentration-compactness method of P.-L. Lions [19], [20]. A similar idea was used by Freire, Müller, and Struwe [12] in their simplified proof of Bethuel's results [2] on Palais-Smale sequences for the H-surface functional and the harmonic map functional.

Let $u_k \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$, $k = 1, 2, \ldots$, be a weakly convergent sequence of weak solutions of (1.4). Upon passing to a subsequence, we may assume that

$$\begin{aligned} u_k \to u \quad & \text{strongly in } L^n(\mathbb{B}^n, \mathbb{R}^{n+1}) \text{ and a.e.,} \\ \nabla u_k \rightharpoonup \nabla u \quad & \text{weakly in } L^n(\mathbb{B}^n, \mathbb{R}^{n\times(n+1)}), \end{aligned} \tag{3.1}$$

for some $u \in W^{1,n}$; by Theorem 2.2, we may also assume that

$$\nabla u_k \ \to\ \nabla u \quad \text{strongly in } L^q(\mathbb{B}^n, \mathbb{R}^{n\times(n+1)}) \text{ for all } q < n \text{ and a.e.} \tag{3.2}$$

By (3.1) and (3.2),

$$|\nabla u_k|^{n-2}\nabla u_k \ \to\ |\nabla u|^{n-2}\nabla u \quad \text{weakly in } L^1(\mathbb{B}^n, \mathbb{R}^{n\times(n+1)}).$$

Therefore,

$$\lim_{k=\infty} \int_{\mathbb{B}^n} |\nabla u_k|^{n-2}\nabla u_k \cdot \nabla\psi\, dx = \int_{\mathbb{B}^n} |\nabla u|^{n-2}\nabla u \cdot \nabla\psi\, dx \tag{3.3}$$

for all $\psi \in C_0^\infty(\mathbb{B}^n, \mathbb{R}^{n+1})$, and we are left with the task of investigating the convergence of right-hand sides of (1.4).

First, extend each u_k to the ball $B(0, 2)$ so that the trace $u_k\big|_{\partial B(0,2)} = 0$ for $k = 1, 2, \ldots$, and set $u_k \equiv 0$ off $B(0, 2)$. With no loss of generality, one may assume that (3.1) and (3.2) are still valid. Moreover, since the sequence u_k is bounded in $W^{1,n}$, one may invoke estimate (2.2) from Proposition 2.3 to obtain, for each $i = 1, \ldots, n + 1$,

$$\sup_{k\in\mathbb{N}} \big\| du_k^1 \underbrace{\wedge\cdots\wedge}_{du_k^i \text{ omitted}} du_k^{n+1} \big\|_{\mathscr{H}^1(\mathbb{R}^n)} \le M < +\infty. \tag{3.4}$$

To each u_k we associate a vector-valued distribution $T_{u_k} \equiv T_k \in \mathscr{D}'(\mathbb{R}^n, \mathbb{R}^{n+1})$, given by

$$\langle T_k, \psi\rangle = \int_{\mathbb{R}^n} H(u_k)\psi \cdot \frac{\partial u_k}{\partial x_1} \wedge\cdots\wedge \frac{\partial u_k}{\partial x_n}\, dx, \qquad \psi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

Since the coordinates of the cross products are given by the determinants of appropriate minors, we write $T_k = (T_k^1, \ldots, T_k^{n+1})$, where each $T_k^i \in \mathscr{D}'(\mathbb{R}^n)$ and

$$\langle T_k^i, \varphi\rangle = (-1)^{i-1} \int_{\mathbb{R}^n} H(u_k)\varphi\, du_k^1 \underbrace{\wedge\cdots\wedge}_{du_k^i \text{ omitted}} du_k^{n+1} \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^n). \tag{3.5}$$

This notation is in accordance with a rule we follow: the upper index denotes the coordinate of a vector object, while the lower one is a sequence index.

It turns out that the following generalization of [20, Lemma 4.3] holds.

