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Regularity of p -harmonic maps from the p -dimensional ball into a sphere

Paweł Strzelecki¹

We prove that, for $p \geq 2$, all weakly p -harmonic maps $u = (u_1, \dots, u_n)$ from the p -dimensional ball into a sphere, i.e. weak solutions of class $W^{1,p}$ of the constrained elliptic system

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2} \nabla u_i) &= u_i |\nabla u|^p \\ \sum (u_i)^2 &= 1, \end{aligned}$$

are Hölder continuous. This result is an analogue of an earlier theorem of F. Hélein for the case $p = 2$.

1. Introduction

Let $B^p = \{x \in \mathbb{R}^p : \sum (x_i)^2 < 1\}$ denote the unit p -dimensional ball, and write S^{n-1} to denote the unit sphere in \mathbb{R}^n . Define the functional

$$I_p(u) = \int_{B^p} |\nabla u(x)|^p dx \quad \text{for } u \in W^{1,p}(B^p, S^{n-1}).$$

We wish to investigate those mappings which are critical points of I_p .

Definition. By a weakly p -harmonic map (or simply p -harmonic map) we mean here any u belonging to the Sobolev space

$$W^{1,p}(B^p, S^{n-1}) \equiv \{f = (f_1, \dots, f_n) \mid f_i \in W^{1,p}(B^p) \text{ and } \sum_{i=1}^n (f_i(x))^2 = 1 \text{ a.e.}\},$$

and being a critical point of the functional I_p in the class of functions having fixed trace (equal to that of u) on ∂B^p , with respect to variations on S^{n-1} , i.e.

$$\left. \frac{d}{dt} \right|_{t=0} I_p \left(\frac{u + t\psi}{|u + t\psi|} \right) = 0 \quad \text{for all } \psi \in C_0^\infty(B^p, \mathbb{R}^n). \quad (1)$$

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Condition (1) is easily checked to be equivalent to the fact that $u \in W^{1,p}(B^p, S^{n-1})$ is a weak solution to the Euler-Lagrange elliptic system

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u_i) = u_i |\nabla u|^p, \quad i = 1, 2, \dots, n. \quad (2)$$

More precisely, the integral identity

$$\int_B |\nabla u|^{p-2} \nabla u_i \cdot \nabla \psi_i \, dx = \int_B \psi_i u_i |\nabla u|^p \, dx, \quad (3)$$

holds true for all $i = 1, \dots, n$, and for every $\psi = (\psi_1, \dots, \psi_n) \in C_0^\infty(B, \mathbb{R}^n)$. Here and everywhere below,

$$|\nabla u|^2 = \sum_{i=1}^n \sum_{j=1}^p \left(\frac{\partial u_i}{\partial x_j} \right)^2.$$

One can find solutions of (2) which are minimizers of I_p in the class of mappings with fixed boundary values. However, weakly p -harmonic maps do not have to be minimizers of I_p . It is also possible to define and consider p -harmonic maps $u : M^m \rightarrow N^n$ between Riemannian manifolds.

In general, weakly p -harmonic maps do not have to be continuous: the familiar map $x \mapsto x/|x|$ from the unit ball B^n to its boundary $\partial B^n \equiv S^{n-1}$ is singular at 0 and weakly p -harmonic for all $p \in [1, n)$. However, there are lots of results about regularity and partial regularity of weakly p -harmonic maps under various additional assumptions. Let us mention below just a few; the list is obviously far from being complete.

M. Fuchs [7], R. Hardt and F.H. Lin [11], and S. Luckhaus [16] proved independently a theorem stating that *minimizing* p -harmonic maps $u : M^m \rightarrow N^n$ are of class $C^{1,\alpha}$, $0 < \alpha < 1$, outside a set of Hausdorff dimension $m - [p] - 1$ (that was a generalization of an earlier result of R. Schoen and K. Uhlenbeck [19] concerning the case $p = 2$ of minimizing harmonic maps). In the series of his recent papers [12], [13], [14] F. Hélein proved that any weakly harmonic map $f : M \rightarrow N$ defined on a two-dimensional Riemannian manifold M is continuous; [12] contains the proof for $N = S^{n-1}$, [13] concerns the case when N is a compact manifold with a Lie group of isometries acting transitively, and [14] deals with the case of arbitrary compact Riemannian N . By standard elliptic regularity methods, continuity of a weakly harmonic map implies its C^∞ -smoothness. L.C. Evans [3] and F. Bethuel [1] generalized Hélein's result to the case of the so-called *stationary* harmonic maps on n -dimensional manifolds, $n \geq 2$, proving their regularity outside a singular set of $(n-2)$ -dimensional Hausdorff measure zero. The interested reader is referred to the papers mentioned above for brief lists of other results in the field.

