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A Sharp Nonlinear Gagliardo–Nirenberg-Type Estimate and Applications to the Regularity of Elliptic Systems

TRISTAN RIVIÈRE[1] AND PAWEŁ STRZELECKI[2]

[1]Departement Mathematik, ETH Zentrum, Zürich, Switzerland
[2]Institute of Mathematics, Warsaw University, Warszawa, Poland

We prove the inequality

$$\int |\nabla u|^{s+2}\, dx \leq C(n, s)\|u\|_{BMO}^2 \int |\nabla u|^{s-2}|\nabla^2 u|^2\, dx, \quad s \geq 2,$$

and to give a sample of possible applications, we show how it can be used to obtain ε-regularity results for the solutions of a wide class of nonlinear degenerate elliptic systems

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = G(x, u, \nabla u),$$

where G grows as $|\nabla u|^p$.

Keywords *BMO*; Gagliardo–Nirenberg inequality; p-harmonic map; ϵ-regularity.

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1. Introduction

The seminal paper of Coifman et al. (1993) has triggered numerous new applications of Hardy spaces and the space *BMO* of functions of bounded mean oscillation to nonlinear partial differential equations. These applications include various problems of variational and geometric origin: harmonic and p-harmonic maps, H-systems, wave maps; (see, e.g., Bethuel, 1993; Evans, 1991; Hélein, 1998, 2002; Toro and Wang, 1995) and *many* others. In a vague sense, a common feature of all those works is that the duality of *BMO* and the Hardy space $\mathscr{H}^1$ are much more subtle than the duality of L^∞ and L^1. In many instances, the local *BMO* norm of a function

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Address correspondence to Tristan Riviere, Departement Mathematik, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland; Fax: 41-1-632-1085; E-mail: riviere@math.ethz.ch

or mapping is small, while the L^∞ norm is just finite. Moreover, in the critical case $kp = n$ of Sobolev imbedding theorem, $W^{k,p}$ is not contained in L^∞; we have instead, $W^{k,p} \subset BMO$ (in fact, $W^{k,p} \subset VMO$, i.e., Sobolev functions have vanishing mean oscillations in the borderline cases; this follows easily from the Poincaré inequality).

Recently, Meyer and Rivière (2003) have proved the interpolation inequality

$$\|\nabla f\|^2_{L^4(\Omega)} \le C(\Omega)\|f\|_{BMO(\Omega)}\|f\|_{W^{2,2}(\Omega)}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain (see also Cohen et al., 1998 for a related improved version of the Sobolev inequality). This inequality is sharper than the corresponding case of the classical Gagliardo–Nirenberg inequality

$$\|\nabla f\|^2_{L^4}(\Omega) \le C(\Omega)\|f\|_{L^\infty(\Omega)}\|f\|_{W^{2,2}(\Omega)};$$

this is why it was possible to use (1.1) to obtain the main result of Meyer and Rivière (2003), a theorem on partial regularity of a class of Yang–Mills connections.

The proof of (1.1) in Meyer and Rivière (2003) employs Littlewood–Paley decomposition, interpolation in Triebel–Lizorkin spaces F^s_{pq}, and the link between BMO and homogeneous Besov spaces. Pumberger generalized this argument to show that

$$\|\nabla f\|^2_{L^{2p}(\Omega)} \le C(p, \Omega)\|f\|_{BMO(\Omega)}\|f\|_{W^{2,p}(\Omega)}, \quad p > 1. \tag{1.2}$$

He also used (1.2) to prove that the singular set $\operatorname{Sing} u$ of a stationary harmonic map $u \in W^{1,p}(\Omega, \mathcal{N})$, $p > 2$, satisfies $H^{n-p}(\operatorname{Sing} u) = 0$.

Different, simpler proofs of (1.1) and (1.2) have been given in Strzelecki (2003), where the argument is based on the Fefferman duality theorem $(\mathcal{H}^1)^* = BMO$ and the well-known trick used by Coifman et al. (1993) to prove that $\det Df \in \mathcal{H}^1$ for every mapping $f \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$.

In this paper, we obtain another interpolation inequality of this type. In the simplest case of compactly supported smooth functions on $\mathbb{R}^n$, it reads

$$\int_{\mathbb{R}^n} |\nabla u|^{s+2}\,dx \le C(n, s)\|u\|^2_{BMO}\int_{\mathbb{R}^n} |\nabla u|^{s-2}|\nabla^2 u|^2\,dx, \quad s \ge 2. \tag{1.3}$$

A more precise local variant, which forms one of the main results of this paper, is stated below, in Section 3, as Theorem 3.1. It can be viewed as a sharp borderline case of interpolation inequalities due to Gagliardo, Nirenberg—and also Campanato (1982), who proved in Cor. 3II a weaker version of (1.2): if $u \in W^{2,p} \cap BMO$, then ∇u is in the Marcinkiewicz space weak–L^{2p} (see his later paper Campanato (1982) for similar results and their applications to elliptic systems).

Moreover, we give an application of (1.3) to nonlinear elliptic systems of the form

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u^i\right) = G^i(x, u, \nabla u), \quad i = 1, \ldots, N, \tag{1.4}$$

where $u = (u^1, \ldots, u^N)\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$, and $p > 2$. Throughout the whole paper we assume that

$$G\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$$

is of class C^1 and satisfies, for some constant Λ, the growth condition

$$|G(x, u, \nabla u)| \leq \Lambda |\nabla u|^p \tag{1.5}$$

(i.e., G grows critically with the gradient of u).

This is a wide class of systems that includes p-harmonic maps into compact Riemannian manifolds and H-systems in higher dimensions.

We are interested in the so-called ε-regularity of those weak solutions u that belong to $W^{2,p}$ *and* have bounded mean oscillations. (These assumptions are quite strong. However, note that we just assume (1.5); no other structure conditions are imposed.)