LEMMA 3.1
Under all the above assumptions, there exists a subsequence $k' \to +\infty$ such that for all $i = 1, 2, \ldots, n+1$,

$$T^i_{k'} \to T^i + \sum_{j \in J} a_{ji} \delta_{x_{ji}} \in \mathscr{D}'(\mathbb{R}^n),$$

where
(1) $T \equiv T_u$ *is associated to* u *by (3.5), with all indices* k *omitted;*
(2) J *is at most countable,* $a_{ji} \in \mathbb{R}$, $x_{ji} \in B(0,2)$, *and* $\sum_{j \in J} |a_{ji}| < +\infty$.

Proof
We proceed as in [20], adding the duality of Hardy space and BMO as a necessary ingredient. We consider only $i = n+1$; for other i's, the reasoning is similar.

Step 1. Assume that $u \equiv 0$. Fix $\varphi \in C_0^\infty(\mathbb{R}^n)$. We begin with an estimate of

$$\big|\langle T_k^{n+1}, \varphi^{n+1}\rangle\big| = \Big|\int_{\mathbb{R}^n} H(u_k)\varphi^{n+1}\, du_k^1 \wedge \cdots \wedge du_k^n\Big|. \tag{3.6}$$

(A word of caution: here and in the sequel various upper indices added to φ always denote *powers*.) Using the telescoping sum

$$\varphi^n du_k^1 \wedge \cdots \wedge du_k^n - d(\varphi u_k^1) \wedge \cdots \wedge d(\varphi u_k^n) = \sum_{j=0}^{n-1} (\omega_k^j - \omega_k^{j+1}),$$

where

$$\omega_k^j = \varphi^{n-j} d(\varphi u_k^1) \wedge \cdots \wedge d(\varphi u_k^j) \wedge du_k^{j+1} \wedge \cdots \wedge du_k^n, \quad j = 0, 1, \ldots, n-1,$$

we obtain

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} H(u_k)\varphi^{n+1}\, du_k^1 \wedge \cdots \wedge du_k^n - \int_{\mathbb{R}^n} H(u_k)\varphi\, d(\varphi u_k^1) \wedge \cdots \wedge d(\varphi u_k^n)\Big| \\
\le \sum_{j=0}^{n-1} \Big|\int_{\mathbb{R}^n} H(u_k)\varphi\, (\omega_k^j - \omega_k^{j+1})\Big| \\
\le \sum_{j=0}^{n-1} \Big|\int_{\mathbb{R}^n} H(u_k)\varphi^{n-j} u_k^{j+1}\, d(\varphi u_k^1) \wedge \cdots \wedge d(\varphi u_k^j) \\
\wedge d\varphi \wedge du_k^{j+2} \wedge \cdots \wedge du_k^n\Big|.
\end{aligned} \tag{3.7}$$

We estimate each term in the last sum using Hölder inequality with n exponents equal to n, noting first that the factors $H(u_k)$, φ^{n-j}, and $d\varphi$ are in L^∞. Since u_k is supported in $B(0,2)$, an application of Minkowski and Poincaré inequalities yields

$$\begin{aligned}\Big(\int_{\mathbb{R}^n} |\nabla(\varphi u_k^l)|^n\,dx\Big)^{1/n} &\le \Big(\int_{B(0,2)} |\nabla\varphi|^n |u_k|^n\,dx\Big)^{1/n} \\ &\quad + \Big(\int_{B(0,2)} |\varphi|^n |\nabla u_k|^n\,dx\Big)^{1/n} \\ &\le C(n)\|\varphi\|_{C^1}\Big(\int_{B(0,2)} |\nabla u_k|^n\,dx\Big)^{1/n}\end{aligned}$$

for each index $l = 1, \dots, n$. Using this observation, one easily checks that

$$\begin{aligned}&\text{the right-hand side of (3.7)} \\ &\quad \le C(n,\varphi)\|H\|_{L^\infty}\Big(\int_{\mathbb{R}^n} |u_k|^n\,dx\Big)^{1/n}\Big(\int_{\mathbb{R}^n} |\nabla u_k|^n\,dx\Big)^{1-1/n} \\ &\quad = \mathrm{o}(1) \quad \text{as } k\to\infty, \text{ by (3.1).}\end{aligned} \tag{3.8}$$