In this paper we give a short proof of an analogue of the main result of [12], namely of the following

Theorem 1. *Let u be a weakly p -harmonic map from the p -dimensional ball unit ball $B^p \equiv \{x \in \mathbb{R}^p : |x| < 1\}$ into the sphere S^{n-1} of arbitrary dimension ($p \geq 2$). Then, u is necessarily Hölder continuous on B^p .*

Remark. M. Fuchs in his recent paper [8] has proved (among other things) the same result independently and with different methods.

In the case $p = 2$ this is precisely Hélein's theorem. Note that for *minimizing* p -harmonic maps u our Theorem 1 follows immediately e.g. from [11, Theorem 1]. However, this restrictive assumption about the map u is not needed here.

Our proof heavily relies on the results of Coifman, Lions, Meyer, and Semmes [2] and the fact that the right-hand side of (2) turns out to be an element of the local Hardy space $\mathcal{H}_{\text{loc}}^1$, a proper subspace of L^1 (for $p = 2$ this was noticed and exploited by Hélein [12], [13], [14]). This is a starting point of our proof. In Section 2, for the reader's convenience, we recall an important theorem from [2] and state explicitly some of its consequences (well known in the folklore).

The idea of the remaining part of the proof resembles slightly that of Evans [3]. Here is a brief sketch of our reasoning. First, note that $u \in W^{1,p}(B^p)$ implies $u \in BMO$. Then, exploit the duality between $\mathcal{H}^1(\mathbb{R}^p)$ and $BMO(\mathbb{R}^p)$ to obtain, by appropriate choice of test functions in (3), the inequality

$$\int_{B(x,r)} |\nabla u(y)|^p dy \leq \lambda \int_{B(x,2r)} |\nabla u(y)|^p dy, \quad 0 < \lambda < 1,$$

valid for all sufficiently small radii r , and finally apply Dirichlet growth theorem [10], [17]. Section 3 contains all necessary details of that proof.

Notation. For a measurable function w and a measurable set A of positive Lebesgue measure, we write

$$[w]_A \equiv \int_A w(x) dx := \frac{1}{\text{meas } A} \int_A w(x) dx$$

to denote the average of w over A .

The standard Sobolev space of functions of class $L^p(\Omega)$ having their first order distributional partial derivatives in $L^p(\Omega)$ is denoted by $W^{1,p}(\Omega)$. If V is a normed vector space of finite dimension m with a fixed basis, then $W^{1,p}(\Omega, V)$ is the space of mappings

$$u = (u_1, \dots, u_m): \Omega \rightarrow V$$

with all coordinates u_i of class $W^{1,p}(\Omega)$.

$\Lambda^\ell(\mathbb{R}^n)$ denotes the $\binom{n}{\ell}$ -dimensional space of all ℓ -covectors in \mathbb{R}^n (with the standard norm and basis).

In all the calculations, C denotes a general constant (depending only on the dimension and integrability exponents) which may change its value from one line to another.

Throughout Section 2, by a slight abuse of notation, p stands for an arbitrary real from $(1, \infty)$, not necessarily an integer.

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2. Prerequisites for the proof

Definition. A measurable function $f \in L^1(\mathbb{R}^n)$ belongs to the *Hardy space* $\mathcal{H}^1(\mathbb{R}^n)$ if and only if

$$f_* := \sup_{\varepsilon > 0} |\varphi_\varepsilon * f| \in L^1(\mathbb{R}^n).$$

Here, $\varphi_\varepsilon(x) := \varepsilon^{-n} \varphi(x/\varepsilon)$, and φ is a fixed function of class $C_0^\infty(B(0,1))$ with $\int \varphi(y) dy = 1$. The definition does not depend on the choice of φ (see [5]).