It is well known that for numerous nonlinear elliptic problems "small scaled energy of a solution u [on some ball] implies its regularity [on a smaller ball]"; see, e.g., Evans (1991) and Bethuel (1993) for stationary harmonic maps, or Toro and Wang (1995), and Strzelecki (1996) for stationary p-harmonic maps. In these papers, smallness of energy yields, via the monotonicity formulae, smallness of BMO norm of u, and this in turn leads to Morrey-type inequalities for the gradient. One also has, however, to show that the right-hand side of the system has special, determinant-like structure, and this is usually a nontrivial task. For $p = 2$, this amounts to the construction of a Coulomb moving frame; for $p \neq 2$; owing to this difficulty, partial regularity results for stationary p-harmonic maps are available only for symmetric targets (see Section 1.1 below).

Our aim is to show that ε-regularity results can be obtained for the general system (1.4) in a relatively simple way *once* it is known that the solution $u \in W^{2,p} \cap BMO$. No extra information on the structure of the nonlinear term $G(x, u, \nabla u)$ is necessary. Inequality (1.3) allows one to control the influence of the critical term.

(The reader should, however, bear in mind that even in the case of the nonconstrained p-Laplace system $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, the $W^{2,p}$ regularity of solutions is not known. Typically, when $G \equiv 0$, one uses the difference quotient technique to obtain some information about second-order derivatives. However, this yields only $|\nabla u|^{(p-2)/2}\nabla u \in W^{1,2}_{\text{loc}}$; see Uhlenbeck's famous paper Uhlenbeck (1977). When $G \not\equiv 0$, the critical growth prevents the use of the difference quotient technique. In that case not even $|\nabla u|^{p/2} \in W^{1,2}_{\text{loc}}$ can be shown.)

Here are the precise statements of these ε-regularity results.

Theorem 1.1. *Assume that $u \in W^{2,p}(\Omega, \mathbb{R}^N) \cap BMO(\Omega, \mathbb{R}^N)$ is a weak solution of* (1.4). *For each $q \in (1, \infty)$ there exists a constant $\varepsilon_0 = \varepsilon_0(n, N, p, q, \Lambda)$ such that if a ball $B_\varrho \subset \Omega$ and $\|u\|_{BMO(B_\varrho)} < \varepsilon_0$, then*

$$\left(\left(\frac{\varrho}{2}\right)^q ⨍_{B_{\varrho/2}} |\nabla u|^q \, dx\right)^{1/q} \leq C(n, N, p, \Lambda)\left(\varrho^p ⨍_{B_\varrho} |\nabla u|^p \, dx\right)^{1/p}. \tag{1.6}$$

Theorem 1.2. *Assume that $u \in W^{2,p}(\Omega, \mathbb{R}^N) \cap BMO(\Omega, \mathbb{R}^N)$ is a weak solution of* (1.4). *There exists a constant $\varepsilon_1 = \varepsilon_1(n, N, p, \Lambda)$ such that if $B_\varrho \subset \Omega$ and*

$$\max\left\{\|u\|_{BMO(B_\varrho)}, \left(\varrho^p ⨍_{B_\varrho} |\nabla u|^p \, dx\right)^{1/p}\right\} < \varepsilon_1,$$

then $|\nabla u| \in L^\infty(B_{\varrho/4})$ *and*

$$\operatorname*{ess\,max}_{B_{\varrho/4}} |\nabla u| \le \frac{C(n, N, p, \Lambda)}{\varrho}\left(\varrho^p \fint_{B_\varrho} |\nabla u|^p\,dx\right)^{1/p}. \tag{1.7}$$

These two theorems are closely related to results of Duzaar and Mingione (2004) who generalize to the p-harmonic case Simon's approach to the regularity theory of harmonic maps and prove the p-harmonic approximation lemma. There are some notable differences between their and our approach. In Duzaar and Mingione (2004, Sections 3 and 4) the authors prove Hölder continuity of a solution u assuming that $u \in W^{1,p}$ has small excess $E(u, B_\varrho) = \varrho^{-n}\int_{B_\varrho} |u - u_{B_\varrho}|^p\,dx$, and that a Caccioppoli (= reverse Poincaré) inequality is satisfied. The latter condition can be derived for *minimizing* p-harmonic maps. We require more, namely the existence of $D^2u \in L^p$, and the smallness condition is also a bit stronger. However, owing to the strength of the inequality (1.3), Theorems 1 and 2 can be applied to *arbitrary*—not just minimizing—p-harmonic maps in $W^{2,p}$.

Let us explain this in some detail below.

1.1. *Partial Regularity of p-Harmonic Maps in $W^{2,p}$*

Let $\mathcal{N}$ be a compact d-dimensional Riemannian manifold isometrically embedded in $\mathbb{R}^N$. Recall that $u \in W^{1,p}(\Omega, \mathcal{N})$ is *weakly p-harmonic* iff u is a critical point of $E_p(u) := \int_\Omega |\nabla u|^p\,dx$ with respect to variations in the range, i.e.,

$$\left.\frac{d}{dt}\right|_{t=0} E_p[\pi \circ (u + t\varphi)] = 0 \quad \text{for each } \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N), \tag{1.8}$$

where π stands for the nearest point projection of a tubular neighborhood of $\mathcal{N}$ onto $\mathcal{N}$. A standard computation (see, e.g., Fuchs, 1994 or Hélein, 2002) shows that (1.8) reads

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) \perp T_u\mathcal{N} \quad \text{in the sense of } \mathcal{D}'(\Omega, \mathbb{R}^N), \tag{1.9}$$

or equivalently

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}A(u)(\nabla u, \nabla u), \tag{1.10}$$

where A denotes the second fundamental form of $\mathcal{N} \subset \mathbb{R}^N$. A p-harmonic map u is said to be *stationary* iff it is a critical point of E_p w.r.t. variations of the domain.

Maps that minimize E_p are regular (i.e., C^∞, when $p = 2$, and $C^{1,\alpha}$, when $p \ne 2$) outside a closed singular set $\operatorname{Sing} u$, which has Hausdorff dimension at most $m - [p] - 1$; see Schoen and Uhlenbeck (1982) for $p = 2$, and Hardt and Lin (1987), Fuchs (1989, 1990), and Luckhaus (1988) for $p \ne 2$ (in this case, Duzaar and Mingione, 2004 offer a new, clear proof). The estimate of the dimension of $\operatorname{Sing} u$ is sharp.