Therefore, (3.6), (3.7), and (3.8) lead to

$$\begin{aligned}\big|\langle T_k^{n+1}, \varphi^{n+1}\rangle\big| &\le \mathrm{o}(1) + \Big|\int_{\mathbb{R}^n} H(u_k)\varphi\, d(\varphi u_k^1)\wedge\cdots\wedge d(\varphi u_k^n)\Big| \\ &\le \mathrm{o}(1) + |H(0)|\Big|\int_{\mathbb{R}^n} \varphi\, d(\varphi u_k^1)\wedge\cdots\wedge d(\varphi u_k^n)\Big| \\ &\quad + \Big|\int_{\mathbb{R}^n} \big(H(u_k)-H(0)\big)\varphi\, d(\varphi u_k^1)\wedge\cdots\wedge d(\varphi u_k^n)\Big|.\end{aligned}$$

By Reshetnyak's lemma, the first integral tends to zero as $k\to\infty$. The second one, by the version of Fefferman's duality theorem stated in Lemma 2.4, does not exceed

$$C\,A_k\,\big(B_k\big)^n,$$

where

$$A_k = \big\|\nabla\big((H(u_k)-H(0))\varphi\big)\big\|_{L^n(\mathbb{R}^n)}, \qquad B_k = \|\nabla(\varphi u_k)\|_{L^n(\mathbb{R}^n)}.$$

Since H is Lipschitz, the Minkowski inequality gives

$$\begin{aligned}A_k &\le \big\|(H(u_k)-H(0))\nabla\varphi\big\|_{L^n(\mathbb{R}^n)} + C\big\|\varphi\nabla u_k\big\|_{L^n(\mathbb{R}^n)} \\ &\le \mathrm{o}(1) + C\|\varphi\nabla u_k\|_{L^n(\mathbb{R}^n)}.\end{aligned}$$

A similar estimate holds for B_k. Thus, we finally obtain

$$\big|\langle T_k^{n+1}, \varphi^{n+1}\rangle\big| \le \mathrm{o}(1) + C(n,H)\Big(\int_{\mathbb{R}^n} |\varphi|^n|\nabla u_k|^n\,dx\Big)^{1+1/n}. \tag{3.9}$$

Note now that all T_k^{n+1} and $|\nabla u_k|^n\,dx$ are in fact *uniformly* bounded (signed) Radon measures supported in $B(0,2) \subset \mathbb{R}^n$. Hence, passing to a subsequence, we may assume that there exist $d\nu, d\mu \in \mathscr{M}(\mathbb{R}^n)$ which are weak limits of, respectively, T_k^{n+1} and $|\nabla u_k|^n\,dx$. Thus, upon letting $k \to +\infty$ in (3.9), we obtain

$$\Big|\int_{\mathbb{R}^n} \varphi^{n+1}\,d\nu\Big| \le C(n,H)\Big(\int_{\mathbb{R}^n} |\varphi|^n\,d\mu\Big)^{1+1/n}. \tag{3.10}$$

Applying P.-L. Lions [19, Lemma 1.2], we conclude that $d\nu = \sum_{j\in J} a_j \delta_{x_j}$, with J being at most countable, and $\sum_{j\in J} |a_j| < +\infty$.

Step 2. Assume now that $u \not\equiv 0$. To estimate the difference between $\langle T_k^{n+1}, \varphi\rangle$ and $\langle T^{n+1}, \varphi\rangle$, we apply the Jones-Journé theorem. This is possible in our setting since

$$du_k^1 \wedge \cdots \wedge du_k^n \to du^1 \wedge \cdots \wedge du^n \quad \text{a.e.}$$

in light of (3.2), the sequence of the wedge products $du_k^1 \wedge \cdots \wedge du_k^n$ is bounded in $\mathscr{H}^1(\mathbb{R}^n)$ due to (3.4), and finally $du^1 \wedge \cdots \wedge du^n \in L^1(\mathbb{R}^n)$ by Hölder inequality. Thus, Theorem 2.5 yields

$$\lim_{k=\infty} \int_{\mathbb{R}^n} H(u)\varphi\, du_k^1 \wedge \cdots \wedge du_k^n = \int_{\mathbb{R}^n} H(u)\varphi\, du^1 \wedge \cdots \wedge du^n \tag{3.11}$$

since $\varphi H(u) \in \mathrm{VMO}(\mathbb{R}^n)$ for $\varphi \in C_0^\infty$, $H \in \mathrm{Lip}$, and $u \in W^{1,n}$. (One checks this by a simple application of Poincaré inequality.)

