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On biharmonic maps and their generalizations

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Abstract. We give a new proof of regularity of biharmonic maps from four-dimensional domains into spheres, showing first that the biharmonic map system is *equivalent* to a set of bilinear identities in divergence form. The method of reverse Hölder inequalities is used next to prove continuity of solutions and higher integrability of their second order derivatives. As a byproduct, we also prove that a weak limit of biharmonic maps into a sphere is again biharmonic. The proof of regularity can be adapted to biharmonic maps on the Heisenberg group, and to other functionals leading to fourth order elliptic equations with critical nonlinearities in lower order derivatives.

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1. Introduction

In a recent paper [8] Chang, Wang and Yang initiate the study of biharmonic maps. These are defined as critical points (with respect to variations in the range) of the functional

$$E(u) = \int_{\mathbb{B}^m} \sum_{\alpha=1}^{k+1} |\Delta u^\alpha|^2 \, dx \,. \tag{1.1}$$

Related functionals leading to fourth order elliptic equations with critical nonlinearities are connected with intriguing problems in four dimensional conformal geometry; see e.g. [6] and [39] for more information.

In [8] the authors consider the model case of maps $u \in W^{2,2}(\mathbb{B}^m, \mathbb{S}^k)$, where $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ is the standard unit sphere. They prove that in dimension $m = 4$ all biharmonic maps are smooth in the interior of $\mathbb{B}^4$, and for $m \geq 5$ stationary biharmonic maps are smooth off a singular set Σ_u with $\mathcal{H}^{m-4}(\Sigma_u) = 0$.

To achieve this goal, they extend a method which they used earlier in [7] to prove Hélein's theorem [18] on regularity of harmonic maps into spheres without relying on Hardy space methods. In the case of biharmonic maps the argument

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becomes quite intricate; one first has to rewrite the right hand side of the Euler equation

$$
\begin{aligned}
&-\Delta^2 u^\alpha \qquad (1.2)\\
&= u^\alpha \sum_{\beta=1}^{k+1} \Big(\big(\Delta u^\beta\big)^2 + \Delta\big(\big|\nabla u^\beta\big|^2\big) + 2\nabla u^\beta \cdot \nabla \Delta u^\beta \Big), \quad \alpha = 1, \dots, k+1,
\end{aligned}
$$

as a linear combination of six different nonlinear terms in "divergence form". Then, u is decomposed into $u = u_0 + u_1 + u_2 + u_3$, where $\Delta^2 u_0 = 0$, and $\Delta^2 u_i = \nabla^i F_i$ for $i = 1, 2, 3$ and suitably chosen F_i. Chang, Wang and Yang use a clever argument based on singular integral estimates for the auxiliary linear equations. In dimension $m = 4$ this leads to a decay estimate

$$
\left(\fint_{B_r} |u - u_{B_r}|^{p^*} \, dx \right)^{1/p^*} + r \left(\fint_{B_r} |\nabla u|^p \, dx \right)^{1/p} \le C r^\beta
$$

for $\beta \in (0, 1)$ and $r < r_0$. The exponent p is strictly smaller than 4 (i.e. the natural one). This gives Hölder continuity of u. Next, using estimates in Morrey–Campanato spaces and linear regularity theory, one obtains the following.

Theorem 1.1 (Chang et al., [8]) *Every biharmonic map $u \in W^{2,2}(\mathbb{B}^4, \mathbb{S}^k)$ is of class C^∞ in the interior of $\mathbb{B}^4$.*

In dimensions $m \ge 5$ one has to add a monotonicity formula to control the decay of the BMO norm of u on a large set; this leads to the aforementioned partial regularity.

A careful inspection of the initial part of the proof in [8, pages 1115–1118] reveals that a single identity in "diveregence form", namely

$$
\operatorname{div} E^{\alpha\beta} = 0 \qquad \text{in } \mathcal{D}'(\mathbb{B}^m), \tag{1.3}
$$

where $E^{\alpha\beta} = u^\beta \nabla \Delta u^\alpha - u^\alpha \nabla \Delta u^\beta - \Delta u^\alpha \nabla u^\beta + \Delta u^\beta \nabla u^\alpha$, is equivalent to the equation of biharmonic maps into spheres. This equivalence is not explicitly stated in [8], and does not seem to be fully exploited there.

This is our starting point. We show that (1.3) is an equivalent form of the biharmonic map equation, and this observation implies immediately that a $W^{2,2}$-weak limit of biharmonic maps is also a biharmonic map. Next, we use (1.3) to rewrite the biharmonic map equation as

$$
N(u^\alpha) = \sum_{\beta=1}^{k+1} \nabla u^\beta \cdot E^{\alpha\beta}, \qquad \alpha = 1, 2, \dots, k+1\,. \tag{1.4}
$$

Here, $N(u^\alpha)$ is a quasilinear fourth order elliptic operator. For the purposes of this introduction it is convenient to imagine that

$$
N(u^\alpha) = \Delta^2 u^\alpha - \operatorname{div}\left(|\nabla u|^2 \nabla u^\alpha\right) + \text{ an unimportant perturbation.}
$$

The left hand side of (1.4) has a familiar form, reminiscent of various applications of Hardy space methods – growing out from the famous paper of Coifman, Lions, Meyer and Semmes [9] — to nonlinear elliptic equations: harmonic and p-harmonic maps, surfaces of prescribed mean curvature, H-systems in higher dimensions etc., see e.g. Hélein's book [20] for numerous comments and references. However, the situation here is slightly different. For every map $u \in W^{2,2}(\mathbb{B}^m, \mathbb{S}^k)$ we have $|\nabla u|^2 = -\sum_\alpha u^\alpha \Delta u^\alpha \le |\Delta u|$, so that $\nabla u \in L^4$, but it is not a priori clear whether $E^{\alpha\beta} \in L^{4/3}$ (as we do not know if $D^3 u$ exists in $L^{4/3}$) – and [9, Theorem II.1, p. 2] cannot be directly applied to conclude that $N(u^\alpha)$ indeed belongs to a local Hardy space.

Nevertheless, we set off from (1.4) to give a new proof of Theorem 1.1. In contrast to Chang, Wang and Yang, we work with natural exponents. The crucial step is to show that $|D^2u|^2$ satisfies a weak reverse Hölder inequality. This leads to Hölder continuity of solutions. Moreover, one can use the improved integrability of $|D^2u|^2$ and $|\nabla u|^4$ to show that the right hand side of (1.2) belongs locally to L^p for some $p > 1$. A rather standard bootstrap argument allows then to show that u is in fact a classical solution; smoothness follows from Schauder theory.

The main difficulty we have to overcome is the estimate of the right hand side of (1.4). At first glance, this expression – when appropriately interpreted via integration by parts – behaves, roughly speaking, like an L^1 function comparable to $|\Delta u|^2$. A toy example of the H-surface equation $\Delta u = 2H u_{x_1} \wedge u_{x_2}$, where $u \in W^{1,2}(\mathbb{B}^2, \mathbb{R}^3)$, shows that in an analogous situation instead of the trivial L^∞–L^1 estimate

$$\left| \int_B \varphi \cdot u_{x_1} \wedge u_{x_2} \right| \le \sup_B |\varphi| \int_B |\nabla u|^2 \, dx, \qquad \varphi \in C_0^\infty(B) \tag{1.5}$$

one may (and should) use the Wente inequality

$$\left| \int_B \varphi \cdot u_{x_1} \wedge u_{x_2} \right| \le C \|\nabla \varphi\|_{L^2(B)} \int_B |\nabla u|^2 \, dx \, . \tag{1.6}$$

The advantage of (1.6) comes from the fact that points in $\mathbb{R}^2$ have zero capacity: one may find a sequence $(\varphi_j)_{j=1,2,\dots} \subset C_0^\infty$ such that

$$0 \le \varphi_j \le 1, \quad \sup \varphi_j \equiv 1, \quad \|\nabla \varphi_j\|_{L^2} \to 0, \quad \operatorname{diam}(\operatorname{supp} \varphi_j) \to 0.$$

Now, Wente inequality follows from the duality of Hardy space and BMO combined with the results of Coifman et al. [9] and the imbedding $W^{1,2}(\mathbb{R}^2) \subset \mathrm{BMO}$ (which is a direct consequence of Poincaré inequality). However, one does not need here the full strength of Hardy space–BMO duality. It is enough to know that the determinant $du^1 \wedge du^2$ generates a linear functional on $W_0^{1,2}(B)$ for each ball B, and that the norm of this functional does not exceed a constant multiple of $\int_B |\nabla u|^2$.

We follow that track in the case of biharmonic maps. Though we cannot directly conclude that $\Lambda = \nabla u^\beta E^{\alpha\beta}$ is in the Hardy space, it is possible to show that locally Λ can be viewed as a functional $\Lambda \in \left(W_0^{2,2}\right)^*$, with $\|\Lambda\| \le C \|\Delta u\|_{L^2}^{3/2}$. We achieve this in Lemma 3.1; its proof is the *only* place where the cancellation property $\operatorname{div} E^{\alpha\beta} = 0$ is absolutely necessary. The argument is fairly elementary and draws inspiration from [30] and [17]. Whitney decomposition and careful analysis of Riesz potentials serve as main tools.

The rest is fairly routine. Lemma 3.1 is sufficient to obtain a reverse Hölder inequality in a more or less standard way (and would be also sufficient to prove continuity of solutions by Widman's hole filling trick). Continuity and smoothness follow by standard bootstrap methods. One also sees that the linearity of Δ^2 does not play any role in the initial stage: in the proof of continuity of solutions we could replace $N(u^\alpha)$ by any reasonable monotone fourth order operator. This opens way to generalizations: (a) to p-biharmonic maps, defined as critical points of $\int |\Delta u|^p\,dx$ in $W^{2,p}(\mathbb{B}^m, \mathbb{S}^k)$ – these turn out to be regular in dimension $m = 2p$, (b) to biharmonic maps on the Heisenberg group. Here are a couple of results in these directions.

Theorem 1.2 *If $u \in W^{2,p}(\mathbb{B}^m, \mathbb{S}^k)$ is a critical point of the energy functional $E_p(u) = \int_{\mathbb{B}^m} |\Delta u|^p\,dx$, and $2p = m \geq 3$, then u is Hölder continuous in the interior of $\mathbb{B}^m$.*

Theorem 1.3 *Assume that $u \in W^{2,2}_X(\Omega, \mathbb{S}^k)$, where Ω is an open domain in the Heisenberg group $\mathbb{H}_1$, and $W^{2,2}_X$ denotes the Folland–Stein Sobolev space. If u is a critical point of the energy functional $E_2(u) = \int_\Omega |\Delta_X u|^2\,dx$, where Δ_X is the subelliptic Laplace operator, then u is smooth in Ω.*

Theorem 1.4 *Assume that $u \in W^{2,p}_X(\Omega, \mathbb{S}^k)$, where Ω is an open domain in a Carnot group $\mathbb{G}$ of homogeneous dimension Q. If u is a critical point of the energy functional $E_p(u) = \int_\Omega |\Delta_X u|^p\,dx$, where Δ_X is the subelliptic Laplace operator on $\mathbb{G}$, and $2p = Q$, then u is Hölder continuous in Ω.*

Our motivation was, partially, to contribute to the theory of subelliptic generalizations of variational problems associated with nonlinear geometric elliptic systems. This has recently been an active field of research; it seems to be a widespread belief that numerous regularity results regarding (quasi)minima of variational integrals with appropriate growth conditions, and harmonic or p-harmonic maps, have their subelliptic counterparts. To support this judgment, let us just mention here the work of Capogna and Garofalo [5], Jost and Xu [23], C.Y. Wang [34], Xu and Zuilly [38], and Hajłasz and the author [17]; the reader is referred to these papers for a more thorough bibliography. The present paper can be seen as yet another drop in that stream.

However, we believe that the method used here to prove Theorem 1.1 is interesting in its own right. The duality estimate from Lemma 3.1 is applicable in a situation where the duality of $\mathcal{H}^1$ and BMO is not a priori available as a tool. We plan to investigate in future possible extensions of this approach to other higher order elliptic systems with critical nonlinearities.

One can easily imagine more results resembling Theorems 1.2–1.4 above. In fact, we conjecture that if $\mathcal{A}$ is an arbitrary ℓ-th order elliptic operator with constant coefficients acting on $C^\infty(\mathbb{R}^m, \mathbb{R}^{k+1})$, then critical points $u \in W^{\ell,p}(\mathbb{B}^m, \mathbb{S}^k)$ of the energy

$$E_{\mathcal{A}}(u) = \int_{\mathbb{B}^m} |\mathcal{A}u|^p\,dx$$

are regular whenever $p = m/\ell > 1$.