The assumption that u is a local minimum of energy is crucial: there are examples of harmonic maps $u : \mathbb{B}^3 \to S^2$ that are nowhere continuous, see Rivière (1995).

For $p = 2$, *all* weakly harmonic maps into arbitrary compact Riemannian manifolds are smooth. This been proved by Hélein in a series of papers Hélein (1990, 1991a,b) (see also his book Hélein, 2002 and the survey Hélein, 1998), first for $\mathcal{N} = \mathbb{S}^{n-1}$, then for $\mathcal{N}$ being a homogeneous space, and finally for arbitrary compact targets. The proof was based on the duality of Hardy space and BMO. Extending Hélein's methods, Evans (1991) (for round spheres) and then Bethuel (1993) (for arbitrary targets) proved that the $(n-2)$-dimensional Hausdorff measure of the singular set of a stationary harmonic map is equal to zero.

These "harmonic" results have been extended by various authors to the case of $p \neq 2$. If $u \in W^{1,p}(\Omega, \mathcal{N})$ is a stationary weakly p-harmonic map, then $H^{n-p}(\operatorname{Sing} u) = 0$. This result has been proved—but only for symmetric targets $\mathcal{N}$, i.e., round spheres or, more generally, homogeneous spaces—by various authors, see, e.g., Fuchs (1993), Mou and Yang (1996), Strzelecki (1994, 1996), Toro and Wang (1995); Toro and Wang (1995) give the most general version. It remains an open question whether the result holds for general targets.

However, we have the following theorem.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ and let $\mathcal{N}$ be a d-dimensional compact Riemannian manifold isometrically embedded in $\mathbb{R}^N$. Suppose that $2 < p \leq \frac{n}{2}$ and $u \in W^{2,p}(\Omega, \mathcal{N})$ is a p-harmonic map. Then u is locally Lipschitz on an open subset $V \subset \Omega$ such that $\mathcal{H}^{n-2p}(\Omega \backslash V) = 0$. (If $p > n/2$, then $V \equiv \Omega$.)*

Crucial difficulties of the proof are covered by Theorems 1.1 and 1.2; other ingredients are well known to experts. Thus we only give a brief

Sketch of the proof of Theorem 1.3. Recall that by standard Gagliardo–Nirenberg inequalities, $\nabla u \in L^{2p}$ for each $u \in W^{2,p}(\Omega, \mathcal{N})$. Thus by Hölder inequality, the set

$$S_1 = \left\{ x \in \Omega \colon \lim_{r \to 0} r^p \fint_{B(x,r)} |\nabla u|^p \, dy = 0 \right\}$$

contains

$$S_2 = \left\{ x \in \Omega \colon \lim_{r \to 0} r^{2p} \fint_{B(x,r)} |\nabla u|^{2p} \, dy = 0 \right\},$$

and $H^{n-2p}(\Omega \backslash S_2) = 0$ by Frostman's lemma; see Ziemer (1989, Cor. 3.2.3). On some ball around each point in S_1 the map u is locally Lipschitz by Theorem 1.2. Indeed, on some ball around each point in S_1 the scaled energy is small. Since second derivatives of u exist in L^p, one can integrate (1.10) against test vectors $\zeta \frac{\partial u}{\partial x_j}$ to obtain

$$\int_\Omega |\nabla u|^p \operatorname{div} \Phi \, dx = p \sum_{j,k,l} \int_\Omega |\nabla u|^{p-2} \frac{\partial u^j}{\partial x_k} \frac{\partial u^j}{\partial x_l} \frac{\partial \Phi_l}{\partial x_k} \, dx \quad \text{for all } \Phi \in C_0^\infty(\Omega, \mathbb{R}^n).$$

Thus the monotonicity formula is automatically satisfied, $u \in BMO$, and its local BMO norm is controlled by the scaled energy; see, e.g., Evans (1991) (for $p \neq 2$ the computation is identical).

Moreover, one shows in a standard way that S_1 is open, and it is enough to take $V = S_1$. Finally, $H^{n-2p}(\Omega \backslash S_1) \leq H^{n-2p}(\Omega \backslash S_2) = 0$.

The rest of this paper is organized as follows. In Section 2, we derive a rather standard Caccioppoli inequality for the gradient of weak solutions of (1.4). Section 3 contains the proof of the interpolation inequality (1.3) and its local variant, which is tailored suitably for our purposes. Finally, in Section 4 we show how to combine (1.3) and the Caccioppoli inequality to prove Theorems 1.1 and 1.2 via a modified version of Moser's iterative method Moser (1961). (This approach to gradient bounds for systems involving the p-Laplacian was used by Uhlenbeck, 1977, and also by DiBenedetto and Friedman, 1984 in the parabolic context; in our setting, the critical nonlinearity G leads to new difficulties that were absent in those two papers.)

Notation. Barred integrals denote averages, i.e., $\fint_A f\,dx = |A|^{-1}\int_A f\,dx$; sometimes we also write $f_A = \fint_A f\,dx$. $BMO(\mathbb{R}^n)$ stands for the space of functions of bounded mean oscillation; see, e.g., Stein (1993, Chap. 4), with the seminorm

$$\|f\|_{BMO} := \sup_Q \left(\fint_Q |f(y) - f_Q|\,dy\right) < +\infty,$$

the supremum being taken over all cubes Q in $\mathbb{R}^n$. (One can replace the average f_Q by any other constant c_Q; this does not affect the definition.) Primes are used to denote Hölder conjugate exponents, i.e., $p' = p/(p-1)$ for $p \geq 1$, etc. Finally, the letter C stands for a general constant that may change its value even in a single string of estimates. Numbered constants C_i depend only on n, N, p, and the growth constant Λ.

2. Caccioppoli Inequality

In this section we prove the following Caccioppoli inequality for derivatives of solutions of (1.4).