Set

$$\langle S_k, \varphi\rangle :\, = (-1)^n \int_{\mathbb{R}^n} H(u)\varphi\, du_k^1 \wedge \cdots \wedge du_k^n.$$

According to (3.11), we have

$$\langle T_k^{n+1} - T^{n+1}, \varphi\rangle = \mathrm{o}(1) + \langle T_k^{n+1} - S_k, \varphi\rangle \quad \text{as } k \to \infty. \tag{3.12}$$

We now apply the formula

$$A_1A_2\cdots A_n - B_1B_2\cdots B_n = (A_1 - B_1)A_2A_3\cdots A_n + B_1(A_2 - B_2)A_3\cdots A_n + \cdots + B_1\cdots B_{n-1}(A_n - B_n)$$

to write

$$\langle T_k^{n+1} - S_k, \varphi\rangle = (-1)^n \int_{\mathbb{R}^n} \big(H(u_k) - H(u)\big)\varphi\, d(u_k^1 - u^1) \wedge \cdots \wedge d(u_k^n - u^n) + {\sum}', \tag{3.13}$$

where

$$\sum{}' = \sum_{j=0}^{n-1} (-1)^n \int_{\mathbb{R}^n} \big(H(u_k) - H(u)\big)\varphi\, \Omega_j$$

with

$$\Omega_j :\ = \bigwedge_{s=1}^{j} d(u_k^s - u^s) \wedge du^{j+1} \wedge \bigwedge_{t=j+2}^{n} du_k^t.$$

Each of the terms of the sum $\sum'$ can be written as

$$\pm \int_{\mathbb{R}^n} u^{j+1}\, d\big((H(u_k) - H(u))\varphi\big) \wedge \bigwedge_{s=1}^{j} d(u_k^s - u^s) \wedge \bigwedge_{t=j+2}^{n} du_k^t. \tag{3.14}$$

Our aim now is to apply the Jones-Journé theorem to each of the terms in (3.14) to conclude their convergence to zero. The fixed factor u^{j+1} is of class VMO; this follows from Poincaré inequality. Since $\varphi \in C_0^\infty$ and $H \in C^1$ is Lipschitz, $d(\varphi H(u_k)) \to d(\varphi H(u))$ a.e. by (3.2). Thus, again by (3.2),

$$\begin{aligned} g_k &:= d\big((H(u_k) - H(u))\varphi\big) \wedge \bigwedge_{s=1}^{j} d(u_k^s - u^s) \wedge \bigwedge_{t=j+2}^{n} du_k^t \\ &\to g \equiv 0 \quad \text{a.e.} \end{aligned}$$

Moreover, a uniform bound

$$\sup_{k\in\mathbb{N}} \|g_k\|_{\mathscr{H}^1(\mathbb{R}^n)} \le C_0 < +\infty \tag{3.15}$$

follows, via a routine computation, from estimate (2.2) in Proposition 2.3 combined with (3.1). Indeed,

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \big\|\nabla\big((H(u_k) - H(u))\varphi\big)\big\|_{L^n(\mathbb{R}^n)} \\ &\qquad \le \|\varphi\|_{C^1} \sup_{k\in\mathbb{N}} \big(\|\nabla\big(H(u_k) - H(u)\big)\|_{L^n(\mathbb{R}^n)} + \|H(u_k) - H(u)\|_{L^n(\mathbb{R}^n)}\big) \\ &\qquad \le (\|H\|_{L^\infty} + \text{Lip } H) \sup_{k\in\mathbb{N}} \big(\|u_k\|_{W^{1,n}(\mathbb{R}^n)} + \|u\|_{W^{1,n}(\mathbb{R}^n)}\big) \\ &\qquad < +\infty, \end{aligned}$$

and the bounds for $\sup_k \|\nabla u_k - \nabla u\|_{L^n}$ and $\sup_k \|\nabla u_k\|_{L^n}$ follow trivially from (3.1). (Note carefully that we are using only *weak* convergence of the gradients in L^n here.) This yields inequality (3.15).