The notation throughout the paper is more or less standard. Barred integrals denote averages, i.e. $\fint_A f\,dx = |A|^{-1}\int_A f\,dx$, where $|A|$ is the Lebesgue measure. Sometimes we also use the shortcut $f_A = \fint_A f\,dx$. For various exponents p, q, s etc. $\in (1,\infty)$ we write p', q', s' etc. to denote their Hölder conjugates; p_* an exponent for which p is the Sobolev conjugate, i.e. $p_* = mp/(m+p)$ if the dimension is equal to m. The letter C traditionally stands for a general constant which can change its value even in a single string of estimates.

We use upper Greek indices to denote coordinates of various mappings into Euclidean spaces. The summation convention is *not* employed.

In Sect. 5.1, to render the exposition more or less self-contained, we recall briefly a few facts concerning calculus on homogeneous groups.

Added in proof. When this work has been completed and submitted for publication, the author has learned that C.Y. Wang obtained, in a series of recent preprints [35], [36] and [37], another new proof of Theorem 1.1 and of Proposition 2.1. Moreover, using different methods (estimates in Lorentz spaces), C. Y. Wang extended Theorem 1.1 to arbitrary compact Riemannian target manifolds.

2. The Euler equation and weak convergence

Let us consider a map $u \in W^{2,2}(\mathbb{B}^m, \mathbb{S}^k)$ which is a critical point of the functional

$$E(u) = \int_{\mathbb{B}^m} \sum_{\alpha=1}^{k+1} |\Delta u^\alpha|^2\,dx$$

with respect to variations in the range. To derive the Euler–Lagrange equation, we differentiate $E(\pi\circ(u+t\varphi))$ at $t=0$, where $\varphi \in C_0^\infty(\mathbb{B}^m, \mathbb{R}^{k+1})$ is a smooth test map and $\pi(y) = y/|y|$ denotes the nearest point projection onto $\mathbb{S}^k$. This computation yields

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} E(\pi\circ(u+t\varphi)) &= 2\int_{\mathbb{B}^m} \Big\langle \Delta u, \Delta\Big(\frac{d}{dt}\Big|_{t=0}\pi\circ(u+t\varphi)\Big)\Big\rangle\,dx \\ &= 2\int_{\mathbb{B}^m} \big\langle \Delta u, \Delta\big(\varphi - \langle u,\varphi\rangle u\big)\big\rangle\,dx \qquad (2.1) \\ &= 0\,, \end{aligned}$$

or equivalently

$$\int_{\mathbb{B}^m} \Delta u^\alpha \Delta\zeta\,dx = \sum_{\gamma=1}^{k+1}\int_{\mathbb{B}^m} \Delta u^\gamma \Delta(u^\gamma u^\alpha \zeta)\,dx \qquad (2.2)$$
$$\text{for all } \alpha = 1,\dots,k+1 \text{ and } \zeta \in C_0^\infty(\mathbb{B}^m).$$

Note that $\varphi - \langle u,\varphi\rangle u$ in (2.1) is the tangential part of φ. Hence, the geometric interpretation of the Euler equation is clear: $\Delta^2 u$ is orthogonal to $T\mathbb{S}^k$ a.e.

Several integrations by parts show that for sufficiently regular maps (2.2) takes the form (1.2); see [8, page 1115] for details. To prove that all weak solutions of this last system are in fact smooth, the authors of [8] rewrite the right hand side as a linear combination of three different sorts of nonlinear terms in "divergence form".

An inspection of the long formal computations in [8] reveals that the biharmonic map equation is *equivalent* to a system of identities in divergence form. This equivalence is fully analogous to the equivalence between the equation $-\Delta u = |\nabla u|^2 u$ of harmonic maps into $\mathbb{S}^k$ and the system

$$\operatorname{div}(u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha) = 0\,, \qquad \alpha, \beta = 1, \dots, k+1;$$

see [18], [20] for comments on the relationships of these equations to the symmetries of $\mathbb{S}^k$. One can use this form of the biharmonic map equation to suggest a different proof of regularity, and to obtain a simple theorem on weak limits of biharmonic maps.

To make things precise, we need some notation. For $\alpha, \beta = 1, \dots, k+1$ let

$$E^{\alpha\beta} :\, = u^\beta \nabla \Delta u^\alpha - u^\alpha \nabla \Delta u^\beta - \Delta u^\alpha \nabla u^\beta + \Delta u^\beta \nabla u^\alpha. \tag{2.3}$$

For $u \in W^{2,2}(\mathbb{B}^m, \mathbb{S}^k)$ we interpret $E^{\alpha\beta}$ in the sense of distributions as follows:

$$\langle E^{\alpha\beta}, \varphi \rangle :\, = \int_{\mathbb{B}^m} \Big(\Delta u^\beta \big(\operatorname{div}(u^\alpha \varphi) + \varphi \nabla u^\alpha \big) - \Delta u^\alpha \big(\operatorname{div}(u^\beta \varphi) + \varphi \nabla u^\beta \big) \Big) dx \tag{2.4}$$

for test vector fields $\varphi \in C_0^\infty(\mathbb{B}^m, \mathbb{R}^m)$. To shorten the notation, we set

$$L(w, V) :\, = \operatorname{div}(wV) + \nabla w \cdot V \qquad \text{for } w \in C_0^\infty(\mathbb{B}^m),\ V \in C_0^\infty(\mathbb{B}^m, \mathbb{R}^m). \tag{2.5}$$

Note that one has simply $L(w, \nabla\zeta) = \Delta(\zeta w) - \zeta \Delta w$; however, for reasons that will become transparent later, we want to exclude second order derivatives of w from the notation. We also write

$$\Phi(v, w; V) := \Delta v\, L(w, V) - \Delta w\, L(v, V)\,, \tag{2.6}$$

and record the obvious growth estimate

$$|\Phi(u^\alpha, u^\beta; V)| \le 4|\Delta u| \big(|\nabla u|\, |V| + |\operatorname{div} V| \big), \qquad u \in W^{2,2}(\mathbb{B}^m, \mathbb{S}^k)\,. \tag{2.7}$$

In the new notation, (2.4) becomes simply

$$\langle E^{\alpha\beta}, \varphi \rangle = \int_{\mathbb{B}^m} \Phi(u^\beta, u^\alpha; \varphi)\, dx\,, \qquad \varphi \in C_0^\infty(\mathbb{B}^m, \mathbb{R}^m)\,. \tag{2.8}$$

Computing formally the divergence of both sides of (2.3), we obtain $\operatorname{div} E^{\alpha\beta} = u^\beta \Delta^2 u^\alpha - u^\alpha \Delta^2 u^\beta = 0$; the second equality holds since $\Delta^2 u$ is parallel to u. The formal identity $\operatorname{div} E^{\alpha\beta} = 0$ is interpreted as

$$\int_{\mathbb{B}^m} \Phi(u^\beta, u^\alpha; \nabla\zeta)\, dx = 0 \qquad \text{for all } \zeta \in C_0^\infty(\mathbb{B}^m). \tag{2.9}$$

This is the desired equivalent form of the biharmonic map equation.

Lemma 2.1 *Let $u \in W^{2,2}(\mathbb{B}^m, \mathbb{S}^k)$. The following conditions are then equivalent:*

(i) *u is biharmonic, i.e. (2.2) holds;*
(ii) *identity (2.9) holds for all $\alpha, \beta = 1, \ldots, k+1$*

Remark. There is a strong analogy here with harmonic maps into spheres. Namely, condition (2.9) can be written down also for mappings u with lower integrability assumptions. If $u \in W^{2,\frac{3}{2}}(\mathbb{B}^m, \mathbb{S}^k)$, then $|\nabla u| \in L^3$, and formula (2.8) makes sense. The old example $u(x) = |x|^{-1}x$, proposed by Giusti and Miranda [14], and used in the context of harmonic maps ever since Hildebrandt, Kaul and Widman [21], gives a map of class $W^{2,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$ for every $p \in [1, m/2)$, and satisfies (2.9). Indeed, an elementary computation shows that in this case

$$E^{\alpha\beta} = -\frac{2(m-1)}{|x|^2}\left(u^\beta \nabla u^\alpha - u^\alpha \nabla u^\beta\right),$$

and

$$\langle E^{\alpha\beta}, \nabla\zeta\rangle = (m-1)\int_{\mathbb{B}^m}\left(\frac{\partial\zeta}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\left(|x|^{-2}\right) - \frac{\partial\zeta}{\partial x_\beta}\frac{\partial}{\partial x_\alpha}\left(|x|^{-2}\right)\right)dx = 0,$$

as $D^2\zeta$ is symmetric for $\zeta \in C_0^\infty(\mathbb{B}^m)$.

Thus, for maps that are merely in $W^{2,p}$ with some $p < 2 = m/2$ conditions (i) and (ii) of the lemma are obviously not equivalent, and the system (2.9) does not lead to regularity. (In the case of generalized harmonic maps, L. Almeida [2] and R. Moser [26] give ε-regularity results for weak solutions $u\colon \Omega \to \mathbb{S}^k$ of the counterpart of (2.9), i.e. of div $(u^\alpha \nabla u^\beta - u^\beta \nabla u^\alpha) = 0$.)

Proof of Lemma 2.1. (i) $\Rightarrow$ (ii). Inserting $\zeta = u^\beta \varphi$ in (2.2), we obtain

$$\int_{\mathbb{B}^m} \Delta u^\alpha \Delta(u^\beta\varphi)\,dx = \sum_{\gamma=1}^{k+1}\int_{\mathbb{B}^m} \Delta u^\gamma \Delta(u^\gamma u^\alpha u^\beta \varphi)\,dx\,.$$

The right hand side is symmetric with respect to α and β. Thus, switching the role of α and β, and substracting two identities, we check that

$$\begin{aligned} 0 &= \int_{\mathbb{B}^m} \Delta u^\beta \Delta(u^\alpha\varphi)\,dx - \int_{\mathbb{B}^m} \Delta u^\alpha \Delta(u^\beta\varphi)\,dx \\ &= \int_{\mathbb{B}^m} \Delta u^\beta\, L(u^\alpha, \nabla\varphi)\,dx - \int_{\mathbb{B}^m} \Delta u^\alpha\, L(u^\beta, \nabla\varphi)\,dx\,. \end{aligned}$$

This is (2.9).
(ii) $\Leftarrow$ (i). Fix $\zeta \in C_0^\infty(\mathbb{B}^m)$ and insert $\varphi = \varphi^\beta : = \zeta u^\beta$ in (2.9). Summing with respect to β, and using the identities

$$\sum_{\beta=1}^{k+1}\left(u^\beta\right)^2 = 1, \qquad \sum_{\beta=1}^{k+1} u^\beta \nabla u^\beta = 0, \qquad \sum_{\beta=1}^{k+1} u^\beta \Delta u^\beta = -|\nabla u|^2, \tag{2.10}$$

which hold for all maps $u \in W^{2,2}(\mathbb{B}^m, \mathbb{S}^k)$, we obtain

$$\sum_{\beta=1}^{k+1} \operatorname{div}(u^\beta \nabla \varphi^\beta) = \Delta\zeta,$$

$$\sum_{\beta=1}^{k+1} \Delta u^\alpha \nabla u^\beta \cdot \nabla \varphi^\beta = \zeta |\nabla u|^2 \Delta u^\alpha = -\zeta \Delta u^\alpha \sum_{\beta=1}^{k+1} u^\beta \Delta u^\beta .$$

Thus,

$$\begin{aligned} \sum_{\beta=1}^{k+1} \Phi(u^\beta, u^\alpha, \nabla\varphi^\beta) &= \sum_{\beta=1}^{k+1} \big(\Delta u^\beta\, L(u^\alpha, \nabla\varphi^\beta) - \Delta u^\alpha\, L(u^\beta, \nabla\varphi^\beta)\big) \\ &= -\Delta u^\alpha \Delta\zeta + \sum_{\beta=1}^{k+1} \Delta u^\beta \nu^\beta , \end{aligned} \tag{2.11}$$

where $\nu^\beta = \zeta u^\beta \Delta u^\alpha + L(u^\alpha, \nabla\varphi^\beta) = \Delta(\zeta u^\beta u^\alpha)$. Integrating both sides of (2.11), we obtain (2.2). □

This lemma has an immediate consequence. It turns out that the following analogue of weak compactness of harmonic maps into spheres (see [20, Theorem 2.5.1]) is true.

Proposition 2.1 *If $(u_j)_{j=1,2,\ldots} \subset W^{2,2}(\mathbb{B}^m, \mathbb{S}^k)$ is a weakly convergent sequence of biharmonic maps, then $u = \lim u_j$ is also biharmonic.*

Proof. Applying Rellich–Kondrashov's theorem, and passing to a subsequence if necessary, we may assume that $(\nabla u_j)_{j=1,2,\ldots}$ and $(u_j)_{j=1,2,\ldots}$ converge in the strong topology of L^2 to ∇u and u, respectively. Thus, it follows from the definition of Φ that

$$0 = \int_{\mathbb{B}^m} \Phi(u_j^\beta, u_j^\alpha; \nabla\varphi)\, dx \stackrel{j=\infty}{\longrightarrow} \int_{\mathbb{B}^m} \Phi(u^\beta, u^\alpha; \nabla\varphi)\, dx$$

for each $\varphi \in C_0^\infty$. Hence, u satisfies condition (ii) of the last Lemma. □

We conclude this section with one more equivalent form of the biharmonic map equation. It shall be useful in the sequel.