Lemma 2.1. *Assume that $u \in W^{2,p}$ solves* (1.4). *Let $\zeta \in C_0^\infty(\Omega)$ and set $w := |\nabla u|^2$. There exists a constant $C_1 = C_1(n, N, p, \Lambda)$ such that for each $\alpha \geq 0$ we have*

$$\begin{aligned} &\frac{p-2+\alpha}{8}\int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha-2}|\nabla w|^2\,dx + \frac{1}{2}\int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha-1}|\nabla^2 u|^2\,dx \\ &\quad\leq \left(\frac{2(p-1)^2}{p-2+2\alpha} + \frac{1}{2}\right)\int_\Omega |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha}\,dx + C_1(p+\alpha)\int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha+1}\,dx, \end{aligned} \tag{2.1}$$

provided the right hand side is finite.

Proof. Differentiating both sides of (1.4) with respect to x_j, we see that for each test function ϕ_{ij} with compact support,

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla\left(\frac{\partial u^i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}(|\nabla u|^{p-2})\nabla u^i\right]\cdot\nabla\phi_{ij}\,dx = -\int_\Omega \frac{\partial\phi_{ij}}{\partial x_j}G(x, u, \nabla u)\,dx. \tag{2.2}$$

Now insert here $\phi_{ij} = \zeta^2|\nabla u|^{2\alpha}u^i_{x_j}$, where $\alpha \geq 0$, and $\zeta \in C_0^\infty(\Omega)$ is nonnegative, *Left-hand side of* (2.2). A simple but tedious computation leads to the following two equalities (indices i, j are summed):

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla\left(\frac{\partial u^i}{\partial x_j}\right)\cdot\nabla\phi_{ij}\,dx &= \int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha}\left|\nabla\left(\frac{\partial u^i}{\partial x_j}\right)\right|^2 dx + \frac{\alpha}{2}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-1}|\nabla w|^2\,dx\\ &\quad + \int_\Omega \zeta(\nabla\zeta\cdot\nabla w)w^{\frac{p-2}{2}+\alpha}\,dx\\ &=: I_1 + I_2 + I_3; \qquad (2.3)\\ \int_\Omega \frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\right)\nabla u^i\cdot\nabla\phi_{ij}\,dx &= \frac{p-2}{4}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-1}|\nabla w|^2\,dx\\ &\quad + \frac{(p-2)\alpha}{2}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-2}\sum_i\left(\nabla w\cdot\nabla u^i\right)^2 dx\\ &\quad + (p-2)\int_\Omega \zeta w^{\frac{p-2}{2}+\alpha-1}\sum_i(\nabla\zeta\cdot\nabla u^i)(\nabla w\cdot\nabla u^i)\,dx\\ &=: I_4 + I_5 + I_6. \qquad (2.4)\end{aligned}$$

Using the familiar inequality $ab \le \frac{\varepsilon^2 a^2}{2} + \frac{b^2}{2\varepsilon^2}$, we write

$$\begin{aligned}|I_3| + |I_6| &\le (p-1)\int_\Omega \zeta|\nabla\zeta|\,|\nabla w|w^{\frac{p-2}{2}+\alpha}\,dx\\ &\le \frac{(p-1)\varepsilon^2}{2}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-1}|\nabla w|^2\,dx + \frac{p-1}{2\varepsilon^2}\int_\Omega |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha}\,dx. \qquad (2.5)\end{aligned}$$

Choosing ε^2 so that $(p-1)\varepsilon^2/2 = (p-2+2\alpha)/8$, and combining (2.3), (2.4), and (2.5), we obtain finally

$$\begin{aligned}\text{left-hand side of (2.2)} &\ge \frac{p-2+2\alpha}{8}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-1}|\nabla w|^2\,dx + \int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha}|\nabla u^i_{x_j}|^2\,dx\\ &\quad + \frac{(p-2)\alpha}{2}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha-2}\sum_i\left(\nabla w\cdot\nabla u^i\right)^2 dx\\ &\quad - \frac{2(p-1)^2}{p-2+2\alpha}\int_\Omega |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha}\,dx. \qquad (2.6)\end{aligned}$$

Right-hand side of (2.2). Using the growth condition $|G(x,u,\nabla u)| \le \Lambda|\nabla u|^p$, we write

$$\left|\int_\Omega \frac{\partial\phi_{ij}}{\partial x_j}G^i(x,u,\nabla u)\,dx\right| \le C(J_1 + J_2 + J_3), \qquad (2.7)$$

where the constant $C = C(n, N, \Lambda)$ and

$$J_1 = \int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha}\left|\nabla u^i_{x_j}\right|dx, \qquad J_2 = \alpha\int_\Omega \zeta^2 w^{\frac{p-1}{2}+\alpha}\left|\nabla w\right|dx,$$
$$J_3 = \int_\Omega \zeta|\nabla\zeta|w^{\frac{p+1}{2}+\alpha}\,dx. \qquad (2.8)$$

Set

$$J_0 := \int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha+1}\, dx.$$

To absorb all terms that contain second-order derivatives of u, we apply the Cauchy–Schwarz inequality in a familiar way and obtain

$$J_1 \le \frac{\varepsilon_1^2}{2}\int_\Omega \zeta^2 w^{\frac{p-2}{2}+\alpha}\left|\nabla u^i_{x_j}\right|^2 dx + \frac{1}{2\varepsilon_1^2}J_0,$$

$$J_2 \le \frac{\alpha\varepsilon_2^2}{2}\int_\Omega \zeta^2 w^{\frac{p}{2}+\alpha-2}\left|\nabla w\right|^2 dx + \frac{\alpha}{2\varepsilon_2^2}J_0.$$

Finally,

$$J_3 \le \frac{C}{2}J_0 + \frac{1}{2C}\int_\Omega |\nabla\zeta|^2 w^{\frac{p}{2}+\alpha}\, dx. \tag{2.9}$$

Making appropriate choices of $\varepsilon_1, \varepsilon_2 > 0$, we combine the estimates of J_1, J_2, J_3 with (2.6) and easily complete the proof of the lemma. □

3. Interpolation Inequality

The following theorem provides the main new ingredient in our proof of boundedness of the gradient.

