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A new proof of regularity of weak solutions of the H-surface equation

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Abstract. We give a new proof of a theorem of Bethuel, asserting that arbitrary weak solutions $u \in W^{1,2}(\mathbb{B}, \mathbb{R}^3)$ of the H-surface system $\Delta u = 2H(u)u_{x_1} \wedge u_{x_2}$ are locally Hölder continuous provided that H is a bounded Lipschitz function. Contrary to Bethuel's, our proof completely omits Lorentz spaces. Estimates below natural exponents of integrability are used instead. (The same method yields a new proof of Hélein's theorem on regularity of harmonic maps from surfaces into arbitrary compact Riemannian manifolds.) We also prove that weak solutions with continuous trace are continuous up to the boundary, and give an extension of these results to the equation of hypersurfaces of prescribed mean curvature in $\mathbb{R}^{n+1}$, this time assuming in addition that $|\nabla H(y)|$ decays at infinity like $|y|^{-1}$.

1 Introduction

In this note, we consider the regularity of weak solutions of the H-surface equation

$$\Delta u = 2H(u)u_{x_1} \wedge u_{x_2} \,, \tag{1.1}$$

where u is a mapping from the unit disc $\mathbb{B} \subset \mathbb{R}^2$ into $\mathbb{R}^3$, and $H\colon \mathbb{R}^3 \to \mathbb{R}$ is a bounded Lipschitz function. It is well known that those classical solutions of (1.1) which are conformal, i.e. satisfy the conditions

$$|u_{x_1}|^2 - |u_{x_2}|^2 = \langle u_{x_1} \,,\, u_{x_2} \rangle = 0 \,,$$

parametrize (away from branch points) surfaces of prescribed mean curvature H; at a point $u(x)$, the mean curvature is equal to $H(u(x))$.

We say that $u = (u^1, u^2, u^3) \in W^{1,2}(\mathbb{B}, \mathbb{R}^3)$ is a weak solution of (1.1) if and only if

$$\int_{\mathbb{B}} \nabla u \nabla \varphi \, dx = -2 \int_{\mathbb{B}} \varphi H(u) u_{x_1} \wedge u_{x_2} \, dx \qquad \text{for every } \varphi \in C_0^\infty(\mathbb{B}, \mathbb{R}^3). \tag{1.2}$$

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This is the only condition imposed on u. By a standard limit argument, (1.2) holds also for all compactly supported φ of class $L^\infty \cap W^{1,2}(\mathbb{B}, \mathbb{R}^3)$.

The regularity results for solutions of the H-surface equation, under various additional assumptions, form a part of the vast regularity theory of nonlinear elliptic systems with a right hand side growing critically with the gradient of solution. (Another prominent example is supplied by harmonic maps.) A thorough discussion of known facts is beyond the scope of this paper; let us just mention the pioneering works of Wente [26] and Hildebrandt and Kaul [15], and the papers of Grüter [11] and Bethuel [1], who considered, respectively, conformal weak solutions for bounded H, and arbitrary weak solutions for bounded and Lipschitz H. For more information we refer the reader to Chapter 7 of the monograph [4] and the references therein. A good survey of existence results can be found e.g. in Duzaar and Steffen [7].

The purpose of this note is twofold. First of all, we give a new proof (without any use of Lorentz spaces) of Bethuel's theorem [1] on interior regularity of weak solutions of (1.1).

Theorem 1.1 (Bethuel) *Assume that $H\colon \mathbb{R}^3 \to \mathbb{R}$ is a bounded Lipschitz function. Then, every weak solution $u \in W^{1,2}(\mathbb{B}, \mathbb{R}^3)$ of equation (1.1) is Hölder continuous on compact subsets of* $\mathbb{B}$.

We deduce this theorem from interior Morrey type estimates, working with exponents $2 - \varepsilon_0 < p < 2$, where ε_0 is a small (absolute) constant. We show that for such exponents $r^{p-2} \int_{B_r} |\nabla u|^p \, dx$ goes to zero faster than some positive power of r. A rough description of the method of proof — which has its origins in the works of T. Iwaniec in the theory of quasiconformal and quasiregular mappings— is given at the end of the Introduction.

Once it is known that weak solutions are continuous, well-known methods lead to the following result.

Corollary 1.2 *Let u, H be as in Theorem 1.1. Then u is of class $C^{2,\beta}_{\mathrm{loc}}(\mathbb{B}, \mathbb{R}^3)$ for any $\beta < 1$.*

Our second aim is to prove that weak solutions with continuous trace $u\big|_{\partial\mathbb{B}}$ are continuous up to the boundary. We deduce this from interior decay estimates, combining them with Morrey's Dirichlet Growth Theorem in a scale invariant version and the ACL property of Sobolev functions.

Theorem 1.3 *Let u, H be as in Theorem 1.1. If the trace $\psi = u\big|_{\partial\mathbb{B}}\colon \partial\mathbb{B} \to \mathbb{R}^3$ is continuous, then $u \in C^0(\overline{\mathbb{B}}, \mathbb{R}^3)$.*

In this generality that result, apparently, seems to be unknown in the existing literature. Under more restrictive assumptions on H, and using a different method, a similar theorem has been obtained by Jakobowsky [19].

Higher boundary regularity of *conformal* weak solutions follows then from known theorems, see [4], Vol. 2, Theorem 7.3.2.

One of the main difficulties in the proof of Theorem 1.1 stems from the fact that for u of class $W^{1,2}$ the right hand side $H(u)u_{x_1} \wedge u_{x_2}$ is only an integrable function.

(A priori, u does not have to be minimizing, bounded or conformal; no extra decay conditions on H are imposed, etc.) Although $u_{x_1} \wedge u_{x_2}$ has its coordinates in the Hardy space, multiplication by $H \circ u$ destroys this special structure. Besides, since there is no imbedding of $W^{1,2}$ into L^∞, the gradient of $(u - \mathrm{const})H(u)$ does not have to be square integrable — and there is no apparent reason why $(u-\mathrm{const})H(u)$ should belong to BMO. Thus, even sophisticated tools like the duality of Hardy space and BMO do not justify the use of $\zeta(u - \mathrm{const})$, where $\zeta \in C_0^\infty(\Omega)$, as a test function in (1.2).

To overcome this difficulty, Bethuel [1] has been using Lorentz spaces. He proved that for a small, fixed number $\alpha \in (0, 1)$ the decay inequality

$$\|u\|_{L^{2,\infty}(B(x,\alpha r))} \leq \frac{1}{2}\|u\|_{L^{2,\infty}(B(x,r))}$$

holds for all $x \in \Omega$ and all sufficiently small radii r. His proof heavily relies on various properties of Lorentz spaces, on fine-tuned imbedding theorems, and on Wente estimates for the equation

$$\begin{cases} -\Delta v = f_{x_1} g_{x_2} - f_{x_2} g_{x_1} & \text{in } U, \\ \quad v = 0 & \text{on } \partial U, \end{cases}$$

where $f, g \in W^{1,2}(U)$, and $U \subset \mathbb{R}^2$. (Strictly speaking, Bethuel combined classical Wente estimates, i.e. L^∞ bounds for v and L^2 bounds for ∇v, with an additional bound for ∇v in the Lorentz space $L^{2,1}$. The last improvement, attributed in [1] to Tartar, can be obtained via the interpolation theory, or via the theory of Hardy spaces.)