As in the proof of Lemma 2.1, we obtain from (2.9)

$$\sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} \Phi(u^\beta, u^\alpha, \varphi\nabla u^\beta)\, dx = -\sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} \Phi(u^\beta, u^\alpha, u^\beta \nabla\varphi)\, dx \tag{2.12}$$

$$= \sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} \Delta u^\alpha\, L(u^\beta, u^\beta\nabla\varphi)\, dx - \sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} \Delta u^\beta\, L(u^\alpha, u^\beta\nabla\varphi)\, dx .$$

Now, $\Delta u^\alpha \sum_{\beta=1}^{k+1} L(u^\beta, u^\beta \nabla\varphi) = \Delta u^\alpha \Delta\varphi$, whereas the second integrand

$$
\begin{aligned}
-&\sum_{\beta=1}^{k+1} \Delta u^\beta\, L(u^\alpha, u^\beta \nabla\varphi) \\
&= \ -\sum_{\beta=1}^{k+1} \Delta u^\beta\, (u^\beta \nabla u^\alpha \cdot \nabla\varphi) - \sum_{\beta=1}^{k+1} \Delta u^\beta\, \mathrm{div}\, (u^\alpha u^\beta \nabla\varphi) \\
&\overset{(2.10)}{=} |\nabla u|^2 u^\alpha \Delta\varphi + 2|\nabla u|^2 \nabla u^\alpha \nabla\varphi - \sum_{\beta=1}^{k+1} u^\alpha \Delta u^\beta \nabla u^\beta \nabla\varphi.
\end{aligned}
$$

Hence, (2.12) leads to

$$
\begin{aligned}
&\int_{\mathbb{B}^4} \big(\Delta u^\alpha + |\nabla u|^2 u^\alpha\big) \Delta\varphi\, dx + 2\int_{\mathbb{B}^4} |\nabla u|^2 \nabla u^\alpha \nabla\varphi\, dx \\
-&\sum_{\beta=1}^{k+1} \int_{\mathbb{B}^4} u^\alpha \Delta u^\beta \nabla u^\beta \nabla\varphi\, dx = \sum_{\beta=1}^{k+1} \int_{\mathbb{B}^4} \Phi(u^\alpha, u^\beta; \varphi\nabla u^\beta)\, dx
\end{aligned}
\tag{2.13}
$$

for every $\alpha = 1, 2, \ldots, k+1$ and every test function $\varphi \in C_0^\infty(\mathbb{B}^4)$. By a density argument, (2.13) holds for all $\varphi \in W_0^{2,2} \cap W^{1,4} \cap L^\infty$. Note that formally (2.13) reads

$$
N(u^\alpha) = \sum_{\beta=1}^{k+1} \nabla u^\beta \cdot E^{\alpha\beta}, \qquad \alpha = 1, 2, \ldots, k+1, \tag{2.14}
$$

with

$$
\begin{aligned}
N(u^\alpha) \ :&= \Delta(\Delta u^\alpha + |\nabla u|^2 u^\alpha) - 2\mathrm{div}\, (|\nabla u|^2 \nabla u^\alpha) \\
&\quad + \sum_{\beta=1}^{k+1} \mathrm{div}\, (u^\alpha \Delta u^\beta \nabla u^\beta), \qquad\qquad (2.15) \\
E^{\alpha\beta} \ :&= u^\beta \nabla\Delta u^\alpha - u^\alpha \nabla\Delta u^\beta - \Delta u^\alpha \nabla u^\beta + \Delta u^\beta \nabla u^\alpha.
\end{aligned}
$$

Here $N(u^\alpha)$ is a quasilinear elliptic operator, equal to Δ^2 plus a lower order perturbation. Due to the identity $\sum_\alpha u^\alpha \nabla u^\alpha = 0$ some terms cancel when the system (2.14) is tested with $\varphi \approx u^\alpha$, and one can treat $N(u^\alpha)$ more or less in the same way as $\Delta^2 u^\alpha - \mathrm{div}\, (|\nabla u|^2 \nabla u^\alpha)$; see the next Section for details.

The right hand side has the familiar "div-curl" form. However, one cannot directly apply [9, Theorem II.1, p. 2] to conclude from (2.14) that $N(u^\alpha)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$, since it is not a priori assumed that $D^3 u \in L^{4/3}$.

3. Proof of Theorem 1.1

In this Section we present a proof of regularity of biharmonic maps from $\mathbb{B}^4$ into $\mathbb{S}^k$. We use the biharmonic map equation in the form (2.14), and a direct approach to regularity. To obtain continuity of solutions, and higher integrability of Δu and ∇u, we employ the method of weak reverse Hölder inequalities. Higher regularity follows from a classical bootstrap argument, relying on L^p theory of elliptic systems with constant coefficients.

3.1. Weak reverse Hölder inequalities

The derivation of a reverse Hölder inequality for biharmonic maps is also fairly standard, up to the following lemma, which allows us to cope with the critical nonlinearity.

Lemma 3.1 *Assume that $a \in \mathbb{B}^4$ and $2r < \operatorname{dist}(a, \partial\mathbb{B}^4)$. Let $u \in W^{2,2}(\mathbb{B}^4, \mathbb{S}^k)$ be a weakly biharmonic map, and let Φ be defined by (2.5), (2.6) . Then, for every test function $\varphi \in W_0^{2,2}(B(a, 4r/3))$ and all $\alpha, \beta \in \{1, 2, \ldots, k+1\}$ we have*

$$\left| \int_{B_{2r}} \Phi(u^\beta, u^\alpha; \varphi \nabla u^\beta)\, dx \right| \le C \left(\int_{B_{2r}} |\Delta \varphi|^2\, dx \right)^{\frac{1}{2}} \left(\int_{B_{2r}} |\Delta u|^2\, dx \right)^{\frac{3}{4}}, \quad (3.1)$$

where C is an absolute constant.

Remarks. (1) Observe that for $\varphi \approx u$ we obtain here the obvious term $\int |\Delta u|^2\, dx$ times a factor which tends to zero as $r \to 0$. (2) The lack of homogeneity (note the exponent $\frac{3}{4}$ in the right side) is due to the trivial bound $|u - u_B| \le 2$, which is employed in the proof of Lemma 3.1 to simplify some computations.

A fairly elementary, self-contained proof is given in the last Section of the paper. We shall now take this (crucial) lemma for granted, and go ahead to the next one.

Lemma 3.2 *Every weakly biharmonic map $u \in W^{2,2}(\mathbb{B}^4, \mathbb{S}^k)$ satisfies the inequality*

$$\begin{aligned} ⨍_{B_r} |D^2 u|^2\, dx \le C \left(⨍_{B_{2r}} |D^2 u|^{8/5}\, dx \right)^{5/4} \\ + C \left(\int_{B_{2r}} |\Delta u|^2\, dx \right)^{1/4} ⨍_{B_{2r}} |D^2 u|^2\, dx \end{aligned}$$

for all balls $B_r \equiv B(a, r) \subset B_{2r} \equiv B(a, 2r) \subset \mathbb{B}^4$, where C is some absolute constant.

Proof. We test (2.13) with $\varphi \equiv \varphi^\alpha : = \zeta^2 (u^\alpha - T_1 u^\alpha)$, where ζ is a standard smooth cutoff function, $\zeta \equiv 1$ on $B(a, r)$, $\zeta \equiv 0$ off $B(a, 4r/3)$, $|D^k \zeta| \le C r^{-k}$ for $k = 1, 2$, and $T_1 u^\alpha$ denotes the averaged Taylor polynomial of u^α, i.e.

$$T_1 u^\alpha(x) = ⨍_{B_{2r}} u^\alpha(y)\, dy + \left(⨍_{B_{2r}} \nabla u^\alpha(y)\, dy \right) \cdot (x - a)\,.$$

Next, we sum the resulting equalities over $\alpha = 1, 2, \ldots, k+1$.

Estimates of left hand side of (2.13) are a bit lengthy but otherwise completely routine. The leading terms (those which contain $\zeta^2 \left(\Delta u^\alpha\right)^2$ or $\zeta^2 |\nabla u|^2\ |\nabla u^\alpha|^2$ or $\zeta^2 |\nabla u|^2 u^\alpha \Delta u^\alpha$) sum to

$$\int_{B_{2r}} \zeta^2 (|\Delta u|^2 + |\nabla u|^4)\, dx$$

(recall that $\sum u^\alpha \Delta u^\alpha = -|\nabla u|^2$). Besides that, there are lots of lower order terms. We estimate each of them in a standard manner, applying Hölder and Sobolev inequalities, and choosing the exponents appropriately. Here are some details. First, for $p = 4$ and $p' = 4/3$,

$$\begin{aligned}
&\left|\int_{B_{2r}} (\Delta u^\alpha + |\nabla u|^2 u^\alpha)(u^\alpha - T_1 u^\alpha)\Delta(\zeta^2)\,dx\right| \\
&\le Cr^2 \left(\fint_{B_{2r}} |\Delta u|^{p'}\,dx\right)^{1/p'} \left(\fint_{B_{2r}} |u - T_1 u|^p\,dx\right)^{1/p} \qquad \text{as } |\nabla u|^2 \le |\Delta u| \\
&\le Cr^4 \left(\fint_{B_{2r}} |\Delta u|^{4/3}\,dx\right)^{3/4} \left(\fint_{B_{2r}} |D^2 u|^{4/3}\,dx\right)^{3/4} \qquad \text{as } p' = (p_*)_* = \tfrac{4}{3} \\
&\le Cr^4 \left(\fint_{B_{2r}} |D^2 u|^{4/3}\,dx\right)^{3/2}.
\end{aligned}$$

Proceeding in a similar way, we obtain

$$\begin{aligned}
&\left|\int_{B_{2r}} \Big(|\Delta u| + |\nabla u|^2\Big)\nabla(\zeta^2)\cdot\nabla(u^\alpha - T_1 u^\alpha)\,dx\right| \\
&\quad + \left|\int_{B_{2r}} \Big(|\Delta u| + |\nabla u|^2\Big)|\nabla u|\,|\nabla(\zeta^2)|\,|u^\alpha - T_1 u^\alpha|\,dx\right| \\
&\le Cr^4 \left(\fint_{B_{2r}} |D^2 u|^s\,dx\right)^{2/s} \qquad \text{for } s = \tfrac{8}{5},
\end{aligned}$$

and

$$\left|\int_{B_{2r}} |\nabla u|^2 u^\alpha (u^\alpha - T_1 u^\alpha)\Delta(\zeta^2)\,dx\right| \le Cr^4 \left(\fint_{B_{2r}} |D^2 u|^p\,dx\right)^{2/p} \qquad \text{for } p = \tfrac{4}{3}.$$

Finally, for $m^\alpha = \fint_{B_{2r}} \nabla u^\alpha\,dx$ we have

$$\left||m^\alpha| \int_{B_{2r}} \Big(|\Delta u| + |\nabla u|^2\Big)|\nabla u|\zeta^2\,dx\right| \le Cr^4 \left(\fint_{B_{2r}} |D^2 u|^q\,dx\right)^{2/q} \qquad \text{for } q = \tfrac{3}{2}.$$

After a quick glance at the above inequalities – choosing the largest of all exponents appearing in the right hand sides and applying Hölder inequality – one concludes that

$$\begin{aligned}
\sum_{\alpha=1}^{k+1} \langle N(u^\alpha), \varphi^\alpha\rangle \ge \int_{B_{2r}} &\zeta^2(|\Delta u|^2 + |\nabla u|^4)\,dx \\
&- Cr^4 \left(\fint_{B_{2r}} |D^2 u|^{8/5}\,dx\right)^{5/4}.
\end{aligned} \tag{3.2}$$

Next, one replaces the term $\int \zeta^2|\Delta u|^2$ by a positive multiple of $\int_{B_r} |D^2u|^2$. To achieve this, set, for sake of brevity, $v^\alpha :\ = u^\alpha - T_1 u^\alpha$ and check that

$$\begin{aligned}
&\int_{B_{2r}} \zeta^2|\Delta u^\alpha|^2\,dx = \int_{\mathbb{R}^4} \zeta^2|\Delta v^\alpha|^2\,dx\\
&\geq \frac{1}{8}\int_{\mathbb{R}^4} |\Delta(\zeta v^\alpha)|^2\,dx - \int_{\mathbb{R}^4} (\Delta\zeta)^2(v^\alpha)^2\,dx - \int_{\mathbb{R}^4} |\nabla\zeta|^2|\nabla v^\alpha|^2\,dx\\
&\geq \frac{1}{8}\int_{\mathbb{R}^4} |\Delta(\zeta v^\alpha)|^2\,dx - Cr^4\Bigl(\fint_{B_{2r}} |D^2u^\alpha|\,dx\Bigr)^2 \qquad (3.3)\\
&\quad - Cr^4\Bigl(\fint_{B_{2r}} |D^2u^\alpha|^{4/3}\,dx\Bigr)^{3/2}\\
&\geq \frac{1}{8}\int_{\mathbb{R}^4} |\Delta(\zeta v^\alpha)|^2\,dx - Cr^4\Bigl(\fint_{B_{2r}} |D^2u^\alpha|^{8/5}\,dx\Bigr)^{5/4}
\end{aligned}$$

(we used Sobolev inequality to obtain the third line). Now, an easy Fourier transform argument shows that for all i, j we have

$$\begin{aligned}
\int_{\mathbb{R}^4} |\Delta(\zeta v^\alpha)|^2\,dx &\geq \int_{\mathbb{R}^4} \bigl|\partial^2_{ij}(\zeta v^\alpha)\bigr|^2\,dx \qquad (3.4)\\
&\geq \int_{B_r} \bigl|\partial^2_{ij}(\zeta v^\alpha)\bigr|^2\,dx = \int_{B_r} \bigl|\partial^2_{ij}u^\alpha\bigr|^2\,dx\,.
\end{aligned}$$

Hence, using (3.2) and (3.3), and dropping the nonnegative term $|\nabla u|^4$ in the integrand, we obtain

$$\sum_{\alpha=1}^{k+1}\langle N(u^\alpha),\varphi^\alpha\rangle \geq \lambda_0\int_{B_r} |D^2u|^2\,dx - Cr^4\Bigl(\fint_{B_{2r}} |D^2u|^{8/5}\,dx\Bigr)^{5/4}, \qquad (3.5)$$

where $\lambda_0 > 0$ is an absolute positive constant.