Hence, an application of the Jones-Journé theorem to the integrals (3.14) is justified; each of these integrals goes to zero as $k \to \infty$. Therefore, $\sum'$ in (3.13) goes to zero and we deduce that

$$\langle T_k^{n+1} - S_k, \varphi\rangle = \langle V_k, \varphi\rangle + \mathrm{o}(1) \quad \text{as } k \to \infty, \tag{3.16}$$

where

$$\langle V_k, \varphi\rangle :\ = (-1)^n \int_{\mathbb{R}^n} \big(H(u_k) - H(u)\big)\varphi\, d(u_k^1 - u^1) \wedge \cdots \wedge d(u_k^n - u^n).$$

Reasoning precisely as in the first step of the proof of the lemma, we obtain

$$|\langle V_k, \psi^{n+1}\rangle| \le C(n, H)\Big(\int_{\mathbb{R}^n} |\psi|^n |\nabla(u_k - u)|^n\Big)^{1+1/n} + \mathrm{o}(1), \quad \psi \in C_0^\infty(\mathbb{R}^n).$$

Hence, again by [19, Lemma 1.2],

$$V_k \to d\nu = \sum_{j\in J} a_j \delta_{x_j}, \tag{3.17}$$

with J at most countable, $a_j \in \mathbb{R}$, $\sum_{j\in J} |a_j| < +\infty$, and $x_j \in B(0, 2)$. Combining (3.17) with (3.16) and (3.12), we complete the proof of the whole lemma. □

As an immediate application, we obtain the following.

COROLLARY 3.2
If $u_k \in W^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$ are weak solutions of the H-system (1.4), and $u_k \rightharpoonup u$ weakly in $W^{1,n}$, then

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = H(u)\frac{\partial u}{\partial x_1} \wedge \cdots \wedge \frac{\partial u}{\partial x_n} + \sum_{j\in J} a_j \delta_{x_j}, \tag{3.18}$$

with J at most countable and

$$(x_j)_{j\in J} \subset \mathbb{B}^n, \qquad a_j \in \mathbb{R}^{n+1}, \qquad \sum_{j\in J} |a_j| < +\infty.$$

To complete the whole proof, it remains now to remove the singularities at x_j. To this end, fix $j_0 \in J$ and select a sequence of test maps $\varphi_l \in C_0^\infty(\mathbb{B}^n, \mathbb{R}^{n+1})$ such that

$$\varphi_l \to 0 \quad \text{on } \mathbb{R}^n \setminus \{x_{j_0}\}, \qquad 0 \le |\varphi_l| \le C, \qquad \int_{\mathbb{B}^n} |\nabla \varphi_l|^n \to 0,$$

and

$$\langle a_{j_0}\delta_{x_{j_0}}, \varphi_l\rangle = |a_{j_0}|.$$

(Such a choice of (φ_l) is possible since $W^{1,n}(\mathbb{B}^n)$ contains unbounded functions.) Testing (3.18) with φ_l, we obtain

$$\mathrm{o}(1) = \mathrm{o}(1) + |a_{j_0}| + \sum_{j \neq j_0} \langle a_j \delta_{x_j}, \varphi_l \rangle,$$

and thus $|a_{j_0}| = 0$, since by the dominated convergence theorem the last sum goes to zero as $l \to \infty$. The proof of Theorem 1.1 is complete now. □

Remark. It would be interesting to know whether for $n \geq 3$ a counterpart of Theorem 1.1 holds for n-harmonic maps from $\Omega \subset \mathbb{R}^n$ into arbitrary compact Riemannian manifolds (as it does for $n = 2$, and for target manifolds that are round spheres or, more generally, compact symmetric spaces).

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