We present a different proof. Some theory of Hardy spaces – notably the duality of Hardy space and the space BMO of functions having bounded mean oscillation – is still present in the argument. Fefferman's duality theorem serves here as a replacement for the isoperimetric inequality that has been used much earlier (see e.g. [26], [12]) for similar purposes. Apart of that we use only standard estimates in various L^p and Sobolev spaces. No knowledge of Lorentz spaces, and no nonstandard imbedding theorems are necessary.

As we have already said, $\zeta(u - \mathrm{const})$ is not an admissible test function, and there is no direct way to obtain decay estimates for $\int_{B(x,r)} |\nabla u|^2$. To circumvent this problem, we adapt an idea that originates – in a different context of quasiregular mappings – in a series of papers of T. Iwaniec and his various coauthors; see e.g. [16], [17], [18]. This idea is to work below the natural exponents of integrability. One employs test functions φ for which $\nabla\varphi$ behaves, modulo harmless divergence free terms, roughly speaking like $|\nabla u|^{-\varepsilon}\nabla u$, where ε is a small positive parameter. Standard L^q-estimates for Hodge decomposition yield various bounds for φ and its gradient. In particular, φ is Hölder continuous by Sobolev imbedding theorem. (Due to a stability theorem of Iwaniec [16, Theorem 8.2], asserting that for small ε the divergence free term is small in L^q, this idea has proved to be a fruitful and powerful one in the theory of quasiregular mappings. We hope that it can find new applications also in the regularity theory of nonlinear elliptic systems of variational origin; this belief was part of the motivation for writing the present note.)

Plugging such φ into the weak form of the equation, one immediately obtains integrals of $|\nabla u|^p$, with $p = 2 - \varepsilon$, from terms which correspond to the Laplacian. It is less clear that the nonlinear expression on the right hand side can also be estimated in terms of such integrals. To achieve this, we perform one integration by parts and apply the Fefferman duality theorem [23, Chapter 4], combined with a result of Coifman, Lions, Meyer and Semmes [3]. Due to the small parameter ε, the norms of φ in L^∞ and its gradient in L^2 (or even slightly above L^2) are bounded by norms of $|\nabla u|$ *below* L^2. It follows from Poincaré inequality that the norm of u in BMO can also be controlled by appropriate maximal functions of $|\nabla u|^p$. (In this respect, duality of $\mathcal{H}^1$ and BMO is better suited for our purposes than isoperimetric inequality.) Finally, integrals of $|\nabla u|^2$ are (locally, on small balls) dominated by small constants, and it is possible to apply a variant of the hole-filling trick. See Section 2 for details.

Boundary continuity of weak solutions follows from uniform interior decay estimates and standard properties of Sobolev functions. The details are given in Section 3. In Section 4, we briefly sketch another application of the method of Section 2 (i.e., a new proof of Hélein's theorem on regularity of harmonic maps from surfaces into compact Riemannian manifolds), and discuss a generalization to the equation of hypersurfaces of prescribed mean curvature in $\mathbb{R}^{n+1}$. The method of Section 2 does not work in the latter case, and we need an extra decay assumption for ∇H.

The *notation* is mostly standard. $W^{1,p}$ stands for the usual Sobolev space of those functions in L^p whose first order distributional partial derivatives also belong to L^p. For definition and an excellent discussion of basic properties of the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and the space of functions of bounded mean oscillation $\mathrm{BMO}(\mathbb{R}^2)$ we refer to Chapters 3 and 4 of Stein's monograph [23]. A ball in $\mathbb{R}^2$ with center a and radius r is denoted by $B(a,r)$. The barred integral $⨍_E f\,dx$ denotes the average value of a function f over a measurable set E, $⨍_E f \equiv [f]_E := |E|^{-1}\int_E f\,dx$. If $E = B(a,r)$ is a ball, we use the abbreviation $[f]_{B(a,r)} \equiv [f]_{a,r}$. The letter C denotes a general constant that may change even in a single string of estimates; writing $C(p,\varepsilon,\ldots)$ etc. we emphasize sometimes that C depends on p, ε etc. Primes are used to denote Hölder conjugate exponents.

2 Interior estimates

Continuity of solutions is derived from a Morrey type lemma. To state it concisely, we need some notation.

For $p \in (1,2]$, $a \in \mathbb{B}$, and $r \le 1 - |a| \equiv \operatorname{dist}(a, \partial\mathbb{B})$ we set

$$J_p(a,r) := \frac{1}{r^{2-p}} \int_{B(a,r)} |\nabla u(y)|^p\,dy, \tag{2.1}$$

and

$$\mathcal{M}_p(a,r) := \sup_{\substack{z\in B(a,r),\\ \rho<r-|a-z|}} J_p(z,\rho) \tag{2.2}$$

(the supremum is taken over all balls contained in $B(a,r)$). Note that $\mathcal{M}_p(a,\cdot)$ is a monotone function of r, and, by Hölder inequality,

$$\mathcal{M}_p(a,r) \le \pi^{1-\frac{p}{2}} \left(\int_{B(a,r)} |\nabla u(x)|^2 \, dx \right)^{p/2} .$$

Therefore, $\mathcal{M}_p(a,r)$ goes to zero as r tends to zero.

We sometimes write $\mathcal{M}_p(a,r;u)$ to indicate that $\mathcal{M}_p$ depends on u.

Lemma 2.1 *There exist numbers $\varepsilon_0 \in (0,\frac{1}{4})$ and $\lambda_0 \in (0,1)$ with the following property: for every exponent p with $\varepsilon := 2-p \in (0,\varepsilon_0)$ one can find an $R_0 = R_0(\varepsilon,u) > 0$ such that the maximal function $\mathcal{M}_p(a,r)$ of a weak solution u of (1.1) satisfies the decay inequality*

$$\mathcal{M}_p(a,r) \le \lambda_0 \mathcal{M}_p(a,4r) \tag{2.3}$$

for all $a \in \mathbb{B}$ and all $r \le \frac{1}{4}\min(R_0, 1-|a|)$.

Proof of Lemma 2.1. Fix a point $a \in \mathbb{B}$ and a radius $r \le \frac{1}{4}(1-|a|)$. Let $\zeta \in C_0^\infty(B(a,2r))$ be a standard cutoff function with $\zeta \equiv 1$ on $B(a,r)$, $|\nabla\zeta| \le 2/r$. Set

$$\tilde{u} = \zeta(u - [u]_A),$$

where $[u]_A$ is the mean value of u on the annulus $A := B(a,2r) \setminus B(a,r)$.