Estimates of the right hand side. We apply directly Lemma 3.1. This is the only place where the special structure of the biharmonic map equation – and cancellation properties resulting from it – are used. Invoking Sobolev and Hölder inequalities to get rid of the derivatives of ζ, and splitting the different terms with the help of Young inequality, we obtain

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{k+1}\left|\int_{B_{2r}} \Phi(u^\alpha,u^\beta;\varphi^\alpha\nabla u^\beta)\,dx\right|\\
&\leq C\sum_{\alpha=1}^{k+1}\Bigl(\int_{B_{2r}} |\Delta\varphi^\alpha|^2\,dx\Bigr)^{1/2}\Bigl(\int_{B_{2r}} |\Delta u|^2\,dx\Bigr)^{3/4}\\
&\leq Cr^4\Bigl(\fint_{B_{2r}} |\Delta u|^2\,dx\Bigr)\Bigl(\int_{B_{2r}} |\Delta u|^2\,dx\Bigr)^{1/4} + Cr^4\Bigl(\fint_{B_{2r}} |D^2u|^{8/5}\,dx\Bigr)^{5/4}.
\end{aligned}$$

Combining this inequality with (3.5) we conclude the proof. $\square$

Now, by the absolute continuity of the integral, the coefficient $\left(\int_{B_{2r}} |\Delta u|^2\right)^{1/4}$ is uniformly small for small radii. Thus, we may apply the Gehring–Giaquinta–Modica higher integrability lemma, see [13, Chapter V, Thm. 1.2] or [3, Chapter 1.3], to conclude that $|D^2u|^2 \in L^s_{\mathrm{loc}}$ for some $s > 1$. Since $|\nabla u|^2 \leq |\Delta u|$ a.e., we obtain $|\nabla u| \in L^{4s}_{\mathrm{loc}}$. Continuity of u follows from Sobolev imbedding theorem. More precisely, $u \in C^{\gamma_0}$ for $\gamma_0 = 1 - 2/s$.

Remark. The above reasoning can be slightly modified. Namely, using Poincaré inequalities instead of Sobolev inequalities in all estimates of lower order terms – and remembering that all derivatives of the cutoff function ζ are nonzero only on the annulus $B_{2r} \setminus B_r$ – one obtains, via a standard hole-filling trick, a decay estimate of the form

$$\int_{B_r} |D^2u|^2\, dx \leq \lambda \int_{B_{2r}} |D^2u|^2\, dx\,, \qquad 0 < r < r_0,$$

where r_0 is a small number and $\lambda < 1$. This leads to "Dirichlet growth" estimates in a scale invariant form. We hope to exploit this observation in a further study, to obtain boundary regularity of biharmonic maps.

3.2. From Hölder continuity to smoothness

We proceed here as in the familiar proof of smoothness of harmonic maps on planar domains (once higher integrability of the gradient is established).

First of all, some information about third order derivatives of u is necessary. Now, combining (2.13) with the growth estimate (2.7), and performing one integration by parts, we check that

$$\left|\int_{\mathbb{B}^4} \Delta u^\alpha \Delta\varphi\, dx\right| \leq C\big(\|D^2u\|^{3/2}_{L^2}\|\nabla\varphi\|_{L^4} + \|D^2u\|^2_{L^2}\|\varphi\|_\infty\big), \qquad (3.6)$$
$$\varphi \in C^\infty_0(\mathbb{B}^4).$$

By Sobolev imbedding theorem, (3.6) implies that the distribution $\Delta^2 u^\alpha$ extends to a continuous linear functional on all $W^{1,q}$ with $q > 4$. Setting $w^\alpha = \Delta u^\alpha$, and invoking the representation of elements of $\big(W^{1,q}\big)^*$ given e.g. in [1, Theorem 3.8, p. 48], one can use boundedness of Riesz transforms in $L^{q'}$ to check that the gradients $\nabla(w^\alpha * \varphi_\varepsilon)$ of the smooth convolution approximation $w^\alpha * \varphi_\varepsilon$ of w^α satisfy the Cauchy condition in $L^{q'}_{\mathrm{loc}}$. Thus, $\Delta u^\alpha \in W^{1,p}_{\mathrm{loc}}$ for every $p = q' < 4/3$. Combining this statement with higher integrability of ∇u and D^2u we see that the right hand side of the Euler equation (1.2), i.e.

$$R_\alpha(u){:}\ = u^\alpha \sum_{\beta=1}^{k+1}\Big(\big(\Delta u^\beta\big)^2 + \Delta\big(|\nabla u^\beta|^2\big) + 2\nabla u^\beta\cdot\nabla\Delta u^\beta\Big)$$

is integrable with a power strictly greater than 1. Hence,

$$-\Delta^2 u^\alpha = R_\alpha(u) \in L^{p_0}_{\mathrm{loc}} \qquad \text{for some } p_0 > 1. \qquad (3.7)$$

We may assume that $2^{k+3}/(2^{k+3}-1) < p_0 < 2^{k+2}/(2^{k+2}-1)$ for some fixed $k \geq 1$. Invoking now continuity of Riesz transforms on L^p and Sobolev imbedding theorem, we conclude that

$$D^4 u \in L^{p_0}_{\mathrm{loc}}, \quad D^i u \in L^{\kappa_i p_0}_{\mathrm{loc}} \ (i=1,2,3) \qquad \text{if } p_0 < \tfrac{4}{3}, \tag{3.8}$$

where $\kappa_i \colon = 4/(4-(4-i)p_0)$. Thus, by Hölder inequality

$$-\Delta^2 u^\alpha = R_\alpha(u) \in L^{p_1}_{\mathrm{loc}} \qquad \text{for} \quad p_1 = \frac{p_0}{2-p_0}\,. \tag{3.9}$$

Repeating k times the reasonig which leads from (3.7) to (3.9), we obtain

$$-\Delta^2 u^\alpha = R_\alpha(u) \in L^{p_k}_{\mathrm{loc}} \qquad \text{for} \quad p_k = \frac{p_0}{2^k-(2^k-1)p_0} \in \left(\tfrac{8}{7}, \tfrac{4}{3}\right).$$

Thus, $D^4 u^\alpha \in L^{p_k}_{\mathrm{loc}}$, and the condition $p_k \in \left(\frac{8}{7}, \frac{4}{3}\right)$ combined with Sobolev imbedding yields

$$D^3 u^\alpha \in L^{q_1}_{\mathrm{loc}}, \quad D^2 u^\alpha \in L^{s_1}_{\mathrm{loc}}, \quad D u^\alpha \in L^{r_1}_{\mathrm{loc}}, \quad -\Delta^2 u^\alpha = R_\alpha(u) \in L^{t_1}_{\mathrm{loc}},$$

with some exponents $q_1 > 8/5$, $s_1 > 8/3$, $r_1 > 8$, $t_1 > 4/3$. In the next step we obtain

$$D^3 u^\alpha \in L^{q_2}_{\mathrm{loc}}, \quad D^2 u^\alpha \in L^{s_2}_{\mathrm{loc}}, \quad D u^\alpha \in C^{\gamma_1}, \quad R_\alpha(u) \in L^{t_2}_{\mathrm{loc}},$$

for $q_2 = 4t_1/(4-t_1) > 2$, $s_2 = 4q_2/(4-q_2) > 4$, $\gamma_1 = 1 - 4/s_2 > 0$, $t_2 > 2$. Another iteration yields

$$D^3 u^\alpha \in L^{q_3}_{\mathrm{loc}}, \quad D^2 u^\alpha \in C^{\gamma_2}, \quad -\Delta^2 u^\alpha = R_\alpha(u) \in L^{q_3}_{\mathrm{loc}},$$

where $q_3 = 4t_2/(4-t_2) > 4$, $\gamma_2 = 1 - 4/q_3 > 0$. Hence, $u \in W^{4,q_3}$, and $D^3 u \in C^{\gamma_3}$ with $\gamma_3 = \gamma_2 = 1 - 4/q_3 > 0$.

Therefore, $u \in C^{3,\gamma_3}$ is a classical solution of $-\Delta^2 u^\alpha = R_\alpha(u)$, where the right hand side is also Hölder continuous. Smoothness of u follows now from Schauder theory.

4. *p*-biharmonic maps

The regularity proof described in Sects. 2 and 3 can be generalized to other functionals leading to fourth order elliptic systems with critical nonlinearities in lower order terms. What really matters is the structure of these nonlinearities; the semi-linearity of the whole system is not that important.

In this section, we present the proof of Theorem 1.2. Consider the functional

$$E_p(u) = \int_{\mathbb{B}^m} |\Delta u|^p \, dx \tag{4.1}$$

defined for maps $u \in W^{2,p}(\mathbb{B}^m, \mathbb{S}^k)$, with $m \geq 3$ and $2 < 2p \leq m$. We say that $u \in W^{2,p}(\mathbb{B}^m, \mathbb{S}^k)$ is *p-biharmonic* if and only if

$$\frac{d}{dt}\Big|_{t=0} E(\pi \circ (u + t\varphi)) = 0 \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{B}^m, \mathbb{R}^{k+1}), \tag{4.2}$$

where $\pi(y) = y/|y|$. A simple computation shows that this condition is equivalent to

$$\int_{\mathbb{B}^m} |\Delta u|^{p-2} \Delta u^\alpha \Delta\zeta \, dx = \sum_{\gamma=1}^{k+1} \int_{\mathbb{B}^m} |\Delta u|^{p-2} \Delta u^\gamma \Delta(u^\gamma u^\alpha \zeta) \, dx \qquad \text{for } \alpha = 1, \dots, k+1 \text{ and } \zeta \in C_0^\infty(\mathbb{B}^m). \tag{4.3}$$

Geometrically, (4.3) means that $\Delta(|\Delta u|^{p-2}\Delta u)$ is orthogonal to $T\mathbb{S}^k$ a.e.

As in Sect. 2, we exhibit a set of identities “in divergence form” which are equivalent to (4.3).

Lemma 4.1 *A map $u \in W^{2,p}(\mathbb{B}^m, \mathbb{S}^k)$ is p-biharmonic if and only if*

$$\int_{\mathbb{B}^m} \Phi_p(u^\beta, u^\alpha; \nabla\zeta) \, dx = 0 \qquad \textit{for all } \alpha, \beta \textit{ and all } \zeta \in C_0^\infty(\mathbb{B}^m), \tag{4.4}$$

where

$$\Phi_p(u^\beta, u^\alpha; V) := |\Delta u|^{p-2}\, \Phi(u^\beta, u^\alpha; V) \tag{4.5}$$

for Φ defined by (2.6), (2.5).