Theorem 3.1. *Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be nonnegative, $s \ge 2$. There exists a constant $C = C(n)$, depending only on n, such that*

$$\int \psi^{s+2}|\nabla u|^{s+2}\, dx \le Cs^2\|u\|^2_{BMO}\left\{\int \psi^{s+2}|\nabla u|^{s-2}|\nabla^2 u|^2\, dx + \|\nabla\psi\|^2_{L^\infty}\int \psi^s|\nabla u|^s\, dx\right\} \tag{3.1}$$

for each function $u \in W^{2,1}_{\mathrm{loc}} \cap BMO$ for which the right-hand side is finite.

The case $s = 2$ of this theorem, obtained by Littlewood–Paley theory, was applied by Meyer and Rivière (2003) in the proof of regularity of a certain class of Yang–Mills connections; see Meyer and Rivière (2003). We do not know how to adapt the Fourier analytic proof from Meyer and Rivière (2003) to encompass the current situation, where a power of $|\nabla u|$ plays the role of a weight.

Since the argument below is based on Fefferman's duality theorem, let us recall that the Hardy space $\mathscr{H}^1(\mathbb{R}^n)$ consists precisely of those $g \in L^1(\mathbb{R}^n)$ for which

$$g_* := \sup_{\varepsilon>0}|\varphi_\varepsilon * g| \in L^1(\mathbb{R}^n).$$

Here and below, $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ for a fixed $\varphi \in C_0^\infty(B(0,1))$ such that $\varphi \ge 0$ and $\int \varphi(y)\, dy = 1$. The definition does not depend on the choice of φ (see Fefferman and Stein, 1972); $\mathscr{H}^1(\mathbb{R}^n)$ is a Banach space with the norm $\|g\|_{\mathscr{H}^1} = \|g\|_{L^1} + \|g_*\|_{L^1}$, and its dual space $(\mathscr{H}^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n)$; see Fefferman (1971), Fefferman and Stein (1972), or Stein's monograph (Stein, 1993, Chap. 4).

Proof of Theorem 3.1. The structure of the argument is quite simple and modelled on Strzelecki (2003). First, we apply a trick similar to that used, e.g., by Coifman et al. (1993) to prove that the Jacobian $\det Df$ of $f \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ belongs to the Hardy space. That is, integration by parts and the Sobolev inequality yield estimates below natural exponents. This is enough to conclude that $\operatorname{div}(|\nabla u|^s \nabla u)$ is not just integrable but also belongs to the Hardy space. (A cancellation condition, which was a must in Coifman et al., 1993, is replaced here by the existence of second-order derivatives of u.) Fefferman's duality theorem closes the argument.

Here are the details.

Set $a := s/2$. Integrating by parts, we write

$$\int_{\mathbb{R}^n} \psi^{2a+2} |\nabla u|^{2a+2}\, dx = \left| \int_{\mathbb{R}^n} u \operatorname{div} V\, dx \right|, \tag{3.2}$$

where $V := \psi^{2a+2}|\nabla u|^{2a}\nabla u$. We shall show that $g := \operatorname{div} V \in \mathscr{H}^1(\mathbb{R}^n)$, i.e., that

$$g_* := \sup_{\varepsilon>0} |\varphi_\varepsilon * g| \in L^1(\mathbb{R}^n). \tag{3.3}$$

Fix $x \in \mathbb{R}^n$ and $\varepsilon > 0$. We split $V = V_1 + V_2 + V_3$ and, accordingly, $g = g_1 + g_2 + g_3$, where $g_i = \operatorname{div} V_i$, setting

$$\begin{aligned}
V_1 &= \psi^{a+1}|\nabla u|^{a+1}\left(\psi^{a+1}|\nabla u|^{a-1}\nabla u - \fint_{B_\varepsilon} \psi^{a+1}|\nabla u|^{a-1}\nabla u\, dy\right),\\
V_2 &= \psi^{a+1}|\nabla u|^{a+1}\left(\fint_{B_\varepsilon} \psi^{a+1}|\nabla u|^{a-1}\nabla u\, dy - \psi \fint_{B_\varepsilon} \psi^{a}|\nabla u|^{a-1}\nabla u\, dy\right),\\
V_3 &= \psi^{a+2}|\nabla u|^{a+1} \fint_{B_\varepsilon} \psi^{a}|\nabla u|^{a-1}\nabla u\, dy
\end{aligned}$$

We have $|\varphi_\varepsilon * g| \le \sum_{i=1}^{3} |\varphi_\varepsilon * g_i|$.

Step 1. *Estimates of* $g_{1,*} = \sup_{\varepsilon>0} |\varphi_\varepsilon * g_1|$. Set $f = \psi^{a+1}|\nabla u|^{a+1}$ and $h = \psi^{a+1} |\nabla u|^{a-1}\nabla u$. Integrating by parts, and applying the Hölder inequality with exponents $2n/(n+1)$ and $2n/(n-1)$ and then the Sobolev inequality, we obtain

$$\begin{aligned}
|g_1 * \varphi_\varepsilon| &\le \frac{C(n)}{\varepsilon} \fint_{B_\varepsilon} |f||h - h_{B_\varepsilon}|\, dy\\
&\le \frac{C(n)}{\varepsilon}\left(\fint_{B_\varepsilon} |f|^{2n/(n+1)}\, dy\right)^{(n+1)/2n} \left(\fint_{B_\varepsilon} |h - h_{B_\varepsilon}|^{2n/(n-1)}\, dy\right)^{(n-1)/2n}\\
&\le C(n)\left(\fint_{B_\varepsilon} |f|^{2n/(n+1)}\, dy\right)^{(n+1)/2n} \left(\fint_{B_\varepsilon} |\nabla h|^{2n/(n+1)}\, dy\right)^{(n+1)/2n}.
\end{aligned} \tag{3.4}$$