Throughout this section ε and p are constrained by the condition $\varepsilon = 2-p$.

Step 1. Choice of test functions. Set

$$G^k := |\nabla\tilde{u}|^{-\varepsilon} \nabla\tilde{u}^k \,, \qquad k = 1,2,3. \tag{2.4}$$

Note that G^k is a vector field of class $L^q(\mathbb{R}^2,\mathbb{R}^2)$ for all $1 \le q \le 2/(1-\varepsilon)$. In what follows, we restrict our attention only to

$$0 < \varepsilon < \frac{1}{4}, \qquad 2-\varepsilon \le q \le \frac{2}{1-\varepsilon} \,. \tag{2.5}$$

Applying Hodge decomposition, we write

$$G^k = \nabla v^k + \beta^k \,. \tag{2.6}$$

Here, $\beta^k \in L^q(\mathbb{R}^2,\mathbb{R}^2)$ is a divergence-free vector field, and v^k belongs to Sobolev class $W^{1,q}(\mathbb{R}^2)$ for all q satisfying (2.5). With no loss of generality we can also assume that

$$\fint_{B(a,2r)} v^k(x) \, dx = 0 \qquad \text{for } k = 1,2,3. \tag{2.7}$$

By well-known estimates, the inequality

$$\|\nabla v^k\|_{L^q(\mathbb{R}^2)} + \|\beta^k\|_{L^q(\mathbb{R}^2)} \le C\|G^k\|_{L^q(\mathbb{R}^2)} \tag{2.8}$$

holds for all $1 < q \le 2/(1-\varepsilon)$. (Papers of Iwaniec [16], Iwaniec–Martin [17], and Iwaniec–Sbordone [18], are an excellent source of information about L^q estimates for Hodge decomposition.) Moreover, by a standard application of Poincaré inequality,

$$\|G^k\|_{L^q} \le C\left(\int_{B(a,2r)} |\nabla u|^{(1-\varepsilon)q}\, dx\right)^{1/q}. \tag{2.9}$$

We shall use $\varphi = (\varphi^1, \varphi^2, \varphi^3)$, where $\varphi^k := \zeta v^k$, $k = 1, 2, 3$, as a test vector for the H-surface system (1.1). Since $\nabla\varphi$ is integrable with power $q_0 = 2/(1-\varepsilon) > 2 =$ dimension, Sobolev imbedding theorem implies that φ is Hölder continuous with exponent $1 - 2/q_0 = \varepsilon$, and, of course, bounded.

Before proceeding further, we have to make

A remark about constants. The constant $C = C_q$ in (2.8) depends on q. However, since we impose the restrictions (2.5), all exponents q for which inequality (2.8) shall ever be applied in this paper belong to some fixed interval, say $[\frac{7}{4}, \frac{8}{3}]$. Therefore, by Riesz-Thorin convexity theorem, one can use (2.8) with a fixed constant, e.g.

$$C = \max(C_{7/4}, C_{8/3})\,.$$

Tracing constants in Poincaré inequality, one can also check that for all q that satisfy (2.5) inequality (2.9) holds true with a constant C that does not depend on ε. Thus, assuming (2.5), we have

$$\|\nabla v^k\|_{L^q(\mathbb{R}^2)} \le C\left(\int_{B(a,2r)} |\nabla u|^{(1-\varepsilon)q}\, dx\right)^{1/q} \tag{2.10}$$

with some *absolute* constant C.

Step 2. Estimates of the test function. We shall need the following lemma.

Lemma 2.2 *The function $\varphi = \zeta v$, where v is defined by (2.6) and (2.4), satisfies the following inequalities:*

$$\|\nabla\varphi\|_{L^2(B(a,2r))} \le C r^\varepsilon J_p(a, 2r)^{(p-1)/p}\,, \tag{2.11}$$

$$\|\varphi\|_{L^\infty(B(a,2r))} \le C(\varepsilon) r^\varepsilon J_p(a, 2r)^{(p-1)/p}\,. \tag{2.12}$$

Remark. The constant in (2.12) blows up to infinity as ε tends to zero.

Proof of Lemma 2.2. We first prove (2.11). As $|\nabla\zeta| \le 2/r$ and $⨍_{B(a,2r)} v = 0$, we have

$$\begin{aligned}\|\nabla\varphi\|_{L^2(B(a,2r))} &\le \sup|\zeta|\, \|\nabla v\|_{L^2(B(a,2r))} + \frac{C}{r}\left(\int_{B(a,2r)} |v|^2\, dx\right)^{1/2} \\ &\le C\|\nabla v\|_{L^2(B(a,2r))}\end{aligned}$$

(by Poincaré inequality). Now, applying (2.10), and then Hölder inequality with exponents $\frac{p'}{2} = (2-\varepsilon)/(2-2\varepsilon)$ and $(\frac{p'}{2})' = (2-\varepsilon)/\varepsilon$, we obtain

$$\|\nabla v\|_{L^2(B(a,2r))} \le C\left(\int_{B(a,2r)} |\nabla u|^{(1-\varepsilon)2}\, dx\right)^{1/2}$$

$$\le C r^{(2-\varepsilon)/\varepsilon} \left(\int_{B(a,2r)} |\nabla u|^{2-\varepsilon}\, dx \right)^{(1-\varepsilon)/(2-\varepsilon)}$$
$$= C r^{\varepsilon} J_p(a,2r)^{(p-1)/p} .$$

To check (2.12) note that $|\varphi(x)| \le |v(x)|$. As v is continuous with zero mean value, $\|\varphi\|_{L^\infty(B(a,2r))} \le \left|\operatorname{osc}_{B(a,2r)} v\right|$. To estimate the oscillation, we employ the fact that $\nabla v \in L^{q_1}$ for $q_1 = p' = \frac{2-\varepsilon}{1-\varepsilon} > 2$. The Sobolev imbedding theorem, combined with estimate (2.10), yields

$$\begin{aligned}\left| \operatorname*{osc}_{B(a,2r)} v \right| &\le C(\varepsilon) r^{1-2/q_1} \|\nabla v\|_{L^{q_1}(B(a,2r))} \\ &\le C(\varepsilon) r^{1-2/q_1} \left(\int_{B(a,2r)} |\nabla u|^{(1-\varepsilon)q_1}\, dx \right)^{1/q_1} .\end{aligned}$$

As before, the right hand side is equal to $C(\varepsilon) r^{\varepsilon} J_p(a,2r)^{(p-1)/p}$. □

The same proof yields the following.