The proof is almost identical to the proof of Lemma 2.1; we omit the details. Using (4.4), we rewrite the equation in a form analogous to (2.14). Inserting $\zeta = u^\beta\varphi$ in (4.4) and summing with respect to β, we obtain

$$\begin{aligned}\sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} \Phi_p(u^\beta, u^\alpha, \varphi\nabla u^\beta) \, dx &= -\sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} \Phi_p(u^\beta, u^\alpha, u^\beta\nabla\varphi) \, dx \\ &= J_1 + J_2 + J_3 + J_4 \,,\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
J_1 &= -\sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} |\Delta u|^{p-2} \Delta u^\beta \nabla u^\alpha \cdot u^\beta \nabla\varphi \, dx \\
&= \int_{\mathbb{B}^m} |\Delta u|^{p-2} |\nabla u|^2 \nabla u^\alpha \cdot \nabla\varphi \, dx \qquad \text{as } |\nabla u|^2 = -(u, \Delta u), \\
J_2 &= -\sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} |\Delta u|^{p-2} \Delta u^\beta \operatorname{div}(u^\alpha u^\beta \nabla\varphi) \, dx \,, \\
J_3 &= \sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} |\Delta u|^{p-2} \Delta u^\alpha \nabla u^\beta \cdot u^\beta \nabla\varphi \, dx = 0 \qquad \text{as } u \perp \nabla u, \\
J_4 &= \sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} |\Delta u|^{p-2} \Delta u^\alpha \operatorname{div}(u^\beta u^\beta \nabla\varphi) \, dx \\
&= \int_{\mathbb{B}^m} |\Delta u|^{p-2} \Delta u^\alpha \Delta\varphi \, dx \,.
\end{aligned}$$

Now,

$$J_2 = -\sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} |\Delta u|^{p-2} u^\alpha \Delta u^\beta \nabla u^\beta \nabla \varphi \, dx$$
$$+ J_1 + \int_{\mathbb{B}^m} |\Delta u|^{p-2} |\nabla u|^2 u^\alpha \Delta \varphi \, dx \, .$$

Thus, (4.6) leads to the identity

$$\begin{aligned} &\int_{\mathbb{B}^m} |\Delta u|^{p-2} \left(\Delta u^\alpha + |\nabla u|^2 u^\alpha \right) \Delta \varphi \, dx \\ &\quad + \int_{\mathbb{B}^m} |\Delta u|^{p-2} \Big(2|\nabla u|^2 \nabla u^\alpha - \sum_{\beta=1}^{k+1} u^\alpha \Delta u^\beta \nabla u^\beta \Big) \cdot \nabla \varphi \, dx \\ &= \sum_{\beta=1}^{k+1} \int_{\mathbb{B}^m} \Phi_p(u^\beta, u^\alpha, \varphi \nabla u^\beta) \, dx \end{aligned} \tag{4.7}$$

for every $\alpha = 1, 2, \ldots, k+1$ and every test function $\varphi \in C_0^\infty(\mathbb{B}^m)$. By a density argument, (4.7) holds for all

$$\varphi \in W_0^{2,p} \cap W^{1,2p} \cap L^\infty \, .$$

From now on we assume that $m = 2p$. Our goal now will be to derive a reverse Hölder inequality for u. To estimate the right hand side of (4.7) we need an analogue of Lemma 3.1.

Lemma 4.2 *Assume that $m = 2p$, $a \in \mathbb{B}^m$ and $0 < 2r < \operatorname{dist}(a, \partial \mathbb{B}^m)$. Let $u \in W^{2,p}(\mathbb{B}^m, \mathbb{S}^k)$ be a weakly p-biharmonic map, and let Φ_p be defined as in Lemma 4.1. Then, for every test function $\varphi \in W_0^{2,2}(B(a, 4r/3))$ and all $\alpha, \beta \in \{1, 2, \ldots, k+1\}$ we have*

$$\left| \int_{B_{2r}} \Phi_p(u^\beta, u^\alpha; \varphi \nabla u^\beta) \, dx \right| \le C \Big(\int_{B_{2r}} |\Delta \varphi|^p \, dx \Big)^{\frac{1}{p}} \Big(\int_{B_{2r}} |\Delta u|^p \, dx \Big)^{1 - \frac{1}{2p}} . \tag{4.8}$$

Remark. For smooth φ, using trivial growth properties of Φ_p, one can estimate $\int \Phi_p(u^\beta, u^\alpha; \varphi \nabla u^\beta)$ by const $\cdot \int |\Delta u|^p$. Note that (4.8) is much better for $\varphi \approx u$ and for small balls.

We postpone the proof of this Lemma to the last Section and pass directly to reverse Hölder inequalities for $|\Delta u|^p$.

Lemma 4.3 *For $m = 2p$ every p-biharmonic map $u \in W^{2,p}(\mathbb{B}^m, \mathbb{S}^k)$ satisfies the inequality*

$$\fint_{B_r} |D^2 u|^{\frac{m}{2}} \, dx \le C \left(\fint_{B_{2r}} |D^2 u|^{\frac{m}{2} \cdot \frac{m}{m+1}} \, dx \right)^{\frac{m+1}{m}}$$
$$+ C \left(\int_{B_{2r}} |D^2 u|^{\frac{m}{2}} \, dx \right)^{\frac{1}{m}} \fint_{B_{2r}} |D^2 u|^{\frac{m}{2}} \, dx \, .$$

The constant C depends on the dimension only.

Proof. The reasoning is very similar to the proof of Lemma 3.2; thus, we omit some of the details. One tests (4.7) with

$$\varphi^\alpha :\ = \zeta^p(u^\alpha - T_1 u^\alpha),$$

where $T_1 u^\alpha$ stands, as before, for the averaged first order Taylor polynomial of u^α. Next, the summation with respect to α is performed. Those terms from the left hand side which contain critical powers of u and its derivatives, i.e.

$$|\Delta u|^{p-2}(\Delta u^\alpha)^2, \quad \text{or} \quad |\Delta u|^{p-2}|\nabla u|^2|\nabla u^\alpha|^2, \quad \text{or} \quad |\Delta u|^{p-2}|\nabla u|^2 u^\alpha \Delta u^\alpha,$$

sum to

$$\int_{\mathbb{B}^m} \zeta^p(|\Delta u|^p + |\Delta u|^{p-2}|\nabla u|^4)\,dx \geq \int_{B_{2r}} \zeta^p |\Delta u|^p\,dx.$$

Keeping in mind that $\sum_\alpha u^\alpha \nabla u^\alpha = 0$, we check that the lower order terms resulting from the left-hand side of (4.7) do not exceed a constant multiple of $I_1+I_2+I_3+I_4$, where

$$\begin{aligned}
I_1 &= \frac{1}{r}\int_{B_{2r}} |\Delta u|^{p-1}|\nabla(u - T_1 u)|\,dx\,,\\
I_2 &= \frac{1}{r}\int_{B_{2r}} |\Delta u|^{p-1}|\nabla u||u - T_1 u|\,dx\,,\\
I_3 &= \frac{1}{r^2}\int_{B_{2r}} |\Delta u|^{p-1}|u - T_1 u|\,dx\,,\\
I_4 &= \left|\fint_{B_{2r}} \nabla u\,dx\right| \cdot \int_{B_{2r}} |\Delta u|^{p-1}|\nabla u|\,dx\,.
\end{aligned}$$

Applying Hölder inequality and Sobolev imbedding, we obtain

$$\begin{aligned}
I_1 &\leq C r^{m-1}\left(\fint_{B_{2r}} |\Delta u|^{s'(p-1)}\,dx\right)^{1/s'}\left(\fint_{B_{2r}} |\nabla u - (\nabla u)_{B_{2r}}|^s\,dx\right)^{1/s}\\
&\leq C r^m\left(\fint_{B_{2r}} |\Delta u|^{s'(p-1)}\,dx\right)^{1/s'}\left(\fint_{B_{2r}} |D^2 u|^{s_*}\,dx\right)^{1/s_*}\\
&\leq C r^m\left(\fint_{B_{2r}} |D^2 u|^{\frac{m}{2}\cdot\frac{m}{m+1}}\,dx\right)^{\frac{m+1}{m}} \qquad \text{when } s = m^2/(m+2). \qquad (4.9)
\end{aligned}$$

To deal with I_2, recall that $|\nabla u| \le |\Delta u|^{\frac{1}{2}}$, and estimate

$$\begin{aligned} I_2 &\le Cr^{m-1}\Big(\fint_{B_{2r}} |\Delta u|^{s'(p-\frac{1}{2})}\,dx\Big)^{1/s'}\Big(\fint_{B_{2r}} |u-T_1u|^s\,dx\Big)^{1/s} \\ &\le Cr^{m}\Big(\fint_{B_{2r}} |\Delta u|^{s'(p-\frac{1}{2})}\,dx\Big)^{1/s'}\Big(\fint_{B_{2r}} |\nabla u-(\nabla u)_{B_{2r}}|^{s_*}\,dx\Big)^{1/s_*} \\ &\le Cr^{m}\Big(\fint_{B_{2r}} |\Delta u|^{s'(p-\frac{1}{2})}\,dx\Big)^{1/s'}\Big(\fint_{B_{2r}} |\Delta u|^{s_*/2}\,dx\Big)^{1/s_*} \\ &= Cr^{m}\Big(\fint_{B_{2r}} |\Delta u|^{\frac{m}{2}\cdot\frac{m}{m+1}}\,dx\Big)^{\frac{m+1}{m}} \qquad \text{when } s=m^2. \end{aligned} \tag{4.10}$$

The integrals I_3 and I_4 satisfy estimates analogous to (4.9) (to deal with I_4, one needs only Hölder inequality). We leave the details to the reader.

Next, to obtain a counterpart of (3.5) with exponent 2 replaced by $m/2$, one has to use boundedness of Riesz transforms on $L^{m/2}$. The resulting estimate of the left hand side of (4.7) is combined with Lemma 4.1; this completes the whole argument. □

It follows from Lemma 4.2 that $|D^2u|^{m/2}$ satisfies the assumptions of Theorem 1.2 in [13, Chapter V]. Thus, $|D^2u|^{m/2}$ is integrable with some power $s > 1$, and since $|\nabla u|^m \le |\Delta u|^{m/2}$, we conclude that $|\nabla u|$ is integrable with a power larger than the dimension, m. Hence, by Sobolev imbedding theorem, u is Hölder continuous. This completes the proof of Theorem 1.2.

5. Mappings on Carnot groups

In this section we show that the theorem of Chang, Wang and Yang can be generalized to biharmonic maps on the Heisenberg group. The method of proof remains unchanged; we follow the pattern from Sect. 3. Analogously, Theorem 1.2 has a counterpart on general Carnot groups.

Basic definitions, notations and most important facts concerning calculus on Carnot groups are gathered in Sect. 5.1. For more detailed discussions of these topics we refer to Folland and Stein [12], Gromov [15], and Varopoulos, Saloff-Coste and Coulhon [33]. The proof of Theorem 1.3 is presented in Sect. 5.2, and the proof of Theorem 1.4 – in Sect. 5.3.

5.1. Basic concepts

The space, distance and measure. A *Carnot group*, or a *stratified group* $\mathbb{G}$ is a connected and simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ is stratified in the following sense: $\mathfrak{g} = \bigoplus_{i=1}^s V_i$, where $[V_1, V_i] = V_{i+1}$, with $V_i\colon = \{0\}$ for $i > s$. We assume that $V_s \neq \{0\}$; $\mathbb{G}$ is then nilpotent of step s.

Every Carnot group is diffeomorphic to $\mathbb{R}^n$ for $n := \sum_{j=1}^s \dim V_s$. (The exponential map is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$; so, fixing a basis (X_ℓ) of $\mathfrak{g}$, one obtains a natural identification $\mathbb{G} \equiv \mathbb{R}^n$, $\mathbb{G} \ni g = \exp(\sum x_\ell X_\ell) \mapsto x =$

$(x_\ell) \in \mathbb{R}^n$.) However, the natural distance associated to the stratification of $\mathfrak{g}$ is, in general, not equivalent to any Riemannian metric on $\mathbb{R}^n$.

It is clear from the definition that every basis of V_1 generates the whole of $\mathfrak{g}$. In what follows, we fix a basis $X_1, X_2, \ldots, X_l$ of V_1, and extend this basis to a fixed basis $X_1, \ldots, X_l, \ldots, X_n$ of the whole Lie algebra $\mathfrak{g}$. For every $j = 1, \ldots, n$ we denote by d_j the length of X_j as a commutator of $(X_i)_{i \le l}$. Each X_j is identified with the corresponding left invariant vector field on $\mathbb{G}$. The Lebesgue measure on $\mathbb{R}^n$ coincides with the bi-invariant Haar measure on $\mathbb{G}$ (we identify $\mathbb{G} \equiv \mathbb{R}^n$ in the way described above, using the basis (X_j)).