Equivalently, one can define $\mathcal{H}^1(\mathbb{R}^n)$ as the space of those elements of $L^1(\mathbb{R}^n)$, for which all the Riesz transforms $R_j f$, $j = 1, 2, \dots, n$, are also of class $L^1(\mathbb{R}^n)$. The reader is referred to [9] or [20] for more details. Let us just mention here that $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space with the norm

$$\|f\|_{\mathcal{H}^1} = \|f\|_{L^1} + \|f_*\|_{L^1}.$$

Moreover, the condition $f \in \mathcal{H}^1(\mathbb{R}^n)$ implies $\int f(y) dy = 0$.

C. Fefferman [4], [5] proved that the dual of $\mathcal{H}^1(\mathbb{R}^n)$ is equal to the space of functions of bounded mean oscillation, $BMO(\mathbb{R}^n)$. More precisely, there exists a constant C such that

$$\int_{\mathbb{R}^n} h(y)\psi(y) dy \leq C \|h\|_{\mathcal{H}^1} \|\psi\|_{BMO}, \quad (4)$$

for all functions $h \in \mathcal{H}^1(\mathbb{R}^n)$ and $\psi \in BMO(\mathbb{R}^n)$.

The interesting paper of S. Müller [18] inspired some of the research reported in [2], in particular the following remarkable theorem.

Theorem 2 (Coifman, Lions, Meyer, Semmes). *Let $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, and assume that $H \in L^{p/(p-1)}(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the condition $\operatorname{div} H = 0$ in $\mathcal{D}'(\mathbb{R}^n)$. Then, $\nabla u \cdot H \in \mathcal{H}^1(\mathbb{R}^n)$, and*

$$\|\nabla u \cdot H\|_{\mathcal{H}^1} \leq C \cdot \|\nabla u\|_{L^p} \cdot \|H\|_{L^{p/(p-1)}} \quad (5)$$

for some constant C depending only on n and p .

The estimate (5) was not explicitly stated in [2], but follows from the proof presented there (cf. also [3, Section 2]).

Let us now make explicit a corollary of the above theorem (more or less well known to specialists).

Corollary 3. *Let Ω be a ball in \mathbb{R}^n . Assume that $u \in W^{1,p}(\Omega)$, $1 < p < \infty$, and that $H \in L^{p/(p-1)}(\Omega, \mathbb{R}^n)$ satisfies the condition $\operatorname{div} H = 0$ in $\mathcal{D}'(\Omega)$. Then, one can find a function $h \in \mathcal{H}^1(\mathbb{R}^n)$ such that*

$$h(x) = \nabla u(x) \cdot H(x), \quad x \in \Omega,$$

and

$$\|h\|_{\mathcal{H}^1} \leq C \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \|H\|_{L^{p/(p-1)}(\Omega)}.$$

The constant C does not depend on the size of Ω .

Proof. This is a simple consequence of Theorem 2 and the results of [15, Section 4]. The idea is to extend ∇u to a gradient field on \mathbb{R}^n and H to a divergence free vector field on \mathbb{R}^n without increasing too much the appropriate L^s -norms, and then to apply Theorem 2.

Denote $q = p/(p - 1)$. Take the bounded, linear operator

$$T : L^q(\Omega, \Lambda^\ell(\mathbb{R}^n)) \rightarrow W^{1,q}(\Omega, \Lambda^{\ell-1}(\mathbb{R}^n)), \quad \ell = 1, 2, \dots, n$$

satisfying

$$\omega = T(d\omega) + d(T\omega), \quad \|T\| \leq C(n, q)$$

for all forms $\omega \in L^q(\Omega, \Lambda^\ell(\mathbb{R}^n))$ such that $d\omega \in L^q(\Omega, \Lambda^{\ell+1}(\mathbb{R}^n))$. (See [15, Section 4] for the precise definition of T .)