Thus, upon taking the supremum w.r.t. ε,

$$|g_{1,*}| \le C(n)\big(M|f|^{2n/(n+1)}\big)^{(n+1)/2n}\big(M|\nabla h|^{2n/(n+1)}\big)^{(n+1)/2n}. \tag{3.5}$$

As $2n/(n+1) < 2$, the Hardy–Littlewood maximal theorem now yields

$$\begin{aligned}\|g_{1,*}\|_{L^1} &\le C(n)\|f\|_{L^2}\|\nabla h\|_{L^2}\\ &= C(n)\left(\int \psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}\left(\int \left|\nabla(\psi^{a+1}|\nabla u|^{a-1}\nabla u)\right|^2 dx\right)^{1/2}.\end{aligned} \tag{3.6}$$

Step 2. *Estimates of* $g_{2,*} = \sup_{\varepsilon>0}|\varphi_\varepsilon * g_2|$. By the mean value theorem,

$$\left|\psi \fint_{B_\varepsilon} \psi^a|\nabla u|^{a-1}\nabla u\,dy - \fint_{B_\varepsilon}\psi^{a+1}|\nabla u|^{a-1}\nabla u\,dy\right| \le 2\varepsilon\|\nabla\psi\|_{L^\infty}\fint_{B_\varepsilon}\psi^a|\nabla u|^a\,dy. \tag{3.7}$$

Integrating by parts and applying the above inequality, we obtain

$$|g_2 * \varphi_\varepsilon| \le \frac{C(n)}{\varepsilon}\varepsilon\|\nabla\psi\|_{L^\infty}\fint_{B_\varepsilon}\psi^a|\nabla u|^a\,dy\cdot\fint_{B_\varepsilon}\psi^{a+1}|\nabla u|^{a+1}\,dy,$$

whence

$$|g_{2,*}| \le C(n)\|\nabla\psi\|_{L^\infty}M(\psi^a|\nabla u|^a)M(\psi^{a+1}|\nabla u|^{a+1})$$

and, by the maximal theorem,

$$\|g_{2,*}\|_{L^1} \le C(n)\|\nabla\psi\|_{L^\infty}\left(\int\psi^{2a}|\nabla u|^{2a}\,dx\right)^{1/2}\left(\int\psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}. \tag{3.8}$$

Step 3. *Estimates of* $g_{3,*} = \sup_{\varepsilon>0}|\varphi_\varepsilon * g_3|$.

Here we do not integrate the divergence by parts but rather estimate the convolution directly, writing

$$|\varphi_\varepsilon * g_3(x)| \le C(n)\fint_{B_\varepsilon}\psi^a|\nabla u|^a\,dy\fint_{B_\varepsilon}\left|\nabla\left(\psi^{a+2}|\nabla u|^{a+1}\right)\right|dy.$$

Computing the gradient in the second integral, we obtain

$$\begin{aligned}|g_{3,*}| &\le C(n)(a+2)\|\nabla\psi\|_{L^\infty}M(\psi^a|\nabla u|^a)M(\psi^{a+1}\nabla u|^{a+1})\\ &\quad + C(n)(a+1)M(\psi^a|\nabla u|^a)M(\psi^{a+2}|\nabla u|^a|\nabla^2 u|)\\ &=: F_1 + F_2.\end{aligned} \tag{3.9}$$

The term F_1 is in L^1, since both maximal functions are in L^2. Moreover, the L^1 norm of F_1 can be added to the right-hand side of (3.8). One just has to increase the constant in (3.8) by a factor $\approx (a+2)$. To cope with F_2, note that

$$M(\psi^a|\nabla u|^a) \in L^{(2a+2)/a}, \qquad M(\psi^{a+2}|\nabla u|^a|\nabla^2 u|) \in L^{(2a+2)/(a+2)}. \tag{3.10}$$

This follows easily from the Hölder inequality and the Hardy–Littlewood maximal theorem. For $a \ge 1$ we have $(2a+2)/a \in [2,4]$ and the conjugate $(2a+2)/(a+2) \in [4/3,2]$. Thus the constants in two applications of the maximal theorem needed to obtain (3.10) were uniformly bounded by a factor $C = C(n)$.

Hence

$$
\begin{aligned}
\|F_2\|_{L_1} &\leq C(n)(a+1)\|\psi^a|\nabla u|^a\|_{L^{(2a+2)/a}} \left\|\psi^{a+2}|\nabla u|^a|\nabla^2 u|\right\|_{L^{(2a+2)/(a+2)}} \\
&\leq C(n)(a+1)\|\psi^a|\nabla u|^a\|_{L^{(2a+2)/a}} \left\|\psi|\nabla u|\right\|_{L^{2a+2}} \left\|\psi^{a+1}|\nabla u|^{a-1}|\nabla^2 u|\right\|_{L^2} \\
&= C(n)(a+1)\left(\int \psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}\left(\int \psi^{2a+2}|\nabla u|^{2a-2}|\nabla^2 u|^2\,dx\right)^{1/2}.
\end{aligned}
$$

Adding the estimate of F_1, we obtain finally $\|g_{3,*}\|_{L^1} \leq (a+2)$ times the right-hand side of (3.8)

$$
C(n)(a+1)\left(\int \psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}\left(\int \psi^{2a+2}|\nabla u|^{2a-2}|\nabla^2 u|^2\,dx\right)^{1/2}. \tag{3.11}
$$

It follows from (3.6), (3.8), and (3.11) that $g = \operatorname{div} V \in \mathscr{H}^1(\mathbb{R}^n)$. Thus by Fefferman's duality theorem (3.2) yields

$$
\int_{\mathbb{R}^n} \psi^{2a+2}|\nabla u|^{2a+2}\,dx \leq C(n)\|u\|_{BMO}\|g\|_{\mathscr{H}^1}.
$$