There is a natural distance, the so-callled *Carnot-Carathéodory* (or CC for short) distance, associated with this family of vector fields. Namely, an absolutely continuous curve $\gamma\colon [0,T] \to \mathbb{G}$ is called *admissible* if for every $t \in [0,T]$ we have

$$\dot\gamma(t) = \sum_{j=1}^{l} c_j(t) X_j(\gamma(t)), \quad \text{where} \quad \sum_{j=1}^{l} c_j(t)^2 \le 1.$$

The CC distance $\rho(x,y)$ of two points $x, y \in \mathbb{G}$ is defined as the infimum of those $T > 0$ for which there exist an admissible curve $\gamma\colon [0,T] \to \mathbb{G}$ joining $x = \gamma(0)$ and $y = \gamma(T)$. In other words, $\rho(x,y)$ is the shortest possible travel time from x to y with at most unit speed, along curves tangent to $\operatorname{span}(X_1, \ldots, X_l)$. The metric ρ is well defined, as every two points of $\mathbb{G}$ can be joined by a piecewise smooth admissible curve (this is the accessibility theorem of Chow and Rashevsky).

Throughout the rest of Sect. 5, $B(x,r)$ always stands for a ball in the Carnot-Carathéodory metric.

The CC metric is in general not equivalent to the Euclidean metric. However, it turns out that for every bounded set $E \subset \mathbb{G}$ there exists a constant $C > 0$ such that

$$C^{-1}|x-y| \le \rho(x,y) \le C|x-y|^{1/s}, \qquad x, y \in \mathbb{G}\,,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n \equiv \mathbb{G}$.

There is a one-parameter family of dilations δ_r, $r > 0$, on $\mathbb{G}$. One sets $\delta_r X = r^i X$ for $X \in V_i$; this map extends to a linear automorphis of $\mathfrak{g}$, and, using the exponential map, to an automorphism of $\mathbb{G}$. The CC metric is left invariant and commutes with the dilations δ_r, i.e. $\rho(\delta_r x, \delta_r y) = r\rho(x,y)$ for $x, y \in \mathbb{G}$.

The *homogeneous dimension* Q of $\mathbb{G}$ is defined as

$$Q = \sum_{j=1}^{s} j \dim V_j\,. \tag{5.1}$$

In Sobolev and Poincaré inequalities on $\mathbb{G}$, this number plays a role analogous to the Euclidean dimension in the classical case. This is due to the behaviour of Lebesgue measure under the dilations δ_r: we have

$$|\delta_r(E)| = r^Q |E| \text{ for } E \subset \mathbb{G}; \quad |B(x,r)| = Cr^Q \text{ for all } x \in \mathbb{G},\ r > 0. \tag{5.2}$$

Here is the simplest nontrivial example of a Carnot group.

Example (Heisenberg group). In $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$, with points denoted by (x_1, x_2, t), or (z, t), where $z = x_1 + ix_2$, consider the multiplication

$$(z_1, t_1)(z_2, t_2) = \Big(z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}(z_1 \bar{z}_2) \Big) .$$

The vector fields

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial t}, \qquad X_3 \equiv T = \frac{\partial}{\partial t} \tag{5.3}$$

form the basis of all left-invariant vector fields. Since X_1, X_2, and $[X_1, X_2] = -4T$ span the tangent space $\mathbb{R}^3$ at every point, the Lie algebra $\mathfrak{h} = V_1 \oplus V_2$, where $V_1 = \mathrm{span}\,(X_1, X_2)$ and $V_2 = \mathrm{span}\, T$. We have $d_1 = d_2 = 1$ and $d_3 = 2$. The homogeneous dimension of $\mathbb{H}_1$ is $Q = 2 + 2 = 4$ (it turns out to be the Hausdorff dimension of $\mathbb{H}_1$ w.r.t. the CC metric associated to X_1, X_2).

In order to get a glimpse of the behaviour of ρ on $\mathbb{H}_1$, consider the so-called homogeneous norm $\|(z, t)\| = (t^2 + |z|^4)^{1/4}$. One can show that $d(x, y) := \|x^{-1}y\|$ is a metric which is bi-Lipschiz equivalent to ρ. In particular $\rho(0, x) \approx d(0, x) = (t^2 + |z|^4)^{1/4}$ if $x = (z, t)$. Thus, the CC balls are definitely non-isotropic: they become flatter and flatter in the t direction as the radius tends to zero.

Sobolev spaces, polynomials and Sobolev–Poincaré inequalities. We follow the notation introduced above. Let $\Omega \subset \mathbb{G}$ be a bounded open domain. A function $u\colon \Omega \to \mathbb{R}$ belongs to the Folland–Stein Sobolev space $W^{2,p}_X(\Omega)$ if and only if $u \in L^p(\Omega)$ and the distributional derivatives

$$X_i u,\ X_i X_j u \in L^p(\Omega) \qquad \text{for all } 1 \le i, j \le l = \dim V_1$$

(recall that $X_1, \ldots, X_l$ form a basis of V_1). We write $Xu = (X_1 u, \ldots, X_l u)$, $X^2 u = (X_i X_j u)_{i,j=1,\ldots,l}$, and set

$$|Xu| = \left(\sum_{i=1}^{l} |X_i u|^2 \right)^{1/2}, \qquad |X^2 u| = \left(\sum_{1 \le i,j \le l} |X_i X_j u|^2 \right)^{1/2}.$$

The space $W^{2,p}_X(\Omega, \mathbb{S}^k)$ is defined as the set of those $u = (u^1, \ldots, u^{k+1}) \in \Big(W^{2,p}_X(\Omega) \Big)^{k+1}$ for which the pointwise constraint $|u|^2 = 1$ is satisfied a.e.

It is known that counterparts of Poincaré inequalities and Sobolev inequalities hold on Carnot groups; Hajłasz and Koskela [16] provide a very useful and readable survey of this topic, and give rich references to earlier original works. If $B \subset \mathbb{G}$ is a CC ball of radius r, then

$$\left(\fint_B |u - u_B|^p \, dx \right)^{1/p} \le Cr \left(\fint_B |Xu|^{p_*} \, dx \right)^{1/p_*} \tag{5.4}$$

Here, $p \in [\frac{Q}{Q-1}, \infty)$, and $p_* \colon = Qp/(p + Q)$, where Q denotes the homogeneous dimension of $\mathbb{G}$. The Poincaré inequality, see Jerison [22],

$$\left(\fint_B |u - u_B|^p \, dx \right)^{1/p} \le Cr \left(\fint_B |Xu|^p \, dx \right)^{1/p}, \qquad 1 \le p < \infty, \tag{5.5}$$

follows from (5.4) and Hölder inequality.

To state higher order Sobolev inequalities, one needs an appropriate generalization of the notion of the Taylor polynomial. Now, to define polynomials on $\mathbb{G}$, one takes the basis $(\xi_j)_{j=1,\dots,n}$ of $\mathfrak{g}^*$ which is dual to $(X_j)_{j=1,\dots,n}$. Let $\eta_j \colon = \xi_j \circ \exp^{-1}$; these are global coordinates on $\mathbb{G}$. A function $P\colon \mathbb{G} \to \mathbb{R}$ is a polynomial if, by definition, $P \circ \exp$ is a polynomial on the Lie algebra $\mathfrak{g}$. Equivalently, P is a polynomial if

$$P(x) = \sum_{J=(j_1,\dots,j_n)} c_J \eta^J(x), \qquad \eta^J \colon = \eta_1^{j_1} \cdots \eta_n^{j_n},$$

where the coefficients c_J are real and vanish for all but finitely many $J \in \mathbb{N}^n$. The order, or homogeneous degree, of P is equal to $\max\{d(J) : c_J \neq 0\}$, where $d(J) = \sum_{\ell=1}^n j_\ell d_\ell$ for $J = (j_1, \dots, j_n)$, d_ℓ being the length of X_ℓ as a commutator.

The set of all polynomials on $\mathbb{G}$ of homogeneous degree $< k$ is denoted by $\mathcal{P}_k$. To carry out the computations from Sect. 3 also in the subelliptic case, we need a few properties of $\mathcal{P}_2$. Lu [25, Section 2] proves that for every ball B contained in a fixed bounded domain $\Omega \subset \mathbb{G}$ there exists a linear map

$$\pi_B \colon W_X^{2,1}(\Omega) \to \mathcal{P}_2$$

such that

$$\sup_{x\in B} |\pi_B u(x)| \le C \fint_B |u(x)|\, dx\,,$$

with a constant C independent of u and B, and moreover $\pi_B P = P$ for every $P \in \mathcal{P}_2$. It follows from [25, Theorem 2.8] that for every $1 \le q \le \infty$ we have

$$\|X^i(\pi_B u)\|_{L^q(B)} \le C\|X^i u\|_{L^q(B)}, \qquad i = 0, 1, \tag{5.6}$$

with a constant independent of u and B.

Lu also proves the following result.

Theorem 5.1 (Lu, [25]) *If $B \subset \mathbb{G}$ is a metric ball of radius r, then for $j = 0, 1$ and $j < i \le 2$ we have*

$$\left(\fint_B \left|X^j\big(u - \pi_B u\big)\right|^q dx\right)^{1/q} \le C r^{i-j} \left(\fint_B \left|X^i u\right|^p dx\right)^{1/p},$$

if $\frac{Q}{i-j} > p \ge 1$, $X^i u \in L^p(\Omega)$, and $\frac{1}{q} = \frac{1}{p} - \frac{(i-j)}{Q}$. The constant C is independent of u and B.

Let

$$\Delta_X = X_1^2 + \ldots + X_l^2$$

denote the sublaplacian on $\mathbb{G}$. It follows from the results of Folland [11], see also Foland and Stein [12], that Δ_X has a fundamental solution $\Gamma\colon \mathbb{G}\times\mathbb{G}\to\mathbb{R}$ such that Γ is smooth away from the diagonal and

$$\Big(\Delta_X \int_{\mathbb{G}} \Gamma(\cdot,y)\varphi(y)\,dy\Big)(x) = \varphi(x) \qquad \text{for all } \varphi\in C_0^\infty(\mathbb{G}). \tag{5.7}$$

Besides that, $G(x,y) = X_iX_j\Gamma(x,y)$ is a sigular integral kernel of homogeneous degree zero (each of the differentiations can be performed either with respect to x or to y). This yields the following.

Theorem 5.2 *If $\Delta_X f = g$ for some $g\in L^p(\mathbb{G})$, $1<p<\infty$, then $X_iX_jf\in L^p(\mathbb{G})$ for all $i,j=1,\dots,l$. Moreover,*

$$\|X_iX_jf\|_{L^p(\mathbb{G})} \le C\|g\|_{L^p(\mathbb{G})}\,. \tag{5.8}$$

This is the desired boundedness of ‘Riesz transforms’ on L^p. More general results of that type can be found e.g. in Folland [11, Theorem 6.1], and Rothschild and Stein [28, Theorem 16].

5.2. Biharmonic maps on the Heisenberg group

For a bounded domain $\Omega\subset\mathbb{H}_1$, and $u\in W_X^{2,2}(\Omega,\mathbb{S}^k)$, we set

$$E_2(u) = \int_\Omega |\Delta_X u|^2\,dx\,.$$

Here,

$$\Delta_X :\, = {X_1}^2 + {X_2}^2$$

is the sub-laplacian on $\mathbb{H}_1$. We say that $u\in W_X^{2,2}(\Omega,\mathbb{S}^k)$ is a *subelliptic biharmonic map* if and only if

$$\frac{d}{ds}\Big|_{s=0} E(\pi\circ(u+s\varphi)) = 0 \qquad \text{for every } \varphi\in C_0^\infty(\Omega,\mathbb{R}^{k+1}). \tag{5.9}$$

As before, $\pi(y) = y/|y|$ denotes the nearest point projection onto $\mathbb{S}^k$.

To write the Euler equation in the form resembling (2.9), set

$$\begin{aligned} L_X(w,V) &:\, = X\cdot(wV) + Xw\cdot V \quad \text{for } w\in C_0^\infty(\Omega),\ V\in C_0^\infty(\Omega,\mathbb{R}^2);\\ \Phi_X(v,w;V) &:\, = \Delta_X v\,L_X(w,V) - \Delta_X w\,L_X(v,V)\,. \end{aligned} \tag{5.10}$$

We have of course $L_X(w,X\zeta) = \Delta_X(\zeta w) - \zeta\Delta_X w$; as before we do not want to use second order derivatives of w in the notation for L_X.

Mimicking the computations from Sect. 2, one easily proves the following lemma.

Lemma 5.1 *Assume that $u\in W_X^{2,2}(\Omega,\mathbb{S}^k)$. Then the following conditions are equivalent:*

(i) u *is a subelliptic biharmonic map;*
(ii) *The identity*

$$\int_\Omega \Phi_X(u^\beta, u^\alpha; X\zeta)\, dx = 0 \tag{5.11}$$

holds for all $\zeta \in C_0^\infty(\Omega)$ *and all* $\alpha, \beta = 1, \dots, k+1$*;*
(iii) *The identity*

$$\begin{aligned}\int_\Omega (\Delta_X u^\alpha + |Xu|^2 u^\alpha)\Delta_X \varphi\, dx + 2\int_\Omega |Xu|^2 Xu^\alpha X\varphi\, dx \\ - \sum_{\beta=1}^{k+1}\int_\Omega u^\alpha \Delta_X u^\beta Xu^\beta X\varphi\, dx = \sum_{\beta=1}^{k+1}\int_\Omega \Phi_X(u^\alpha, u^\beta; \varphi X u^\beta)\, dx\end{aligned} \tag{5.12}$$

holds for all $\varphi \in C_0^\infty(\Omega)$ *and all* $\alpha = 1, \dots, k+1$.