Corollary 2.3 *Both inequalities (2.11) and (2.12) of Lemma 2.2 are valid if one replaces φ by v.* □

Step 3. Estimates of the left-hand side. We compute (Latin indices are summed from 1 to 3, Greek indices from 1 to 2):

$$\begin{aligned}\int_\Omega \nabla u \nabla \varphi\, dx &= \int_{B(a,2r)} \frac{\partial u^k}{\partial x_\alpha} \frac{\partial \varphi^k}{\partial x_\alpha}\, dx \\ &= \int_{B(a,2r)} \frac{\partial \tilde u^k}{\partial x_\alpha} \frac{\partial v^k}{\partial x_\alpha}\, dx \\ &\quad + \int_{B(a,2r)} \frac{\partial \zeta}{\partial x_\alpha} \left(\frac{\partial u^k}{\partial x_\alpha} v^k - \frac{\partial v^k}{\partial x_\alpha} (u - [u]_A) \right) dx =: I_1 + I_2 .\end{aligned}$$

Since $\operatorname{div} \beta^k = 0$ in the sense of distributions,

$$\begin{aligned} I_1 &= \int_{B(a,2r)} \frac{\partial \tilde u^k}{\partial x_\alpha} \frac{\partial v^k}{\partial x_\alpha}\, dx \;=\; \int_{B(a,2r)} \frac{\partial \tilde u^k}{\partial x_\alpha} \left(\frac{\partial v^k}{\partial x_\alpha} + \beta^k_\alpha \right) dx \\ &= \int_{B(a,2r)} |\nabla \tilde u(x)|^p\, dx \;\ge\; \int_{B(a,r)} |\nabla u(x)|^p\, dx \end{aligned}$$

(in the second line we use the definition of G^k, v^k and β^k, and then shrink the domain of integration).

As $\nabla \zeta$ is supported only in the annulus A,

$$|I_2| \le \frac{C}{r} \int_A \Big(|v|\, |\nabla u| + |\nabla v|\, |u - [u]_A| \Big) dx .$$

We apply here Hölder inequality with exponents $p' = (2-\varepsilon)/(1-\varepsilon)$ and $p = 2 - \varepsilon = (1-\varepsilon)p'$, and then Poincaré inequality to estimate integrals of $|v|^{p'}$ and

$|u-[u]_A|^p$. A bound for $\int|\nabla v|^{p'}$ follows from the Hodge decomposition estimate (2.10) Finally, we use Cauchy inequality to separate integrals over the annulus from those over $B(a,2r)$. Here is the whole (routine) computation:

$$\begin{aligned}|I_2| &\le \frac{C}{r}\left[\left(\int_{B(a,2r)}|v|^{p'}\,dx\right)^{1/p'}\left(\int_A|\nabla u|^p\,dx\right)^{1/p}\right.\\ &\qquad\left.+\left(\int_{B(a,2r)}|\nabla v|^{p'}\,dx\right)^{1/p'}\left(\int_A|u-[u]_A|^p\,dx\right)^{1/p}\right]\\ &\le C\left(\int_{B(a,2r)}|\nabla v|^{p'}\,dx\right)^{1/p'}\left(\int_A|\nabla u|^p\,dx\right)^{1/p}\\ &\le C\left(\int_{B(a,2r)}|\nabla u|^{p}\,dx\right)^{1/p'}\left(\int_A|\nabla u|^p\,dx\right)^{1/p}\\ &\le C\int_A|\nabla u|^p\,dx+\frac{1}{8}\int_{B(a,2r)}|\nabla u|^p\,dx\,. \end{aligned}\tag{2.13}$$

Both estimates of I_1 and I_2 yield

$$\int_\Omega\nabla u\nabla\varphi\,dx\ge\int_{B(a,r)}|\nabla u|^p\,dx-C_0\int_A|\nabla u|^p\,dx-\frac{1}{8}\int_{B(a,2r)}|\nabla u|^p\,dx\,,\tag{2.14}$$

with some absolute constant C_0. (Note that no estimates of the divergence free field β^k were necessary.)

Step 4. Estimates of right-hand side. This is the heart of proof. The integral

$$-2\int_\Omega H(u)\varphi u_{x_1}\wedge u_{x_2}\,dx$$

is a linear combination (with coefficients ± 2) of terms

$$J_{lkj}=\int_\Omega H(u)\varphi^l\det\left(Du^k,Du^j\right)dx\,,$$

with (l,k,j) being a permutation of $(1,2,3)$. We show how to estimate $J=J_{123}$; other terms can be handled in the same way. For sake of brevity, let $w:=H(u)\varphi^1$. Integrating by parts, we obtain

$$|J|=\left|\int_{\mathbb{R}^2}\zeta_1(u^2-[u^2]_{a,2r})\det\left(Dw,Du^3\right)dx\right|$$

where ζ_1 is a function of class $C_0^\infty(B(a,3r))$ with $\zeta_1\equiv 1$ on $B(a,2r)$ and $|\nabla\zeta_1|\le 2/r$. To bound this integral, we employ a theorem of Coifman, Lions, Meyer and Semmes [3, Theorem II.1] combined with the duality of Hardy space and the space of functions of bounded mean oscillation [23, Chapter IV]. This yields[1]

$$\begin{aligned}|J| &\le C\|\zeta_1(u^2-[u^2]_{a,2r})\|_{\mathrm{BMO}(\mathbb{R}^2)}\,\|\det\left(Dw,Du^3\right)\|_{\mathcal{H}^1(\mathbb{R}^2)}\\ &\le C\|\zeta_1(u-[u]_{a,2r})\|_{\mathrm{BMO}(\mathbb{R}^2)}\,\|Dw\|_{L^2(B(a,2r))}\,\|\nabla u\|_{L^2(B(a,2r))}\end{aligned}\tag{2.15}$$

[1] The reasoning requires some care; one first extends the gradients of w and u^3 to the whole space, without increasing their norms too much, and then applies Theorem II.1 of [3] and Fefferman duality theorem [9]. For a more detailed explanation, we refer to [24], [22].

Employing Lemma 2.2, we quickly check that

$$\begin{aligned}\|Dw\|_{L^2(B(a,2r))} &\le \|H\|_\infty \|\nabla\varphi\|_{L^2(\mathbb{R}^2)} + \|\varphi\|_\infty \|\nabla(H\circ u)\|_{L^2(B(a,2r))}\\ &\le C(H,\varepsilon) r^\varepsilon (1+\|\nabla u\|_{L^2(B(a,2r))}) J_p(a,2r)^{(p-1)/p}\\ &\le C(H,\varepsilon) r^\varepsilon \mathcal{M}_p(a,4r)^{(p-1)/p}\,. \end{aligned}\tag{2.16}$$

(With no loss of generality one can assume here that $\int_{B(a,2r)} |\nabla u|^2\,dx \le 1$. Such estimate holds anyway when $2r < R_0$, with R_0 to be specified later.) We also claim that

$$\|\zeta_1(u-[u]_{a,2r})\|_{\mathrm{BMO}(\mathbb{R}^2)} \le C\mathcal{M}_p(a,4r)^{1/p}\,. \tag{2.17}$$

Inserting (2.17) and (2.16) into the right-hand side of (2.15), and estimating other terms J_{lkj} in the same way, we obtain

$$2\left|\int_\Omega H(u)\varphi u_{x_1}\wedge u_{x_2}\,dx\right| \le C_1(H,\varepsilon) r^\varepsilon \mathcal{M}_p(a,4r)\|\nabla u\|_{L^2(B(a,2r))}\,, \tag{2.18}$$

with $C_1(H,\varepsilon)$ that is finite for every positive ε but blows up to infinity as ε goes to zero. It remains to verify (2.17).