Let, for $1 \leq s < \infty$, E_s be the extension operator,

$$E_s : W^{1,s}(\Omega, \Lambda^\ell(\mathbb{R}^n)) \rightarrow W_{loc}^{1,s}(\mathbb{R}^n, \Lambda^\ell(\mathbb{R}^n)),$$

such that $\|\nabla E_s(u)\|_{L^s(\mathbb{R}^n)} \leq C(n, s)\|\nabla u\|_{L^s(\Omega)}$. Identify the vector field H with the $(n - 1)$ -form ω ,

$$\omega = \sum_{j=1}^n (-1)^{j-1} H_j \underbrace{dx_1 \wedge \dots \wedge dx_n}_{dx_j \text{ omitted}}, \quad d\omega \equiv \operatorname{div} H \cdot dx_1 \wedge \dots \wedge dx_n.$$

It is easy to see that $h = \nabla E_p(u) \cdot d(E_q T(\omega))$ has all the desired properties. Here, as before, we identify the $(n - 1)$ -form $d(E_q T(\omega))$ with a (divergence free) vector field of class $L^q(\mathbb{R}^n)$. \square

3. Proof of Theorem 1

Let us begin with a straightforward calculation proving that the right-hand side of each equation of system (2) can be extended to a function $h_i \in \mathcal{H}^1(\mathbb{R}^p)$. In the case $p = 2$ this crucial observation is due to F. Hélein.

Write $V_i = |\nabla u|^{p-2} \nabla u_i$. The condition $\sum (u_k)^2 = 1$ implies that $\sum u_k \nabla u_k = 0$, hence

$$V_i = \sum_{k=1}^n u_k (u_k V_i - u_i V_k) \quad \text{for } i = 1, 2, \dots, n.$$

Now, note that (2) implies

$$\operatorname{div} (u_k V_i - u_i V_k) = \nabla u_k \cdot V_i - \nabla u_i \cdot V_k + u_k \operatorname{div} V_i - u_i \operatorname{div} V_k = 0,$$

and therefore

$$\operatorname{div} V_i = \sum_{k=1}^n \nabla u_k \cdot (u_k V_i - u_i V_k). \tag{6}$$

This is a starting point for the following energy decay estimate.

Lemma 1. *Let $u \in W^{1,p}(B^p, S^{n-1})$ be a weak solution of (2). Then, there exist $\lambda \in (0, 1)$ and $r_0 > 0$ such that*

$$\int_{B^p(x,r)} |\nabla u(y)|^p dy \leq \lambda \int_{B^p(x,2r)} |\nabla u(y)|^p dy \tag{7}$$

for all $x \in B^p(0, 1)$ and all $r < \frac{1}{2} \min(r_0, \operatorname{dist}(x, \partial B^p))$.

Proof. Fix $x \in B^p$ and $r < \frac{1}{2} \operatorname{dist}(x, \partial B^p)$. Use Corrolary 3 to construct, for each $i = 1, 2, \dots, n$, a function $h_i \in \mathcal{H}^1(\mathbb{R}^p)$ satisfying

$$-\operatorname{div} (|\nabla u|^{p-2} \nabla u_i) = h_i \quad \text{on } B^p(x, 2r),$$

and such that

$$\|h_i\|_{\mathcal{H}^1} \leq C \int_{B^p(x,2r)} |\nabla u(y)|^p dy. \tag{8}$$

Therefore, for all test functions ψ_i with support contained in $B^p(x, 2r)$,

$$\int_{B^p} |\nabla u(y)|^{p-2} \nabla u_i(y) \cdot \nabla \psi_i(y) dy = \int_{B^p} h_i(y) \psi_i(y) dy. \tag{9}$$

Write A_r to denote the annulus $B^p(x, 2r) \setminus B^p(x, r)$. Choose $\psi_i = \eta^p(u_i - [u_i]_{A_r})$, where $\eta \in C_0^\infty(B^p(x, 2r))$ satisfies

$$0 \leq \eta(y) \leq 1, \quad |\nabla \eta(y)| \leq \frac{C}{r}, \quad \eta(y) \equiv 1 \text{ for } y \in B^p(x, r).$$

Then, identity (9) leads after a routine calculation to

$$\int_{B^p(x,r)} |\nabla u(y)|^p dy \leq J_1 + J_2,$$

where

$$J_1 = \sum_{i=1}^n \left| \int_{\mathbb{R}^p} h_i(y) \psi_i(y) dy \right|,$$

$$J_2 = p \sum_{i=1}^n \int_{A_r} |\nabla u(y)|^{p-1} \cdot |\nabla \eta(y)| \cdot |u_i(y) - [u_i]_{A_r}| dy.$$