Since (5.11) is continuous with respect to weak convergence in $W_X^{2,2}$, we have the following.

Proposition 5.1 *If a sequence of subelliptic biharmonic maps converges weakly in* $W_X^{2,2}(\Omega, \mathbb{S}^k)$*, then its limit is also a subelliptic biharmonic map.*

The proof is analogous to the proof of Proposition 2.2.

Sketch of proof of Theorem 1.3. The general pattern of proof is the same as in Sect. 3. To avoid too much repetition, we briefly indicate necessary changes.

We test the equation (5.12) with $\zeta^2(u^\alpha - T(u^\alpha))$, where ζ is a cutoff function, equal to 1 on a small metric ball B and vanishing off $2B$, with standard estimates for derivatives w.r.t. X_i, up to the second order, and $T(u^\alpha) = \pi_{2B} u^\alpha$ is Lu's projection polynomial described above.

Step 1. To estimate the right hand side, one proves a counterpart of Lemma 3.1, going through the proof from Sect. 6 and replacing all "Euclidean ingredients" by their subelliptic counterparts. (See the comments after the proof of Lemma 4.2.) This yields

$$\left|\int_B \Phi_X(u^\alpha, u^\beta; \varphi X u^\beta)\, dx\right| \le C\left(\int_{2B} |X^2\varphi|^2\, dx\right)^{1/2}\left(\int_{2B} |X^2 u|^2\, dx\right)^{3/4} \tag{5.13}$$

for concentric Carnot-Carathédory balls $B = B(a,r) \subset 2B = B(a,2r) \subset \Omega$, and test functions $\varphi \in C_0^\infty(B(a, 4r/3))$.

Step 2. Using inequality (5.13), and mimicking the proof of Lemma 3.2, one proves that $|X^2 u|$ satisfies a family of weak reverse Hölder inequalities. All lower order terms are estimated with the help of inequality (5.6) and Theorem 5.1.

Since the homogeneous dimension of $\mathbb{H}_1$ is equal to 4, all integrability exponents match those from Sect. 3.1.

Step 3. A modification of standard arguments, see Zatorska-Goldstein [40] for details, shows that a variant of Gehring–Giaquinta–Modica higher integrability lemma (in the form applied earlier in Sect. 3.1 and in Sect. 4) holds for the CC metric. Thus, $|X^2 u|$ is locally integrable with a power greater than 2. This means that Xu is integrable with a power greater than the homogeneous dimension. Hence, u is Hölder continuous (see e.g. [16, Theorem 5.1, part 3] for an appropriate version of Sobolev imbedding).

Step 4. Adapting the argument presented at the beginning of Sect. 3.2, one proves that u has third order order derivatives $X_i X_j X_k u \in L^p_{\mathrm{loc}}$ for all $i, j, k \in \{1, 2\}$ and all $p < 4/3$. We write (5.9) in the form

$$\int_\Omega \Delta_X u^\alpha \Delta_X \zeta \, dx = \sum_{\gamma=1}^{k+1} \int_\Omega \Delta_X u^\gamma \Delta_X (u^\gamma u^\alpha \zeta) \, dx \tag{5.14}$$
$$\text{for all } \alpha = 1, \ldots, k+1 \text{ and } \zeta \in C_0^\infty(\Omega).$$

Integrating the right hand side by parts, and using counterparts of identities (2.10), we obtain

$$-{\Delta_X}^2 u^\alpha = \sum_{\gamma=1}^{k+1} R_{\gamma,\alpha}(u), \tag{5.15}$$

where

$$\begin{aligned} R_{\gamma,\alpha}(u) = (\Delta_X u^\gamma) \Delta_X (u^\alpha u^\gamma) - 2X \cdot \big((\Delta_X u^\gamma) X (u^\alpha u^\gamma) \big) \\ + \Delta_X (|X u^\gamma|^2 u^\alpha) \end{aligned}$$

belongs locally to L^p for some $p > 1$. The bootstrap argument from Sect. 2 can be repeated. Once it is known that $X_i X_j X_k u$ are Hölder continuous, smoothness of u follows from Schauder estimates and Folland's results [11] (alternatively, one can rewrite (5.15) as a second order system system of $2k + 2$ equations for $U = (u^\alpha, w^\alpha)$, where $w^\alpha : \; = \Delta_X u^\alpha$, and apply the main theorem of Xu and Zuilly [38]). $\square$

5.3. *Other functionals*

It should be clear by now that the proof of Theorem 1.4 is fully analogous to the proof of Theorem 1.2. All necessary tools have been described in Sect. 5.1.

We only indicate the most important steps and leave all other details to an interested reader.

Definitions. Let Ω be a bounded domain in a Carnot group $\mathbb{G}$ of homogeneous dimension Q. Define

$$E_p(u) = \int_\Omega \left(\sum_{\alpha=1}^{k+1} |\Delta_X u|^2 \right)^{p/2} dx \tag{5.16}$$

for maps $u \in W^{2,p}_X(\Omega, \mathbb{S}^k)$. We say that $u \in W^{2,p}_X(\Omega, \mathbb{S}^k)$ is a subelliptic p-biharmonic map if and only if

$$\frac{d}{ds}\Big|_{s=0} E_p(\pi \circ (u + s\varphi)) = 0 \qquad \text{for every } \varphi \in C_0^\infty(\Omega, \mathbb{R}^{k+1}), \tag{5.17}$$

where $\pi(y) = y/|y|$.

Recall that $l = \dim V_1$, where $V_1 \subset \mathfrak{g}$ is the generating subspace of $\mathfrak{g}$. For vector fields $V \in C_0^\infty(\Omega, \mathbb{R}^l)$ we write

$$\Phi_{p,X}(u^\beta, u^\alpha; V) :\, = |\Delta_X u|^{p-2}\, \Phi_X(u^\beta, u^\alpha; V), \tag{5.18}$$

where Φ_X is defined by (5.10).

Various forms of the Euler equation. It is easily proved that (5.17) is equivalent to each of the following identities:

$$\int_\Omega \Phi_{p,X}(u^\beta, u^\alpha; X\zeta)\, dx = 0 \quad \text{for all } \zeta \in C_0^\infty(\Omega), 1 \le \alpha, \beta \le k+1; \tag{5.19}$$

$$\begin{aligned} &\int_\Omega |\Delta u_X|^{p-2} \left(\Delta_X u^\alpha + |Xu|^2 u^\alpha\right) \Delta_X \varphi\, dx \\ &\qquad + \int_\Omega |\Delta_X u|^{p-2} \Big(2|Xu|^2 X u^\alpha - \sum_{\beta=1}^{k+1} u^\alpha \Delta_X u^\beta X u^\beta \Big) \cdot X\varphi\, dx \\ &= \sum_{\beta=1}^{k+1} \int_\Omega \Phi_{p,X}(u^\beta, u^\alpha, \varphi X u^\beta)\, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega),\, \alpha = 1, \dots, k+1. \end{aligned} \tag{5.20}$$

(One can safely mimick the proof of Lemma 2.1, and copy the reasoning from Sect. 4, used to prove (4.7). Commutativity of Euclidean derivatives was not employed there.) As before, (5.19) provides the necessary cancellation condition to estimate the critical term on the right side of (5.20).

Reverse Hölder inequalities. Hole filling. Assume that $2p = Q$. One can then replace m by Q and Euclidean derivatives by subelliptic ones in Lemma 4.1 and Lemma 4.2; no essential changes are needed in the proofs. Thus, we obtain higher integrability of $|X^2 u|$ and $|Xu|$; Hölder continuity of u follows.

As remarked at the end of Sect. 3.1, it is also possible to use a variant of hole filling and obtain a decay estimate of the form

$$\int_{B_r} |X^2 u|^{Q/2}\, dx \le C r^\gamma, \qquad 0 < r < r_0,$$

with some $\gamma > 0$. Hölder continuity of u, in light of the pointwise inequality $|\Delta_X u| \ge |Xu|^2$, follows then from a well known variant of Morrey's lemma on Carnot groups.

6. Proofs of the crucial estimates

In this section we give self-contained proofs of Lemma 3.1 and Lemma 4.2. The key arguments depend heavily on cancellation properties analogous to those observed by Coifman, Lions, Meyer and Semmes in [9], and widely used in PDE theory afterwards. In fact, for a biharmonic map $u \in W^{2,2}(\mathbb{B}^4, \mathbb{S}^k)$ with $\nabla \Delta u \in L^{4/3}$ the inequality stated in Lemma 3.1 follows directly from the results of [9] and the duality of Hardy space and BMO. However, the argument given at the beginning of Sect. 3.2 implies only that $D^3 u \in L^p$ for all $p < 4/3$, and the author does not know how to obtain $D^3 u \in L^{4/3}$ a priori, *before* proving reverse Hölder inequalities, continuity of u, etc. Thus, a slightly different way is chosen, so that the existence of third order distributional derivatives of u is not needed in the proof of Lemma 3.1.

The general strategy is as follows. We represent the smooth test function φ as $\varphi = \Gamma * \Delta\varphi$, where Γ stands for the fundamental solution of the Laplace operator. Then, applying Fubini's theorem and writing the integral which represents $\Gamma * \Delta\varphi$ as the outer one, we use cancellation properties of the equation and Sobolev inequality to bound the inner integral by some Riesz potential. An application of the classical fractional integration theorem closes the argument.

Before entering into more details, we recall a standard technique, namely the Whitney decomposition of an open set $\Omega \subset \mathbb{R}^n$. We shall need it for $\Omega := \mathbb{R}^n \setminus \{y\}$.

For $x \in \Omega$ set $r_x = |x-y|/250$. Then $\{B(x, r_x/4)\}_{x \in \mathbb{R}^n \setminus \{y\}}$ is a covering of $\Omega = \mathbb{R}^n \setminus \{y\}$. We select a maximal subfamily of pairwise disjoint balls $\{B(x_i, r_{x_i}/4)\}_{i\in I}$. Next, we set $B_i := B(x_i, r_{x_i})$ and $r_i := r_{x_i}$. We also fix $\eta \in C_0^\infty(B(0,1))$ with $\eta \equiv 1$ on $B(0, \frac{3}{4})$.

One then easily checks the following properties.

(i) $\bigcup_{i\in I} B(x_i, \frac{3}{4} r_i) = \mathbb{R}^n \setminus \{y\}$ (this follows from maximality).
(ii) $\sum_{i\in I} \chi_{B(x_i, 2r_i)} \le N$, where the constant N depends only on the dimension, i.e. no point of $\mathbb{R}^n$ belongs to more than N balls $B(x_i, 2r_i)$.
(iii) $\sum_{i \in I} \theta_i \equiv 1$ on $\mathbb{R}^n \setminus \{y\}$, where $\theta_i \in C_0^\infty(B_i)$ is given by

$$\theta_i(x) := \frac{\eta_i(x)}{\sum_{j\in I} \eta_j(x)} \qquad \text{with} \quad \eta_i(x) := \eta\big((x-x_i)/r_i\big) .$$

(Note that the sum in the denominator is locally finite – in fact, for any point x at most N terms are nonzero.)
(iv) $|D^k \theta_i| \le C r_i^{-k}$ for $k = 1, 2$.

We shall also need the following simple result.