Step 5. Proof of the BMO estimate (2.17). Recall that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^2)} := \sup_{z\in\mathbb{R}^2,\rho>0} \fint_{B(z,\rho)} \big|f-[f]_{z,\rho}\big|\,dx\,.$$

Let $f = \zeta_1(u-[u]_{a,2r})$. We consider two cases. If $\rho > r/2$, then

$$\fint_{B(z,\rho)} \big|f-[f]_{z,\rho}\big|\,dx \le \frac{2}{\pi\rho^2}\int_{B(z,\rho)} |f|\,dx \le \frac{8}{\pi r^2}\int_{B(a,3r)} \big|u-[u]_{a,2r}\big|\,dx\,.$$

By Poincaré and Hölder inequalities, the last integral does not exceed $\mathcal{M}_p(a,3r)^{1/p}$.

If, on the other hand, $\rho \le r/2$, then we can assume that $B(z,\rho)\subset B(a,4r)$, since otherwise f vanishes on $B(z,\rho)$. Set $g = u-[u]_{a,2r}$. For $x\in B(z,\rho)$, by triangle inequality,

$$\begin{aligned}|f(x)-[f]_{z,\rho}| &\le |\zeta_1(x)g(x)-\zeta_1(x)[g]_{z,\rho}| + |\zeta_1(x)[g]_{z,\rho} - [\zeta_1 g]_{z,\rho}|\\ &\le |g(x)-[g]_{B(z,\rho)}| + \frac{C}{r\rho}\int_{B(z,\rho)} |g|\,dy \end{aligned}\tag{2.19}$$

(note that $|\zeta_1(x)-\zeta_1(y)| \le C|x-y|/r$). As $\nabla g = \nabla u$, we have, by Poincaré and Hölder inequalities,

$$\fint_{B(z,\rho)} |g-[g]_{z,\rho}|\,dx \le C\left(\rho^{p-2}\int_{B(z,\rho)} |\nabla g|^p\,dx\right)^{1/p} \le C\mathcal{M}_p(a,4r)^{1/p}. \tag{2.20}$$

The second term on the right-hand side of (2.19), $C(r\rho)^{-1}\int_{B(z,\rho)}|g|\,dy$, is simply constant, and does not change after averaging. Applying consecutively Schwarz, Poincaré–Sobolev, and Hölder inequalities, we estimate it as follows:

$$\begin{aligned}\frac{C}{r\rho}\int_{B(z,\rho)}|g|\,dy &\le \frac{C}{r}\Big(\int_{B(z,\rho)}|g|^2\,dy\Big)^{1/2}\\ &\le \frac{C}{r}\Big(\int_{B(a,4r)}|u-[u]_{a,2r}|^2\,dy\Big)^{1/2}\\ &\le \frac{C}{r}\int_{B(a,4r)}|\nabla u|\,dy \;\le\; C\mathcal{M}_p(a,4r)^{1/p}\,. \qquad (2.21)\end{aligned}$$

The desired BMO estimate (2.17) follows at once from (2.20) and (2.21).

Step 6. Choice of ε_0. We combine inequalities (2.14) and (2.18). Dividing both sides of the resulting inequality by $r^{\varepsilon}=r^{2-p}$, we obtain

$$\begin{aligned}J_p(a,r) \le \frac{1}{4}J_p(a,2r) &+ C_0\Big(2^{\varepsilon}J_p(a,2r)-J_p(a,r)\Big)\\ &+ C_1(\varepsilon)\mathcal{M}_p(a,4r)\|\nabla u\|_{L^2(B(a,2r))}\end{aligned} \qquad (2.22)$$

Fix $\varepsilon_0>0$ so that $2^{\varepsilon_0}C_0\le C_0+\frac{1}{4}$. Thus, for all $\varepsilon\in(0,\varepsilon_0)$, we have

$$(C_0+1)J_p(a,r)\le\Big(C_0+\frac{1}{2}\Big)J_p(a,2r)\;+\;C_1(\varepsilon)\mathcal{M}_p(a,4r)\|\nabla u\|_{L^2(B(a,2r))}\,.$$

Choosing $R_0(\varepsilon)$ so small that $\|\nabla u\|_{L^2(B(a,2r)\cap\mathbb{B})}\le\big(4C_1(\varepsilon)\big)^{-1}$ for every $a\in\mathbb{B}$ and every $r<\frac{1}{2}R_0(\varepsilon)$, we obtain for such radii

$$J_p(a,r)\le\lambda_0\mathcal{M}_p(a,4r)\qquad\text{with}\quad\lambda_0=\frac{C_0+\frac{3}{4}}{C_0+1}<1.$$

For any smaller ball $B(z,\rho)\subset B(a,r)$ we have $B(z,4\rho)\subset B(a,4r)$. Thus,

$$J_p(z,\rho)\le\lambda_0\mathcal{M}_p(z,4\rho)\le\lambda_0\mathcal{M}_p(a,4r)\,.$$

Upon taking the supremum over all $B(z,\rho)\subset B(a,r)$, we obtain (2.3). □

Fixing an $\varepsilon\in(0,\varepsilon_0)$, where ε_0 is specified in Lemma 2.1, and using the decay inequality (2.3), one shows, via a standard iterative argument, that for each compact subset $K\subset\Omega$ there exists a constant C_K such that

$$\int_{B(a,\rho)}|\nabla u|^{2-\varepsilon}\,dy\le C_K\rho^{\varepsilon+\mu}\qquad\text{for all } a\in K \text{ and all } \rho<\operatorname{dist}(K,\partial\Omega). \qquad (2.23)$$

Here, $\mu=\log_4(1/\lambda_0)>0$, where λ_0 is the constant from Lemma 2.1. Hence, the assumptions of Morrey's Dirichlet Growth lemma (see, e.g., Giaquinta's monograph [10], Chapter 3, pages 64–65) are satisfied. Thus it follows that u is (locally) of class C^γ with $\gamma := \mu/(2-\varepsilon)$. More precisely, when $0 < r \le R \le R_0$ and $R \le 1 - |a|$, the following scale invariant inequalities hold:

$$\mathcal{M}_p(a,r) \le \lambda_0^{-1}\left(\frac{r}{R}\right)^\mu \mathcal{M}_p(a,R)\,, \tag{2.24}$$

$$|u(x)-u(y)| \le C\left(\frac{|x-y|}{R}\right)^\gamma \mathcal{M}_p(a,R)^{1/p}, \qquad x,y \in B(a,\tfrac{R}{2}). \tag{2.25}$$

Interior Hölder continuity follows.