In the second sum, all the integrations are performed over the annulus A_r since $\nabla \eta$ vanishes on $B^p(x, r)$. We apply Hölder inequality and then Poincaré inequality to estimate J_2 in the following way

$$\begin{aligned}
 J_2 &\leq \frac{C}{r} \left(\int_{A_r} |\nabla u(y)|^p dy \right)^{1-1/p} \sum_{i=1}^n \left(\int_{A_r} |u_i(y) - [u_i]_{A_r}|^p dy \right)^{1/p} \\
 &\leq C \int_{A_r} |\nabla u(y)|^p dy.
 \end{aligned}
 \tag{10}$$

To deal with J_1 , we shall prove that $\psi_i \in BMO(\mathbb{R}^p)$ and

$$\|\psi_i\|_{BMO(\mathbb{R}^p)} \leq C \left(\int_{B^p(x,2r)} |\nabla u(y)|^p dy \right)^{1/p}. \tag{11}$$

Indeed, take a cube $Q \subset \mathbb{R}^p$. We apply Poincaré inequality two times: first, to estimate the integral over cube, and then to estimate the one over $Q \cap \{\nabla \eta \neq 0\}$, a subset of A_r . This calculation gives:

$$\begin{aligned}
 \int_Q |\psi_i(y) - [\psi_i]_Q| dy &\leq \left(\int_Q |\psi_i(y) - [\psi_i]_Q|^p dy \right)^{1/p} \\
 &\leq C (\text{diam } Q) \left(\int_Q |\nabla \psi_i(y)|^p dy \right)^{1/p} \\
 &\leq \frac{C}{r} \left(\int_{Q \cap \{\nabla \eta \neq 0\}} |u_i(y) - [u_i]_{A_r}|^p dy \right)^{1/p} + C \left(\int_{Q \cap \{\eta \neq 0\}} |\nabla u_i(y)|^p dy \right)^{1/p} \\
 &\leq C \left(\int_{B^p(x,2r)} |\nabla u(y)|^p dy \right)^{1/p},
 \end{aligned}$$

and (11) is proved.

To conclude the proof, note that (4), (8), and (11) imply that

$$J_1 \leq C \sum_{i=1}^n \|h_i\|_{\mathcal{H}^1(\mathbb{R}^p)} \|\psi_i\|_{BMO(\mathbb{R}^p)} \leq C \left(\int_{B^p(x,2r)} |\nabla u(y)|^p dy \right)^{1+1/p}. \tag{12}$$

Hence, denoting $I(x, r) = \int_{B^p(x,r)} |\nabla u(y)|^p dy$, we obtain from (10) and (12) the inequality

$$I(x, r) \leq C_0 \cdot (I(x, 2r))^{1+1/p} + C_0 \cdot (I(x, 2r) - I(x, r)),$$

or, equivalently,

$$I(x, r) \leq \frac{C_0}{C_0 + 1} I(x, 2r) \left(1 + (I(x, 2r))^{1/p} \right). \tag{13}$$

Now, use absolute continuity of the integral to find r_0 such that for all $z \in B^p$ and all $r < \frac{1}{2} \min(r_0, \text{dist}(z, \partial B^p))$ the integral $I(z, 2r)$ does not exceed $(2C_0)^{-p}$. Then, (13) implies that

$$I(x, r) \leq \lambda I(x, 2r), \quad \lambda := \frac{2C_0 + 1}{2C_0 + 2} \in (0, 1).$$

This completes the proof of Lemma 1. \square

Proof of Theorem 1. By iterations of inequality (7), Lemma 1 implies that for some positive constants C and β we have

$$\int_{B^p(x,r)} |\nabla u(y)|^p dy \leq C \cdot r^\beta \quad (14)$$

for all $x \in B^p(0, 1)$ and all sufficiently small r . Inequality (14) allows us to apply Morrey's Dirichlet growth theorem (see [17, Theorem 3.5.2] or [10, Chapter 3, pages 64–65]) and conclude that u is uniformly Hölder continuous with exponent $\alpha = \beta/p$ on compact subsets of B^p .

Remark. Theorem 1 is of course not a final result; we expect that it is possible to generalize the results of Bethuel [1] and prove that stationary p -harmonic maps $u : M^m \rightarrow N^n$ between arbitrary compact Riemannian manifolds are of class $C^{1,\alpha}(V)$ for some open $V \subset M^m$ with $H^{m-p}(M \setminus V) = 0$ (or maybe even with $\dim(M \setminus V) < m - p$).

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