Invoking again (3.6), (3.8), and (3.11) to estimate $\|g\|_{\mathscr{H}^1}$, cancelling the factor

$$
\left(\int \psi^{2a+2}|\nabla u|^{2a+2}\,dx\right)^{1/2}
$$

and squaring the result, we conclude the whole argument. □

4. Gradient Bounds

As we already mentioned in the introduction, the strategy behind the proof of ε-regularity is quite simple: Theorem 3.1 is used to absorb the integral of $|\nabla u|^{p+2+2\alpha}$, which appears in the right-hand side of Caccioppoli inequality (2.1), and then Moser iteration works. There are, however, some technical difficulties. First of all, this absorption trick can be applied only finitely many times. This is because the constant in Theorem 3.1 blows up to ∞ as $p + 2\alpha = s \to \infty$. Hence we perform only finitely many steps of classical Moser iteration, absorbing the unpleasant term with $|\nabla u|^{p+2+2\alpha}$ as long as the BMO norm of u is small enough. This yields $\nabla u \in L^q$ for some large q, with appropriate bounds, and then it turns out that the gain of integrability given by the Sobolev imbedding is more important than the influence of the critical term in the Caccioppoli inequality. A modified version of Moser iteration shows then that $\nabla u \in L^\infty$ and gives an appropriate bound for all solutions with small scaled energy.

One has to control all the constants carefully, so we give all details, dividing the computation into two natural stages.

4.1. *Initial L^q Estimates*

Let C_1 and C_2 denote the constants in, respectively, (2.1) and (3.1). We impose the following smallness condition: Assume that

$$
C_1 C_2 (p+2\alpha)^3 \|u\|^2_{BMO} \leq \frac{1}{2} \tag{4.1}
$$

for all $\alpha \in [0, \alpha_{\max}]$, where $\alpha_{\max} = \alpha_{\max}(n, p)$ is some sufficiently large number that shall be specified later.

Pick two nonnegative functions $\zeta, \psi \in C_0^\infty$ so that $\zeta^2 = \psi^{p+2\alpha+2}$, $\psi \equiv 1$ on a ball B_r, $\psi \equiv 0$ off a larger concentric ball B_R, and $|\nabla\psi| \le 2(R-r)^{-1}$. Next, write down the Caccioppoli inequality (2.1), and apply Theorem 3.1 (setting there $s = p + 2\alpha$) to estimate the nasty term

$$C_1(p+\alpha)\int_\Omega \zeta^2 |\nabla u|^{p+2\alpha+2}\, dx.$$

The smallness condition (4.1) allows us to absorb the resulting integral of $\zeta^2|\nabla u|^{p+2\alpha-2}|\nabla^2 u|^2$ in the second term of the left-hand side of the Caccioppoli inequality. This yields

$$\int_{B_r} w^{\frac{p}{2}+\alpha-2}|\nabla w|^2\, dx \le \frac{C_3(p+2\alpha)^2}{(R-r)^2}\int_{B_R} w^{\frac{p}{2}+\alpha}\, dx. \tag{4.2}$$

As before, $w \equiv |\nabla u|^2$. The constant C_3 depends only on n, N, p, and Λ.

Set $\kappa = n/(n-2)$. From now on we assume that $r < R \le 2r$. Applying the Sobolev inequality,

$$\left(\int_{B_r} |f|^{2\kappa}\, dx\right)^{1/\kappa} \le C(n)\int_{B_r} |\nabla f|^2\, dx + C(n)\, r^{-2}\int_{B_r} f^2\, dx, \tag{4.3}$$

to $f := w^{(p+2\alpha)/4}$, we obtain from (4.2) the reverse Hölder inequality

$$\left(\int_{B_r} |\nabla u|^{(p+2\alpha)\kappa}\, dx\right)^{1/\kappa} \le \frac{C_4(p+2\alpha)^2}{(R-r)^2}\int_{B_R} |\nabla u|^{p+2\alpha}\, dx. \tag{4.4}$$

We shall now iterate this inequality, starting from $\alpha = 0$. We insert here $R = \varrho_j$, $r = \varrho_{j+1}$, $\alpha \equiv \alpha_j$, where the sequences (ϱ_j) and (α_j) are defined by

$$\begin{aligned} \varrho_j &= \varrho\left(\frac{1}{2} + \frac{1}{2^j}\right), \quad j = 1, 2, \ldots, \\ \alpha_j &= \frac{p}{2}(\kappa^{j-1} - 1), \quad j = 1, 2, \ldots \end{aligned} \tag{4.5}$$

We also set $q_j := p + 2\alpha_j$ and

$$I(r, q) = \left(r^q \fint_{B_r} |\nabla u|^q\, dx\right)^q. \tag{4.6}$$

Using this notation, we rewrite (4.4) as

$$\begin{aligned} I(\varrho_{j+1}, q_{j+1}) &\le (C_3(n,p)q_j^2)^{1/q_j}\left(\frac{\varrho_{j+1}}{\varrho_j - \varrho_{j+1}}\right)^{2/q_j} I(\varrho_j, q_j) \\ &\le C_4\big(n, p\big)^{j/\kappa^j} I(\varrho_j, q_j)\,. \end{aligned} \tag{4.7}$$

We iterate (4.7) as long as the smallness condition (4.1) is satisfied, to obtain

$$\left(\left(\frac{\varrho}{2}\right)^q \fint_{B_{\varrho/2}} |\nabla u|^q \, dx\right)^{1/q} \le C_5\left(\varrho^p \fint_{B_\varrho} |\nabla u|^p \, dx\right)^{1/p} \quad \text{for all } q \le q_{\max}. \tag{4.8}$$

One can take $C_5 = C_5(n, N, p, \Lambda)$, given by

$$\log C_5 = \log C_4 \sum_{j=1}^{\infty} \frac{j}{\kappa^j}\,. \tag{4.9}$$

This proves Theorem 1.1; one has only to adjust $\alpha_{\max}$ (and hence ε_0) to obtain q large enough.

For the remaining part of the computations that are necessary to prove Theorem 1.2, it is convenient to choose $q_{\max} = p + 6n$, $\alpha_{\max} = 3n$. Obviously, no matter how $\alpha_{\max}$ is chosen, the smallness condition (4.1) fails to be satisfied for some sufficiently large α unless $u \equiv \text{const}$. Therefore, to obtain uniform L^q gradient bounds for large q (including $q = \infty$), we must change the classical Moser argument slightly. This is done in the next subsection.