The proof of higher interior regularity, Corollary 1.2, is standard. First, one shows that $u \in C^{1,\gamma}$ for some $\gamma > 0$. This follows e.g. from a theorem of Tomi [25]. Then, $C^{2,\beta}$ regularity of u (with any $\beta < 1$) follows from a classical L^q and potential theoretic bootstrap reasoning. We omit the details.

3 Boundary regularity

We begin with the following simple lemma. It is a counterpart of [15, Lemma 3], adapted to our purposes: one assumes that Morrey type estimates hold *below* natural exponents.

Lemma 3.1 *Let $u \in W^{1,2}(\mathbb{B},\mathbb{R}^N)$. Assume that for some $p \le 2$ and some $\mu > 0$ the inequality*

$$\mathcal{M}_p(a,r;u) \le C\left(\frac{r}{R}\right)^\mu \mathcal{M}_p(a,R;u)$$

holds for all $a \in \mathbb{B}$ and for all $0 < r < R \le \min(R_0, 1-|a|)$, where R_0 is a fixed positive number. If the trace $\psi\colon = u\big|_{\partial\mathbb{B}}$ is continuous on $\partial\mathbb{B}$, then $u \in C^0(\overline{\mathbb{B}},\mathbb{R}^N)$.

Proof. By Morrey's Dirichlet Growth Theorem, u is Hölder continuous in the interior of the unit disc $\mathbb{B}$. Moreover, the scale invariant inequality (2.25) holds. It remains to show that u is continuous at boundary points.

Write $u(x_1,x_2) = v(\rho,\theta)$, where (ρ,θ) denote standard polar coordinates. Fix a point $y_0 \in \partial\mathbb{B}$, $y_0 = (1,\theta_0)$ in polar coordinates. Set

$$\zeta(\theta)\colon = \psi(\exp(i\theta))\,.$$

Let ω denote the modulus of continuity of ζ. Define

$$\Sigma_p(\delta)\colon = \sup\{\mathcal{M}_p(a,\delta)^{1/p} : \mathrm{dist}\,(a,\partial\mathbb{B}) \ge \delta\}\,.$$

Recall that $\mathcal{M}_p(a,r) \le \mathrm{const}\left(\int_{B(a,r)} |\nabla u|^2\,dx\right)^{p/2}$. Hence, $\Sigma_p(\delta) \to 0$ as $\delta \to 0$.

Let $x = (x_1,x_2) = \rho\exp(i\theta)$ be an interior point of $\mathbb{B}$. We assume that $1-\rho < R_0$. Put $\delta = 1-\rho$. Note that for any $0 < \sigma < 2\pi$ we have

$$\int_\theta^{\theta+\sigma}\int_{1-\delta}^1 |v_r(r,\vartheta)|^2\,r\,dr\,d\vartheta \le \int\limits_{\{x\,:\,1-\delta\le|x|\le1\}} |\nabla u|^2\,dx \;=:\; I(\delta)\,. \tag{3.1}$$

Thus,

$$\int_{\theta}^{\theta+\sigma}\int_{1-\delta}^{1} |v_r(r,\vartheta)|^2\,dr d\vartheta \le \frac{I(\delta)}{1-\delta}\,.$$

Therefore, there exists a set $E_\sigma \subset (\theta, \theta+\sigma)$ having positive one-dimensional Lebesgue measure and such that

$$\int_{1-\delta}^{1} |v_r(r,\theta_1)|^2\,dr \le \frac{I(\delta)}{\sigma(1-\delta)}\,, \qquad \lim_{r\to 1^-} v(r,\theta_1) = \zeta(\theta_1) \tag{3.2}$$

for every $\theta_1 \in E_\sigma$. For such θ_1, applying Schwarz inequality and (3.2), we obtain

$$\begin{aligned} |\zeta(\theta_1) - v(\rho,\theta_1)| = |v(1,\theta_1) - v(\rho,\theta_1)| \;&\le\; \int_\rho^1 |v_r(r,\theta_1)|\,dr \qquad (3.3)\\ &\le (1-\rho)^{1/2}\left(\int_\rho^1 |v_r(r,\theta_1)|^2\,dr\right)^{1/2}\\ &\le \left(\frac{\delta}{\sigma}\right)^{1/2}\frac{\sqrt{I(\delta)}}{(1-\delta)^{1/2}}\,. \end{aligned}$$

We set now $\sigma = \frac{1}{4}\delta$. By (2.25), for any point $x' \in B(x, \frac{1}{2}\delta)$ one has

$$|u(x) - u(x')| \le C_0\left(\frac{|x-x'|}{\delta}\right)^{\gamma} \mathcal{M}_p(x,\delta)^{1/p} \le C_0\left(\frac{|x-x'|}{\delta}\right)^{\gamma} \Sigma_p(\delta)\,. \tag{3.4}$$

Pick $x' \in B(x, \frac{1}{2}\delta)$ with radial coordinate equal to that of x, and angular coordinate $\theta_1 \in E_\sigma$. By triangle inequality,

$$|u(x) - \psi(y_0)| \le |u(x) - u(x')| + |u(x') - \psi(x'/|x'|)| + |\psi(x'/|x'|) - \psi(y_0)|\,.$$

We estimate the first and the second term on the right-hand side, applying (3.4) and (3.3), respectively. Finally, continuity of ζ and choice of θ_1 yield

$$|\psi(x'/|x'|) - \psi(y_0)| = |\zeta(\theta_1) - \zeta(\theta_0)| \le \omega(|\theta_1 - \theta_0|) \le \omega\left(|\theta - \theta_0| + \tfrac{\delta}{4}\right)\,.$$

Combining all estimates, and plugging in $\delta = 1 - \rho$, we obtain

$$|u(x) - \psi(y_0)| \le C_0 \Sigma_p(1-\rho) + 2\left(\frac{I(1-\rho)}{\rho}\right)^{1/2} + \omega\left(|\theta-\theta_0| + \tfrac{1-\rho}{4}\right)\,.$$

For $x \to y_0$, i.e. for $\rho \to 1$ and $\theta \to \theta_0$, the right hand side of the last inequality tends to zero. The proof is complete now. $\square$

I am grateful to Stefan Hildebrandt who carefully explained to me a simple proof of this lemma in the case $p = 2$.

By Lemma 2.1 of the previous Section, all weak solutions of the H-surface system satisfy uniform Morrey-type decay estimates on (small) balls contained in $\mathbb{B}$. Therefore, every weak solution u which has continuous trace on $\partial\mathbb{B}$, automatically satisfies the assumptions of Lemma 3.1 for some $p > 2$. Theorem 1.3 follows immediately.