Lemma 6.1 *With the above notation, for every* $\gamma \in (0, n)$ *we have*

$$\sum_{i\in I} r_i^\gamma \fint_{B_i} |f(x)|\, dx \le C \int_{\mathbb{R}^n} \frac{|f(x)|\, dx}{|y-x|^{n-\gamma}} .$$

Sketch of proof. Set $I_k = \{i \in I : |x_i - y| \in (2^k, 2^{k+1}]\}$ and split the sum on the left-hand side into

$$\sum_{k\in\mathbb{Z}} \sum_{i\in I_k} r_i^\gamma \fint_{B_i} |f(x)|\,dx\,.$$

For a fixed k the set I_k has at most $c = c(n)$ elements and all balls B_i with $i \in I_k$ are contained in the annulus $A_k := B(y, 2^{k+2}) \setminus B(y, 2^{k-1})$. If $x \in B_i$, then $r_i \approx |x - y| \approx |x_i - y|$. Using these observations and (i)-(iv) above, we conclude that

$$\sum_{i\in I_k} r_i^\gamma \fint_{B_i} |f(x)|\,dx \le C \int_{A_k} \frac{|f(x)|\,dx}{|y-x|^{n-\gamma}}\,;$$

summation over k yields the result. □

We are now ready to give the

Proof of Lemma 3.1. We may assume that the test function ϕ is of class C_0^∞. Recall that

$$\Phi(u, w; V) = \Delta u L(w, V) - \Delta w L(u, V), \tag{6.1}$$

where

$$L(\phi, V) = \nabla\phi \cdot V + \operatorname{div}(\phi V).$$

Write

$$\varphi(x) = \int_{\mathbb{R}^4} \Gamma(x, y)\Delta\varphi(y)\,dy,$$

where $\Gamma(x, y) = \mathrm{const}|x - y|^{-2}$ is the fundamental solution of the Laplace operator. Differentiating under the integral sign, we obtain

$$\begin{aligned} L(v, \varphi\nabla u^\beta) &= \varphi\nabla v \cdot \nabla u^\beta + \operatorname{div}(\varphi v \nabla u^\beta) \\ &= \int_{\mathbb{R}^4} \Delta\varphi(y)\left[\Gamma(\cdot, y)\nabla v \cdot \nabla u^\beta + \operatorname{div}(v\Gamma(\cdot, y)\nabla u^\beta)\right] dy \\ &= \int_{\mathbb{R}^4} \Delta\varphi(y)\, L\left(v, \Gamma(\cdot, y)\nabla u^\beta\right) dy\,. \end{aligned} \tag{6.2}$$

Thus, by Fubini theorem, (6.1) and (6.2) lead to

$$\int_{B_{4r/3}} \Phi(u^\alpha, u^\beta; \varphi\nabla u^\beta)\,dx = \int_{B_{4r/3}} \Delta\varphi(y)\Psi(y)\,dy\,, \tag{6.3}$$

where

$$\Psi(y) := \int_{\mathbb{R}^4} \Phi(u^\alpha, u^\beta; \zeta_1\Gamma(\cdot, y)\nabla u^\beta)\,dx\,; \tag{6.4}$$

here, ζ_1 is a cutoff function of class $C_0^\infty(B_{3r/2})$ with $\zeta_1 \equiv 1$ on some neighbourhood of $B_{4r/3}$ and $|D^k\zeta_1| \le Cr^{-k}$ for $k = 1, 2$. To conclude the whole proof, it remains to show that

$$\Psi \in L^2(B_{4r/3})\,, \qquad \|\Psi\|_{L^2(B_{4r/3})} \le C\left(\int_{B_{2r}} |\Delta u|^2\,dx\right)^{3/4}. \tag{6.5}$$

To establish (6.5), we fix $y \in B_{4r/3}$ and estimate Ψ by two suitably chosen Riesz potentials. To this end, we pick the partition of unity $\theta_i \equiv \theta_i^y \in C_0^\infty$ described above. Write B_i to denote the ball which contains the support of θ_i. Now, recalling the crucial cancellation property (2.9) of Φ, we obtain

$$\begin{aligned}
|\Psi(y)| &= \left|\sum_{i\in I}\int_{B_i} \Phi(u^\alpha, u^\beta; \theta_i\zeta_1\Gamma(\cdot,y)\nabla u^\beta)\,dx\right| \\
&= \left|\sum_{i\in I}\int_{B_i} \Phi\left(u^\alpha, u^\beta; (u^\beta - u^\beta_{B_i})\nabla\big[\theta_i\zeta_1\Gamma(\cdot,y)\big]\right)\,dx\right| \\
&= \left|\sum_{i\in I_1}\int_{B_i} \Phi\left(u^\alpha, u^\beta; (u^\beta - u^\beta_{B_i})\nabla\big[\theta_i\zeta_1\Gamma(\cdot,y)\big]\right)\,dx\right|,
\end{aligned} \tag{6.6}$$

where $I_1 \subset I$ denotes the set of those indices $i \in I$ for which $\theta_i\zeta_1 \not\equiv 0$, i.e. $B_i \cap B_{3r/2}$ is nonempty. Using triangle inequality, one checks that in this case

$$|x_i - y| \le r_i + \frac{3r}{2} + \frac{4r}{3} = \frac{|x_i - y|}{250} + \frac{17r}{6}\,.$$

Thus $\operatorname{diam} B_i = |x_i - y|/125 \le r/40$. Therefore, $B_i \subset B_{2r}$ for all $i \in I_1$. Moreover, for $k = 1, 2$ and for all $x \in B_i$, $i \in I_1$ we have

$$\left|D^k\big[\theta_i\zeta_1\Gamma(\cdot,y)\big]\right| \approx |x-y|^{-2-k} \approx r_i^{-2-k} \approx r_i^{2-k}|B_i|^{-1}\,. \tag{6.7}$$

Thus, using (2.7) to estimate Φ, and invoking the trivial bounds $|u| = 1$, $|u - u_{B_i}| \le 2$, we obtain from (6.6)

$$\begin{aligned}
|\Psi(y)| \;&\le\; C\sum_{i\in I_1}\fint_{B_i} |\Delta u|\,|u - u_{B_i}|\,dx + C\sum_{i\in I_1} r_i\fint_{B_i} |\Delta u|\,|\nabla u|\,dx \\
&=:\; S_1(y) + S_2(y)\,.
\end{aligned} \tag{6.8}$$

Now, we estimate each sum by (a power of) a suitable Riesz potential, using Sobolev inequality, the pointwise estimate $|\nabla u|^2 \le |\Delta u|$, and Lemma 6.1. To deal with S_1, we choose $p = 6$ (so that $p' = 6/5$ and $p_* = 12/5 = 2p'$) and estimate

$$
\begin{aligned}
|S_1(y)| &\le C \sum_{i\in I_1} \Big(\fint_{B_i} |\Delta u|^{p'}\,dx \Big)^{1/p'} \Big(\fint_{B_i} |u-u_{B_i}|^p\,dx \Big)^{1/p} \\
&\le C \sum_{i\in I_1} r_i \Big(\fint_{B_i} |\Delta u|^{p'}\,dx \Big)^{1/p'} \Big(\fint_{B_i} |\nabla u|^{p_*}\,dx \Big)^{1/p_*} \\
&\le C \sum_{i\in I_1} r_i \Big(\fint_{B_i} |\Delta u|^{p_*/2}\,dx \Big)^{3/p_*} \qquad \text{as } p' = p_*/2 \text{ and } |\nabla u|^2 \le |\Delta u| \\
&\le C \Big(\sum_{i\in I_1} r_i^{p_*/3} \fint_{B_i} |\Delta u|^{p_*/2}\,dx \Big)^{3/p_*} \qquad \text{as } 3/p_* > 1 \\
&\le C \Big(\int_{\mathbb{R}^4} \frac{|\Delta u(x)|^{6/5}\chi_{B_{2r}}(x)}{|y-x|^{4-4/5}}\,dx \Big)^{5/4} \qquad \text{by Lemma 6.1.}
\end{aligned}
$$

Applying now the fractional integration theorem, we conclude that $S_1^{4/5} \in L^q$ for $\frac{1}{q} = \frac{6}{10} - \frac{4}{4\cdot 5} = \frac{2}{5}$, i.e. $S_1 \in L^2$, and moreover

$$
\|S_1\|_{L^2(B_{4r/3})} \le C \Big(\int_{B_{2r}} |\Delta u|^2\,dx \Big)^{3/4}. \tag{6.9}
$$

S_2 can be estimated directly by a Riesz potential of $|\Delta u|^{3/2}$; another application of fractional integration theorem yields

$$
\|S_2\|_{L^2(B_{4r/3})} \le C \Big(\int_{B_{2r}} |\Delta u|^2\,dx \Big)^{3/4}. \tag{6.10}
$$

Gathering (6.9) and (6.10) we infer that $\Psi \in L^2$; estimate (6.5) follows. The proof of Lemma 3.1 is complete now. $\square$

Note that the particular form of Φ was not really important here. We only used the cancellation property (2.9), linearity of Φ in the last argument, and natural growth estimates. This is precisely the reason why Lemma 4.2 holds.

Sketch of proof of Lemma 4.2. Since the reasoning is very similar to the previous proof, we only indicate a few changes briefly. As before, using the kernel $\Gamma(x,y) = c_m|x-y|^{2-m}$ to represent φ, we obtain

$$
\int_{B_{4r/3}} \Phi_p(u^\alpha, u^\beta; \varphi\nabla u^\beta)\,dx = \int_{B_{4r/3}} \Delta\varphi(y)\Psi_p(y)\,dy\,, \tag{6.11}
$$

where

$$
\begin{aligned}
\Psi_p(y) \,:\, &= \int_{\mathbb{R}^4} \Phi_p(u^\alpha, u^\beta; \zeta_1\Gamma(\cdot,y)\nabla u^\beta)\,dx \\
&= -\sum_{i\in I_1} \int_{B_i} \Phi_p\left(u^\alpha, u^\beta; (u^\beta - u^\beta_{B_i})\nabla\big[\theta_i\zeta_1\Gamma(\cdot,y)\big]\right)\,dx
\end{aligned}
$$

(Whitney decomposition and the cancellation property (4.4) of Φ_p were used to obtain the last equality). Thus,

$$|\Psi_p(y)| \le C(S_1(y) + S_2(y)),$$

where

$$S_1(y) = \sum_{i\in I_1} \fint_{B_i} |\Delta u|^{p-1} |u - u_{B_i}|\, dx\,, \quad S_2(y) = \sum_{i\in I_1} r_i \fint_{B_i} |\Delta u|^{p-1} |\nabla u|\, dx\,.$$

As in the estimate of S_1 in the previous proof, we apply Hölder inequality with exponents $s' = m(m-1)/(m-2)(m+1)$, $s = m(m-1)/2$ (so that $s_*/2 = (p-1)s'$), then invoke Sobolev inequality, and finally use Lemma 6.1 to obtain

$$\begin{aligned}
|S_1(y)| &\le C \sum_{i\in I_1} \left(\fint_{B_i} |\Delta u|^{(p-1)s'}\, dx \right)^{1/s'} \left(\fint_{B_i} |u - u_{B_i}|^s\, dx \right)^{1/s} \\
&\le C \sum_{i\in I_1} r_i \left(\fint_{B_i} |\Delta u|^{(p-1)s'}\, dx \right)^{1/s'} \left(\fint_{B_i} |\nabla u|^{s_*}\, dx \right)^{1/s_*} \\
&\le C \sum_{i\in I_1} r_i \left(\fint_{B_i} |\Delta u|^{s_*/2}\, dx \right)^{1+\frac{1}{m}} \\
&\le C \left(\int_{B(a,2r)} \frac{|\Delta u(x)|^{\frac{m}{2}\cdot\frac{m-1}{m+1}}}{|y-x|^{m^2/(m+1)}}\, dx \right)^{1+\frac{1}{m}}.
\end{aligned}$$

Thus, by fractional integration theorem,

$$\|S_1\|_{L^{m/(m-2)}(B_{4r/3})} \le C \|\Delta u\|^{(m-1)/2}_{L^{m/2}(B_{2r})}. \tag{6.12}$$

One more application of fractional integration theorem shows that

$$\|S_2\|_{L^{m/(m-2)}(B_{4r/3})} \le C \|\Delta u\|^{(m-1)/2}_{L^{m/2}(B_{2r})}. \tag{6.13}$$

Combining (6.12) with (6.13), we infer that $\Psi_p \in L^{m/(m-2)}$ and obtain the desired estimate of the norm. Inserting this estimate in (6.11), we complete the proof of Lemma 4.2. $\square$

Remark. Note that in both proofs above we use *only* the following tools:

- the triangle inequality for the Euclidean metric;
- the doubling property of Lebesgue measure;
- the Whitney decomposition (of the complement of a single point);
- standard growth estimates of the fundamental solution of the Laplacian;
- the Sobolev inequality on balls;
- the fractional integration theorem for Riesz potentials;
- (last but not least) the cancellation property of Φ, resp. Φ_p.

We refer to Hajłasz and Koskela [16] for a thorough discussion of generalized Riesz potentials and Sobolev inequalities on stratified groups and in metric spaces. The fundamental solution Γ of Δ_X satisfies, together with its derivatives with respect to both variables x and y, the growth estimates

$$|\Gamma(x,y)| \le C\frac{\rho(x,y)^2}{|B(x,\rho(x,y))|} \qquad \text{for } n \ge 3,$$

and, for $n \ge 2$,

$$|X_i\Gamma(x,y)| \le C\frac{\rho(x,y)}{|B(x,\rho(x,y))|}, \qquad |X_iX_j\Gamma(x,y)| \le \frac{C}{|B(x,\rho(x,y))|},$$

where B denotes a ball in the Carnot–Carathéodory metric; see e.g. Sánchez-Calle [29, Theorem 1], or [27]. These inequalities lead to a counterpart of (6.7), which is needed for a proof of (5.13) (a closely related argument is given in the proof of Lemma 3.2 in [17]).

Thus, all the computations in this Section can in fact be repeated in a subelliptic setting. It is clear that in both proofs presented above the *topological* dimension of $\mathbb{R}^m$ plays no role; it is the *homogeneous* dimension which is decisive.

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