4.2. *Boundedness of the Gradient*

We shall show now that if the scaled energy of u is small, i.e.,

$$\left(\varrho^p \fint_{B_\varrho} |\nabla u|^p \, dx\right)^{1/p} < \varepsilon_1(n, p, N, \Lambda), \tag{4.10}$$

in which $\varepsilon_1(n, p, N, \Lambda)$ will be specified later on, then $|\nabla u| \in L^\infty(B_{\varrho/4})$. All numbered constants C_i below depend only on n, N, p, and Λ. We fix a ball B_ϱ for which *both* smallness conditions (4.1) and (4.10) hold.

Let us return to the Caccioppoli inequality (2.1). Dropping the second term in the left-hand side of (2.1), and choosing a standard cutoff function ζ, we obtain

$$\frac{p-2+\beta}{8} \int_{B_r} w^{\frac{p}{2}+\beta-2} |\nabla w|^2 \, dx \;\le\; \frac{C_6}{(R-r)^2} \int_{B_R} w^{\frac{p}{2}+\beta} \, dx + C_6(p+2\beta) \int_{B_R} w^{\frac{p}{2}+\beta+1} \, dx. \tag{4.11}$$

As before, $w = |\nabla u|^2$ and $B_r \subset B_R$ denote two concentric balls. We shall use this inequality only for $\beta \ge 2n$ and for $r < R < 2r$, assuming always that $B_R \subset B_{\varrho/2}$. Combining (4.11) with the Sobolev inequality (4.3), we get

$$\left(\int_{B_r} |\nabla u|^{(p+2\beta)\kappa}\right)^{1/\kappa} \le C_7(p+2\beta)^2 \left\{\frac{1}{(R-r)^2} \int_{B_R} |\nabla u|^{p+2\beta} \, dx + \int_{B_R} |\nabla u|^{p+2\beta+2} \, dx\right\}. \tag{4.12}$$

Note that for β large enough, in particular for all $\beta \ge 2n$, we have $(p+2\beta)\kappa > p + 2\beta + 2$. Thus for such β inequality (4.12) yields indeed higher integrability of ∇u.

Multiplying both sides of (4.12) by $r^{p+2\beta+2-n}$ and applying the Hölder inequality to get two equal integrands in the right-hand side of (4.12), we see easily that

$$I(r,(p+2\beta)\kappa)^{p+2\beta} \le C_8(p+2\beta)^2\left\{\frac{r^2}{(R-r)^2} I(R, p+2\beta+2)^{p+2\beta} + I(R, p+2\beta+2)^{p+2\beta+2}\right\}, \tag{4.13}$$

where $I(\varrho, q)$ is defined by (4.6).

We shall iterate (4.13) for a sequence of nested concentric balls and for a sufficiently chosen sequence of exponents $s = p + 2\beta + 2 \to \infty$. Namely, let $r = r_{j+1}$ and $R = r_j$, where

$$r_j = \frac{\varrho}{2}\left(\frac{1}{2} + \frac{1}{2^j}\right) \quad \text{for } j = 1, 2, \ldots,$$

and set $\beta := \beta_j$, $s_j := p + 2\beta_j + 2$, where the sequence (β_j) is defined recursively by the formulae

$$\beta_1 = 2n \tag{4.14}$$

$$\beta_{j+1} = \frac{p}{2}(\kappa - 1) + \beta_j\kappa - 1, \quad j = 1, 2, \ldots, \tag{4.15}$$

so that for each j we have

$$s_{j+1} = p + 2\beta_{j+1} + 2 = (p + 2\beta_j)\kappa.$$

It is easy to check that

$$\beta_{j+1} = \frac{p}{2}(\kappa^j - 1) + \beta_1\kappa^j - \left(1 + \kappa + \cdots + \kappa^{j-1}\right).$$

Thus

$$(p + 3n)\kappa^j < p + 2\beta_{j+1} < (p + 4n)\kappa^j \quad \text{for all } j. \tag{4.16}$$

Using these observations and (4.13), it is easy to see that *as long as* the threshold condition

$$I(r_j, s_j) \le 1 \tag{4.17}$$

is satisfied, we have

$$\begin{aligned} I(r_{j+1}, s_{j+1}) &\le \left(C_8(p+2\beta_j)^2\right)^{1/(p+2\beta_j)}\left(8^j + 1\right)^{1/(p+2\beta_j)} I(r_j, s_j) \\ &\le C_9^{j/\kappa^j} I(r_j, s_j). \end{aligned} \tag{4.18}$$

Now, for $j = 1$ we have $r_1 = \varrho/2$ and $s_1 = p + 2\beta_1 + 2 < q_{\max} = p + 6n$. Hence, for $j = 1$, (4.17) clearly holds, in light of (4.8) and (4.10); one has only to choose ε_1 not greater than $1/C_5$.

By induction, *if* (4.17) is satisfied for $j = 1, 2, \ldots, k$, then

$$\begin{aligned} I(r_{k+1}, s_{k+1}) &\leq C_9^{\sum_{j=1}^{k} j/\kappa^j} I(r_1, s_1) \\ &\leq C_{10} C_5 \left(\varrho^p \fint_{B_\varrho} |\nabla u|^p \, dx \right)^{1/p}, \end{aligned} \tag{4.19}$$

where $\log C_{10} / \log C_9 = \sum_{j=1}^{\infty} j/\kappa^j$. Thus choosing $\varepsilon_1(n, p, N, \Lambda) = (C_{10} C_5)^{-1} > 0$ in (4.10), we obtain the threshold condition (4.17) for each j. Then, upon passing to the limit $k \to \infty$ in (4.19), we conclude that

$$\operatorname*{ess\,max}_{B_{\varrho/4}} |\nabla u| \leq \frac{C}{\varrho} \left(\varrho^p \fint_{B_\varrho} |\nabla u|^p \, dx \right)^{1/p}. \tag{4.20}$$

This completes the proof of Theorem 1.2.

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