4 Remarks and comments

4.1 An application to harmonic maps

The method described in detail in Section 2 can be also applied to obtain a new proof of regularity of harmonic maps from surfaces into compact Riemannian manifolds, [13], [14, Chapter 4]. The original proof of Hélein relies on the properties of Lorentz spaces. The construction of test functions through the Hodge decomposition of $|\nabla u|^{-\varepsilon}\nabla u$ (or, strictly speaking, of the coefficients of its projections onto the tangent space of image manifold) can replace Lorentz spaces also in this case. We sketch the reasoning briefly.

Let $u \in W^{1,2}(\mathbb{B}, \mathcal{N})$ be a weakly harmonic map from the two dimensional disc $\mathbb{B}$ into a smooth, compact, closed n-dimensional Riemannian manifold $\mathcal{N}$. We assume, as it is usually done, that $\mathcal{N}$ is isometrically embedded in a Euclidean space $\mathbb{R}^d$. Weak harmonicity of u means that $\Delta u \perp T\mathcal{N}$, or, to be precise, that

$$\sum_{\alpha=1}^{2} \int_{\mathbb{B}} \left\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial \varphi}{\partial x_\alpha} \right\rangle dx = 0 \tag{4.1}$$

for every $\varphi \in L^\infty \cap W_0^{1,2}(\mathbb{B}, \mathbb{R}^d)$ with $\varphi(x) \in T_{u(x)}\mathcal{N}$ a.e.

Theorem 4.1 (Hélein) *Every weakly harmonic map $u \in W^{1,2}(\mathbb{B}, \mathcal{N})$ is continuous (and therefore smooth).*

It is known [14, Lemma 4.1.2] that without loss of generality one can assume that $\mathcal{N}$ is diffeomorphic to an n-dimensional torus. Then, to prove Theorem 4.1 via the method of Section 2, one checks that the conclusion of Lemma 2.1 holds true also for weakly harmonic maps. We fix $a \in \mathbb{B}$ and $r < \frac{1}{4}\mathrm{dist}\,(a, \partial\mathbb{B})$. To construct test functions for the system (4.1), we employ the Coulomb moving frame [14, Lemma 4.1.3], i.e. a map

$$e = (e_1, e_2, \ldots, e_n) : B(a, 4r) \to \mathbb{R}^{n\times d}$$

that satisfies the following conditions:

(i) $(e_i(x))_{i=1,\ldots,n}$ is an orthonormal basis of $T_{u(x)}\mathcal{N}$ for a.e. $x \in B(a, 4r)$;

(ii) $\int_{B(a,4r)} |\nabla e|^2 dx \le C \int_{B(a,4r)} |\nabla u|^2 dx$;

(iii) $\sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha} \left\langle \frac{\partial e_i}{\partial x_\alpha}, e_j \right\rangle = 0$ for all $i, j = 1, \ldots, n$.

Lorentz space estimates for de, [14, Lemma 4.1.7], are not necessary here.

Test functions and decay estimates. As in Section 2, we set $A = B(a, 2r)\backslash B(a, r)$ and $\tilde{u} = \zeta(u-[u]_A)$, where $\zeta \in C_0^\infty(B(a, 2r))$, $\zeta \equiv 1$ on $B(a, r)$, and $|\nabla\zeta| \le 2/r$. Introduce differential forms

$$\omega_i := \sum_{\alpha=1}^{2} |d\tilde{u}|^{-\varepsilon} \left\langle \frac{\partial \tilde{u}}{\partial x_\alpha}, e_i \right\rangle dx_\alpha \equiv |d\tilde{u}|^{-\varepsilon} \langle d\tilde{u}, e_i \rangle \,, \qquad i = 1, ..., n,$$

and apply the Hodge decomposition [17, Theorem 6.1] to write

$$\omega_i = dv_i + d^*\beta_i ,$$

where $v_i \in W^{1,p'}(\mathbb{R}^2)$, $\beta_i \in W^{1,p'}(\mathbb{R}^2, \Lambda^2)$, and p' stands for the Hölder conjugate of $p = 2 - \varepsilon$.

By (i) above, we have

$$\int_{B(a,r)} |du|^p \, dx \le \sum_{i=1}^{n} \int_{\mathbb{R}^2} \langle d\tilde{u}, e_i \rangle \cdot (dv_i + d^*\beta_i) \, ;$$

the dot denotes the standard scalar product of 1-forms. Now,

$$\left| \int_{\mathbb{R}^2} \langle d\tilde{u}, e_i \rangle \cdot d^*\beta_i \right| \le C_0 \|du\|_{L^2(B(a,2r))} \|du\|^p_{L^p(B(a,2r))} .$$

This follows from the duality of Hardy space and BMO combined with [3, Theorem II.1]. (One has to take the $L^{p'}$-bounds of $d^*\beta_i$ into account, see [17].) To estimate the crucial term

$$T_i := \int_{\mathbb{R}^2} \langle d\tilde{u}, e_i \rangle \cdot dv_i \, ,$$

we employ (4.1) with $\varphi = \zeta(v_i - \mathrm{const})e_i$. One checks that

$$T_i = -\sum_{\alpha,j} \int_{\mathbb{R}^2} \left\langle \frac{\partial u}{\partial x_\alpha}, \left\langle \frac{\partial e_i}{\partial x_\alpha}, e_j \right\rangle e_j \right\rangle \zeta(v_i - \mathrm{const}) \, dx$$
$$+ \text{ integrals (over } A\text{) that contain derivatives of } \zeta.$$

Performing one integration by parts, we can repeat almost verbatim the proofs of inequalities (2.13) and (2.18) to verify that

$$|T_i| \le C_1 \int_A |du|^p \, dx + C_2(\varepsilon) r^\varepsilon \mathcal{M}_p(a, 4r) \|\nabla u\|_{L^2(B(a,4r))} .$$

Next, we proceed as in the last part proof of Section 2 to obtain interior Morrey type decay estimates. This yields Hölder continuity; higher regularity of u follows from a classical bootstrap reasoning.

4.2 Problems involving the n-Laplace operator

If, instead of assuming that H is just Lipschitz and bounded, one adds a decay assumption

$$(1 + |y|)|\nabla H(y)| \le C, \tag{4.2}$$

(which was used already by E. Heinz in [12]), then the whole proof of local regularity of weak solutions of (1.1) can be shortened drastically. Namely, one checks that $(u - \mathrm{const})H(u)$ belongs to BMO, and therefore $\zeta(u - \mathrm{const})$ can be used as a test function. In this case, one obtains directly a decay estimate for $\int_{B_r} |\nabla u|^2$ (see [24], where a related hole-filling trick is explained carefully).

In fact, assuming (4.2), one can prove the following.

Theorem 4.2 *Let $H\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $n > 2$, be a bounded Lipschitz function. Assume that (4.2) holds for all $y \in \mathbb{R}^{n+1}$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^1.*

Then, every weak solution $u \in W^{1,n}(\Omega, \mathbb{R}^{n+1})$ of the system

$$\mathrm{div}\,(|\nabla u|^{n-2}\nabla u) = H(u)u_{x_1} \wedge \ldots \wedge u_{x_n} \tag{4.3}$$

is locally Hölder continuous in Ω, and if the trace $u\big|_{\partial\Omega}$ is continuous, then u is continuous up to the boundary of Ω.

It is known that under less demanding assumptions on H one can prove regularity for *bounded* weak solutions (Duzaar and Fuchs [5]) or *conformal* weak solutions (Mou and Yang [21]). For *minimizing* weak solutions a similar theorem has been obtained by Duzaar and Grotowski [6]. Mou and Yang conjecture that weak solutions are of class $C^{1,\alpha}$ if $H \circ u$ is just bounded, and the problem is wide open for $n > 2$. (For $n = 2$ the question is also not yet fully settled; see, however, the paper of Bethuel and Ghidaglia [2].)

To prove Theorem 4.2, one takes $\zeta(u - \mathrm{const})$ as a test function. By (4.2), $\zeta(u - \mathrm{const})H(u)$ is of class $W^{1,n} \subset \mathrm{BMO}$, and the hole-filling argument [24] can be carried out. This leads to interior Hölder continuity. Then, continuity up to the boundary can be obtained upon introducing minor changes to the proof from Section 3. We leave the details as an exercise for interested readers.

(The method of Section 2 breaks down. It is easy to see that the power of maximal function $\mathcal{M}_p$ which enters into the estimates of the right hand side is equal to $\frac{1}{n-\varepsilon} + \frac{1-\varepsilon}{n-\varepsilon}$. For $n > 2$ this exponent is smaller than 1, and the resulting inequality cannot be iterated in a reasonable way.)

A similar reasoning implies that a counterpart of Theorem 4.2 holds for weak solutions $u \in W^{1,n}(\Omega, \mathbb{R}^{n+1})$ of the system

$$\mathrm{div}\,(|\nabla u|^{n-2}\nabla u) = H(x)u_{x_1} \wedge \ldots \wedge u_{x_n}, \tag{4.4}$$

where H is a function of class $W^{1,q}(\mathbb{R}^n)$ for some $q > n$. Again, we omit the details.

We hope that modifications of the method of Section 2, combined with a stability result of Iwaniec [16, Theorem 8.2], can be used to cope with regularity questions for other nonlinear conformally invariant problems involving the n-Laplacian.

References

1. Bethuel, F.: Un résultat de régularité pour des solutions de l'equation des surfaces à courbure moyenne prescrite. C. R. Acad. Sci. Paris **314** (1992), 1003–1007.
2. Bethuel, F., Ghidaglia, J.-M.: Improved regularity of solutions to elliptic equations involving Jacobians and applications. J. Math. Pures Appl. **72** (1993), 441–474.
3. Coifman, R., Lions, P. L., Meyer, Y., Semmes, S.: Compensated compactness and Hardy spaces. J. Math. Pures Appl. **72** (1993), 247–286.
4. Dierkes, U., Hildebrandt, S., Küster, A., Wohlrab, O.: Minimal Surfaces, vol. I and II. Grundlehren der mathematischen Wissenschaften 295, 296. Springe-Verlag 1992.
5. Duzaar, F., Fuchs, M.: Einige Bemerkungen über die Regularität von stationären Punkten gewisser geometrischer Variationsintegrale. Math. Nachr. **152** (1991), 39–47.

6. Duzaar, F., Grotowski, J.F.: Existence and regularity for higher dimensional H-systems. Duke Math. J. **101** (2000), 459–489.
7. Duzaar, F., Steffen, K.: The Plateau problem for parametric surfaces with prescribed mean curvature. Geometric analysis and the calculus of variations, 13–70, Internat. Press, Cambridge, MA, 1996.
8. Fefferman, C., Stein, E.: H^p spaces of several variables, Acta Math. **129** (1972), 137–193.
9. Fefferman, C.: Characterizations of bounded mean oscillation, Bull. AMS **77** (1971), 585–587.
10. Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Annals of Mathematics Studies, 105. Princeton University Press, Princeton, NJ, 1983.
11. Grüter, M.: Regularity of weak H-surfaces. J. Reine Angew. Math. **329** (1981), 1–15.
12. Heinz, E.: Ein Regularitätssatz für schwache Lösungen nichtlinearer elliptischer Systeme. Nachr. Akad. Wiss. Göttingen, Math.–Ph. Klasse, 1975, 1–13.
13. Hélein, F.: Regularité des applications faiblement harmoniques entre une surface et une variètè riemannienne, C. R. Acad. Sci. Paris **312** (1991), 591–596.
14. Hélein, F.: Harmonic maps, conservation laws and moving frames. Editions Frontieres, Diderot Publishers, Paris 1997.
15. Hildebrandt, S., Kaul, H.: Two-dimensional variational problems with obstructions, and Plateau's problem for H-surfaces in a Riemannian manifold. Comm. Pure Appl. Math. **25** (1972), 187–223.
16. Iwaniec, T.: p-harmonic tensors and quasiregular mappings. Ann. of Math. **136** (1992), no. 3, 589–624.
17. Iwaniec, T., Martin, G.: Quasiregular mappings in even dimensions. Acta Math. **170** (1993), no. 1, 29–81.
18. Iwaniec, T., Sbordone, C.: Weak minima of variational integrals. J. Reine Angew. Math. **454** (1994), 143–161.
19. Jakobowsky, N.: On interior and boundary regularity of weak solutions to a certain quasilinear elliptic system. Math. Z. **209** (1992), 1–15.
20. Maz'ya, V.G.: Sobolev spaces. Springer-Verlag 1985.
21. Mou, L.; Yang, P.: Multiple solutions and regularity of H-systems. Indiana Univ. Math. J. **45** (1996), no. 4, 1193–1222.
22. Semmes, S.: A primer on Hardy spaces and some remarks on a theorem of Evans and Müller. Comm. P.D.E. 19 (1994), 277–319.
23. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory integrals. Princeton University Press, Princeton, New Jersey 1993.
24. Strzelecki, P.: Regularity of p-harmonic maps from p-dimensional ball into a sphere. Manuscr. Math. **82** (1994), 407–415.
25. Tomi, F.: Ein einfacher Beweis eines Regularitässatzes für schwache Lösungen gewisser elliptischer Systeme. Math. Z. **112** (1969), 214–218.
26. Wente, H.: An existence theorem for surfaces of constant mean curvature. J. Math. Anal. Appl. **26** (1969), 318-344.
27. Wang, C.: Regularity of high-dimensional H-systems. Nonlinear Anal. **38** (1999), 675–686.

Note added in proof. When this paper has already been accepted for publication, the author has learned that the first part of Theorem 4,2 (interior regularity) had been obtained earlier by Wang [27].